How large can the domination numbers of a graph be?

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Abstract

A vertex v in a graph G dominates itself as well as its neighbors. A set S of vertices in G is (1) a dominating set if every vertex of G is dominated by some vertex of S, (2) an open dominating set if every vertex of G is dominated by a vertex of S distinct from itself, and (3) a double dominating set if every vertex of G is dominated by at least two distinct vertices of S. The minimum cardinality of a set S satisfying (1), (2), and (3), respectively, is the domination number, open domination number, and double domination number of G, respectively. We consider the problem of determining the maximum value of each of these domination numbers among all graphs of a given order and size.

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1 The Maximum Domination Number of a Graph with Prescribed Order and Size

In graph theory we have often been intrigued by how large or how small the value of a graphical parameter can be under various constraints. We discuss three problems of this type, where the parameters involved are various domination numbers and the constraints are given order and size. We refer to books [1, 3] for concepts not defined here.

In a graph G a vertex v is said to *dominate* itself as well as its neighbors. A set S of vertices in G is a *dominating set* for G if every vertex of G is dominated by some vertex of S. A dominating set of minimum cardinality is a *minimum dominating set* and its cardinality is the *domination number* $\gamma(G)$. A comprehensive treatise now exists by Haynes, Hedetniemi and Slater [4] on domination.

The first question that we address concerns the largest domination number of a graph of given order n and size m. It is possible to give a complete answer to this question with the significant help of a result of Vizing [5].

Theorem A (Vizing) Let G be a graph of order n and size m. If $\gamma(G) \geq 2$, then

$$m \leq \left\lfloor \frac{(n-\gamma(G))(n-\gamma(G)+2)}{2} \right\rfloor . \tag{1}$$

We write $\max(\gamma; n, m)$ for the largest domination number of a graph of order n and size m.

Theorem 1.1 *For integers $n \geq 1$ and m with $0 \leq m \leq \binom{n}{2}$,*

$$\max(\gamma; n, m) = \left\lfloor n+1-\sqrt{1+2m} \right\rfloor .$$

Proof. The result is certainly true for $n = 1$, so we consider $n \geq 2$. If $m \geq \binom{n}{2} - \lfloor (n-1)/2 \rfloor$, then every graph G of order n and size m contains a vertex of degree $n-1$ and so $\gamma(G) = 1$. Hence the result holds here as well.

Thus it remains to consider a graph G of order n and size m, where $n \geq 2$ and $0 \leq m < \binom{n}{2} - \lfloor (n-1)/2 \rfloor$. Consequently, $\max(\gamma; n, m) \geq 2$. Solving for $\gamma(G)$ in (1), we obtain $\gamma(G) \leq \left\lfloor n+1-\sqrt{1+2m} \right\rfloor$, showing that

$$\max(\gamma; n, m) \leq \left\lfloor n+1-\sqrt{1+2m} \right\rfloor .$$

To verify the reverse inequality, we construct a graph G of order n and size m with $\gamma(G) \geq \left\lfloor n+1-\sqrt{1+2m} \right\rfloor = n+1-\left\lceil \sqrt{1+2m} \right\rceil$. Let $k = \left\lceil \sqrt{1+2m} \right\rceil$ so that

$$2 \leq \max(\gamma; n, m) \leq n+1-k;$$

hence $k < n$.

We first construct a graph H according to the parity of k. When k is odd, let M be a perfect matching of K_{k+1} and let $H = K_{k+1} - M$, so $\gamma(H) = 2$. For k

even, let M be a maximum matching of K_{k+1} and let e be an edge incident with the unmatched vertex of K_{k+1}. Let $M' = M \cup \{e\}$. In this case, $H = K_{k+1} - M'$. Here also $\gamma(H) = 2$.

Now define $G' = H \cup \overline{K}_{n-k-1}$. Then $\gamma(G') = n + 1 - k$. Certainly, G' has order n. Now we determine its size. If k is odd, then the size of G' is

$$\frac{k^2-1}{2} = \frac{\left(\left\lceil\sqrt{1+2m}\right\rceil\right)^2 - 1}{2} \geq \frac{1+2m-1}{2} = m.$$

On the other hand, if k is even, then the size of G' is

$$\frac{k^2-2}{2} = \frac{\left(\left\lceil\sqrt{1+2m}\right\rceil\right)^2 - 2}{2} > \frac{1+2m-2}{2} = m - \frac{1}{2}. \tag{2}$$

The inequality in (2) is strict since $\left\lceil\sqrt{1+2m}\right\rceil$ is even. Therefore, the size of G' is at least m. We now remove edges from G', if necessary, so that the resulting graph G has size m. Then G has order n, size m and $\gamma(G) \geq n+1-k$ since the removal of edges can never decrease the domination number. So $\max(\gamma; n, m) \geq n + 1 - \left\lceil\sqrt{1+2m}\right\rceil$.
■

2 The Maximum Open Domination Number of a Graph with Prescribed Order and Size

Next we investigate the maximum open domination number among all graphs of a given order and size. A vertex is said to *openly dominate* each of its neighbors (but not itself). A set S of vertices in a graph G is an *open dominating set* if every vertex of G is openly dominated by a vertex of S. An open dominating set of minimum cardinality is a *minimum open dominating set* and its cardinality is the *open domination number* $\rho(G)$. The open domination number is also referred to as the *total domination number* (see [4]). We note that no graph with isolates has an open dominating set. Hence in this section no graph contains an isolate. If G is a graph of order n and size m such that $\rho(G)$ is defined, then, necessarily, $m \geq \lceil n/2 \rceil$. Furthermore, the neighbor of each end-vertex must belong to every open dominating set. For integers $n \geq 2$ and m with $\lceil n/2 \rceil \leq m \leq \binom{n}{2}$, we write $\max(\rho; n, m)$ for the maximum open domination number among all graphs of order n and size m. We begin by presenting a lower bound for $\max(\rho; n, m)$.

Theorem 2.1 *For integers $n \geq 2$ and m with $\lceil n/2 \rceil \leq m \leq \binom{n}{2}$,*

$$\max(\rho; n, m) \geq 2\left\lfloor \frac{1}{2}(n-1) - \sqrt{\frac{1}{2}\left(m - \frac{1}{2}n + \frac{1}{2}\right)} \right\rfloor + 2.$$

Proof. Let $k = \left\lfloor \frac{1}{2}(n-1) - \sqrt{\frac{1}{2}\left(m - \frac{1}{2}n + \frac{1}{2}\right)} \right\rfloor$ and let $H = kK_2 \cup K_{n-2k}$. Since $n - 2k \geq 2$, it follows that $\rho(H) = 2k+2$. Certainly H has order n and size $k + \binom{n-2k}{2}$.

Some routine algebra shows that $k + \binom{n-2k}{2} \geq m$. We may remove edges from H, if necessary, to obtain a graph G of order n, size m and without isolates. Then $\rho(G) \geq \rho(H) \geq 2k+2$. ∎

Certainly, for all integers $n \geq 2$ and m with $\lceil n/2 \rceil \leq m \leq \binom{n}{2}$,

$$2 \leq \max(\rho; n, m) \leq n.$$

We consider this upper bound for $\max(\rho; n, m)$. For even $n \geq 2$, the graph $(n/2)K_2$ has open domination number n. Therefore, $\max(\rho; n, n/2) = n$. Thus for any graph G of order n and size m with $n/2 < m \leq \binom{n}{2}$, G contains a component G_1 having vertices of degree 2 or more. Let v be a vertex of minimum degree in G_1. Then the set $S = V(G) - \{v\}$ is an open dominating set for G. Therefore, $\rho(G) < n$ and $\max(\rho; n, m) < n$. Consequently,

$$\max(\rho; n, m) = n \text{ if and only if } m = n/2. \tag{3}$$

We now turn to the lower bound $\max(\rho; n, m) \geq 2$. We are unable to determine all m for which equality holds. However, to determine a collection of integers m for which $\max(\rho; n, m) = 2$, we present the following lemma.

Lemma 2.2 *Let G be a connected graph of order n, size m, and diameter 2. If G has no vertices of degree $n-1$, then*

$$m \geq \left\lceil \frac{3n-5}{2} \right\rceil.$$

Proof. Since every connected graph of order n with $1 \leq n \leq 3$ has a vertex of degree $n-1$, we consider a connected graph G of order $n \geq 4$ and size m with maximum degree $\Delta(G) = k \leq n-2$. As G is connected with diameter 2, it follows that $k \geq 2$. Let u be a vertex of degree k and denote the closed neighborhood of u by $N[u] = N(u) \cup \{u\}$. Let $W = V(G) - N[u]$. Then $|W| = n-k-1 \geq 1$. Let $w \in W$. We consider two cases.

Case 1. $k = n-2$. Then $W = \{w\}$. Let $x \in N(u)$. Since $d(x,w) \leq 2$, it follows that either $xw \in E(G)$ or x is adjacent to some neighbor of w. Therefore, there are at least $n-2$ edges not incident with u and so

$$m \geq 2(n-2) > \frac{3n-5}{2}.$$

Case 2. $k \leq n-3$. Then $|W| \geq 2$. Let $X = N[w] \cap W$ and $Y = W - X$. So $W = X \cup Y$ and possibly $Y = \emptyset$. Since $w \notin N[u]$, it follows that $d(u,w) = 2$. So w is adjacent to at least one vertex in $N(u)$. Thus $|X| = |N[w] \cap W| \leq k$. This implies that

$$|Y| = |W| - |X| \geq n - 2k - 1.$$

We proceed by counting the following three types of edges of G.

(1) There are k edges between u and $N(u)$. For each $v \in N(u)$, $d(v,w) \leq 2$. So either $vw \in E(G)$ or v is adjacent to a neighbor of w. Hence there are at least k

edges either in $\langle N(u)\rangle$ or between $N(u)$ and $N[w]$. Therefore, there are at least $2k$ edges that are either within $\langle N[u]\rangle$ or between $N(u)$ and $N[w]$.

(2) For each $x \in X - \{w\}$, $xw \in E(G)$. So there are at least $|X| - 1$ edges in $\langle X\rangle$.

(3) Let $y \in Y$. Since $d(u, y) = 2$, it follows that y is adjacent to a vertex of $N(u)$. If $\deg y = 1$, then since $d(y, z) \leq 2$ for all $z \in V(G)$, the neighbor of y must have degree $n-1$, which contradicts the fact that $\Delta(G) \leq n-2$. Therefore, $\deg y \geq 2$ for every vertex $y \in Y$. Thus, each $y \in Y$ is adjacent to a vertex of $N(u)$ and to at least one other vertex (perhaps also in $N(u)$). Counting the number of edges incident with the vertices of Y gives at least $|Y|$ edges between Y and $N(u)$ and at least $\frac{1}{2}|Y|$ additional edges since these edges may join two vertices of Y.

Combining (1), (2), (3), we have

$$\begin{aligned} m &\geq 2k + (|X| - 1) + \left(|Y| + \frac{1}{2}|Y|\right) \\ &= 2k - 1 + |W| + \frac{1}{2}|Y| \\ &= 2k - 1 + n - k - 1 + \frac{1}{2}|Y| \\ &\geq 2k - 1 + n - k - 1 + \frac{1}{2}(n - 2k - 1) \\ &= \frac{3n-5}{2}. \end{aligned}$$

■

We are now able to present the desired result.

Theorem 2.3 *For integers $n \geq 3$ and m with $m > (n^2 - 4n + 5)/2$,*

$$\max(\rho; n, m) = 2.$$

Proof. Assume, to the contrary, that $\max(\rho; n, m) > 2$. Then there exists a graph G of order n and size m with $\rho(G) \geq 3$. Since G has no isolates, $\Delta(\overline{G}) \leq n-2$. If u and v are two nonadjacent vertices of $\overline{G}$, then $uv \in E(G)$. Since $\{u, v\}$ is not an open dominating set of G, there exists a vertex w in G such that $uw, vw \notin E(G)$, implying $uw, vw \in E(\overline{G})$ and so $d_{\overline{G}}(u, v) = 2$. Therefore, $\overline{G}$ is connected and $diam(\overline{G}) \leq 2$. Since $\Delta(\overline{G}) \leq n-2$, it follows that G is not complete and so $diam(\overline{G}) = 2$. By Lemma 2.2, the size of $\overline{G}$ is at least $(3n-5)/2$, contradicting the fact that $m > (n^2 - 4n + 5)/2$. ■

While the preceding theorem gives only a sufficient condition for $\max(\rho; n, m) = 2$, we show next that if the bound on m is lowered by $(n-5)/2$, then $\max(\rho; n, m) \neq 2$.

Lemma 2.4 *For $n \geq 6$,*

$$\max\left(\rho; n, \frac{n^2 - 5n + 10}{2}\right) \geq 3.$$

Proof. Let G be the graph obtained by subdividing some edge of K_{n-2} twice. Since G has order n, size $(n^2 - 5n + 10)/2$, and open domination number 3, the desired result follows. ■

We next show, for a given n, that $\max(\rho; n, m)$ is a decreasing function of m.

Theorem 2.5 *For each fixed integer $n \geq 2$,*

$$\max(\rho; n, m+1) \leq \max(\rho; n, m).$$

Proof. Let $\max(\rho; n, m) = r$. Assume, to the contrary, that there exists a graph G of order n and size $m+1$ with double domination number $\rho(G) > r$. Since $m \geq \lceil n/2 \rceil$, it follows that that $m + 1 \geq \lceil n/2 \rceil + 1$. We now show that there exists an edge e of G such that $\rho(G - e)$ is defined. If this is not the case, then necessarily, G consists of k components, each of which is a star and so $\rho(G) = 2k$. Since $m+1 \geq \lceil n/2 \rceil + 1$, either G contains a component isomorphic to $K_{1,s}$ for some $s \geq 3$ or two components isomorphic to $K_{1,2}$. In the first case, we construct a graph G' by replacing $K_{1,s}$ by $K_{1,s-2} \cup K_2$, and in the second case, G' is obtained by replacing $2K_{1,2}$ by $3K_2$. However, G' has order n, size m, and $\rho(G') = 2k + 2 > \rho(G)$, which is impossible. Hence, as claimed, G contains an edge $e = uv$ with $\deg u, \deg v \geq 2$. Therefore, $G - e$ is a graph of order n and size m for which $\rho(G - e)$ is defined. Let S be a minimum open dominating set of $G - e$. Then S is a open dominating set of G as well. Hence

$$r < \rho(G) \leq \rho(G - e) \leq r,$$

which is a contradiction. ■

The table on the next page gives exact values of $\max(\rho; n, m)$ for small values of n $(2 \leq n \leq 8)$ and for $\lceil n/2 \rceil \leq m \leq \binom{n}{2}$. Several values of $\max(\rho; n, m)$ are easy to compute, some with the aid of (3), Theorem 2.3, or the following lemma. Since the proof of this lemma is straightforward, we omit the proof.

Lemma 2.6 *(a) Let G be a graph of order n and size m with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. If k is the smallest positive integer such that $d_1 + d_2 + \cdots + d_k \geq n$, then* $\max(\rho; n, m) \geq \rho(G) \geq k$.

(b) If G is a graph with components $G_1, G_2, \cdots, G_k$, then $\rho(G) = \sum_{i=1}^{k} \rho(G_i)$.

(c) If G is a graph of order n, then $\rho(G) \leq n - \Delta(G) + 1$.

In the remainder of this section, we verify a few of the less obvious entries in Table 1. These entries are indicated in bold.

Theorem 2.7 $\max(\rho; 7, 12) = 3$.

Proof. If G is a graph of order 7 and size 12 containing a vertex v with $\deg v \geq 5$, then v and at most two other vertices openly dominate all vertices of G. So we assume that $\Delta(G) \leq 4$. Since G has size 12, it follows that $\Delta(G) = 4$. The number of vertices of degree 4 is necessarily at least 3.

Let u and v be two vertices of degree 4 in G. Assume first that $uv \in E(G)$. If $N(u) \cup N(v) = V(G)$, then $\{u, v\}$ is an open dominating set of G. On the other hand,

n	2	3	4	4	5	5	5	5	6	6
m	1	2-3	2	3-6	3	4	5	6-10	3	4
$\max(\rho; n, m)$	2	2	4	2	4	4	3	2	6	4
n	6	6	6	6	6	7	7	7	7	7
m	5	6	7	8	9-15	4	5	6	7	8
$\max(\rho; n, m)$	4	4	4	3	2	6	6	5	4	4
n	7	7	7	7	7	8	8	8	8	8
m	9	10	11	12	**13**-21	4	5	6	7	8
$\max(\rho; n, m)$	4	4	4	**3**	2	8	6	6	6	6
n	8	8	8	8	8	8	8	8	8	8
m	9	10	11	12	13	14	15	16	17	**18**-28
$\max(\rho; n, m)$	4	4	4	4	4	4	4	4	**3**	2

Table 1: $\max(\rho; n, m)$ for $2 \leq n \leq 8$

if $|N(u) \cup N(v)| = 6$, then let $V(G) - (N(u) \cup N(v)) = \{w\}$, where x is a neighbor of w. Then $\{u, v, x\}$ is an open dominating set for G. Otherwise, $|N(u) \cup N(v)| = 5$, where, say, $V(G) - (N(u) \cup N(v)) = \{w, y\}$. Let x and z be (not necessarily distinct) neighbors of w and y, respectively. Then $\{u, x, z\}$ is an open dominating set for G, in this case, $\rho(G) \leq 3$.

Next assume that $d(u, v) \geq 2$. If there exists a vertex $w \notin N(u) \cup N(v)$, then let x be a neighbor of w. In this case, $\{u, x\}$ is a minimum open dominating set for G. Otherwise, let x be a common neighbor of u and v. Then $\{u, v, x\}$ is an open dominating set for G. So, in this case, we also have $\rho(G) \leq 3$. By Lemma 2.4, $\max(\rho; 7, 12) = 3$. ■

By Theorem 2.3 $\max(\rho; 7, m) = 2$ if $m \geq 14$. However, $\max(\rho; 7, 13) = 2$ as well, as we now show.

Theorem 2.8 $\max(\rho; 7, 13) = 2$.

Proof. We show that every graph of order 7 and size 13 has open domination number 2. Let G be a graph of order 7 and size 13. Necessarily, $\Delta(G) \geq 4$. If $\Delta(G) = 6$, then $\rho(G) = 2$ by Lemma 2.6(c). Hence either $\Delta(G) = 5$ or $\Delta(G) = 4$.

Assume first that $\Delta(G) = 5$. Let v be a vertex with $\deg v = 5$. Then there exists a unique vertex $w \notin N[v]$. Let x be a neighbor of w. Since $x \in N(v)$, the set $\{v, x\}$ is an open dominating set for G and $\rho(G) = 2$.

Hence we may assume that $\Delta(G) = 4$. Since the size of G is 13, G has at least five vertices of degree 4. There must be two nonadjacent vertices u and v of degree 4, for otherwise the size of G is not 13. Necessarily, $d(u, v) = 2$. If $N(u) = N(v)$, then there exists a unique vertex $w \notin N[u] \cup N[v]$. Let x be a neighbor of w. Then $\{u, x\}$ is an open dominating set for G and so $\rho(G) = 2$. Thus, we may assume that $N(u) \neq N(v)$. Let $N(u) - N(v) = \{u_1\}$ and $N(v) - N(u) = \{v_1\}$. Since the size of $\langle N(u) \cap N(v) \rangle$ is at most 3, at least one of u_1 and v_1, say u_1, is adjacent to a vertex $x \in N(u) \cap N(v)$. Then $\{v, x\}$ is an open dominating set for G and $\rho(G) = 2$. ■

We now verify two numbers of the type $\max(\rho; 8, m)$.

Theorem 2.9 $\max(\rho; 8, 17) = 3$.

Proof. Let G be a graph of order 8 and size 17. We show that $\rho(G) \leq 3$. Necessarily, $\Delta(G) \geq 5$. If $\Delta(G) \geq 6$, then by Lemma 2.6(c), $\rho(G) \leq 3$. Hence we may assume that $\Delta(G) = 5$. In fact, G contains at least two vertices u and v of degree 5 which have at least two common neighbors. Let x be one of these. If $N[u] \cup N[v] = V(G)$, then $\{u, v, x\}$ is an open dominating set for G.

Assume next that there is exactly one vertex w not belonging to $N[u] \cup N[v]$. If w is adjacent to a vertex $y \in N(u) \cap N(v)$, then $\{u, v, y\}$ is an open dominating set for G. Otherwise, $uv \in E(G)$ and w is adjacent to a vertex y belonging to exactly one of $N(u)$ and $N(v)$ and once again $\{u, v, y\}$ is an open dominating set.

Finally, assume that there are exactly two vertices w and z not belonging to $N[u] \cup N[v]$. Let w' and z' be neighbors of w and z, respectively. Then $\{u, w', z'\}$ is an open dominating set for G. Therefore, $\rho(G) \leq 3$ and so $\max(\rho; 8, 17) \leq 3$. By Lemma 2.4, $\max(\rho; 8, 17) = 3$. ∎

By Theorem 2.3 $\max(\rho; 8, m) = 2$ if $m \geq 19$. We next show that $\max(\rho; 8, 18) = 2$ also has the value 2.

Theorem 2.10 $\max(\rho; 8, 18) = 2$.

Proof. It suffices to show that every graph of order 8 and size 18 has open domination number 2. Let G be a graph of order 8 and size 18. Necessarily, $\Delta(G) \geq 5$. If $\Delta(G) = 7$, then $\rho(G) = 2$ by Lemma 2.6(c). Hence either $\Delta(G) = 6$ or $\Delta(G) = 5$.

Assume first that $\Delta(G) = 6$. Let v be a vertex with $\deg v = 6$. Then there exists a unique vertex $w \notin N[v]$. Let x be a neighbor of w. Certainly $x \in N(v)$. Then $\{v, x\}$ is an open dominating set for G and $\rho(G) = 2$.

Hence we may assume that $\Delta(G) = 5$. If G has a vertex of degree 1, then G contains seven vertices of degree 5 and one vertex of degree 1. This graph is unique and is shown in Figure 1. Furthermore, $\rho(G) = 2$. Otherwise, $\delta(G) \geq 2$.

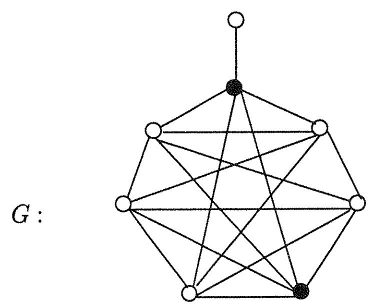

Figure 1: The graph with degree sequence 5,5,5,5,5,5,5,1

Let v be a vertex of degree 5. Then there are two vertices, say u and w, not belonging to $N[v]$. Since the size of G is 18 and $\Delta(G) = 5$, it follows that $\deg u +$

$\deg v \geq 6$. If u and w are mutually adjacent to a neighbor z of v, then $\{v, z\}$ is an open dominating set for G and so we may assume that no such vertex z exists. This implies that $uw \in E(G)$ since $\deg u + \deg v \geq 6$ and that every vertex of $N(v)$, with at most one exception, is a neighbor of u or w.

Let $U = N(u) \cap N(v)$ and $W = N(w) \cap N(v)$. Thus U and W are disjoint. If every pair of vertices $x \in U$ and $y \in W$ are not adjacent, then the size of $\langle N(v) \rangle$ is at most 6. Since the size of $\langle N(v) \cup N(w) \rangle$ is at most 6 as well, the size of G is at most 17, which is impossible.

Hence we may assume that there are vertices of U adjacent to vertices of W. If there is a vertex $x \in U$ adjacent to a vertex $y \in W$ such that every vertex of $N(v)$ is neighbor of x or y, then $\{x, y\}$ is an open dominating set.

Assume, without loss of generality, that $|U| \leq |W|$. Of course, $|U| \geq 1$, $|W| \geq 1$, and $4 \leq |U| + |W| \leq 5$. Hence $|U| = 1$ or $|U| = 2$. We prove the case $|U| = 1$, $|W| = 3$ only since the other cases are similar. Let u_1 be the vertex of $N(v)$ adjacent to u and w_1, w_2, w_3 the vertices of $N(v)$ adjacent to w. By hypothesis, we may assume that $u_1w_1 \in E(G)$. In this case, there is some vertex x of $N(v)$ that is adjacent to neither u_1 nor w_1, where possibly $x = w_2$ or $x = w_3$. Assume, without loss of generality, that $x \neq w_2$. Since the size of G is 18, we must have $u_1w_2 \in E(G)$. So there is a vertex y of $N(w)$ (possible $x = y$) that is adjacent to neither u_1 nor w_2. But then the size of $\langle N(v) \rangle$ is at most 7 and so the size of G is at most 17. This produces a contradiction. ■

3 The Maximum Double Domination Number of a Graph with Prescribed Order and Size

We now turn our attention to the double domination number of a graph. A set S of vertices of a graph G is a *double dominating set* if every vertex of G is dominated by two distinct vertices of S. A double dominating set of minimum cardinality is a *minimum double dominating set* and this cardinality is the *double dominating number* $dd(G)$. Two basic observations are that (1) no graph with isolates has a double dominating set and (2) if v is an end-vertex of a graph G without isolates, then every double dominating set of G contains both v and its neighbor. The first observation implies that if G is a graph of order n and size m for which $dd(G)$ is defined, then $m \geq \lceil n/2 \rceil$, which is the same requirement for $\rho(G)$ to be defined. Throughout this section we therefore assume that every graph under consideration is without isolates. For integers $n \geq 2$ and m with $\lceil n/2 \rceil \leq m \leq \binom{n}{2}$, we write $\max(dd; n, m)$ for the maximum double domination number among all graphs of order n and size m. Certainly, $2 \leq \max(dd; n, m) \leq n$.

The *corona* G^0 of a graph G of order k is defined in [2] as that graph obtained from G by joining a new vertex to each vertex of G. Thus the order of G^0 is $2k$. It is straightforward to see that $dd(G^0) = 2k$ since the vertex set of G^0 is its unique double dominating set. With the aid of the corona of a graph, we can determine precisely those m for which $\max(dd; n, m) = n$.

Theorem 3.1 *For each integer $k \geq 1$,*
(a) $\max(dd; 2k, m) = 2k$ *if and only if* $k \leq m \leq k + \binom{k}{2}$,
(b) $\max(dd; 2k+1, m) = 2k+1$ *if and only if* $k+1 \leq m \leq k + \binom{k}{2} + 1$.

Proof. We only verify (a) since the proof of (b) is similar. Let $k \leq m \leq k + \binom{k}{2}$. Let H be a graph of order k and size $m - k \geq 0$ and let $G = H^0$. Then G has order $2k$, size m and $dd(G) = 2k$. Therefore, $\max(dd; 2k, m) = 2k$.

For the converse, let $\max(dd; 2k, m) = 2k$. Let G be a connected graph of order $2k$ and size m with $dd(G) = 2k$. Certainly, $m \geq k$, for otherwise G contains isolates. Now assume, to the contrary, that $m \geq k + \binom{k}{2} + 1$. Observe that if G has k or more end-vertices, then $m \leq k + \binom{k}{2}$. So G has less than k end-vertices. Therefore, there exists a vertex v of G that is neither an end-vertex of G nor adjacent to an end-vertex of G. In particular, $\deg v \geq 2$. The set $S = V(G) - \{v\}$ is a double dominating set of G and $dd(G) < 2k$, a contradiction. ∎

We now determine for $n \geq 2$, all integers m for which $\max(dd; n, m) = 2$. Before presenting this result, we note that a graph G of order $n \geq 2$ has double domination number 2 if and only if G contains $K_2 + \overline{K}_{n-2}$ as a (spanning) subgraph, that is, G has double domination number 2 if and only if G contains two vertices of degree $n-1$.

Theorem 3.2 *Let $n \geq 2$. Then*

$$\max(dd; n, m) = 2 \quad \textit{if and only if} \quad m > (n-1)^2/2. \tag{4}$$

Proof. First, we show that if G is a graph of order $n \geq 2$ and size $m > (n-1)^2/2$, then $dd(G) = 2$. We claim that G contains two vertices of degree $n-1$. Assume, to the contrary, that G contains at most one vertex of degree $n-1$. Then the size of G is at most $[n-1+(n-1)(n-2)]/2 = (n-1)^2/2$, which is a contradiction. Thus G contains two vertices of degree $n-1$ and so $\max(dd; n, m) = 2$.

For the converse, we show that if $n/2 \leq m \leq (n-1)^2/2$, then there exists a graph G of order n and size m with $dd(G) > 2$. If n is odd, let H be an $(n-3)$-regular graph of order $n-1$; while if n is even, let H be a graph of order $n-1$ containing $n-2$ vertices of degree $n-3$ and one vertex of degree $n-4$. That such graphs exist is shown in [1, Chap. 9]. Define $G' = K_1 + H$. Then G' has order n, size $\lfloor (n-1)^2/2 \rfloor$, and exactly one vertex of degree $n-1$ and so $dd(G') > 2$. If $m < \lfloor (n-1)^2/2 \rfloor$, then we remove edges from G', producing a graph G of order n, size m, and $dd(G) \geq dd(G') > 2$. This gives the desired result. ∎

As with open domination, for a fixed n, it follows that $\max(dd; n, m)$ is a monotonically decreasing function of m.

Theorem 3.3 *For each fixed integer n,*

$$\max(dd; n, m+1) \leq \max(dd; n, m).$$

Proof. Let $\max(dd; n, m) = r$. Assume, to the contrary, that there exist a graph G of order n and size $m+1$ with double domination number $dd(G) > r$. By Theorem 3.1, $m > k + \binom{k}{2}$ if $n = 2k$ and $m > k + \binom{k}{2} + 1$ if $n = 2k + 1$. As we have seen in the proof of Theorem 3.1, there exists an edge $e = uv$ of G such that $\deg u, \deg v \geq 2$. Therefore, $G - e$ is a graph of order n and size m for which $dd(G - e)$ is defined. Let S be a minimum double dominating set of $G - e$. Then S is a double dominating set of G as well. Hence

$$k < dd(G) \leq dd(G - e) \leq k,$$

which is a contradiction. ∎

The table below gives exact values of $\max(dd; n, m)$ for $2 \leq n \leq 6$ and $\lceil n/2 \rceil \leq m \leq \binom{n}{2}$. Many of these values are computed with aid of the following lemma, whose routine proof is omitted.

Lemma 3.4 *(a) Let G be a graph of order n and size m with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. If k is the smallest positive integer such that $k + d_1 + d_2 + \cdots + d_k \geq 2n$, then $\max(dd; n, m) \geq dd(G) \geq k$.*

(b) If G is a graph with components $G_1, G_2, \cdots, G_k$, then $dd(G) = \sum_{i=1}^{k} dd(G_i)$.

n	2	3	3	4	4	4	4	4	5	5
m	1	2	3	2	3	4	5	6	3	4
$\max(dd; n, m)$	2	3	2	4	4	3	2	2	5	5
n	5	5	5	5	5	5	6	6	6	6
m	5	6	7	8	9	10	3	4	5	6
$\max(dd; n, m)$	4	4	3	3	2	2	6	6	6	6
n	6	6	6	6	6	6	6	6	6	
m	7	8	9	10	11	12	13	14	15	
$\max(dd; n, m)$	5	4	4	4	3	3	2	2	2	

Table 2: $\max(dd; n, m)$ for $2 \leq n \leq 6$

We conclude the paper by illustrating the verification of an entry in Table 2.

Theorem 3.5 $\max(dd; 6, 8) = 4$.

Proof. First, we note that the graph of Figure 2 has order 6, size 8, and double domination number 4. Incidentally, this graph is the unique graph of order 6 and size 8 with degree sequence 3, 3, 3, 3, 3, 1.

G :

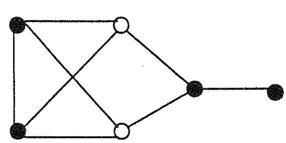

Figure 2: A graph G of order 6 and size 8 with $dd(G) = 4$

Hence it remains to show that the double domination number of every graph of order 6 and size 8 is at most 4. Let G be a graph of order 6 and size 8. Then $3 \leq \Delta(G) \leq 5$. We consider three cases.

Case 1 $\Delta(G) = 3$. In this case, the degree sequence of G is either 3, 3, 3, 3, 2, 2 or 3, 3, 3, 3, 3, 1. We have already seen that the graph G of Figure 2 is the unique graph with the latter degree sequence. Assume, therefore, that the degree sequence of G is 3, 3, 3, 3, 2, 2 and that u and v are the two vertices of degree 2. If $uv \in E(G)$, then $N(u) \cup N(v)$ is a double domination set; while if $uv \notin E(G)$, then $V(G) - \{u, v\}$ is a double domination set.

Case 2 $\Delta(G) = 4$. We consider two subcases.

Subcase 2.1 *The graph G has a unique vertex v of degree 4.* Let $V(G) - N(v) = \{u\}$. Then $1 \leq \deg u \leq 3$. If $\deg u = 3$, then let $x \in N(u) \cap N(v)$ and $N(v) - N(u) = \{y\}$. The set $\{u, v, x, y\}$ is a double dominating set. If $\deg u = 2$, then either G is the graph of Figure 3 (a) and $N(v)$ is a double dominating set, or $N(u) \cup \{v, z\}$ is a double dominating set, where z is a vertex of minimum degree in $N(v) - N(u)$. There are three graphs of the latter type, shown in Figure 3 (b)-(d).

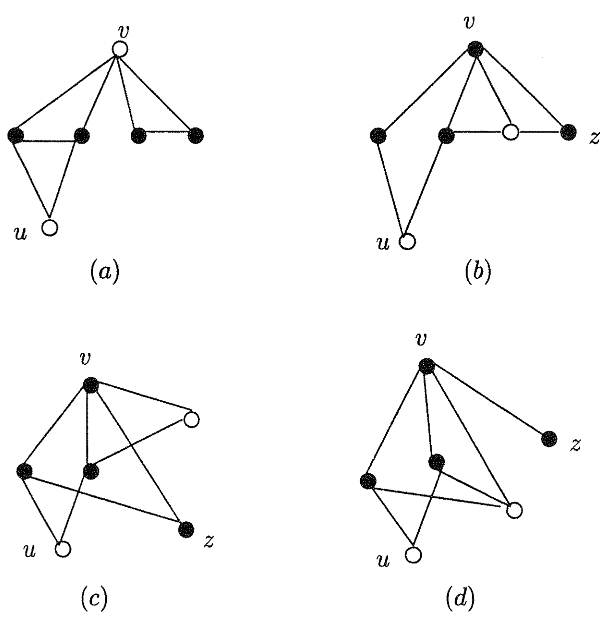

Figure 3: Graphs G with $\deg u = 2$

If $\deg u = 1$, then there are exactly two such graphs (shown in Figure 4). A double dominating set is indicated in each.

Subcase 2.2 *There are at least two vertices of degree 4 in G.* If G contains two nonadjacent vertices of degree 4, then $G = K_{2,4}$ and the two vertices of degree 4 and any other vertex is a double dominating set. Otherwise, G contains two adjacent

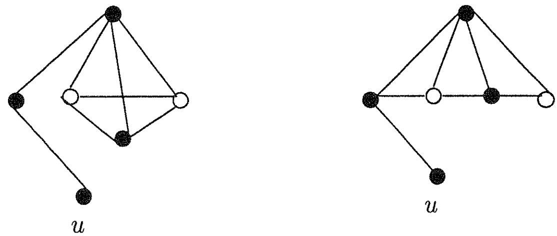

Figure 4: Graphs G with $\deg u = 1$

vertices u and v of degree 4. If $|N(u) \cap N(v)| = 2$, then $V(G) - (N(u) \cap N(v))$ is a double dominating set. If $|N(u) \cap N(v)| = 3$, then there is a unique vertex x with $d(v, x) = 2$. Let v, w, x be a path in G. Then $\{u, v, w, x\}$ is a double dominating set.

Case 3 $\Delta(G) = 5$. Let v be a vertex of G with $\deg v = 5$. Since the size of G is 8, there exists a vertex $u \in N(v)$ with $\deg u \geq 3$. Since at most two vertices do not belong to $N(u)$, the set $(V(G) - N(u)) \cup \{u, v\}$ is a double dominating set of cardinality at most 4. ∎

That $\max(dd; 6, 9) = \max(dd; 6, 10) = 4$ is a direct consequence of Theorem 3.3 and the fact that the graph G of Figure 5 has double domination number 4.

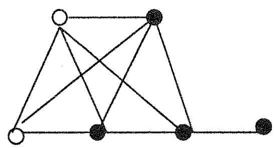

Figure 5: A graph G of order 6, size 10, and $dd(G) = 4$

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