Applications of line geometry, III: The quadric Veronesean and the chords of a twisted cubic

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Abstract

The chords of a twisted cubic in $PG(3,q)$ are mapped via their Plücker coordinates to the points of a Veronese surface lying on the Klein quadric in $PG(5,q)$. This correspondence over a finite field gives a cap in $PG(5,q)$, that is, a set of points no three of which are collinear. The dual structure, namely the axes of the osculating developable, is also mapped to a Veronese surface. The two surfaces can be combined to give a larger cap.

The constructions can be extended to the chords and axes of an arbitrary $(q+1)$-arc in $PG(3,q)$ when q is even. An alternative construction for the cap associated to a twisted cubic is given for q odd.

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1. Introduction and Notation

The chords of a twisted cubic in $PG(3,q)$ are mapped via their Plücker coordinates to the points of a Veronese surface lying on the Klein quadric in $PG(5,q)$. This correspondence over a finite field gives a cap in $PG(5,q)$, that is, a set of points no three of which are collinear. The dual structure, namely the axes of the osculating developable, is also mapped to a Veronese surface. The two surfaces can be combined to give a larger cap.

The constructions can be extended to the chords and axes of an arbitrary $(q+1)$-arc in $PG(3,q)$ when q is even. An alternative construction for the cap associated to a twisted cubic is given for q odd.

The following notation is used:

γ is the Galois field $GF(q)$ of order $q=p^h$, $h \geq 1$;
γ^+ is $\gamma \cup \{\infty\}$;
γ' is a quadratic extension of γ;
$\bar{\gamma}$ is the algebraic closure of γ;
$PG(n,q)$ is the projective space of n dimensions over γ;
$\mathbf{P}(X)$ is the point of $PG(3,q)$ with coordinate vector $X=(x_0,x_1,x_2,x_3)$;
$\pi(U)$ is the plane of $PG(3,q)$ with equation $UX^t=0$, where $U=(u_0,u_1,u_2,u_3)$;
$l=I(L)=\mathbf{P}(X)\mathbf{P}(Y)$ is the line of $PG(3,q)$ with coordinate vector
$L=(l_{01},l_{02},l_{03},l_{12},l_{31},l_{23})$, where $l_{ij}=x_iy_j-x_jy_i$;
$\mathbf{U}_0,\mathbf{U}_1,\mathbf{U}_2,\mathbf{U}_3$ are the vertices of the tetrahedron of reference in $PG(3,q)$;
$\mathbf{U}$ is the unit point;
$\mathcal{H}^5$ is the Klein quadric of $PG(5,q)$ with equation $X_0X_5+X_1X_4+X_2X_3=0$;
$\mathcal{V}_2^4$ is the Veronese surface of $PG(5,q)$.

2. Preliminaries

Consider a twisted cubic of $PG(3,q)$ in its canonical form:

$$\mathcal{C}=\{P(t)=\mathbf{P}(t^3,t^2,t,1) : t\in\gamma^+\},$$

where $t=\infty$ gives the point $\mathbf{U}_0$. A *chord* of $\mathcal{C}$ is a line of $PG(3,q)$ joining either a pair of real points of $\mathcal{C}$, possibly coincident, or a pair of complex conjugate points of $\mathcal{C}$. By a *real point* of $\mathcal{C}$ we mean a point of $\mathcal{C}$ defined over γ, and by *complex conjugate points* of $\mathcal{C}$, we mean points $P(t_1)$ and $P(t_2)$, such that t_1 and t_2 are in γ' conjugate over γ. Let $l(t_1,t_2)=P(t_1)P(t_2)$. Then

$$\begin{aligned} l(t_1,t_2) &= I({t_1}^2{t_2}^2, t_1t_2(t_1+t_2), {t_1}^2+t_1t_2+{t_2}^2, t_1t_2, -(t_1+t_2), 1) \\ &= I({\alpha_2}^2, \alpha_1\alpha_2, {\alpha_1}^2-\alpha_2, \alpha_2, -\alpha_1, 1) \end{aligned}$$

where $\alpha_1=t_1+t_2$ and $\alpha_2=t_1t_2$.

The criteria for the three types of chords are that the polynomial $x^2-\alpha_1x+\alpha_2$ has 2, 1 or 0 roots in γ. If $x^2-\alpha_1x+\alpha_2$ has 2 roots in γ, that is, $P(t_1)$ and

$P(t_2)$ are distinct real points of $\mathcal{C}$, then we will say that $l(t_1, t_2)$ is a *real chord* of $\mathcal{C}$; if $x^2 - \alpha_1 x + \alpha_2$ has one root in γ, that is, $P(t_1)$ and $P(t_2)$ are coincident, then $l(t_1, t_2)$ is a *tangent* to $\mathcal{C}$; if $x^2 - \alpha_1 x + \alpha_2$ has no roots in γ, namely $P(t_1)$ and $P(t_2)$ are complex conjugate points of $\mathcal{C}$, then $l(t_1, t_2)$ is an *imaginary chord* of $\mathcal{C}$. Note that imaginary chords are defined over γ. If $t_1 = t_2 = t$, then

$$l(t) = l(t,t) = I(t^4, 2t^3, 3t^2, t^2, -2t, 1)$$

is the tangent to $\mathcal{C}$ at the point $P(t)$.

At each point $P(t)$ of $\mathcal{C}$, there is an *osculating plane*

$$\pi(t) = \pi(1, -3t, 3t^2, -t^3),$$

which meets $\mathcal{C}$ only in $P(t)$. Such osculating planes form the *osculating developable* Γ to $\mathcal{C}$. In particular Γ is the dual of $\mathcal{C}$. For $p \neq 3$, dual to the chords of $\mathcal{C}$ are the *axes* of Γ. An *axis* of Γ is a line of $PG(3,q)$, which is the intersection of a pair of real planes of Γ, possibly coincident, or of a pair of complex conjugate planes of Γ (also called (2−)complex conjugate planes).

Let $l'(v_1, v_2) = \pi(v_1) \cap \pi(v_2)$. Then

$$\begin{aligned} l'(v_1, v_2) &= I({v_1}^2{v_2}^2, v_1v_2(v_1+v_2), 3v_1v_2, ({v_1}^2 + v_1v_2 + {v_2}^2)/3, -(v_1+v_2), 1) \\ &= I({\beta_2}^2, \beta_1\beta_2, 3\beta_2, ({\beta_1}^2 - \beta_2)/3, -\beta_1, 1), \end{aligned}$$

where $\beta_1 = v_1 + v_2$ and $\beta_2 = v_1 v_2$. We will call $l'(v_1, v_2)$ a *real axis*, a *generator* or an *imaginary axis* of Γ, according as $x^2 - \beta_1 x + \beta_2$ has two, one or zero roots in γ. Note that the generator of Γ in $\pi(t)$ is

$$l'(v,v) = l(t,t) = I(t^4, 2t^3, 3t^2, t^2, -2t, 1) = l(t);$$

that is, a generator of Γ is self−dual with respect to the null polarity defined by the linear complex $\mathcal{A}$ to which the tangents to $\mathcal{C}$ belong (see [5, Theorem 21.1.2]).

If $p \neq 3$, from [5, Lemma 21.1.4] we have that

$|\mathcal{K}_1| = q(q+1)/2$, where $\mathcal{K}_1$ is the set of all real chords of $\mathcal{C}$;
$|\mathcal{K}_2| = q+1$, where $\mathcal{K}_2$ is the set of all tangents to $\mathcal{C}$;
$|\mathcal{K}_3| = q(q-1)/2$, where $\mathcal{K}_3$ is the set of all imaginary chords of $\mathcal{C}$.

So the total number of chords of $\mathcal{C}$ is $q^2 + q + 1$. Dually, the total number of axes of Γ is $q^2 + q + 1$.

3. The construction of the Veronese surface of $PG(5, q)$ from the chords of $\mathcal{C}$

3.1. Construction I

Suppose that $p \neq 3$ and consider the generic chord of $\mathcal{C}$:

$$l(t_1, t_2) = I({\alpha_2}^2, \alpha_1\alpha_2, {\alpha_1}^2 - \alpha_2, \alpha_2, -\alpha_1, 1), \tag{1}$$

where, as above, $\alpha_1 = t_1 + t_2$ and $\alpha_2 = t_1 t_2$. We set $\alpha_1 = v/w$ and $\alpha_2 = u/w$. By substituting in (1), we obtain

$$l(t_1, t_2) = I(u^2/w^2, uv/w^2, v^2/w^2 - u/w, u/w, -v/w, 1)$$

or equivalently,

$$l(t_1, t_2) = l(u, v, w) = I(u^2, uv, v^2 - uw, uw, -vw, w^2) \tag{2}$$

for $u, v, w \in \gamma$. Now, the Veronese surface of $PG(5, q)$ has parametric equations (see [6, Ch. 25])

$$X_0 = u^2, \quad X_1 = uv, \quad X_2 = v^2, \quad X_3 = uw, \quad X_4 = vw, \quad X_5 = w^2$$

for all $u, v, w \in \gamma$. Also the Veronese surface is embedded in $\mathcal{H}^5$ by the linear map

$$\begin{aligned} \Phi_- :& (X_0, \dots, X_5) \mapsto (X_0, X_1, X_2 - X_3, X_3, -X_4, X_5) \\ & (u^2, uv, v^2, uw, vw, w^2) \mapsto (u^2, uv, v^2 - uw, uw, -vw, w^2). \end{aligned}$$

Hence, the Plücker coordinates of a chord of $\mathcal{C}$, considered as homogeneous projective coordinates of $PG(5, q)$, represent a point of a Veronese surface $\mathcal{V}_2^4$, embedded in $\mathcal{H}^5$.

Dually, the generic axis of Γ is

$$l'(v_1, v_2) = I({\beta_2}^2, \beta_1\beta_2, 3\beta_2, ({\beta_1}^2 - \beta_2)/3, -\beta_1, 1). \tag{3}$$

Set $\beta_2 = u/w$ and $\beta_1 = v/w$. By substituting in (3), we obtain that

$$l'(u, v, w) = I(u^2, uv, 3uw, (v^2 - uw)/3, -vw, w^2). \tag{4}$$

The Plücker coordinates of an axis of Γ, considered as projective coordinates of $PG(5, q)$, represent a point of a Veronese surface $\overline{\mathcal{V}_2^4}$ embedded in $\mathcal{H}^5$. This surface $\overline{\mathcal{V}_2^4}$, when embedded in $\mathcal{H}^5$, is the image of $\mathcal{V}_2^4$ under the linear transformation

$$(X_0, \dots, X_5) \mapsto (X_0, X_1, 3X_3, X_2/3, X_4, X_5).$$

Since $p \neq 3$, the two surfaces $\mathcal{V}_2^4$ and $\overline{\mathcal{V}_2^4}$ are distinct.

3.2. Construction II

Now, we present another construction involving the Veronese surface of $PG(5,q)$, q odd, and the chords of $\mathcal{C}$. Consider the embedding

$$\Phi_+ : (u^2, uv, v^2, uw, vw, w^2) \mapsto (u^2, uv, v^2 - uw, uw, vw, w^2).$$

Then $Z = (u^2, uv, v^2 - uw, uw, vw, w^2)$, for $u, v, w \in \gamma$, is a point of a Veronese surface $\mathcal{V}_2^4$ embedded in $\overline{\mathcal{H}^5} : X_0X_5 - X_1X_4 + X_2X_3 = 0$. In the open set $u = 1$ of $PG(5, \bar{\gamma})$, the tangent plane to $\mathcal{V}_2^4$ at Z is

$$T_Z(\mathcal{V}_2^4) = span\left(Z, \frac{\partial Z}{\partial v}, \frac{\partial Z}{\partial w}\right),$$

where $Z = (1, v, v^2 - w, w, vw, w^2)$. It follows that

$$\frac{\partial Z}{\partial v} = (0, 1, 2v, 0, w, 0),$$
$$\frac{\partial Z}{\partial w} = (0, 0, -1, 1, v, 2w).$$

The generic tangent line to $\mathcal{V}_2^4$ at Z is of the form $P(Z)P(V)$, where $V = \lambda\frac{\partial Z}{\partial v} + \mu\frac{\partial Z}{\partial w}$, with $\lambda, \mu \in \gamma$, $(\lambda, \mu) \neq (0, 0)$ and $VQV^t = 0$, where

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the symmetric matrix associated to $\overline{\mathcal{H}^5}$. It follows that

$$V = (0, \lambda, 2v\lambda - \mu, \mu, \lambda w + \mu v, 2\mu w)$$

with $\lambda(\lambda w + \mu v) = \mu(2\lambda v - \mu)$; that is,

$$\lambda^2 w - \lambda\mu v + \mu^2 = 0. \tag{5}$$

Since the discriminant of this quadratic form is not zero, there are two tangent lines (real or complex conjugate) at the generic point of $\mathcal{V}_2^4$. To these tangent lines through Z correspond two pencils in $PG(3,q)$ each containing the line z corresponding to Z and another line in the neighbourhood of z. By putting $\lambda = 1$ in (5) and by using the Plücker embedding, we find that the points $\mathbf{P}(Z)$ and $\mathbf{P}(V)$ are spanned by the rows of

$$\begin{pmatrix} 1 & 0 & -w & -vw \\ 0 & 1 & v & v^2 - w \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \mu & 0 & -2\mu w \\ 0 & 0 & 1 & 2v - \mu \end{pmatrix}.$$

These two lines of $PG(3,q)$ intersect in a point

$$R = \mathbf{P}\,(1, \quad \mu, \quad \mu v - w, \quad (v^2 - w)\mu - vw\,).$$

Using the relation (5), we find that

$$R = \mathbf{P}\,(1, \quad \mu, \quad \mu^2, \quad \mu^3\,)$$

with $\mu \in \gamma$. This means that $\mathcal{V}_2^4$ represents the congruence of the chords of a twisted cubic $\mathcal{C}$, where $\mathcal{C}$ is the set of the focal points of such a congruence; see [7].

4. A new family of caps

In [3], it has been shown that by starting from the set of all conics of $PG(2,q)$, q odd, which are inscribed in a triangle, it is possible to construct a cap of $PG(5,q)$ of size $2q^2 - q + 2$. Such a cap turns out to be the union of two Veronese surfaces of $PG(5,q)$ which meet in the union of three conics pairwise intersecting in one point. Now, we have seen that by starting from the chords and axes of a twisted cubic $\mathcal{C}$, it is possible to construct two Veronese surfaces of $PG(5,q)$, which we have called $\mathcal{V}_2^4$ and $\overline{\mathcal{V}_2^4}$.

Our aim in this section is to construct a new family of caps of $PG(5,q)$, $q = p^h$, $p \neq 3$, embedded in the Klein quadric $\mathcal{H}^5$, by glueing $\mathcal{V}_2^4$ and $\overline{\mathcal{V}_2^4}$ together along their intersection.

Lemma 4.1. *The Veronese surfaces $\mathcal{V}_2^4$ and $\overline{\mathcal{V}_2^4}$ meet in $q+1$ points, belonging to a parabolic quadric $\mathcal{P}_4$, namely, the intersection of $\mathcal{H}^5$ by a non–tangent prime.*

Proof. The surfaces $\mathcal{V}_2^4$ and $\overline{\mathcal{V}_2^4}$ meet in $q+1$ points, since the tangents to $\mathcal{C}$ are self–dual with respect to the null polarity defined by $\mathcal{A}$. Since, for $p \neq 3$, the tangents to $\mathcal{C}$ lie in a general linear complex (see [5, Th. 21.1.2 (ii)]), their images under the Plücker embedding are $q+1$ points on a parabolic quadric $\mathcal{P}_4$, obtained by cutting $\mathcal{H}^5$ by a non–tangent prime. □

Proposition 4.2. *The set $K = \mathcal{V}_2^4 \cup \overline{\mathcal{V}_2^4}$ is a $(2q^2+q+1)$–cap embedded in $\mathcal{H}^5$.*

Proof. From Lemma 4.1, $|K| = 2q^2 + q + 1$. By way of contradiction, suppose that there exist three collinear points P_1,P_2,P_3 on K and let l be the line containing them. By Bezout's theorem, l is contained in $\mathcal{H}^5$. The three points above cannot belong to $\mathcal{V}_2^4$ or $\overline{\mathcal{V}_2^4}$, since both $\mathcal{V}_2^4$ and $\overline{\mathcal{V}_2^4}$ are caps [6, Lemma 25.2.5]. By virtue of the Plücker embedding, the collinearity of the points P_1, P_2, P_3 on $\mathcal{H}^5$ means that the corresponding lines in $PG(3,q)$ belong to the same pencil. So, in our setting, there would be two chords of $\mathcal{C}$ and one axis of Γ or two axes of Γ and one chord of $\mathcal{C}$ belonging to the same pencil.

Suppose that we have two concurrent real chords of $\mathcal{C}$, necessarily meeting in one point P of $\mathcal{C}$ since no plane contains four points of a twisted cubic. Since $\mathcal{C}$

is fixed by a 3-transitive group [5, Lemma 21.1.3], let $P = \mathbf{U}_3$. Denote by π the real plane containing these two chords. If there was an axis l of Γ through P, this would be a generator of Γ, which is excluded by the first part of the proof. This means that l is a chord; this contradiction completes the proof. □

Remarks

(a) The chords of $\mathcal{C}$ form a $(1,3)-$congruence, namely a line congruence of order 1 (the number of chords through a general point of $PG(3,q)$) and class 3 (the number of chords of $\mathcal{C}$ in a general plane). Dually, the axes of Γ form a $(3,1)-$congruence. For more details, see [1, p. 49].

From Proposition 4.2, it follows that the chords and axes through a point are contained in at least one quadratic cone and the chords and axes in a plane are contained in at least one dual conic.

(b) Some tests performed for low values of q show that the caps constructed above are far from being complete.

Corollary 4.3. *The $(2q^2+q+1)-$cap K has a collineation group isomorphic to $PGL(2,q) \rtimes C_2$, namely, the semidirect product of $PGL(2,q)$ by a cyclic group of order two.*

Proof. The collineation group of $\mathcal{C}$ is $PGL(2,q)$; it acts $3-$transitively on the points of $\mathcal{C}$ [5, p. 234], and partitions the chords of $\mathcal{C}$ into three orbits [5, Lemma 21.1.4], namely the real chords, the tangents and the imaginary chords. So $PGL(2,q)$ leaves $\mathcal{V}_2^4$ invariant. The null polarity defined by the general linear complex containing the tangents to $\mathcal{C}$ induces an involutory collineation, which interchanges chords and axes [5, Th. 21.1.2]. It follows that $PGL(2,q)$ also leaves $\overline{\mathcal{V}_2^4}$ invariant. □

5. A plane representation of the chords of $\mathcal{C}$ for q odd

Again, consider a twisted cubic $\mathcal{C}$ of $PG(3,q)$ in its canonical form:

$$x_0 = t^3, \quad x_1 = t^2, \quad x_2 = t, \quad x_3 = 1,$$

$t \in \gamma^+$. We recall that $\mathcal{C}$ is the complete intersection of three quadrics $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$ of $PG(3,q)$, $q \geq 7$ [5, Lemma 21.1.6 (i)], where

$$Y_1 = x_0x_2 - {x_1}^2, \quad Y_2 = x_0x_3 - x_1x_2, \quad Y_3 = x_1x_3 - {x_2}^2.$$

Associate to a point $\mathbf{P}(X)$ of $PG(3,q)$, $X = (x_0, x_1, x_2, x_3)$, the point with homogeneous coordinates (Y_1, Y_2, Y_3) of a projective plane π isomorphic to $PG(2,q)$. Hence a point $(A_1, A_2, A_3) \in \pi$ corresponds to those points of $PG(3,q)$ such that

$$(Y_1, Y_2, Y_3) = (A_1, A_2, A_3). \tag{6}$$

But, we observe that the quadrics $Y_i = 0$ are linearly independent and contain $\mathcal{C}$. This means that the points (6) are on the intersection of two quadrics through

$\mathcal{C}$, say $Y_1 = 0$ and $Y_3 = 0$. This intersection consists of the twisted cubic $\mathcal{C}$ and residually of one chord of $\mathcal{C}$, say l. In particular, if l is the chord of $\mathcal{C}$ joining the points $P(t_1)$ and $P(t_2)$, associate to l the point of π given by

$$Y_1 = t_1 t_2, \quad Y_2 = t_1 + t_2, \quad Y_3 = 1.$$

In this way, we obtain a one-to-one correspondence between the chords of $\mathcal{C}$ and the points of π; call this correspondence Ψ. In particular, for the tangents to $\mathcal{C}$, we have $t_1 = t_2$ and so their images under Ψ are the points of the conic ω of π given parametrically as

$$(Y_1, Y_2, Y_3) = (t^2, 2t, 1)$$

and with equation

$$Y_2{}^2 - 4Y_1Y_3 = 0.$$

It follows that

(i) the images in π of the real and imaginary chords of $\mathcal{C}$ are respectively the external and internal points of ω;

(ii) the image of a regulus of chords is a line in π;

(iii) $\Psi^{-1}(\omega)$ is the quartic surface Ω containing the points on the tangents of $\mathcal{C}$ [5, Lemma 21.1.10], where

$$\Omega : \; (x_0x_3 - x_1x_2)^2 - 4(x_0x_2 - x_1{}^2)(x_1x_3 - x_2{}^2) = 0;$$

(iv) the chords of $\mathcal{C}$ can be partitioned into $q+1$ reguli sharing one chord;

(v) if the axes of Γ are represented as the lines of π, then the null polarity defined by the linear complex containing the tangents to $\mathcal{C}$ corresponds to the polarity induced by the conic ω in π.

A collection of q^2+q+1 nondegenerate conics in a projective plane $PG(2,q)$ that mutually intersect in exactly one point is called a *projective bundle* [2]. So these conics can be considered as the lines of another projective plane. In particular, a *circumscribed bundle* is a set $\mathcal{B}$ of q^2+q+1 nondegenerate conics containing the three vertices of a triangle defined over a cubic extension of γ. There is a connection between the set of chords of a twisted cubic and a projective circumscribed bundle of a projective plane as is shown as follows.

Let $\mathcal{Q}=\{\mathcal{F}_{\lambda,\mu,\nu} | \lambda, \mu, \nu \in \gamma\}$ be the net of quadrics through the twisted cubic $\mathcal{C}$ and let π_0 be a plane meeting $\mathcal{C}$ in three conjugate points Q_1,Q_2,Q_3; that is, the parameters of the three points are conjugate over γ in a cubic extension. Let $\mathcal{T}$ be the set of chords of $\mathcal{C}$. The plane π_0 meets a quadric $\mathcal{F}_{\lambda\mu\nu}$ in a nondegenerate conic through Q_1,Q_2,Q_3. Let $\mathcal{N}$ be the net of conics $\pi_0 \cap \mathcal{F}_{\lambda\mu\nu}$. Then any two conics in $\mathcal{N}$ meet residually in a real point P. This gives a mapping from $\mathcal{N}$ to π_0. A chord of $\mathcal{C}$ maps to a point of π_0 simply as the intersection of the line with the plane. Hence we have a map

$$\phi : \mathcal{T} \to \mathcal{N}.$$

In fact, there is a natural bijection between any two of the four sets $\mathcal{Q}$, $\mathcal{N}$, $\mathcal{T}$, π_0, all of which have size q^2+q+1 as is shown in the following diagram:

$$\begin{array}{ccc} \mathcal{Q} & \leftrightarrow & \mathcal{N} \\ \updownarrow & \boxtimes & \updownarrow \\ \mathcal{T} & \leftrightarrow & \pi_0 \end{array}$$

6. The chords of $(q+1)$-arcs in $PG(3,q)$, q even

We now consider the analogous properties for the chords of an arbitrary $(q+1)$-arc in $PG(3,q)$, q even.

In $PG(3,q)$, $q=2^h$, a $(q+1)$-arc is projectively equivalent to a set

$$\mathcal{C}(m) = \{P(t) = \mathbf{P}(t^{m+1}, t^m, t, 1) : t \in \gamma^+\},$$

where $m = 2^n$, $(n,h) = 1$ [5, Th. 21.3.15].

The osculating developable of $\mathcal{C}(m)$ is $\Gamma(m) = \{\pi(t) = \pi(1, t, t^m, t^{m+1}) : t \in \gamma^+\}$.

The chord $P(r)P(s)$ has Plücker coordinates

$$l(r,s) = I(r^m s^m, \frac{rs(r^m+s^m)}{r+s}, \frac{r^{m+1}+s^{m+1}}{r+s}, \frac{rs(r^{m-1}+s^{m-1})}{r+s}, \frac{r^m+s^m}{r+s}, 1).$$

The axis $\pi(u)\cap\pi(v)$ is

$$l'(u,v) = I(u^m v^m, \frac{uv(u^m+v^m)}{u+v}, \frac{uv(u^{m-1}+v^{m-1})}{u+v}, \frac{u^{m+1}+v^{m+1}}{u+v}, \frac{u^m+v^m}{u+v}, 1).$$

The tangent at $P(t)$ is $l(t) = l(t,t) = I(t^{2m}, 0, t^m, t^m, 0, 1)$, and this coincides with the generator of $\Gamma(m)$ in $\pi(t)$.

The tangents form a regulus lying on $\mathcal{H}^3 : X_0X_3 + X_1X_2 = 0$ whose corresponding null polarity $\mathcal{U}$ is $\mathbf{P}(a_0,a_1,a_2,a_3) \longleftrightarrow \pi(a_3,a_2,a_1,a_0)$. So $\mathcal{U}$ interchanges $\mathcal{C}(m)$ and $\Gamma(m)$.

Theorem 6.1. *The chords and axes of the $(q+1)$-arc $\mathcal{C}(m)$ and its osculating developable $\Gamma(m)$ form a $(2q^2+q+1)$-cap on $\mathcal{H}^5$.*

Proof. The arguments of Proposition 4.2 can be copied if $(n,2h)=1$ since then $\mathcal{C}(m)$, $m=2^n$, defines a $(q+1)$-arc in $PG(3,q)$ and a (q^2+1)-arc in $PG(3,q^2)$. Hence, we only consider the case $(n,2h)=2$; that is, h is odd and n is even.

When $(n,2h)=2$, then $\mathcal{C}(m)$ does not define an arc in $PG(3,q^2)$. We first determine the maximum number of points of $\mathcal{C}(m)$, extended to γ', in a plane of $PG(3,q)$.

Let $\pi = \pi(a_0,a_1,a_2,a_3)$. This intersects $\mathcal{C}(m)$ where $a_0t^{m+1}+a_1t^m+a_2t+a_3 = 0$.

By letting $t = t'+\alpha$, with α a solution of this equation in γ', we can reduce this equation to one with $a_3=0$. So it suffices to study an equation $a_0t^{m+1}+a_1t^m+a_2t=0$.

If $a_0 = 0$, then $a_1t^m + a_2t = 0$ if and only if $t = 0$ or $a_1t^{m-1} + a_2 = 0$. The latter equation has at most three solutions in γ' since $(m-1, 2^{2h}-1) = 3$.

So there are at most 4 distinct solutions; together with the solution $t = \infty$, this gives at most 5 solutions in γ'.

When $a_0 \neq 0$, by introducing homogeneous coordinates (t, l) such that $(t, 1) \equiv t$, by making the equation homogeneous to $a_0t^{m+1} + a_1t^ml + a_2tl^m = 0$ and by interchanging t and l, an equation $a_0l^{m+1} + a_1tl^m + a_2t^ml = 0$ is obtained. Letting $l = 1$, the equation $a_0 + a_1t + a_2t^m = 0$ is obtained. So, again, in γ', there are at most 5 solutions.

We now check that no three chords lie in a pencil.

If a plane of $PG(3, q)$ contains exactly 5 points in $PG(3, q^2)$, at least one of them is defined over γ. If three of them are defined over γ, by the 3-transitivity of the group of $\mathcal{C}(m)$ [5, p. 249], we can assume that this plane is $X_1 = X_2$. This does contain the two points $\mathbf{P}(\omega^2, \omega, \omega, 1), \mathbf{P}(\omega, \omega^2, \omega^2, 1)$, where $\omega^2 + \omega + 1 = 0$, in $PG(3, q^2)$ but no three of the real chords and the imaginary chord are concurrent.

If the plane contains one point of $PG(3, q)$, and the tangent to $\mathcal{C}(m)$ at that point, then the plane contains at most one extra point of $\mathcal{C}(m)$. For, we can assume that this point is $\mathbf{U}_3$, and the planes through the tangent line $X_0 = X_1 = 0$ to this point contain at most one other point of $\mathcal{C}(m)$. If the plane contains one point of $PG(3, q)$, and two pairs of complex conjugate points in $PG(3, q^2)$, then the plane only contains two imaginary chords.

If the plane contains exactly four points of $\mathcal{C}(m)$ in $PG(3, q^2)$, then if these four points consist of two pairs of complex conjugate points, there are only two imaginary chords in the plane, and similarly, if the plane contains two real points, and two conjugate imaginary points, there are again only two chords in the plane.

If a plane contains exactly three points, one of them is real. If all three are real, they form a 3-cap, and if only one is real, then this plane only contains an imaginary chord.

This shows that no plane contains three concurrent chords. So the chords form a $(q^2 + q + 1)$-cap.

From the null polarity $\mathcal{U}$, also the axes define a $(q^2 + q + 1)$-cap on $\mathcal{H}^5$.

Consider now the chords and the axes. Suppose they do not define a $(2q^2+q+1)$-cap. Assume that three points are collinear where two correspond to chords of $\mathcal{C}(m)$.

If the two chords are real chords, suppose they are $\mathbf{U}_0\mathbf{U}_3$ and $\mathbf{U}_3\mathbf{U}$. The third point then must correspond to an axis passing through $\mathbf{U}_3$. The only axis passing through this point is its tangent to $\mathcal{C}(m)$, but this does not lie in a pencil with two real chords.

If the two chords are one real chord and a tangent, let the tangent be $X_0 = X_1 = 0$ and the chord be $\mathbf{U}_0\mathbf{U}_3$. Again, there is no other chord or axis passing through $\mathbf{U}_3$ and lying in the plane of the real chord and tangent.

Assume the two chords are one real chord and an imaginary chord. By [4, Th. 5], an imaginary chord and an imaginary axis never intersect. So the third line is a real axis.

Since an imaginary bisecant cannot pass through a real point of $\mathcal{C}(m)$, assume that the real chord is $\mathbf{U}_0\mathbf{U}_3$ and that the real axis lies in the plane $X_0 + X_1 + X_2 + X_3 = 0$. Then the vertex of the pencil is $\mathbf{P}(1,0,0,1)$. The planes of the osculating developable through $\mathbf{P}(1,0,0,1)$ satisfy $t^{m+1} = 1$, and there must be at least a second solution since there is a real axis passing through $\mathbf{P}(1,0,0,1)$. So $(m+1, 2^h - 1) > 1$; hence $(m+1, 2^h - 1) \geq 5$, since h is odd. This would imply that there are at least 5 planes through $\mathbf{P}(1,0,0,1)$. This is false.

When the two chords are imaginary chords, by [4, Th. 5], the third line must be a real axis. Suppose it is the intersection of $X_0 = 0$ and $X_3 = 0$. Then the two imaginary chords intersect in the same point of $X_0 = X_3 = 0$; assume that this point is $\mathbf{P}(0,1,1,0)$.

Then, three points $\mathbf{P}(0,1,1,0), P(t_1), P(t_2)$, with $t_2 = t_1^q$ and $t_1 \in \gamma' \backslash \gamma$, are collinear if and only if $(t_1 + t_2)^{m-1} = 1$ and $t_1^{m+1} = t_2^{m+1}$. Since $(n, 2h) = 2$, necessarily $t_1 + t_2 \in \{1, \omega, \omega^2\}$. Hence $t_1 + t_2 = 1$, since $\omega \notin \gamma$, which shows that $t_1^q + t_1 + 1 = 0$.

Then $t_1^{m+1} = t_2^{m+1}$ implies $t_1^m + t_1 + 1 = 0$; so $t_1^m = t_1^q$ which implies $t_1 = 1$ since $m = 2^n$, $(n, h) = 1$.

This shows that the chords and axes of $\mathcal{C}(m)$ and $\Gamma(m)$ define a $(2q^2+q+1)$-cap on $\mathcal{H}^5$. □

Theorem 6.2. *The $(2q^2+q+1)$-cap constructed in Theorem 6.1 is the intersection of the hypersurfaces*

$$(X_2 + X_3)^{m-1} X_5 + X_4^m = 0,$$
$$(X_2 + X_3)^{m-1} X_0 + X_1^m = 0,$$
$$X_0 X_5 + X_1 X_4 + X_2 X_3 = 0.$$

Proof. By [4], the coordinates of the chords and axes can be rewritten as

$$I(R^{2m}\rho^m, R^{m+1}\rho, R^m(1+\rho+\rho^2+\rho^4+\ldots+\rho^{m/2}), R^m(\rho+\rho^2+\ldots+\rho^{m/2}), R^{m-1}, 1)$$

and

$$I(U^{2m}\mu^m, U^{m+1}\mu, U^m(\mu+\mu^2+\ldots+\mu^{m/2}), U^m(1+\mu+\mu^2+\ldots+\mu^{m/2}), U^{m-1}, 1).$$

For a point lying in the intersection of the hypersurfaces, if $X_5 = 0$, then the points are $(1, t^{m-1}, 0, t^m, 0, 0)$, $(0,0,0,1,0,0)$, $(1, t^{m-1}, t^m, 0, 0, 0)$, $(0,0,1,0,0,0)$ which correspond to the axes in $X_3 = 0$ and the chords through $\mathbf{U}_0$.

If $X_5 = 1$ and $X_4 = 0$, then $(X_0, \ldots, X_5) = (t^{2m}, 0, t^m, t^m, 0, 1)$. If $X_5 = 1$ and $X_4 \neq 0$, letting $X_4 = U^{m-1}$ implies $X_2 + X_3 = U^m$ and $U^{m^2-m} X_0 = X_1^m$. Substituting $X_1 = U^{m+1}\mu$ implies $X_2 + X_3 = U^m$, $X_0 = U^{2m}\mu^m$, $X_2 X_3 = U^{2m}(\mu^m + \mu)$.

So X_2, X_3 are solutions to $X^2 + U^m X + (\mu^m + \mu)U^{2m} = 0$. Hence the caps form the intersection of the hypersurfaces. □

Remark. The last result is not valid for odd characteristic. In $PG(5,q)$, q odd, the Klein quadric is the only quadric containing the $(2q^2+q+1)$-cap constructed.

7. An alternative construction in $PG(5,q)$, q odd

To obtain a result similar for q odd to Theorem 6.2, consider the chords

$$I(s^2, rs, r^2 - s, s, -r, 1)$$

of the cubic $\mathcal{C}$ and the axes

$$I(v^2, uv, v, u^2 - v, -u, 1)$$

of the developable Γ, which correspond to each other under the null polarity

$$\mathbf{P}(x_0, x_1, x_2, x_3) \longleftrightarrow \pi(-x_3, x_2, -x_1, x_0)$$

defined by the linear complex $l_{03} = l_{12}$ of $PG(3,q)$ or equivalently by the section $X_2 = X_3$ of $\mathcal{H}^5$ in $PG(5,q)$.

Lemma 7.1. *The set of points*

$$\begin{aligned}&\{(s^2, rs, r^2 - s, s, -r, 1) : r, s \in \gamma\} \cup\\ &\{(v^2, uv, v, u^2 - v, -u, 1) : u, v \in \gamma\} \cup\\ &\{(s^2, s, 1, 0, 0, 0), (s^2, s, 0, 1, 0, 0) : s \in \gamma\} \cup \{(1, 0, 0, 0, 0, 0)\}\end{aligned}$$

is the intersection of the quadrics

$$\begin{aligned}&X_0X_5 + X_1X_4 + X_2X_3 = 0,\\ &(X_2 + X_3)X_5 - X_4^2 = 0,\\ &(X_2 + X_3)X_0 - X_1^2 = 0.\end{aligned}$$

Proof. To prove this, the arguments of Theorem 6.2 can be used. □

Theorem 7.2. *The set considered in Lemma 7.1 is a $(2q^2 + 2)$-cap of $PG(5,q)$, q odd.*

Proof. Suppose three of the points are collinear on a line l; then this line is contained in the intersection of the three quadrics. So it consists only of points defined above.

Since the set consists of two $(q^2 + q + 1)$-caps; necessarily $|l| \leq 4$. So $q = 3$. When $q = 3$, it was checked by computer that the set is a cap.

The size of the cap is $2q^2 + 2$ since the points $(1, 0, 0, 0, 0, 0)$, $(s^2, 0, -s, s, 0, 1)$, $s \in \gamma$, and $(r^4/4, r^3/2, r^2/2, r^2/2, -r, 1)$, $r \in \gamma$, define both chords and axes. □

Remark. The two parts of the $(2q^2 + 2)$-cap intersect in the conic $X_2^2 = X_0X_5$ in the plane $X_1 = X_4 = 0, X_2 + X_3 = 0$, and in the normal rational curve $\{(t^4/4, t^3/2, t^2/2, t^2/2, -t, 1) : t \in \gamma^+\}$ of the hyperplane $X_2 = X_3$.

The conic defines the regulus in $PG(3,q)$ consisting of the lines $\mathbf{U}_0\mathbf{U}_1$, $\mathbf{U}_2\mathbf{U}_3$, $P(t)P(-t)$ for $t \in \gamma\backslash\{0\}$ or $t \in \gamma'\backslash\gamma$ with $t^q = -t$.

The normal rational curve in $X_2 = X_3$ defines the lines $I(s^2, rs, r^2 - s, s, -r, 1)$, where $s = r^2/2 = t_1t_2$, $r = t_1 + t_2$ with $t_1^2 = -t_2^2$.

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