On Minimal Symmetric Automorphism Groups of Finite Symmetric Graphs

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Abstract: Let Γ be a finite connected graph, and G a group of automorphisms of Γ which is transitive on vertices. Suppose that, for a vertex α of Γ, $S \leq G_\alpha^{\Gamma(\alpha)} \leq Aut\ S$ for some simple group S with S acting primitively on the set $\Gamma(\alpha)$ of neighbours of α, and suppose that G is minimal with these properties. Then one of : (i) G is a nonabelian simple group, (ii) Γ is a Cayley graph for a normal subgroup N of G and $G = N.S$, (iii) Γ is bipartite, (iv) Γ is a proper cover of a graph of the same valency and with the same properties. In the special case where Γ has prime valency this is a result of Lorimer. More details of the structure of Γ and G are obtained for graphs which satisfy (ii) or (iii) but are not proper covers as in (iv). Constructions are given for several families of examples.

1. Introduction

Let Γ be a finite connected graph and G a subgroup of the automorphism group $Aut\ \Gamma$ of Γ. Then G is said to be *symmetric* on Γ if G is transitive on the set $\Gamma_1 = \{(\alpha, \beta) \mid \{\alpha, \beta\} \text{ an edge}\}$ of ordered pairs of adjacent vertices of Γ, and G is *minimal symmetric* if it is symmetric on Γ but no proper subgroup of G is symmetric on Γ. Note that symmetric graphs (that is those whose automorphism groups act symmetrically) are vertex transitive and hence are *regular*. The motivation for this paper came from a theorem of Lorimer [4] about minimal symmetric groups G of automorphisms of a connected graph Γ of prime valency v. Lorimer showed, essentially, that one of the following is true for such G, Γ:

(i) G is a nonabelian simple group;

(ii) Γ is a Cayley graph for a normal subgroup N of G and $G = N.Z_v$;

(iii) Γ is bipartite;

(iv) Γ is a proper cover of a graph Σ of valency v such that the group of automorphisms induced by G on Σ is minimal symmetric.

In [4] the result is stated differently: extra assumptions are made about Γ so that the only conclusion is that G is simple. (More information about the bipartite case (iii) is given in [6]). This expanded version of Lorimer's theorem has been stated in this way so as to highlight the inductive role of case (iv) for reducing to graphs satisfying one of (i) to (iii). For this to be a useful reduction we need more information about the graphs arising in cases (i) to (iii) which are not proper covers of smaller examples. The aim of the paper is to look at the graphs arising in cases (ii) and (iii) in detail for a larger class of symmetric graphs than the class with prime valency.

If $G \leq Aut\ \Gamma$ acts symmetrically on a connected graph Γ of prime valency v then, for a vertex α, the stabilizer G_α is transitive on the set $\Gamma(\alpha)$ of v neighbours of α in Γ, since G is transitive on Γ_1. Moreover it was shown by Burnside, see [9], that a transitive permutation group of prime degree v (such as the group $G_\alpha^{\Gamma(\alpha)}$ induced by G_α on $\Gamma(\alpha)$) has a simple normal subgroup, so we have $S \leq G_\alpha^{\Gamma(\alpha)} \leq Aut\ S$ for some simple group S. Note that S might be cyclic of order v. Moreover the group S acts primitively on $\Gamma(\alpha)$. (A permutation group on a set X is said to be *primitive* on X if it is transitive on X and the only partitions of X invariant under the group are the trivial ones, namely $\{X\}$, and $\{\{x\} \mid x \in X\}$.)

We shall be concerned with finite connected graphs with a group G of automorphisms which is symmetric on Γ such that, for a vertex α, $S \leq G_\alpha^{\Gamma(\alpha)} \leq Aut\ S$ with S simple and primitive on $\Gamma(\alpha)$. First we show that the conclusions of Lorimer's theorem hold for such graphs.

Theorem 1 Let Γ be a finite connected graph of valency v and $G \leq Aut\ \Gamma$. Suppose that G is symmetric on Γ and that, for a vertex $\alpha, S \leq G_\alpha^{\Gamma(\alpha)} \leq Aut\ S$ where S is simple and S is primitive on $\Gamma(\alpha)$. If G is minimal with respect to these properties then one of the following holds:

(i) G is a nonabelian simple group;

(ii) Γ is a Cayley graph for a normal subgroup N of G and $G = N.S$;

(iii) Γ is bipartite;

(iv) Γ is a proper cover of a graph Σ of valency v such that the group of automorphisms of Σ induced by G is symmetric, $S \leq G_\sigma^{\Sigma(\sigma)} \leq Aut\ S$ with S primitive on $\Sigma(\sigma)$ (σ a vertex of Σ), and is minimal with respect to these properties.

A more detailed version of this theorem will be proved in section 2. A graph Γ is said to be a *cover* of a graph Σ is there is an epimorphism $\phi : V\Gamma \to V\Sigma$ of the vertex sets which maps edges to edges (that is, if $\{\alpha, \beta\}$ is an edge of Γ then $\{\alpha\phi, \beta\phi\}$ is an edge of Σ) and is such that ϕ induces an isomorphism from the induced subgraph on $\Gamma(\alpha)$ to the induced subgraph on $\Sigma(\alpha\phi)$ for all vertices α of Γ. If ϕ is not a bijection then Γ is said to be a *proper cover* of Σ.

In the rest of the paper we analyse cases (ii) and (iii) of Theorem 1 in more detail assuming that the graphs are not proper covers as in (iv). We construct classes of examples in each case. We do not address the problem of constructing covers of our examples with the required symmetry properties. Some constructions of this type are given by Biggs [**1**].

Problem Given Γ, G, S, v as in Theorem 1 (i) to (iii), find all connected graphs $\bar{\Gamma}$ which are proper covers of Γ and have a group $\bar{G}$ of automorphisms which is symmetric on $\bar{\Gamma}$ and induces the group G on Γ.

Recall that a *Cayley graph* for a group G is a graph $\Gamma = Cay(G, X)$ (where $X \subseteq G, 1_G \notin X$ and if $x \in X$ then $x^{-1} \in X$) with vertex set G such that g and h are adjacent if and only if $gh^{-1} \in X$. Such a graph admits G, acting by right multiplication, as a regular group of automorphisms. Also $Cay(G, X)$ is connected if and only if $< X > = G$. A graph Γ is a Cayley graph if and only if $Aut\ \Gamma$ contains a subgroup N acting regularly on vertices (that is N is transitive and $N_\alpha = 1$), and in that case the vertex set of Γ may be identified with N so that N acts by right multiplication and, if α is identified with $1_N, \Gamma \simeq Cay(N, \Gamma(\alpha))$. We obtain a more detailed description of the structure of Γ and G for case (ii) of Theorem 1 in Theorem 2 below.

Theorem 2 Let Γ, G, S, v be as in Theorem 1 and suppose that case (ii) holds so that N is a normal subgroup of $G = N.S$ and $\Gamma = Cay(N, X)$ for some $X = X^{-1} \subset N, 1_N \notin X, N = < X >$. Suppose also that case (iv) does not hold. Then one of the following happens:

(i) $\Gamma = C_p, N = Z_p, p$ a prime, $v = \mid S \mid = 2$;

(ii) $N = Z_2^k$ and $S \leq GL(k, 2)$ is irreducible;

(iii) $N = S$ is nonabelian, $G = S \times S$, and X is a conjugacy class of involutions of S such that $C_S(x)$ is maximal in S for $x \in X$;

(iv) $N = T^k$ for some nonabelian simple group T, and $G = NS \leq Aut\ T\ wr\ S_k$ such that S projects onto a transitive subgroup of S_k, and X is an S-orbit of involutions in N such that $C_S(x)$ is maximal in S for $x \in X$.

This result will be proved in section 2. Our investigations of the bipartite case lead to Theorem 3, which will be proved in Section 3.

Theorem 3 Let Γ, G, S, v be as in Theorem 1 and suppose that case (iii) holds so that Γ is bipartite, but that case (iv) does not hold. Then G has a minimal normal subgroup $N = T^k$, where T is simple and $k \geq 1$, such that N has two orbits in $V\Gamma, G = < NG_\alpha, a >$ for some 2-element a, and one of the following holds:

(i) $\Gamma = K_{v,v}$, and $S = T$;

(ii) $T = Z_p$ for some prime p and one of

(a) $\Gamma = C_{2p}, N = Z_p, v = \mid S \mid = 2$;

(b) Γ is a Cayley graph for $N.2$, an abelian normal subgroup of $G, p = 2$, and $G_\alpha = S \leq GL(k, 2)$ is irreducible;

(c) $G \leq AGL(k, p)$ and $G/N \leq GL(k, p)$ is irreducible on $N = Z_p^k$;

(iii) T is a nonabelian simple group and N is semiregular and one of

(a) $N = T = S, \Gamma = Cay(S.2, X)$ where $X = X^{-1} \subseteq S.2 \setminus S$ is an S-conjugacy class of involutions such that $C_S(x)$ is maximal in S for $x \in X$; (Here $S.2 \leq Aut\ S$.)

(b) $N = T = S$, and $G = NH$ is contained in the holomorph of N with $N \cap H = 1, S \leq H \leq Aut\ S$ and $H/S = Z_{2^s}$ for some $s \geq 1$;

(c) $G \leq Aut\ N$;

(iv) T is a nonabelian simple group, N is not semiregular, $G \leq Aut\ N, G/N = Z_{2^s}, s \geq 1$, and one of

(a) $N = T$;

(b) $k > 1, k$ divides 2^s, and, for $\alpha \in V\Gamma, N_\alpha$ is a subgroup of a diagonal subgroup of $N = T^k$.

A *diagonal subgroup* of $T_1 \times \ldots \times T_k$, where each $T_i \simeq T$, is a subgroup of the form $\{(t^{\phi_1}, \ldots, t^{\phi_k}) \mid t \in T\} \simeq T$ where the $\phi_i \in Aut\ T$.

A discussion of the graphs which arise, including constructions for several families of examples, is given in section 4.

2. Proofs of Theorem 1 and Theorem 2.

In this section we prove the following technical version of Theorem 1. Then we begin the more detailed analysis of the various cases leading to a proof of Theorem 2.

Proposition 2.1 Let Γ be a finite connected graph of valency v and let $G \leq Aut\ \Gamma$. Suppose that G is symmetric on Γ and, for $\alpha \in V\Gamma, S \leq G_\alpha^{\Gamma(\alpha)} \leq Aut\ S$ with S simple and primitive on $\Gamma(\alpha)$; and let G be minimal with respect to these properties. Let N be a minimal normal subgroup of G, so that $N \simeq T^k$ for some simple group T and integer $k \geq 1$. Then one of the following holds:

(i) $G = N = T$ is a nonabelian simple group;

(ii) $G = NS$ and Γ is a Cayley graph for N;

(iii) Γ is bipartite, N has two orbits on vertices, and either

(a) $\Gamma = K_{v,v}, v = \mid S \mid$ is prime, $S = T, N = S \times S$;

(b) N is semiregular, so $\mid V\Gamma \mid = 2 \mid N \mid$;

(c) $S \leq N_\alpha^{\Gamma(\alpha)} \leq Aut\ S, G = < a, N >, N$ is nonabelian, $G/N = < aN > = Z_{2^s}$ with k dividing $2^s, s \geq 1$.

(iv) Γ is a proper cover of some graph Σ with valency v and the group of automorphisms induced by G on Σ is symmetric, $S \leq G_\sigma^{\Sigma(\sigma)} \leq Aut\ S$ with S primitive on $\Sigma(\sigma)$ (σ a vertex of Σ), and is minimal with respect to these properties.

If a normal subgroup N of G has more than two orbits on vertices then the *quotient graph* Γ_N is defined as the graph with vertices the N-orbits, two N-orbits being adjacent in Γ_N if some vertex in one is joined to some vertex in the other N-orbit. In the proof of this proposition the graph Σ in (iv) will be Γ_N, but in subsequent lemmas Σ may be Γ_C for some other normal subgroup C of G.

Proof The subgroup $N_\alpha^{\Gamma(\alpha)}$ is normal in $G_\alpha^{\Gamma(\alpha)}$. Suppose first that $N_\alpha^{\Gamma(\alpha)} = 1$. Then, by the connectivity of $\Gamma, N_\alpha = 1$. If N has more than two orbits then, by [7, Lemma 1.6 and the remarks following], the quotient graph Γ_N with N-orbits as vertices is such that the group of automorphisms induced by G is G/N; this group is symmetric on Γ_N and the group induced by the stabilizer of an N-orbit on its neighbours in Γ_N is permutation isomorphic to $G_\alpha^{\Gamma(\alpha)}$. In particular Γ_N has valency v and (iv) holds with $\Sigma = \Gamma_N$. If N has two orbits then (iii) (b) holds, while if N is transitive then Γ is a Cayley graph for N and $G = NG_\alpha$. In the latter case G_α acts faithfully on $\Gamma(\alpha)$, and by the minimality of G we have $G_\alpha = S, G = NS$.

So now suppose that $N_\alpha^{\Gamma(\alpha)} \neq 1$, whence $S \leq N_\alpha^{\Gamma(\alpha)} \leq Aut\ S$. The group $M := < N_\beta \mid \beta \in V\Gamma >$ is normal in G and by connectivity has one or two orbits in $V\Gamma$. As N is a minimal normal subgroup and $M \leq N$, it follows that $N = M$ has 1 or 2 orbits. If N is transitive on $V\Gamma$ then, by the minimality of G, we have $G = N$, whence G is a nonabelian simple group. So suppose that N has two orbits. Then Γ is bipartite. If N is abelian then $N_\alpha \simeq S$ has prime order v and by [8, Lemma 1.1], $\Gamma = K_{v,v}$ and $T = S = Z_v, N = S \times S$. So suppose that N is non-abelian. There is a 2-element $a \in G \setminus N$ which interchanges the two N-orbits and, by minimality, $G = < N, a >$ so G/N is cyclic of order 2^s for some $s \geq 1$. Since N is a minimal normal subgroup, $< a >$ permutes the simple direct factors of V transitively so k divides 2^s.

Before starting on a more detailed analysis of the cases in Proposition 2.1 we make the following observations about the structure of a stabilizer G_α. It will be used many times in the subsequent arguments.

Lemma 2.2 If Γ, G, S are as in Proposition 2.1 then G_α has exactly one composition factor isomorphic to S, and every other composition factor has order less than $\mid S \mid$.

Proof For $i \geq 1$ let $G_i(\alpha)$ denote the subgroup of G_α fixing pointwise every vertex distant at most i from α. If $G_i(\alpha) \neq 1$ then, for each vertex β at distance i from $\alpha, G_i(\alpha)^{\Gamma(\beta)}$ is a subgroup of $G_\beta^{\Gamma(\beta)}$ fixing a point of $\Gamma(\beta)$ at distance $i-1$ from α. Consequently every composition factor of $G_i(\alpha)^{\Gamma(\beta)}$ is a composition factor of some proper subgroup of S or of $Out\ S$ and hence has order less than $\mid S \mid$. (This uses the simple group classification in the assertion that $\mid Out\ S \mid < \mid S \mid$.) Thus each composition factor of $G_i(\alpha)/G_{i+1}(\alpha)$ has order less than $\mid S \mid$. This, together with the fact that $G_\alpha^{\Gamma(\alpha)}$ has one composition factor S and all others have order less than $\mid S \mid$, completes the proof.

First we consider the case where the subgroup N of Proposition 2.1 is abelian.

Lemma 2.3 Let Γ, G, S, N, v be as in Proposition 2.1 with N abelian.

(i) If Proposition 2.1(ii) holds then either

  (a) $\Gamma = C_p, p$, a prime, $v = \mid S \mid = 2$, or

  (b) $N = Z_2^k, S \leq GL(k,2), S$ irreducible.

(ii) If Proposition 2.1(iii) holds then either

  (a) $\Gamma = K_{v,v}$, and $v = \mid S \mid$ is prime, or

  (b) $\Gamma = C_{2p}, N = Z_p, p$ a prime, $v = \mid S \mid = 2$, or

  (c) $N = Z_2^k, \Gamma$ is a Cayley graph for $C_G(N) = N.2$, an abelian regular normal subgroup of $G, G = C_G(N).G_\alpha$, and $G_\alpha = S \leq GL(k,2)$ is irreducible, or

  (d) $N = Z_p^k$, and $G \leq AGL(k,p)$ with $G/N \leq GL(k,p)$ irreducible.

Proof In case (ii), $\Gamma = Cay(N, X)$ with $X \subseteq N, X = X^{-1}$, and if α is the vertex identified with 1_N then G_α acts primitively on $\Gamma(\alpha) = X$. For $x \in X, \{x, x^{-1}\}$ is a block of imprimitivity for G_α in $\Gamma(\alpha)$ and hence either $x = x^{-1}$ whence $N = Z_2^k$, or $\Gamma(\alpha) = \{x, x^{-1}\}$ whence $v = 2$ and Γ is a cycle. In the latter case Γ is a cycle of prime length since N is a minimal normal subgroup. In the former case N is self-centralizing in G (see [9]) so $G_\alpha = S \leq GL(k, 2)$ and by the minimality of N, S is irreducible.

In case (iii) either $\Gamma = K_{v,v}, v = \mid S \mid$ prime, or (iii)(b) holds. Suppose the latter, so N is semiregular with two orbits. Let $C = C_G(N)$. If $C = N$ then (ii)(d) holds, so assume that $C > N$. If $C_\alpha \neq 1$ then $\Gamma = K_{v,v}$ with $v = \mid S \mid$ prime. So assume that $C_\alpha = 1$. Then C is regular and abelain and $\Gamma = Cay(C, X)$ for some $X \subset C, X = X^{-1}$. If α is identified with 1_C then $X \subset C \setminus N$, and as above either $\mid X \mid = 2$ whence $\Gamma = C_{2p}, p$ a prime, or X consists of involutions. In the latter case $N = Z_2^k$ and $G_\alpha \leq Aut\ N = GL(k, 2)$. As Γ is connected G_α is faithful on $\Gamma(\alpha)$ and by minimality $G = CG_\alpha = C.S$ and S is irreducible.

From now on we may assume that N is nonabelian, so $N = T^k, T$ a nonabelian simple group. Next we treat the case where the centralizer C of N in G is nontrivial. Then G is not simple and, if C has more than two orbits then Proposition 2.1 (iv) holds with $\Sigma = \Gamma_C$. The following lemma deals with the case where C has at most two orbits.

Lemma 2.4 Let Γ, G, S, N, v be as in Proposition 2.1 with N nonabelian, and suppose that the centralizer C of N in G is nontrivial. Suppose further that Proposition 2.1(iv) does not hold. Then $N = T = S$.

(i) If Proposition 2.1(ii) holds then $C \simeq S, G = N \times C \simeq S \times S$ and $\Gamma = Cay(S, X)$ where X is a conjugacy class of involutions in S such that $C_S(x)$ is maximal in S for $x \in X$.

(ii) If Proposition 2.1(iii) holds then we are in case (iii)(b) and either $C = S$ or $C = S.2 \leq Aut\ S$. Moreover either

  (a) $\Gamma = Cay(S.2, X)$ for $X \subseteq (S.2) \setminus S$ an S-conjugacy class of involutions in $S.2$, such that $C_S(x)$ is maximal in S for $x \in X$, or

  (b) $C = S$ and G is contained in the holomorph of $N = S$ with $G = N.H, N \cap H = 1, S \leq H \leq Aut\ S, H/S = Z_{2^s}$, for some $s \geq 1$.

Proof Since Proposition 2.1(iv) does not hold, C has 1 or 2 orbits. If N were not semiregular then Proposition 2.1 (iii)(c) would hold and we would obtain $\Gamma = K_{v,v}, N = S \times S$, and C would have at least one composition factor S so that $(N \times C)_\alpha$ would have at least two composition factors S contradicting Lemma 2.2. Thus N is semiregular so that we are in case (ii) or case (iii)(b) of Proposition 2.1. Moreover C induces on each N-orbit a group isomorphic to T^k so that $(N \times C)_\alpha$ has a normal subgroup isomorphic to T^k and this normal subgroup must act nontrivially on $\Gamma(\alpha)$. It follows from Lemma 2.2 that $T = S, k = 1$ and either $C = S$, or we are in case (iii)(b) and $C = S.2$. In case (ii) of Proposition 2.1 $\Gamma = Cay(S, X)$ with X a self-inverse conjugacy class of S and as $G_\alpha = S$ is primitive on $\Gamma(\alpha) = X$, it follows that X consists of involutions and $C_S(x)$ is maximal in S for $x \in X$. In case (iii)(b) of Proposition 2.1, if $C = S.2$ then $\Gamma = Cay(C, X)$ with $X \subseteq C \setminus S$ a self-inverse S-conjugacy class in C. Since Proposition 2.1(iv) does not hold, $C = S.2 \leq Aut\ S$ (that is $C \neq S \times Z_2$). As S is primitive on X, X consists of involutions and $C_S(X)$ is maximal in S for $x \in X$. Similarly if $C = S$ and $C_G(C) = N.2 = S.2$ then $\Gamma = Cay(S.2, X)$ for some X satisfying the above properties. If $C_G(C) = N$ then G is contained in the holomorph of $N = S$. Thus $G = N.H$, where $N \cap H = 1, S \leq H \leq Aut\ S$. By minimality $G = < N \times C, y >$ where y is a 2-element which interchanges the bipartite halves, and so $H/S \simeq Z_{2^s}$ for some $s \geq 1$.

In case (ii) of Proposition 2.1, if N has trivial centralizer then $N = T^k$ with T a nonabelian simple group, $\Gamma = Cay(N, X)$, and $G = NS \leq Aut\ N = Aut\ T\ wr\ S_k$. Moreover S must permute the simple direct factors of N and so S must project onto a transitive subgroup of S_k; and as S is primitive on $\Gamma(\alpha) = X$, where $\alpha = 1_N, X$ is an S-orbit of involutions. Theorem 2 follows from Lemmas 2.3 and 2.4, and these remarks.

3. The bipartite case

Let Γ, G, S, v be as in case (iii) of Proposition 2.1, and hence case (iii) of Theorem 1, so that Γ is bipartite and some minimal normal subgroup N of G has two orbits on $V\Gamma$. The cases where N has nontrivial centralizer were treated in

Lemmas 2.3 and 2.4, so we may assume that $N = T^k = T_1 \times \ldots \times T_k$ for some nonabelian simple group T and integer $k \geq 1$, and that $G \leq Aut\ N = Aut\ T\ wr\ S_k$. Let Δ_1 and Δ_2 be the two N-orbits in $V\Gamma$. We shall investigate the action of several normal subgroups of N on Γ.

Lemma 3.1 If the pointwise stabilizer $N_{(\Delta_1)}$ of Δ_1 in N is nontrivial then $\Gamma = K_{v,v}$, and $N = S \times S \leq G \leq Aut\ S\ wr\ Z_2$.

Proof Let $M_1 = N_{(\Delta_1)} \neq 1$. We may assume that $T_1 \leq M_1$. We are in case (iii)(c) of Proposition 2.1, so $G = < a, N >$ and we may assume that $T_i^a = T_{i+1}$, for all i, taking subscripts modulo k. Now $\Delta_1^a = \Delta_2$ so $T_2 \leq M_2 = N_{(\Delta_2)}$. Continuing this line of argument we have $N = M_1 \times M_2$ with $M_1 = T_1 \times T_3 \times \ldots$ and $M_2 = T_2 \times T_4 \times \ldots$ with $k \geq 2$ even. Let $\alpha \in \Delta_1$. Now the M_1-orbits in Δ_2 are of equal length greater than 1, so $M_1^{\Gamma(\alpha)}$ is a nontrivial normal subgroup of $N_\alpha^{\Gamma(\alpha)}$. Since $M_1^{\Gamma(\alpha)}$, a quotient of M_1, is a direct power of copies of T, it follows that $M_1^{\Gamma(\alpha)} \simeq T \simeq S$. by Lemma 2.2, M_1 can have only one composition factor S, and so $k = 2$. By [8, Lemma 1.1] $\Gamma = K_{v,v}$.

We assume from now on that N acts faithfully on each of Δ_1, Δ_2. Next we look at the case where some nonidentity element of T_1 fixes a point, that is T_1 is not semiregular on $V\Gamma$.

Lemma 3.2 Suppose that $N_{(\Delta_1)} = 1$ and that some nonidentity element of T_1 fixes a point. Then $N = T \leq G \leq Aut\ T$.

Note that in any example for which Lemma 3.2 holds, N must have two conjugacy classes of subgroups (containing N_α and $N_\beta, \alpha \in \Delta_1, \beta \in \Delta_2$) which are interchanged by G.

Proof We may assume that $(T_1)_\alpha \neq 1$ for $\alpha \in \Delta_1$. Then, as $(T_1)_\alpha^{\Gamma(\alpha)} \lhd N_\alpha^{\Gamma(\alpha)}$, it follows that $S \leq (T_1)_\alpha^{\Gamma(\alpha)}$. We are in case (iii)(c) of Proposition 2.1: $G = < N, a >$ and we may take $T_i^a = T_{i+1}$ for all i (mod k). If $k > 2$ then some nonidentity element of $T_3 = T_1^{a^2}$ fixes a point of $\Delta_1 = \Delta_1^{a^2}$ and so, as the T_3-orbits in Δ_1 have equal length, it follows that $S \leq (T_3)_\alpha^{\Gamma(\alpha)}$. Thus $N_\alpha \geq (T_1)_\alpha \times (T_3)_\alpha$ has at least two composition factors S, contradicting Lemma 2.2. So $k \le 2$. Suppose that $k = 2$. If $(T_2)_\alpha \ne 1$ then we would obtain a contradiction as above and hence T_2 is semiregular on Δ_1, and so $T_1 = T_2^a$ is semiregular on $\Delta_2 = \Delta_1^a$. Thus $(T_1)_\alpha^{\Gamma(\alpha)}$ is regular. However $S \le (T_1)_\alpha^{\Gamma(\alpha)} \le Aut\ S$ and S is primitive on $\Gamma(\alpha)$, and such primitive groups are not regular (see [**3**]). Thus $k = 1$.

Lemma 3.3 Suppose that $N_{(\Delta_1)} = 1$ and that T_1 is semiregular on $V\Gamma$. Then either N is semiregular on $V\Gamma$ or $k > 1$ and for $\alpha \in V\Gamma, N_\alpha$ is a subgroup of a diagonal subgroup of $N = T_1 \times \ldots \times T_k$.

Recall that a *diagonal subgroup* of $T_1 \times \ldots \times T_k$ is a subgroup of the form $\{(t^{\phi_1}, t^{\phi_2}, \ldots, t^{\phi_k}) \mid t \in T\} \simeq T$ where the $\phi_i \in Aut\ T$.

Proof Suppose that N is not semiregular and that T_1 is semiregular on $V\Gamma$. Then case (iii)(c) of Proposition 2.1 holds so $G = < N, a >$ with $T_i^a = T_{i+1}$ for all i. Let j be the least integer such that $Y = T_1 \times \ldots \times T_j$ is not semiregular. Then $2 \le j \le k$. We may assume that $Y_\alpha^{\Gamma(\alpha)} \ne 1$ for $\alpha \in \Delta_1$. Then Y_α is a subgroup of a diagonal subgroup of $Y = T_1 \times \ldots \times T_j$ since $(T_1 \times \ldots \times T_{j-1})_\alpha = 1$. Now $Y^{a^2} = T_3 \times \ldots \times T_{j+2}$ taking the subscripts modulo k, and Y^{a^2} is such that $(Y^{a^2})_{\alpha^{a^2}}$ acts nontrivially on $\Gamma(\alpha^{a^2})$. Since $\alpha^{a^2} \in \Delta_1^{a^2} = \Delta_1$ it follows that $(Y^{a^2})_\alpha^{\Gamma(\alpha)} \ne 1$. Now $(Y^{a^2})_\alpha$ is a subgroup of a diagonal subgroup of $T_3 \times \ldots \times T_{j+2}$. If $j < k$ then $Y_\alpha \cap (Y^{a^2})_\alpha = 1$ whence $N_\alpha \ge Y_\alpha \times (Y^{a^2})_\alpha$ has at least two composition factors S, contradicting Lemma 2.2. So $j = k$ and $N = Y$, and the lemma is proved.

Theorem 3 now follows from Lemmas 2.3, 2.4, 3.1, 3.2 and 3.3.

4. Constructions and discussions

There are many examples of graphs satisfying the conditions of Theorem 1 with G a nonabelian simple group: a second paper of Lorimer [5] classifies all such cubic graphs on up to 120 vertices. The purpose of this section is to give constructions for classes of graphs satisfying the conditions of Theorem 2 or 3 to illustrate the various types of examples which arise. We begin with Theorem 2. There are many

Cayley graphs on Z_2^k as in Theorem 2(ii), perhaps the best known are the folded cubes.

4.1 Cubes and folded cubes The cube Q_k has vertex set Z_2^k with vertices adjacent if they differ in one coordinate. The vertices of the folded cube $\square_k$ are pairs of antipodal vertices, that is $\{x, y\}$ where $x + y$ is the all-1 vector. Now Q_k satisfies the conditions of Theorem 1 with $G = Z_2^k . A_k$ with the alternating group A_k permuting coordinates, and Q_k is a double cover of $\square_k$ and so is an example of Theorem 1(iv). Factoring out by $< (1, 1, \ldots, 1) > \simeq Z_2$ we obtain the group $Z_2^{k-1} A_k$ acting on $\square_k$. If k is odd then A_k acts irreducibly on the normal subgroup Z_2^{k-1} and Theorem 2(ii) holds, while if k is even then Q_k and $\square_k$ are bipartite and $\square_k$ arises in case (ii)(b) of Theorem 3.

The Cayley graphs arising in case (iii) of Theorem 2 are easily understood and there will be an example for each self-inverse conjugacy class X of a nonabelian simple group S such that $C_S(x)$ is maximal, for $x \in X$. The nature of the graphs satisfying Theorem 2(iv) is a little less clear. We give examples, for most pairs T, S, of Cayley graphs for T^k admitting $T^k.S$ (for some k) satisfying most of the conditions of Theorem 2(iv): the primitivity of S on $\Gamma(\alpha)$ is the most difficult property to guarantee.

4.2 Cayley graphs for T^k Let T and S be simple groups with T nonabelian. Let k be an integer such that S is isomorphic to a transitive subgroup of the symmetric group S_k. (This is equivalent to choosing a proper subgroup M of S and considering the action by right multiplication on the $k = \mid S : M \mid$ right cosets of M in S.) Set $G = T \ wr \ S = T^k.S$. Then $N = T^k$ is a minimal normal subgroup of G. Now we define a Cayley graph on T^k admitting G. Let X be an S-invariant class of involutions in N such that X generates N and define $\Gamma = Cay(N, X)$. Since X generates N, Γ is connected and Γ admits G. In order for this to be an example, S must act primitively on X. If $S = Z_k$ with k prime, and if T has a generating set $x_1, \ldots, x_t$ of $t \leq k$ involutions, then we can take X to be the set of images under S of $(x_1, \ldots, x_t, 1, \ldots 1) \in T^k$. For other groups S a different choice of generating set X for T^k may give an example with S primitive. Note that we require $C_S(x)$ maximal in S for $x \in X$, and $C_S(x)$ must preserve the partition of the k entries determined by equality of x-entries. Thus each element x of X must determine a different such partition. For example suppose that T has a generating set $y_1, \ldots, y_t$ of t involutions, and that the subgroup M of S stabilizing the first entry of elements of N is maximal in S and has t orbits on entries. Then let $x = (x_1, \ldots, x_k) \in N$ be such that $x_i = y_j$ if i is in the jth orbit of M on entries and let X be the set of images of x under S. Then S acts primitively on X since $C_S(x) = M$ is maximal in S, and $< X > = N$, again since the stabilizer of an entry is maximal in S.

These examples are fairly typical of graphs satisfying Theorem 2(iv). For such graphs the group G has a minimal normal subgroup $N = T^k$ and a simple subgroup S which complements N and which permutes the simple direct factors of N transitively. A group G with these properties is a twisted wreath product of T by S (see [2, Section 3]). The groups discussed above were ordinary wreath products, but the problem of constructing examples for twisted wreath products is similar.

Next we consider examples satisfying the conditions of Theorem 3. There are many examples in case (ii)(b), some having been mentioned in 4.1 above. Also constructions of examples in case (ii)(a) are obvious and need no more discussion. We next describe a general construction which applies in cases (ii)(c), (iii)(b) and (iii)(c).

4.3 Examples for Theorem 3(ii) and (iii)

(a) In these cases we have $G = < NG_\alpha, a >$ for some 2-element a interchanging the two halves Δ_1 and Δ_2 of the bipartition, and with N semiregular with orbits Δ_1 and Δ_2. Set $M = G_\alpha$. Then $V\Gamma$ may be identified with the transversal (that is a set of right coset representatives) $N \cup Na$ for M in G, so that $\alpha = 1_N$ and $\Gamma(\alpha) \subseteq Na$, say $\Gamma(\alpha) = Xa$ with $X \subseteq N$. It is difficult to get precise conditions on X in general so let us suppose that a normalizes M and $a^2 \in M$ which is true in many cases. Then the image of the edge $\{1_N, xa\}, x \in X$, under a is $\{a, x^{a^2}\}$ since $Mxa^2 = Mx^{a^2}$, and the image of this edge under $(x^{a^2})^{-1} \in N$ is $\{(x^a)^{-1}a, 1_N\}$ whence $(x^a)^{-1} \in X$. This is condition ($x \in X$ implies $(x^a)^{-1} \in X$) is the condition for Γ to be undirected. An element $m \in M$ sends $xa \in \Gamma(\alpha)$ to $x^{ama^{-1}}a$, so $x^{ama^{-1}} \in X$ whence X is an M-orbit in N on which M acts primitively.

(b) One class of examples satisfying (ii)(c) can be easily obtained as follows. Let $N = Z_p^k$ with p prime, $G = N.S_k$, with S_k permuting entries, $M = S = A_k$ and $a = (1,2)$. Let X be the set of standard basis vectors $x_i = (0^{i-1}10^{k-i})$, $1 \leq i \leq k$. The graph $\Gamma = Cay(N, X)$ with group G satisfies most of the conditions of Theorem 3(ii)(c); unfortunately N is not a minimal normal subgroup. The subgroup $C =< (1,\ldots,1) >$ of N is normal in G and the quotient graph Γ_C and group G/C satisfy Theorem 3(ii)(c) if p does not divide k (for in that case N/C is minimal normal in G/C).

(c) In Theorem 3(iii)(b) G has a normal subgroup $T \times T$ such that $N = T \times \{1\}$ and $G = NH$ with $S = H \cap (T \times T) = \{(t,t) \mid t \in T\} \simeq T$. In this case $H = < M, a > = < S, a >, \mid H : M \mid = 2$, and the above assumptions on a all hold. The subset X is an M-conjugacy class of T (for $T \simeq S \leq M \leq Aut\ T$) and we require $C_T(x)$ maximal in T for $x \in X$ in order that $S = T$ be primitive on $\Gamma(\alpha)$.

(d) In Theorem 3(iii)(c), $N = T^k \leq G \leq Aut\ N$. Let $Cay(N, X)$ be one of the graphs obtained from the construction in 4.2 satisfying Theorem 2(iv) admitting $N.S = T\ wr\ S$ and suppose that S has an automorphism $a \in Aut\ S \setminus S$ of order 2 such that $< S, a >$ still acts transitively of degree k, that is a lies in the normalizer of S in S_k. Let $G = T\ wr\ < S, a > = < NS, a >$. Then a satisfies the assumptions above and the graph with vertex set identified with $N \cup Na$ and with $\Gamma(\alpha) = Xa$ defined above satisfies all the conditions of (iii)(c).

Examples for Theorem 3(iv)(a) may be constructed in a similar manner to 4.3(d) from examples of Theorem 1(i).

4.4 Almost simple bipartite graphs Let Σ be a graph satisfying the conditions of Theorem 1(i) but not (iv) admitting T, a nonabelian simple group, and suppose that T has an automorphism $a \in T$ such that $a^2 \in T$ and a normalizes the stabilizer M of vertex α. Set $G = < T, a >$, identify $V\Sigma$ with the coset space $[T : M]$ and define Γ to be the graph with vertex set $[G : M]$ such that $\Gamma(\alpha) = \{Mxa \mid Mx \in \Sigma(\alpha)\}$. Note that $m \in M$ maps $Mxa \in \Gamma(\alpha)$ to $Mxam = Mx(ama^{-1})\, a \in \Gamma(\alpha)$ (since $ama^{-1} \in M$), so this graph Γ satisfies Theorem 3(iv)(a). The stabilizer M fixes a point in both parts of the bipartition, namely M fixes M and Ma. A graph in case (iv)(a) need not have this property, but if it does then it will arise in the way described above.

Examples in case (iv)(b) of Theorem 3 are the most difficult to describe in general. We give just one class of examples with $S = T$.

4.5 Bipartite graphs with diagonal action In Theorem 3(iv)(b), $N = T^k \leq G = < N, a >$ with a a 2-element. We shall take here $a = (1, 2, \ldots, k)$ of order $k = 2^s$ such that $N = T_1 \times \ldots \times T_k$ and $T_i^a = T_{i+1}$ for all i. So $G = N. < a >$. Let $S = \{(t, \ldots, t) \mid t \in T\}$ and set $M = S. < a^2 > \leq Aut\ T$. Note that $m \in S$ acts on $t = (t_1, \ldots, t_k) \in N$ by $t^m = (t_1^m, \ldots, t_k^m)$. The vertex set of Γ will be the coset space $[G : M]$ which can be identified with the tranversal $V \cup Va$ for M in G where $V = T_1 \times \ldots \times T_{k-1}$. Then for $\alpha = M, G_\alpha = M$ and $\Gamma(\alpha) \subseteq Va$. An element $m \in M \leq Aut\ T$ maps a coset $Mva, v \in V$, to $Mvam = Mv^{ama^{-1}}a$ since $ama^{-1} \in M$, and $v^{ama^{-1}} \in V$. So $\Gamma(\alpha) = Xa$ with $X \subseteq V$ an M-orbit in V such that $C_S(x)$ is maximal in S for $x \in X$. This can be achieved for example by choosing $t \in T$ to be an involution such that $C_T(t)$ is maximal in T and taking $X = \{(t^m, t^m, \ldots t^m, 1) \mid m \in M\} \subseteq V$.

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