Generalized Steiner triple systems with group size $g \equiv 1, 5 \pmod 6$*

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Abstract

Generalized Steiner triple systems, GS$(2,3,n,g)$ are equivalent to maximum constant weight codes over an alphabet of size $g+1$ with distance 3 and weight 3 in which each codeword has length n. The existence of GS$(2,3,n,g)$ has been solved by several authors for $2 \leq g \leq 10$. The necessary conditions for the existence of a GS$(2,3,n,g)$ are $(n-1)g \equiv 0\ (mod\ 2)$, $n(n-1)g^2 \equiv 0\ (mod\ 6)$, and $n \geq g+2$. Recently, D. Wu et al proved that for any given $g \geq 7$, if there exists a GS$(2,3,n,g)$ for all n, $g+2 \leq n \leq 9g+158$, satisfying the above two congruences, then the necessary conditions are also sufficient. In this paper, the result is partially improved. It is shown that for any given g, $g \equiv 1,5\ (mod\ 6)$ and $g \geq 11$, if there exists a GS$(2,3,n,g)$ for all n, $n \equiv 1,3\ (mod\ 6)$ and $g+2 \leq n \leq 9g+4$, then the necessary conditions are also sufficient. As an application, it is proved that the necessary conditions for the existence of a GS$(2,3,n,g)$ are also sufficient for $g = 11$.

1 Introduction

A $(g+1)$-ary *constant weight code* (n, w, d) is a code $C \subseteq (Z_{g+1})^n$ of length n and minimum distance d, such that every $c \in C$ has Hamming weight w. To construct a constant weight code (n, w, d) with $w = 3$, a *group divisible design* (GDD) will be used. A K−GDD is an ordered triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a set of n elements, $\mathcal{G}$ is a collection of subsets of $\mathcal{V}$ called *groups* which partition $\mathcal{V}$, and $\mathcal{B}$ is a set of some subsets of $\mathcal{V}$ called *blocks*, such that each block intersects each group in at most one element and that each pair of elements from distinct groups occurs together in exactly one block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. The group type is the multiset $\{|G| : G \in \mathcal{G}\}$. A k-GDD(g^n) denotes a K-GDD with n groups

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of size g and $K = \{k\}$. If all blocks of a GDD can be partitioned into parallel classes, then the GDD is called *resolvable* GDD and denoted by RGDD, where a parallel class is a set of blocks partitioning the element set $\mathcal{V}$. In a 3-GDD(g^n), let $\mathcal{V} = (Z_{g+1} \setminus \{0\}) \times (Z_{n+1} \setminus \{0\})$ with n groups $G_i \in \mathcal{G}$, $G_i = (Z_{g+1} \setminus \{0\}) \times \{i\}$, $1 \leq i \leq n$ and blocks $\{(a, i), (b, j), (c, k)\} \in \mathcal{B}$. One can construct a constant weight code $(n, 3, d)$ as stated in [5], [7]. From each block we form a codeword of length n by putting an a, b and c in positions i, j and k respectively and zeros elsewhere. This gives a constant weight code over Z_{g+1} with minimum distance 2 or 3. If the minimum distance is 3, then the code is a $(g+1)$-ary *maximum constant weight code* (MCWC) $(n, 3, 3)$ and the 3-GDD(g^n) is called *generalized Steiner triple system*, denoted by GS$(2, 3, n, g)$. It is easy to see that a 3-GDD(g^n) is a GS$(2, 3, n, g)$ iff any two intersecting blocks meet at most two common groups of the GDD. The following result is known.

Lemma 1.1 ([5], [7]) *The following are the necessary conditions for the existence of a GS$(2, 3, n, g)$:*
(1) $(n-1)g \equiv 0 \ (mod\ 2)$;
(2) $n(n-1)g^2 \equiv 0 \ (mod\ 6)$;
(3) $n \geq g+2$.

The necessary conditions are shown to be sufficient by several authors with one exception for $2 \leq g \leq 10$. Hence, we have the following lemma.

Lemma 1.2 ([5], [7], [8], [3], [4], [9], [6]) *The necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient for $2 \leq g \leq 10$ with one exception of $(g, n) = (2, 6)$.*

Blake-Wilson and Phelps [2] proved that the necessary conditions for the existence of a GS$(2, 3, n, g)$ are also asymptotically sufficient for any g. Recently, D. Wu et al [9] proved that for any given $g \geq 7$, if there exists a GS$(2, 3, n, g)$ for all n, $g+2 \leq n \leq 9g+158$, satisfying $(n-1)g \equiv 0 \ (mod\ 2)$ and $n(n-1)g^2 \equiv 0 \ (mod\ 6)$, then the necessary conditions for the existence of a GS$(2, 3, n, g)$ are also sufficient.

Since the existence of GS$(2, 3, n, g)$ has been solved for $g \leq 10$, we need only to consider the case $g \geq 11$. For $g \equiv 1, 5 \ (mod\ 6)$ and $g \geq 11$, let $T_g = \{n$: there exists a GS$(2, 3, n, g)\}$, $B_g = \{n$: n satisfying the necessary conditions listed in Lemma 1.1 $\}$, $M_g = \{n$: $n \in B_g, n \leq 9g+4\ \}$. In this paper, the results of [9] will be partially improved and the following results are obtained.

Theorem 1.3 *For any $g \equiv 1, 5 \ (mod\ 6)$ and $g \geq 11$, if $M_g \subset T_g$, then $B_g = T_g$. That is, the necessary conditions for the existence of a GS$(2, 3, n, g)$ are also sufficient.*

Theorem 1.4 $B_{11} = T_{11}$, *that is, the necessary conditions for the existence of a GS$(2, 3, n, g)$ are also sufficient for $g = 11$.*

Combining Lemma 1.2 and Theorem 1.4 shows that the existence of a GS$(2, 3, n, g)$ is completely determined for any $g \leq 11$.

2 Product Constructions

In product constructions, we will need the concept of both *holey generalized Steiner triple systems* and *disjoint incomplete Latin squares.*

A *holey group divisible design*, $K-HGDD$, is a fourtuple $(\mathcal{V},\mathcal{G},\mathcal{H},\mathcal{B})$, where $\mathcal{V}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into subsets called *groups*, $\mathcal{H}\subset\mathcal{G}$, $\mathcal{B}$ is a set of *blocks* such that a group and a block contain at most one common point and every pair of points from distinct groups, not both in $\mathcal{H}$, occurs in a unique block in $\mathcal{B}$, where $|B|\in K$ for any $B\in\mathcal{B}$. A k-HGDD$(g^{(n,u)})$ denotes a K-HGDD with n groups of size g in $\mathcal{G}$, u groups in $\mathcal{H}$ and $K=\{k\}$. A *holey generalized Steiner triple system*, HGS$(2,3,(n,u),g)$, is a 3-HGDD$(g^{(n,u)})$ with the property that any two intersecting blocks meet at most two common groups.

It is easy to see that if $u=0$ or $u=1$, then a HGS$(2,3,(n+u,u),g)$ is just a GS$(2,3,n,g)$ or a GS$(2,3,n+1,g)$ respectively.

A *Latin square* of side n, LS(n), is an $n\times n$ array based on some set S of n symbols with the property that every row and every column contains every symbol exactly once. An *incomplete Latin square*, ILS$(n+a,a)$, denotes a LS$(n+a)$ "missing" a sub LS(a). Without loss of generality, we may assume that the missing subsquare, or *hole*, is at the lower right corner. We say $(i,j,s)\in$ ILS$(n+a,a)$ if the entry in the cell (i,j) is s. Let A_1, A_2 be two ILS$(n+a,a)$s on the same symbol set. If $(i,j,s_1)\neq(i,j,s_2)$ for any $(i,j,s_1)\in A_1$, $(i,j,s_2)\in A_2$, then we say that A_1 and A_2 are *disjoint*. We use r DILS$(n+a,a)$ to denote r pairwise disjoint ILS$(n+a,a)$s.

For the existence of r DILS$(n+a,a)$, we have the following two lemmas.

Lemma 2.1 ([3]) *There exist $\delta(a)$ DILS$(n+a,a)$, where $\delta(0)=n$ and $\delta(a)=a$ for $1\le a\le n$.*

Lemma 2.2 ([9]) *There exist n DILS$(n+a,a)$ for all $n\equiv 0\pmod 4$ and $0\le a\le n$.*

The following singular indirect product construction for GS$(2,3,n,g)$s is first stated in [3].

Lemma 2.3 *(Singular Indirect Product (SIP)) Let m,n,t,u and a be integers such that $0\le a\le u<n$. Suppose the following designs exist:*
(1) t DILS$(n+a,a)$;
(2) a $3-GDD(g^m)$ with the property that all blocks of the design can be partitioned into t sets $S_0,S_1,\cdots,S_{t-1}$, such that the minimum distance in $S_r, 0\le r\le t-1$, is 3;
(3) a HGS$(2,3,(n+u,u),g)$.
Then there exists a HGS$(2,3,(c,d),g)$, where $c=m(n+a)+u-a, d=ma+u-a$. Further, if there exists
(4) a GS$(2,3,ma+u-a,g)$,
then there exists a GS$(2,3,m(n+a)+u-a,g)$.

Taking $a=0$ in Lemma 2.3, we get the singular direct product construction, which first appeared in [8].

Lemma 2.4 *(Singular Direct Product (SDP)) Let $m, n, t,$ and u be integers such that $0 \leq u < n$. Suppose $t \leq n$ and the following designs exist:*
(1) a $3-GDD(g^m)$ with the property that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3;
(2) a $HGS(2,3,(n+u,u),g)$.
Then there exists a $HGS(2,3,(mn+u,u),g)$. Further, if there exists a $GS(2,3,u,g)$, then there exists a $GS(2,3,mn+u,g)$.

Taking u=0 or 1 in Lemma 2.4, we get the Construction C or D of Etzion in [5] respectively.

Lemma 2.5 *(Direct Product (DP)) Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a 3-$GDD(g^m)$, and suppose there exists a $GS(2,3,n,g)$. Then there exists a $GS(2,3,mn,g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n$.*

Lemma 2.6 ([5]) *Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a 3-$GDD(g^m)$, and suppose there exists a $GS(2,3,n,g)$. Then there exists a $GS(2,3,m(n-1)+1,g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n-1$.*

It is easy to notice that the derived generalized Steiner triple system in Lemma 2.5 and Lemma 2.6 has a sub GS$(2,3,n,g)$. Hence, we have the following.

Lemma 2.7 *Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a 3-$GDD(g^m)$. Suppose there exists a $GS(2,3,n,g)$. Then there exists a $HGS(2,3,(mn,n),g)$ or a $HGS(2,3,(m(n-1)+1,n),g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n$ or $t \leq n-1$ respectively.*

If one uses a 3-RGDD(g^m) in the constructions, then each parallel class becomes an S_r and there are $t = \frac{g(m-1)}{2}$ such classes. The following is stated in [3].

Lemma 2.8 *If there exists a $GS(2,3,n,g)$ and a 3-$RGDD(g^m)$ with $t = \frac{g(m-1)}{2} \leq n$ or $n-1$, then there exists a $GS(2,3,mn,g)$ or a $GS(2,3,m(n-1)+1,g)$ respectively.*

For the existence of a 3-RGDD(g^m), we have the following.

Lemma 2.9 ([1]) *A 3-$RGDD(g^m)$ exists iff $(m-1)g \equiv 0 \ (mod\ 2), mg \equiv 0 \ (mod\ 3)$ and $g^m \neq 2^3, 2^6$ and 6^3.*

By combining Lemmas 2.7-2.9, we have the following.

Lemma 2.10 *For any $g \geq 11$, if there exists a $GS(2,3,n,g)$, then there exists a $GS(2,3,3n,g)$ and a $GS(2,3,3(n-1)+1,g)$. Consequently, there exists a $HGS(2,3,(3n,n),g)$ and a $HGS(2,3,(3(n-1)+1,n),g)$.*

3 Proof of Theorem 1.3

In this section, we will show the proof of Theorem 1.3. First, we need the following lemmas.

Lemma 3.1 *For $g \equiv 1, 5 \pmod 6$ and $g \geq 11$, suppose $v = 18p + j$, $j \in \{1, 3, 7, 9\}$. If $6p + \lfloor \frac{j}{3} \rfloor + \delta(j) \in T_g$, where $\delta(j) = 0$ or 1, and $\delta(j) \equiv j \pmod 3$, then $v \in T_g$.*

Proof. Apply Lemma 2.10 with $n = 6p + \lfloor \frac{j}{3} \rfloor + \delta(j)$, the conclusion then follows. □

Lemma 3.2 *For $g \equiv 1, 5 \pmod 6$ and $g \geq 11$, suppose $v = 54p + j$, $j = 13$ or 15. If $6p + 3 \in T_g$, $18p + j - 12 \in T_g$, and $p \geq \lceil \frac{15-j}{12} \rceil$, then $v \in T_g$.*

Proof. Apply Lemma 2.3 with $m = 3, n = 12p + 4, t = g, u = 6p + 3$ and $a = 6p - \frac{15-j}{2}$. It is easy to check that $a \leq u < n$. Since $p \geq \lceil \frac{15-j}{12} \rceil$, it is easy to see that $a \geq 0$. From Lemma 2.2, there exist n DILS$(n+a, a)$ for $0 \leq a \leq n$. We further have t DILS$(n+a, a)$ since $t \leq u-2 < n$. Thus the condition (1) of Lemma 2.3 is satisfied. For $g \geq 11$, a 3-RGDD(g^3) always exists by Lemma 2.9, which has g parallel classes. So, condition (2) is also satisfied. From $u \in T_g$, we apply Lemma 2.10 to obtain a HGS$(2, 3, (n + u, u), g)$, providing the design in condition (3). Finally, we have $ma + u - a = 18p - 12 + j \in T_g$, the condition (4) is satisfied. Therefore, we have $v \in T_g$. This completes the proof. □

Lemma 3.3 *For $g \equiv 1, 5 \pmod 6$ and $g \geq 11$, suppose $v = 54p + j$, $j \in \{31, 33, 49, 51\}$. If $6p + 7 \in T_g$, $18p + j - 36 \in T_g$, and $p \geq \lceil \frac{43-j}{12} \rceil$, then $v \in T_g$.*

Proof. Apply Lemma 2.3 with $m = 3, n = 12p + 12, t = g, u = 6p + 7$ and $a = 6p - \frac{43-j}{2}$. Then the proof is completed analogously to that of Lemma 3.2. □

Now, we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We need to show that $M_g \subset T_g$ implies that $B_g \subset T_g$. The proof is by induction on n. Suppose $n \in B_g$. If $n \in M_g$, then $n \in T_g$. Otherwise, we have $n \geq 9g + 6$ and distinguish between the following cases:

Case 1: $n = 18p+j \geq 9g+6$, $j \in \{1, 3, 7, 9\}$. It is easy to see that $n \geq 9g+6$ implies $\alpha = 6p+\lfloor \frac{j}{3} \rfloor+\delta(j) \geq 3g+2 > g+2$. It is also easy to verify that $\alpha \in B_g$. If $\alpha \in M_g$, then Lemma 3.1 guarantees that $n \in T_g$ and the proof is complete. Otherwise, we can repeat the induction process taking α as n'.

Case 2: $n = 54p + j \geq 9g + 6$, $j = 13$ or 15. We first claim that $p \geq \lceil \frac{15-j}{12} \rceil$. If not so, then $p < \lceil \frac{15-j}{12} \rceil \leq 1$. Thus $n < 54 + j$. Since $g \geq 11$ and $j = 13$ or 15, we have $n < 69 < 9g + 6$, a contradiction.

Next, it is easy to see that $n \geq 9g + 6$ implies $6p \geq g + \frac{6-j}{9}$. Then it is easily checked that $\alpha = 6p + 3 \geq g + 2$ and $\beta = 18p + j - 12 \geq g + 2$ for $g \geq 11$. Since $\beta \equiv 1$ or $3 \pmod 6$, we see that $\alpha \in B_g$ and $\beta \in B_g$. If we have both $\alpha \in M_g$ and $\beta \in M_g$, then Lemma 3.2 guarantees that $n \in T_g$ and the proof is complete. If at least one of α and β is not in M_g, then we can repeat the induction process taking the number α, β not in M_g as n'.

Case 3: $n = 54p + j \geq 9g + 6$, $j \in \{31, 33, 49, 51\}$. Apply Lemma 3.3, the proof of this case is similar to that of Case 2. We need only to check that $p \geq \lceil \frac{43-j}{12} \rceil$, $6p + 7 \geq g + 2$ and $18p + j - 36 \geq g + 2$. We first claim that $p \geq \lceil \frac{43-j}{12} \rceil$. If not so, then $p < \lceil \frac{43-j}{12} \rceil \leq 1$. Thus $n < 54 + j$. Since $g \geq 11$ and $j \in \{31, 33, 49, 51\}$, we have $n < 54 + 51 \leq 9g + 6$, a contradiction.

Next, it is easy to check that $n \geq 9g+6$ implies $6p+7 \geq g+2$ and $18p+j-36 \geq g+2$ for $g \geq 11$.

After certain steps of induction on n, n' will be small enough so that n' is in M_g, consequently, $n \in T_g$. Case 1 implies the solution for $n \equiv 1, 3, 7$ or 9 $(mod\ 18)$; Cases 2 and 3 imply the solution for $n \equiv 13$ or 15 $(mod\ 18)$. This completes the proof. □

4 Proof of Theorem 1.4

In this section, we will show that the necessary conditions for the existence of a GS$(2, 3, n, 11)$ are also sufficient. From Theorem 1.3, we need only to consider the case $n \in M_{11} = \{n : n \equiv 1, 3\ (mod\ 6)$ and $13 \leq n \leq 103\}$.

For $n \equiv 3$ $(mod\ 6)$, to construct a GS$(2, 3, n, 11)$ in Z_{11n}, it suffices to find a set of generalized base blocks, $\mathcal{A} = \{B_1, \cdots, B_s\}$, $s = \frac{11(n-1)}{2}$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms a GS$(2, 3, n, 11)$, where $\mathcal{V} = Z_{11n}$, $G = \{G_1, G_2, \cdots, G_n\}$, $G_i = \{i + nj : 0 \leq j \leq 10\}$, $1 \leq i \leq n$, and $\mathcal{B} = \{B + 3j : B \in \mathcal{A}, 0 \leq j \leq \frac{11n}{3} - 1\}$. For convenience, we write $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i, x, y\} : \{x, y\} \in S_i\}$. So, for each $\mathcal{A}$ we need only display the corresponding S_i, $1 \leq i \leq 3$.

Lemma 4.1 *There exists a GS$(2, 3, n, 11)$ for all $n \in F_1$, where $F_1 = \{15, 21, 27, 33, 51, 69\}$.*

Proof. For the values $n \in F_1$, with the aid of a computer, we have found a set of generalized base blocks of a GS$(2, 3, n, 11)$. Here, we only list the S_i, $1 \leq i \leq 3$ for $n = 15$. For the remaining values n, the corresponding S_i, $1 \leq i \leq 3$ are listed in the Appendix.

$n = 15$, $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i, x, y\} : \{x, y\} \in S_i\}$,

S_1 = { {24, 95}, {111, 154}, {6, 162}, {90, 148}, {11, 53}, {15, 20}, {139, 159}, {72, 134}, {71, 129}, {2, 109}, {66, 122}, {65, 128}, {14, 67}, {47, 103}, {114, 160}, {39, 138}, {142, 158}, {130, 141}, {22, 63}, {99, 105}, {41, 44}, {55, 133}, {74, 124}, {81, 140}, {80, 85}, {137, 165}, {115, 119}, {9, 157}, {118, 143} };

S_2 = { {38, 106}, {130, 158}, {144, 165}, {116, 143}, {23, 102}, {19, 78}, {15, 83}, {123, 145}, {36, 129}, {114, 155}, {8, 43}, {79, 128}, {90, 110}, {6, 98}, {24, 64}, {12, 63}, {22, 54}, {35, 117}, {101, 149}, {87, 160}, {13, 85}, {74, 76}, {67, 153}, {16, 112}, {3, 91}, {42, 61}, {21, 57}, {66, 150} };

S_3 = { {38, 81}, {6, 118}, {37, 120}, {36, 144}, {139, 142}, {100, 131}, {11, 42}, {35, 122}, {32, 143}, {126, 152}, {15, 133}, {34, 73}, {16, 72}, {66, 161}, {30, 77}, {19, 21}, {52, 119}, {14, 94}, {114, 124}, {67, 101} }. □

Lemma 4.2 *There exists a* $GS(2,3,n,11)$ *for all* $n \in F_2 = \{13, 19, 25, 31\}$.

Proof. With the aid of a computer, we have found a set of base blocks $\mathcal{A}$ of a GS$(2,3,n,11)$ for $n \in F_2$.

For convenience, we write $\mathcal{A} = \{\{1,x,y\} : \{x,y\} \in S\}$. So, for each $\mathcal{A}$ we need only display the corresponding S.

$n = 13$, $S = \{$ {62, 65}, {5, 88}, {130, 136}, {127, 134}, {117, 128}, {114, 126}, {24, 58}, {21, 123}, {32, 129}, {25, 95}, {29, 54}, {34, 63}, {36, 68}, {56, 125}, {44, 81}, {99, 104}, {55, 102}, {52, 100}, {59, 60}, {67, 135}, {3, 39}, {23, 73} $\}$.

$n = 19$, $S = \{$ {28, 108}, {86, 93}, {76, 124}, {42, 133}, {30, 100}, {89, 193}, {43, 152}, {31, 80}, {64, 175}, {82, 138}, {15, 204}, {69, 95}, {54, 166}, {2, 189}, {107, 178}, {126, 144}, {17, 79}, {88, 91}, {29, 75}, {35, 199}, {3, 173}, {55, 163}, {117, 141}, {90, 145}, {151, 174}, {13, 74}, {83, 114}, {143, 195}, {146, 197}, {6, 10}, {44, 84}, {11, 61}, {177, 202} $\}$.

$n = 25$, $S = \{$ {91, 162}, {20, 153}, {99, 113}, {50, 220}, {71, 148}, {65, 118}, {64, 229}, {240, 260}, {44, 234}, {42, 169}, {181, 255}, {140, 194}, {39, 248}, {97, 109}, {173, 174}, {70, 74}, {152, 261}, {6, 9}, {79, 155}, {132, 266}, {136, 138}, {62, 102}, {196, 230}, {63, 252}, {38, 156}, {133, 160}, {192, 225}, {147, 195}, {84, 116}, {31, 66}, {60, 73}, {127, 182}, {8, 177}, {23, 120}, {105, 184}, {46, 69}, {40, 92}, {61, 90}, {10, 68}, {7, 94}, {154, 165}, {27, 45}, {114, 131}, {32, 89} $\}$.

$n = 31$, $S = \{$ {113, 139}, {83, 245}, {183, 272}, {111, 149}, {234, 290}, {267, 270}, {198, 248}, {24, 128}, {80, 231}, {200, 307}, {106, 334}, {188, 227}, {124, 181}, {102, 283}, {103, 222}, {115, 323}, {132, 186}, {14, 177}, {189, 242}, {82, 328}, {77, 273}, {35, 119}, {184, 208}, {254, 332}, {133, 142}, {33, 123}, {137, 338}, {61, 153}, {37, 136}, {165, 195}, {87, 271}, {140, 217}, {75, 97}, {44, 317}, {28, 174}, {12, 305}, {298, 326}, {284, 325}, {175, 255}, {170, 221}, {86, 236}, {321, 336}, {176, 302}, {13, 225}, {278, 341}, {19, 287}, {193, 276}, {3, 214}, {43, 48}, {68, 171}, {138, 199}, {46, 92}, {244, 309}, {226, 335}, {293, 313} $\}$. □

The following lemma is a combination of Theorem 1 and Lemma 7 in [2]. Here, we need a new concept. A *maximum packing with triangles*, MPT(n), is an ordered triple $(\mathcal{P}, \mathcal{T}, \mathcal{L})$, where $\mathcal{P}$ is the vertex set of K_n, $\mathcal{T}$ is a collection of edge disjoint triangles from the edge set of K_n with $|\mathcal{T}|$ as large as possible, and $\mathcal{L}$ is the collection of edges in K_n not belonging to one of the triangles of $\mathcal{T}$. The collection of edges $\mathcal{L}$ is called the *leave*.

Lemma 4.3 *There exists a* $GS(2,3,n,11)$ *for any prime power* $n \equiv 1 \pmod 6$ *and* $n \geq 43$.

Proof. Apply Theorem 1 and Lemma 7 in [2], it suffices to show that there exists a MPT$(11) = (\mathcal{P}, \mathcal{T}, \mathcal{L})$ with 6 partial parallel classes, which is listed below.

$$\mathcal{P} = \{1, 2, \cdots, 11\}, \mathcal{T} = \bigcup_{i=1}^{6} P_i,\ \mathcal{L} = \{\{2,4\},\{4,3\},\{3,10\},\{10,2\}\}.$$
$$P_1 = \{\{4,8,11\},\{1,5,9\},\{2,6,7\}\};\ P_2 = \{\{1,7,8\},\{2,5,11\},\{6,9,10\}\};$$
$$P_3 = \{\{7,10,11\},\{1,2,3\},\{4,5,6\}\};\ P_4 = \{\{2,8,9\},\{3,5,7\},\{1,4,10\}\};$$
$$P_5 = \{\{5,8,10\},\{1,6,11\},\{3,9,11\}\};\ P_6 = \{\{3,6,8\},\{4,7,9\}\}.$$
□

Lemma 4.4 *There exists a* $GS(2,3,v,11)$ *for all* $v \in F_3 = \{37, 39, 45, 55, 57, 63, 75, 81, 91, 93, 99\}$.

Proof. From Lemmas 4.1 and 4.2, we have a GS$(2,3,n,11)$ for all $n \in F = \{13,15,19,21,25,27,31,33\}$. Apply Lemma 2.10 with $n \in F$, we get a GS$(2,3,v,11)$ for all $v \in F_3$. □

Lemma 4.5 *There exists a GS$(2,3,v,11)$ for all $v \in F_4 = \{85,87\}$.*

Proof. There exist 24 DILS$(24+a,a)$ for $a = 0$ or 1 by Lemma 2.2. There exist also a GS$(2,3,13,11)$ and a GS$(2,3,15,11)$ by Lemma 4.2 and Lemma 4.1. From Lemma 2.10, we have a HGS$(2,3,(37,13),11)$. Apply Lemma 2.3 with $m = 3, n = 24, t = 11, u = 13, a = 0$ or 1, we get a GS$(2,3,85,11)$ or a GS$(2,3,87,11)$ respectively. □

Now, we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4: From Theorem 1.3, we need only to consider the values v, such that $v \in M_{11}$. Lemma 4.3 provides a GS$(2,3,v,11)$ for all $v \in F_5 = \{43,49,61,67,73,79,97,103\}$. It is readily checked that the union of F_i, for $1 \le i \le 5$, is the same as M_{11}. The conclusion then follows. □

Appendix

$n = 21$, $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i,x,y\} : \{x,y\} \in S_i\}$,

S_1 = { {135, 202}, {96, 205}, {74, 159}, {61, 171}, {162, 230}, {121, 187}, {79, 214}, {217, 227}, {38, 181}, {99, 213}, {118, 200}, {75, 229}, {176, 184}, {107, 158}, {179, 208}, {36, 53}, {129, 220}, {109, 116}, {41, 128}, {150, 206}, {104, 139}, {120, 157}, {48, 82}, {12, 14}, {113, 222}, {18, 138}, {71, 173}, {23, 223}, {15, 218}, {17, 130}, {21, 110}, {51, 226}, {93, 178}, {133, 212}, {163, 228}, {90, 183}, {153, 199}, {42, 72} };

S_2 = { {70, 142}, {199, 200}, {56, 204}, {166, 168}, {67, 227}, {45, 88}, {57, 175}, {143, 219}, {14, 164}, {72, 180}, {66, 190}, {77, 90}, {41, 172}, {25, 119}, {135, 157}, {17, 84}, {36, 100}, {74, 122}, {140, 187}, {62, 63}, {68, 207}, {27, 40}, {54, 82}, {52, 109}, {38, 79}, {47, 105}, {93, 108}, {5, 94}, {112, 228}, {85, 215}, {18, 159}, {19, 155}, {42, 129} };

S_3 = { {19, 99}, {26, 227}, {135, 216}, {41, 103}, {35, 44}, {133, 137}, {172, 211}, {28, 83}, {51, 229}, {34, 119}, {23, 226}, {6, 75}, {38, 62}, {116, 215}, {91, 110}, {155, 166}, {151, 177}, {53, 183}, {47, 78}, {140, 197}, {9, 10}, {58, 122}, {55, 80}, {198, 207}, {175, 224}, {168, 222}, {161, 205}, {42, 89}, {43, 79}, {185, 212}, {134, 230}, {109, 196}, {61, 202}, {14, 148}, {27, 98}, {48, 81}, {63, 190}, {65, 188}, {11, 132} }.

$n = 27$, $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i,x,y\} : \{x,y\} \in S_i\}$,

S_1 = { {64, 266}, {147, 208}, {29, 116}, {192, 195}, {97, 230}, {186, 222}, {187, 233}, {139, 268}, {117, 131}, {61, 227}, {21, 149}, {76, 115}, {209, 228}, {94, 98}, {69, 154}, {13, 191}, {25, 264}, {124, 216}, {66, 198}, {239, 265}, {23, 26}, {138, 193}, {133, 182}, {46, 119}, {35, 58}, {104, 287}, {86, 256}, {143, 196}, {232, 282}, {174, 194}, {110, 247}, {19, 236}, {85, 100}, {140, 288}, {170, 262}, {9, 44}, {157, 278}, {132, 151}, {141, 295}, {102, 220}, {59, 210}, {99, 108}, {225, 289}, {155, 215}, {114, 137}, {63, 211}, {60, 176}, {181, 188}, {33, 277}, {105, 126} };

S_2 = { {240, 280}, {79, 149}, {37, 66}, {75, 93}, {27, 158}, {26, 244}, {154, 192}, {6, 289}, {142, 179}, {254, 259}, {215, 232}, {15, 92}, {40, 81}, {140, 216}, {16, 138}, {103, 147}, {127, 144}, {234, 249}, {228, 252}, {65, 237}, {36, 38}, {80, 176}, {187, 193}, {156, 257}, {10, 87}, {53, 123}, {188, 260}, {167, 284}, {50, 286}, {4, 95}, {59, 205}, {32, 219}, {111, 247}, {76, 107}, {115, 197}, {11, 270}, {3, 20}, {204, 211}, {102, 266}, {33, 146}, {52, 54}, {100, 224}, {131, 200} };

$S_3 = \{$ {120, 260}, {47, 102}, {72, 205}, {34, 82}, {105, 228}, {44, 170}, {148, 217}, {89, 145}, {28, 253}, {122, 166}, {257, 263}, {25, 101}, {31, 42}, {203, 244}, {4, 171}, {70, 71}, {94, 214}, {52, 180}, {141, 173}, {77, 124}, {161, 190}, {96, 249}, {69, 277}, {226, 290}, {63, 159}, {9, 243}, {14, 189}, {16, 216}, {33, 295}, {8, 29}, {140, 274}, {206, 272}, {129, 233}, {112, 212}, {92, 150}, {118, 288}, {108, 293}, {7, 210}, {182, 194}, {239, 255}, {55, 213}, {40, 251}, {56, 78}, {46, 117}, {185, 267}, {51, 59}, {176, 222}, {49, 220}, {181, 197}, {95, 258} }.

$n = 33$, $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i, x, y\} : \{x, y\} \in S_i\}$,

$S_1 = \{$ {168, 306}, {3, 7}, {147, 225}, {299, 324}, {140, 343}, {136, 339}, {142, 354}, {18, 123}, {48, 222}, {235, 246}, {164, 336}, {282, 333}, {46, 138}, {237, 326}, {61, 323}, {58, 281}, {148, 268}, {33, 40}, {11, 137}, {45, 139}, {179, 301}, {120, 187}, {8, 15}, {106, 214}, {162, 262}, {207, 340}, {212, 249}, {111, 113}, {50, 125}, {16, 362}, {74, 248}, {351, 356}, {37, 193}, {77, 80}, {124, 144}, {175, 194}, {51, 122}, {70, 134}, {26, 273}, {202, 286}, {54, 114}, {105, 185}, {53, 353}, {110, 303}, {43, 55}, {30, 95}, {66, 311}, {126, 183}, {261, 283}, {24, 318}, {36, 314}, {38, 211}, {96, 289}, {19, 145}, {206, 253}, {49, 348}, {63, 200}, {128, 271}, {2, 272}, {47, 215}, {198, 327}, {59, 135}, {56, 169} };

$S_2 = \{$ {163, 224}, {64, 268}, {112, 347}, {212, 217}, {162, 303}, {57, 72}, {109, 189}, {45, 359}, {6, 80}, {308, 317}, {29, 305}, {82, 254}, {21, 339}, {58, 114}, {116, 337}, {56, 215}, {69, 309}, {73, 156}, {84, 137}, {123, 263}, {105, 348}, {31, 210}, {141, 218}, {131, 182}, {127, 296}, {331, 361}, {28, 274}, {32, 198}, {257, 321}, {98, 295}, {119, 164}, {76, 165}, {63, 148}, {208, 298}, {16, 293}, {33, 221}, {213, 343}, {61, 279}, {12, 42}, {179, 330}, {97, 280}, {46, 322}, {150, 325}, {139, 227}, {178, 183}, {83, 275}, {144, 345}, {125, 159}, {158, 235}, {70, 86}, {333, 352}, {44, 262}, {192, 326}, {34, 283}, {25, 234}, {23, 157} };

$S_3 = \{$ {292, 330}, {104, 105}, {217, 289}, {52, 138}, {159, 363}, {239, 259}, {112, 276}, {84, 190}, {11, 345}, {46, 240}, {90, 211}, {95, 358}, {235, 353}, {14, 257}, {62, 181}, {53, 265}, {164, 295}, {131, 146}, {87, 128}, {115, 166}, {55, 314}, {327, 350}, {125, 249}, {12, 110}, {116, 152}, {30, 287}, {233, 268}, {167, 272}, {71, 99}, {298, 307}, {189, 293}, {96, 150}, {193, 308}, {111, 255}, {98, 122}, {73, 76}, {269, 360}, {50, 238}, {86, 175}, {27, 348}, {291, 325}, {66, 251}, {75, 221}, {82, 236}, {91, 197}, {15, 183}, {117, 340}, {58, 174}, {216, 230}, {79, 344}, {, 4, 156}, {139, 320}, {151, 182}, {41, 208}, {100, 127}, {266, 278}, {51, 161} }.

$n = 51$, $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i, x, y\} : \{x, y\} \in S_i\}$,

$S_1 = \{$ {24, 74}, {71, 429}, {21, 58}, {87, 106}, {149, 283}, {116, 314}, {14, 327}, {376, 485}, {138, 548}, {371, 374}, {12, 514}, {216, 284}, {67, 435}, {405, 560}, {96, 320}, {217, 367}, {16, 310}, {418, 478}, {259, 416}, {195, 360}, {294, 510}, {47, 372}, {258, 488}, {408, 440}, {221, 248}, {134, 220}, {44, 319}, {160, 501}, {184, 353}, {541, 550}, {28, 97}, {114, 389}, {378, 387}, {299, 441}, {55, 227}, {247, 339}, {40, 424}, {214, 296}, {168, 226}, {130, 442}, {177, 555}, {62, 251}, {190, 479}, {4, 69}, {218, 430}, {497, 498}, {37, 270}, {370, 426}, {191, 223}, {132, 471}, {292, 344}, {203, 233}, {125, 382}, {275, 540}, {189, 558}, {72, 172}, {439, 445}, {76, 108}, {274, 401}, {255, 444}, {325, 423}, {228, 359}, {236, 365}, {31, 375}, {467, 484}, {153, 352}, {94, 518}, {415, 468}, {286, 489}, {20, 477}, {84, 140}, {420, 475}, {222, 263}, {208, 213}, {300, 538}, {2, 323}, {51, 179}, {9, 123}, {393, 427}, {36, 264}, {30, 481}, {65, 432}, {361, 521}, {231, 463}, {117, 174} };

$S_2 = \{$ {61, 514}, {93, 196}, {68, 295}, {14, 322}, {121, 166}, {134, 286}, {285, 445}, {143, 534}, {96, 112}, {63, 78}, {418, 508}, {235, 433}, {480, 488}, {276, 453}, {375, 482}, {299, 425}, {231, 412}, {416, 503}, {287, 528}, {188, 237}, {177, 537}, {123, 176}, {25, 264}, {9, 420}, {157, 173}, {27, 371}, {120, 165}, {195, 215}, {81, 325}, {10, 458}, {309, 314}, {15, 160}, {26, 540}, {50, 400}, {113, 158}, {91, 290}, {66, 427}, {111, 209}, {48, 248}, {118, 468}, {116, 496}, {232, 557}, {442, 464}, {167, 404}, {186, 318}, {58, 358}, {102, 397}, {94, 421}, {92, 292}, {38, 254}, {483, 484}, {335, 355}, {133, 553}, {293, 362}, {52, 399},{55, 169}, {184, 348}, {114, 478}, {253, 414}, {529, 554}, {183, 463}, {179, 349}, {42, 151}, {31, 271}, {7, 198}, {117, 429}, {30, 341}, {54, 455}, {35, 282}, {303, 387}, {127, 320}, {40, 208}, {126, 561}, {95, 152}, {181, 440}, {72, 338}, {345, 383}, {305, 376}, {37, 302}, {460, 462}, {12, 446}, {187, 234}, {18, 451}, {178, 401}, {17, 261}, {283, 479}, {137, 192}, {44, 542}, {70, 87}, {221, 319}, {103, 443}, {526, 559}, {28, 256}, {69, 109}, {226, 477}, {145, 409}, {185, 247} };

S_3 = { {125, 356}, {418, 458}, {445, 521}, {419, 491}, {243, 409}, {249, 262}, {200, 527}, {211, 253}, {77, 287}, {280, 420}, {268, 394}, {213, 359}, {28, 415}, {85, 92}, {20, 164}, {24, 456}, {27, 291}, {73, 378}, {159, 275}, {166, 395}, {319, 416}, {140, 526}, {126, 285}, {176, 344}, {69, 81}, {82, 428}, {255, 303}, {144, 209}, {365, 522}, {242, 444}, {46, 109}, {506, 545}, {232, 304}, {13, 463}, {182, 487}, {308, 396}, {116, 244}, {502, 520}, {49, 155}, {6, 289}, {237, 296}, {528, 542}, {47, 530}, {95, 560}, {199, 230}, {410, 550}, {151, 316}, {181, 482}, {115, 143}, {175, 393}, {120, 523}, {228, 277}, {154, 338}, {377, 474}, {75, 322}, {439, 533}, {183, 459}, {99, 273}, {72, 100}, {236, 431}, {357, 504}, {261, 292}, {229, 460}, {345, 558}, {270, 385}, {245, 256}, {80, 246}, {65, 538}, {366, 477}, {78, 461}, {70, 161}, {212, 266}, {311, 358}, {14, 293}, {91, 490}, {30, 382}, {483, 501}, {340, 525}, {328, 484}, {215, 429}, {127, 185}, {170, 188}, {165, 434}, {96, 333}, {74, 341}, {122, 426}, {326, 430}, {128, 208}, {254, 465}, {198, 274}, {33, 224}, {392, 531}, {57, 475} }.

$n = 69$, $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i, x, y\} : \ \{x, y\} \in S_i\}$,

S_1 = { {239, 535}, {13, 63}, {249, 563}, {469, 524}, {127, 369}, {316, 715}, {212, 513}, {371, 485}, {110, 459}, {103, 231}, {179, 645}, {60, 759}, {267, 630}, {402, 561}, {589, 704}, {284, 305}, {25, 322}, {490, 504}, {169, 384}, {258, 664}, {279, 565}, {621, 754}, {108, 257}, {532, 716}, {156, 302}, {26, 638}, {349, 562}, {290, 541}, {512, 613}, {531, 633}, {383, 597}, {368, 430}, {235, 625}, {618, 699}, {649, 757}, {648, 733}, {523, 560}, {217, 718}, {323, 673}, {45, 171}, {176, 423}, {247, 707}, {37, 388}, {117, 221}, {372, 694}, {101, 334}, {2, 377}, {416, 519}, {79, 270}, {324, 345}, {612, 671}, {262, 709}, {441, 538}, {536, 577}, {162, 701}, {339, 741}, {71, 745}, {242, 453}, {81, 351}, {264, 362}, {123, 392}, {380, 444}, {57, 672}, {269, 274}, {40, 584}, {131, 616}, {23, 357}, {272, 424}, {330, 404}, {367, 725}, {336, 706}, {174, 688}, {332, 462}, {155, 161}, {69, 381}, {75, 550}, {206, 253}, {154, 268}, {501, 696}, {32, 712}, {118, 546}, {198, 703}, {160, 213}, {12, 266}, {163, 303}, {20, 692}, {95, 355}, {5, 147}, {263, 636}, {293, 667}, {178, 428}, {164, 735}, {340, 350}, {245, 256}, {729, 731}, {413, 600}, {233, 436}, {374, 714}, {210, 750}, {447, 702}, {228, 488}, {182, 288}, {177, 574}, {168, 250}, {254, 385}, {397, 529}, {422, 742}, {50, 497}, {496, 637}, {314, 674}, {170, 593}, {173, 717}, {586, 595}, {6, 36}, {128, 465}, {151, 158}, {100, 457}, {472, 596}, {165, 639}, {722, 758}, {414, 480}, {21, 552}, {260, 575}, {150, 687}, {483, 528}, {149, 329}, {39, 521}, {744, 753}, {34, 551}, {82, 685}, {328, 752}, {3, 343} };

S_2 = { {504, 716}, {74, 164}, {32, 572}, {635, 660}, {119, 655}, {44, 254}, {184, 703}, {50, 223}, {53, 670}, {181, 481}, {41, 200}, {77, 142}, {183, 480}, {113, 131}, {136, 415}, {534, 596}, {147, 316}, {348, 681}, {237, 284}, {272, 745}, {110, 570}, {138, 578}, {148, 443}, {96, 574}, {488, 651}, {177, 482}, {22, 453}, {287, 658}, {193, 239}, {590, 727}, {544, 748}, {283, 738}, {365, 397}, {427, 617}, {48, 539}, {86, 160}, {80, 548}, {410, 522}, {292, 389}, {333, 591}, {150, 567}, {36, 218}, {174, 426}, {24, 649}, {305, 616}, {315, 633}, {95, 109}, {331, 417}, {107, 204}, {461, 559}, {145, 229}, {562, 741}, {558, 694}, {155, 420}, {122, 543}, {104, 290}, {251, 662}, {87, 622}, {127, 225}, {230, 231}, {450, 629}, {286, 611}, {438, 560}, {236, 328}, {234, 484}, {59, 364}, {159, 728}, {214, 756}, {332, 390}, {106, 547}, {121, 310}, {383, 392}, {102, 157}, {78, 378}, {156, 361}, {363, 494}, {33, 530}, {68, 219}, {143, 158}, {515, 685}, {532, 665}, {69, 247}, {421, 516}, {73, 163}, {198, 356}, {5, 243}, {151, 179}, {319, 697}, {263, 675}, {597, 729}, {304, 325}, {56, 83}, {271, 553}, {642, 679}, {258, 721}, {587, 747}, {552, 723}, {648, 701}, {266, 521}, {429, 610}, {39, 93}, {329, 369}, {28, 208}, {45, 245}, {295, 638}, {54, 569}, {296, 424}, {514, 557}, {366, 726}, {440, 464}, {341, 404}, {536, 669}, {171, 535}, {259, 311}, {555, 678} };

S_3 = { {415, 456}, {320, 713}, {286, 407}, {276, 355}, {511, 746}, {558, 738}, {36, 356}, {269, 463}, {53, 377}, {362, 715}, {73, 294}, {554, 689}, {437, 625}, {94, 635}, {268, 469}, {148, 720}, {59, 313}, {246, 623}, {419, 752}, {87, 758}, {178, 596}, {117, 663}, {229, 584}, {75, 529}, {281, 576}, {68, 661}, {568, 600}, {436, 509}, {236, 697}, {47, 490}, {186, 641}, {329, 528}, {234, 297}, {31, 227}, {39, 79}, {108, 251}, {188, 238}, {143, 734}, {29, 482}, {267, 755}, {305, 612}, {432, 445}, {144, 247}, {123, 679}, {691, 754}, {440, 453}, {204, 434}, {37, 195}, {74, 638}, {226, 287}, {52, 112}, {99, 442}, {285, 522}, {54, 290}, {271, 743}, {140, 370}, {594, 683}, {403, 426}, {278, 536}, {154, 670}, {556, 685}, {193, 643}, {70, 376}, {155, 513}, {32, 673}, {98, 291}, {327, 578}, {452, 464}, {371, 487}, {128, 205}, {332, 684}, {557, 637}, {365, 448}, {299, 354}, {90, 266}, {44, 157}, {160, 479}, {295, 383}, {21, 705}, {11, 121}, {337, 631}, {14, 109}, {292, 644}, {170, 619}, {454, 647}, {114, 330}, {264, 526}, {15, 393}, {358, 756}, {580, 610}, {249, 470}, {55, 175}, {397, 502}, {177, 736}, {132, 342}, {228, 245}, {388, 606}, {93, 270}, {158, 523}, {328, 669}, {187, 597}, {110, 733}, {25, 138}, {156, 446}, {306, 472}, {196, 503}, {71, 488}, {97, 408}, {359, 645}, {201, 473}, {103, 577}, {200, 223}, {6, 277}, {318, 745}, {241, 747}, {80, 150}, {199, 680}, {387, 414}, {368, 385}, {282, 654}, {119, 369}, {441, 652}, {192, 353}, {392, 658}, {137, 723}, {653, 714}, {7, 574}}.

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