On the existence of three incomolete idempotent MOLS

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ABSTRACT

It is proved in this paper that for any integer $n>53$, there exist 3IMOILS (three incomplete orthogonal idempotent Latin squares) if and only if $v>4n$.

1. INTRODUCTION

A Latin square of order n is an $n\times n$ array such that every row and every column is a permutation of a set $S=\{1,2,\cdots,n\}$. A transversal in a Latin square is a set of cells, one per row and one per column among which the symbols occur precisely one each. A transversal Latin square is a Latin square whose main diagonal is a transversal. An idempotent Latin square is a Latin square whose symbol is i in the cell (i,i) $(1\le i\le n)$. It is easy to see that the existence of a transversal Latin square is equivalent to the existence of an idempotent Latin square.

Let $\mathbb{H}=\{S_1,S_2,\cdots,S_n\}$ be a set of disjoint subsets of a set S. A holey Latin square having hole set $\mathbb{H}$ is an $|S|\times|S|$ array L. indexed bv S, satisfyino the following properties:

(1) every cell of L either contains a symbol of S or is emoty.

(2) every symbol of S occurs at most once in any row or column of L.

(3) the subarrays indexed bv $S_i\times S_i$ are emoty for $1\le i\le n$ (these subarrays are referred to as holes).

(4) A symbol $w\in S$ occurs in row or column t if and only if $(w,t)\in(S\times S)\setminus(\cup_{1\le i\le n}(S_i\times S_i))$.

The order of L is $s=|S|$. If the holes are pairwise disjoint, the holey Latin square is denoted by $ILS(s;s_1,s_2,\cdots,s_n)$, where "I" stands for incomplete and $s_i=|S_i|$ $(1\le i\le n)$. Two holey Latin squares

L and M on same symbol set S and hole set $\mathbb{H}$ are said to be orthogonal if their superposition yields every ordered pair in $(S\times S)\setminus(\cup_{1\le i\le n}(S_i\times S_i))$. We use the notation kIMOLS$(s:s_1,s_2,\cdots,s_n)$ to denote a set of kILS$(s:s_1,s_2,\cdots,s_n)$ where any two of them are orthogonal. If $\mathbb{H}=\phi$, we obtain kMOLS(s). If $|S_i|=1$ $(1\le i\le n)$, we obtain kMOILS(s). If $\mathbb{H}=\{S_1\}$, we simply write kIMOLS(s,s_1). If $|S_i|=1$ $(2\le i\le n)$, we obtain kIMOILS(s,s_1). It is easy to see that the existence of (k+1)IMOLS(s,s_1) implies the existence of kIMOILS(s,s_1). The existence of kMOLS(s) is equivalent to the existence of kIMOLS(s,1), and the existence of kMOILS(s) is equivalent to the existence of kIMOILS(s:1,1).

kIMOLS and kIMOILS have played an important role in the construction of various kinds of combinatorial designs. In [11], Horton started to look at the existence of kIMOLS. Simple counting shows the following.

Theorem 1.1 (1) If there exist kIMOLS(v,n), then $v\ge(k+1)n$.
(2) If there exist kIMOILS(v,n), then $v>(k+1)n$.

For 2IMOLS, the existence has been completely solved by Heinrich and Zhu in [9].

Theorem 1.2 For any integer $n\ge1$, there exist 2IMOLS(v,n) if and only if $v\ge3n$, except $(v,n)=(6,1)$.

For 2IMOILS, the existence also has been completely solved by combining the results of Heinrich and Zhu [10] and Du [8].

Theorem 1.3 For any integer $n\ge1$, there exist 2IMOILS(v,n) if and only if $v>3n$, except $(v,n)=(6,1)$.

For 3IMOLS, the existence was solved by Zhu in [14] when $n\ge154$. Du [6,7] has lowered the bound and listed 109 pairs of (v,n) as possible exceptions. Most recently, Abel, Colbourn, Yin and Zhang in [1] have further reduced the list to 2 possible exceptions, which we state as follows:

Theorem 1.4 For any integer $n\ge1$, there exist 3IMOLS(v,n) if and only if $v\ge4n$, except $(v,n)=(6,1)$ and possibly except for $(v,n)=(10,1)$ or $(52,6)$.

In this paper we consider 3IMOILS, and prove that for any integer $n\ge53$, there exist 3IMOILS(v,n) if and only if $v>4n$.

Theorem 1.5 For any integer $n>53$, there exist 3IMOILS(v,n) if and only if $v>4n$.

For our purpose, we put

$$E=\{2,3,4,6,10\}.$$

2. PRELIMINARIES

We need the following known construction for IMOILS, which is mainly the working corollary of Theorem 1.1 in [3]. So, we state the following lemma without proof.

Lemma 2.1 Suppose there exist 4MOLS(t), 3MOLS(m) and 3MOILS$(m+1)$, 3MOILS(h) and $1\le h\le t$. Then 3IMOILS$(mt+h,t)$ exist.

For the next construction we need the following result.

Lemma 2.2 (1) there exist 3IMOILS$(v,2)$ for $9\le v\le 11$,

(2) there exist 3IMOILS$(v,8)$ for $v=34$ and 38.

Proof (1) For $v=9$, see Zhu [15]. For $v=10$, see Brouwer [2]. And for $v=11$, see Stinson and Zhu [12].

(2) From Wang [13].

The input designs in Lemma 2.2 (1) are required in the next construction which is mainly the working corollary of Lemma 2.2 in [5]. So, we also state the following lemma without proof.

Lemma 2.3 Suppose there exist 8MOLS(t), 3MOILS$(t+q)$, 3MOILS$(s+q)$ and 3MOILS$(u+q)$ ($0\le s,u\le t$ and $q=0$ or 1). Then there exist 3IMOILS$(7t+s+u+2w+q,2w+q)$ for $0\le w\le t$.

To apply the above lemmas we need some input designs, which we state below.

From Colbourn and Dinitz [4] we have

Lemma 2.4 (1) there exist 3IMOILS(v) for any positive integer $v\notin E$.

(2) there exist 4MOLS(v) for any integer $v>42$.

(3) there exist 4IMOLS$(v,8)$ for any integer $v>53$.

From Lemmas 2.2 to 2.4 we then have

Lemma 2.5 (1) Let n,t,h and v be positive integers such that there exist 8MOLS(t), n even, $2\le n\le 2t$, $5\le h\le 2t$ ($h\ne 21$ if $t=11$) and $v=7t+h+n$. Then there exist 3IMOILS(v,n).

(2) Let n,t,h and v be positive integers such that there exist

8MOLS(t) and 3MOILS(t+1), n odd, $3 \le n \le 2t+1$, $10 \le h \le 2t$ and $v=7t+h+n$. Then there exist 3IMOILS(v,n).

Proof (1) Apply Lemma 2.3 with $q=0$, $s+u=h$ and $n=2w$. We observe that if $h \ge 5$, then we can choose s and u such that both 3MOILS(s) and 3MOILS(u) exist.

(2) Apply Lemma 2.3 with $q=1$, $s+u=h$ and $n=2w+1$. We observe that if $h \ge 10$, then we can choose s and u such that both 3MOILS(s+1) and 3MOILS(u+1) exist.

The following easy lemma by filling in holes is useful.

Lemma 2.6 If there exist 3IMOILS(v,u) and 3IMOILS(u,n), then there exist 3IMOILS(v,n).

3. A GENERALIZED CONSTRUCTION

The construction in Theorem 1.1 of Brouwer and van Rees [3] starts with a kMOLS(t). To generalize this, we start with a kIMOILS(t,s). For simplicity we shall not state its most general form, but only the special case to meet the need of this paper. To state these constructions, suppose kIMOILS(v,n) are based on set S and hole H. A set of $|S|-|H|$ cells is called a holey common transversal if it intersects each row and each column not containing the hole H exactly once and contains in each square every symbol from $S \setminus H$ exactly once. Two holey common transversals are disjoint if they have no cells in common.

Theorem 3.1 Suppose there exist 3IMOILS(t,s) with q disjoint holey common transversals missing the holes of size s. Suppose there exist 3MOLS(m) and 3MOILS(m+1) and $1 \le h \le q$. Then there exist 3IMOILS(mt+h,t) if 3IMOILS(ms+h,s) exist.

Proof We begin with the 3IMOILS(t,s), and fill the h disjoint holey common transversals (containing the main diagonal) with 3IMOILS(m+1:1,1)from 3MOILS(m+1), and the others with 3IMOLS(m,1) from 3MOLS(m). We then obtain the required design by filling the size (ms+h) hole with 3IMOILS(ms+h,s) and permuting rows and columns.

We then have

Corollary 3.2 Suppose there exist 4IMOLS(t,s), 3MOLS(m) and

3MOILS(m+1), 3IMOILS(ms+h,s) and $1 \le h \le s$. Then 3IMOILS(mt+h,t) exist.
Proof Since 3IMOILS(t,s) have an extra orthogonal mate, they have s disjoint holey common transversals each of which is determined by a symbol in hole.

Moreover, we have

Theorem 3.3 Suppose there exist 4IMOLS(t,s), 3MOLS(m) and 3MOILS(m+1) and $1 \le u \le s$. Suppose there exist 3IMOILS(ms+u,s) and 3IMOILS(w+u,u) and $0 \le w \le t-s$. Then 3IMOILS(mt+u+w,t) exist.
Proof Since 3IMOILS(t,s) have an extra orthogonal mate, they have s disjoint holey common transversals and (t−s) disjoint common transversals each of which is determined by a symbol in the extra square. We fill the u disjoint holey common transversals (containing the main diagonal) and w disjoint common transversals with 3IMOILS(m+1:1,1) from 3MOILS(m+1) and the others with 3IMOLS(m,1) from 3MOLS(m). We then obtain the required design by filling the size (ms+u) hole with 3IMOILS(ms+u,s) and the size (w+u) hole with 3IMOILS(w+u,u) and permuting rows and columns.

As a application of Theorem 3.1 we have the following example which we will use later in Theorem 4.4.

Example 3.4 There exist 3IMOILS(v,8) for v=35 and 36.

Proof From 7MOLS(8) we can obtain 3IMOILS(8,1) with 4 disjoint holey common transversals missing the holes of size 1. We then apply Theorem 3.1 with m=4 and h=3 or 4 to obtain the required design.

4. THE PROOF OF THEOREM 1.5

In this section we shall prove Theorem 1.5.

Lemma 4.1 There is a sequence of positive integers

$M=(m_i : i=1,2,3,\cdots)=(23,25,27,29,32,37,41,43,49,53,59,\cdots)$

such that

(1) $m_{i+1}-m_i \le 8$,
(2) $7m_{i+1}+4 \le 9m_i$,
(3) $7m_{i+1}+5 \le 4(2m_i+2)+1$,
(4) $7m_{i+1}+10 \le 4(2m_i+3)+1$, and
(5) there exist 8MOLS(m_i) for all $i \ge 1$.

Proof From existing tables on the number of MOLS (see, for example [4]). It is not difficult to check that such a sequence M exists with $m_{i+1}-m_i\le 8$ and there exist 8MOLS(m_i). Since $m_{i+1}-m_i\le 8$, it is easy to see that $7m_{i+1}+4\le 9m_i$ if $m_i\ge 32$, $7m_{i+1}+5\le 4(2m_i+2)+1$ if $m_i\ge 53$, and $7m_{i+1}+10\le 4(2m_i+3)+1$ if $m_i\ge 59$. For the remaining cases, simple calculation shows that we have the result. This proves the lemma.

Theorem 4.2 There exist 3IMOILS(v,n) whenever $v>5n$ and $n\ge 42$.

Proof Apply Lemma 2.5 with $t\in M$. From Lemma 4.1 we have the result.

Theorem 4.3 There exist 3IMOILS(v,n) whenever $v=4n+h$, $n>42$, $1\le h\le n$ and $h\notin E$.

Proof Apply Lemma 2.1 with t=n and m=4, the required conditions come from Lemma 2.4.

Theorem 4.4 There exist 3IMOILS(v,n) whenever $v=4n+h$, $h\in E$ and $n>53$.
Proof For $h\ne 10$, apply Corollary 3.2 with m=4, t=n with $n>53$ and s=8. The required conditions come from Lemmas 2.2 and 2.4 and Example 3.4.

For k=10, apply Theorem 3.3 with m=4, t=n, s=w=8 and u=2 with $n>53$ and s=8. The required conditions 3IMOILS(10,2) and 3IMOILS(34,8) come from Lemma 2.2, others from Lemmas 2.2 and 2.4.

Proof of Theorem 1.5 The conclusion follows immediately from Theorems 4.2 to 4.4.

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