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# Hamiltonian cycles avoiding sets of edges in a graph

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**Abstract**

Let $G$ be a graph and $H$ be a subgraph of $G$. If $G$ contains a hamiltonian cycle $C$ such that $E(C) \cap E(H)$ is empty, we say that $C$ is an $H$*-avoiding hamiltonian cycle*. Let $F$ be any graph. If $G$ contains an $H$-avoiding hamiltonian cycle for every subgraph $H$ of $G$ such that $H \cong F$, then we say that $G$ is $F$*-avoiding hamiltonian*. In this paper, we give minimum degree and degree-sum conditions which ensure that a graph $G$ is $F$-avoiding hamiltonian for various choices of $F$. In particular, we consider the cases where $F$ is a union of $k$ edge-disjoint hamiltonian cycles or a union of $k$ edge-disjoint perfect matchings. If $G$ is $F$-avoiding hamiltonian for any such $F$, then it is possible to extend families of these types in $G$. Finally, we undertake a discussion of $F$-avoiding pancyclic graphs.

## 1 Introduction

In this paper we consider only graphs without loops or multiple edges. Let $|G| = |V(G)|$ denote the order of $G$. Additionally, let $d(v)$ denote the degree of a vertex $v$ in $G$ and let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of $G$, respectively. Let $\overline{G}$ be the complement of $G$. In this paper, we will consider cycles to have an implicit orientation. With this in mind, given a cycle $C$ and a vertex $x$ on $C$, we let $x^+$ denote the successor of $x$ under this orientation and let $x^-$ denote the predecessor. For any other vertex $y$ on $C$, we let $xC^+y$ denote the path from $x$ to $y$ on $C$ in the direction of the orientation and $xC^-y$ denote the path from $x$ to $y$ on $C$ in the opposite direction.

For any vertex $v$ in $G$, let $N(v)$ denote the set of vertices adjacent to $v$. If $H$ is a subgraph of $G$, we let $N_H(v)$ denote the set of vertices in $V(H)$ adjacent to $v$ and $N(H)$ denote the set of vertices adjacent to at least one vertex in $V(H)$. For a vertex $v$ in $V(G)$ we let $d_H(v)$ denote the degree of $v$ in $H$. If $C$ is a cycle contained in $G$ we let $N_C^+(v)$ denote the set of vertices on the cycle that are successors of vertices in $N_C(v)$ and $N_C^+(H)$ denote the set of vertices on the cycle that are successors of vertices in $N_C(H)$. Similarly we let $N_C^-(v)$ denote the set of vertices on the cycle that are predecessors of vertices in $N_C(v)$ and $N_C^-(H)$ denote the set of vertices on the cycle that are predecessors of vertices in $N_C(H)$.

A spanning cycle in a graph $G$ is called a *hamiltonian cycle*, and if such a cycle exists, we say that $G$ is *hamiltonian.* Hamiltonian graphs have been widely studied, and a good reference for the recent status of such problems is [15]. Let $\sigma_2(G)$ denote the minimum degree sum over all pairs of nonadjacent vertices in $G$. Ore's Theorem [13], one of the classic results pertaining to hamiltonian graphs, states the following.

**Theorem 1.1** (Ore's Theorem 1960). *If $G$ is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then $G$ is hamiltonian.*

A graph is a *butterfly* if it is composed of two complete graphs intersecting in exactly one vertex. If $G$ is isomorphic to a butterfly or if $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$, then $G$ is nonhamiltonian and $\sigma_2(G) = n-1$, demonstrating the sharpness of Ore's Theorem. In fact, it has been noted by several authors [1, 11, 12] that these are the only nonhamiltonian graphs with this property. We will give a new proof of this fact as a corollary to our main result. The class of butterflies and $K_{\frac{n-1}{2},\frac{n+1}{2}}$ will play an important role in the main result of this paper. Dirac's Theorem [5], another classic result, is an immediate corollary of Theorem 1.1.

**Theorem 1.2** (Dirac's Theorem 1952). *If $G$ is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{n}{2}$ then $G$ is hamiltonian.*

Part of the impact of Theorems 1.1 and 1.2 is that the parameters $\sigma_2(G)$ and $\delta(G)$ are often explored as threshold functions for hamiltonicity and other cycle-structural properties. We will refer to the hypotheses of Theorems 1.1 and 1.2, specifically the assumptions that $\sigma_2(G) \geq n$ and $\delta(G) \geq \frac{n}{2}$, as the *Ore condition* and the *Dirac condition*, respectively.

We would like to call attention to a particular class of results pertaining to the cycle structure of a graph. Let $G$ be a graph, and let $S = \{x_1, \ldots, x_k\}$ be a subset of $V(G)$. If there is a cycle in $G$ that contains all of the vertices in $S$, then $S$ is said to be *cyclable.* For instance, if each vertex in $S$ has degree at least $\frac{n}{2}$, then $S$ is cyclable [2, 17]. The problem of putting specified edges or paths in $G$ on to cycles, either arbitrarily or in a prescribed order, has also been considered. For instance, in [9] it is shown (extending a result from [14]) that if $G$ is a graph of order $n$ with $\sigma_2(G) \geq n + k$, and $H$ is any collection of non-trivial paths in $G$ having exactly $k$ edges, then there is a hamiltonian cycle in $G$ containing all of $E(H)$. A good reference for both a classical perspective and recent progress on problems of this type is [16].

In this paper, as a contrast to these results, we are interested in examining conditions on a graph $G$ that assure it will have a hamiltonian cycle that *avoids* given subgraphs.

## 2 $F$-avoiding Hamiltonicity

Let $G$ be a graph and $H$ be a subgraph of $G$. If $G$ contains a hamiltonian cycle $C$ such that $E(C) \cap E(H)$ is empty, we say that $C$ is an *$H$-avoiding hamiltonian cycle*. Let $F$ be any graph. If $G$ contains an $H$-avoiding hamiltonian cycle for every subgraph $H$ of $G$ such that $H \cong F$, then we say that $G$ is *$F$-avoiding hamiltonian*. We note here that $G$ is $F$-avoiding hamiltonian if and only if $G - E(H)$ is hamiltonian for every subgraph $H$ of $G$ such that $H \cong F$. We wish to determine conditions on $G$ and $F$ that assure $G$ is $F$-avoiding hamiltonian.

The *closure* of a graph $G$ of order $n$, denoted $cl(G)$, is obtained by repeatedly connecting nonadjacent vertices $u$ and $v$ such that $d(u) + d(v) \geq n$ until no such pair of vertices exists. The following theorem from [4] will be used several times in this section.

**Theorem 2.1.** *A graph $G$ is hamiltonian if and only if $cl(G)$ is hamiltonian.*

Our first two results give Ore-type conditions that assure $G$ is $F$-avoiding hamiltonian.

**Theorem 2.2.** *Let $G$ be a graph of order $n \geq 3$ and let $F$ be a graph of order $t \leq \frac{n}{2}$ and maximum degree at most $k$. If $\sigma_2(G) \geq n + k$ then $G$ is $F$-avoiding hamiltonian. This result is sharp for all choices of $F$.*

*Proof.* Let $H$ be any subgraph of $G$ that is isomorphic to $F$ and let $G' = G - E(H)$. It suffices to show that $G'$ is hamiltonian. We will, in fact, show that $cl(G')$ is hamiltonian implying the result by Theorem 2.1. Let $v$ be any vertex in $V(G) \setminus V(H)$ and let $w$ be any vertex in $G$ that is not adjacent to $v$. Then $d_{G'}(w) \geq d_G(w) - k$ and $d_{G'}(v) = d_G(v)$, so that

$$d_{G'}(w) + d_{G'}(v) \geq d_G(v) + d_G(w) - k \geq (n + k) - k = n.$$

This implies that $v$ and $w$ are adjacent in $cl(G')$ and, in fact, that $v$ is adjacent to every vertex in $cl(G')$. Hence, $cl(G')$ is isomorphic to a complete graph of order $n - t$ joined to some graph of order $t$. The fact that $n - t \geq \frac{n}{2}$ yields that $cl(G')$ is hamiltonian and, by Theorem 2.1, that $G'$ is hamiltonian as well. The result follows.

Let $H$ be any graph and let $x$ be a vertex of maximum degree in $H$. To see that the theorem is sharp, consider a graph $G$ on $n$ vertices constructed from $K_{n-1}$ and an additional vertex $v$ of degree $\Delta(H)+1$. This graph has $\sigma_2 = n+k-1$ and contains numerous copies of $H$ with $v$ playing the role of $x$. Removing the edges of any of these copies from $G$ leaves a graph that is clearly not hamiltonian, as the degree of $v$ would be one. □

We now turn our attention to the problem of finding $F$-avoiding hamiltonian cycles in a graph $G$ when the order of $F$ is closer to the order of $G$. The following result can be obtained using the techniques like those in the proof of Theorem 2.2, and is also a corollary of the main result of the next section, so here we provide only the sharpness example at this time.

**Theorem 2.3.** *Let $G$ be a graph of order $n \geq 3$ and let $F$ be a graph with maximum degree $k$. If $\sigma_2(G) \geq n+2k$ then $G$ is $F$-avoiding hamiltonian. This result is sharp for all values of $k$.*

To see that the theorem is sharp for every value of $k$, let $n \geq 2k+1$ be an odd integer and let $B$ be any $k$-regular bipartite graph with partite sets of size $\frac{n-1}{2}$. Complete the partite sets of $B$ so that each is a copy of $K_{\frac{n-1}{2}}$, forming a (no longer biparite) graph $B'$. We then create the graph $G$ by taking the join of $B'$ and $K_1$ and we note that $\sigma_2(G) = n+2k-1$. If we remove the edges of the bipartite graph $B$, we are left with two cliques of order $\frac{n+1}{2}$ intersecting in a vertex, which is not hamiltonian. This implies that $G$ is not $B$-avoiding hamiltonian, establishing the desired sharpness.

It is tempting to assume Theorem 2.3 follows trivially from Ore's Theorem, as for any subgraph $H \cong F$, $d_{G-E(H)}(v) \geq d_G(v) - k$ for every vertex $v$ in $G$. This does not imply, however, that $\sigma_2(G - E(H)) \geq n$, as the deletion of $E(H)$ from $G$ creates new pairs of nonadjacent vertices. As these pairs of vertices may have had relatively small degree sum in $G$, $\sigma_2(G - E(H))$ may be significantly smaller than $n$.

In Theorems 2.2 and 2.3, note that in order to assure the existence of a hamiltonian cycle that avoids any nonempty collection of edges in $G$, we must exceed the Ore condition. Interestingly, this is not so when we consider the Dirac condition.

**Theorem 2.4.** *Let $G$ be a graph of order $n \geq 3$ with $\delta(G) \geq \frac{n}{2}$. If $E'$ is any subset of $E(G)$ such that $|E'| \leq \frac{n-3}{4}$ then there is a hamiltonian cycle in $G$ containing no edge from $E'$. This result is sharp.*

*Proof.* It suffices to prove the theorem when $|E'| = \frac{n-3}{4}$. Let $H$ be the subgraph of $G$ induced by $E'$. Note that $\langle V(H)\rangle$ has at most $\frac{n-3}{2}$ vertices. Allowing $G'$ to denote $G - E'$, we proceed by considering $cl(G')$. Each vertex in $G - V(H)$ still has degree at least $\frac{n}{2}$ in $G - E'$, and thus $G - V(H)$ is complete in $cl(G')$. Let $v \in V(H)$ be a vertex of degree $\Delta(H)$. Then

$$|V(H)| \leq \Delta(H) + 1 + 2(|E'| - \Delta(H)) \leq \frac{n-3}{2} - \Delta(H) + 1,$$

as $|V(H)|$ would be maximized in the case where those edges not adjacent to $v$ form a matching in $H$. This implies that $|G - V(H)| = n - |V(H)| \geq \frac{n+3}{2} + \Delta(H) - 1$. Thus, since $G - V(H)$ induces a clique in $cl(G')$ each vertex in $G - V(H)$ has degree at least $\frac{n+3}{2} + \Delta(H) - 2$. We now also note that each vertex in $V(H)$ has degree at least $\frac{n}{2} - \Delta(H)$ in $G'$. Let $x$ and $w$ be arbitrary vertices in $G'$ chosen from $V(H)$ and $G - V(H)$ respectively. After closing $G - V(H)$, we have that

$$d(x) + d(w) \geq \left(\frac{n}{2} - \Delta(H)\right) + \left(\frac{n+3}{2} + \Delta(H) - 2\right) = n - \frac{1}{2}.$$

Since $d(x)$ and $d(w)$ must both be integers, this implies that $d(x) + d(w) \geq n$ in $cl(G')$ for any choice of $x$ and $w$. Consequently, $cl(G')$ contains the join of $K_{|G-V(H)|}$ and $\overline{K_{|V(H)|}}$, which is hamiltonian since $|G - V(H)| > |V(H)|$. Thus, as $cl(G')$ is hamiltonian, $G$ is $H$-avoiding hamiltonian, and the result follows.

To see that the theorem is sharp, let $k \geq 2$ be a positive integer, and let $n = 4k+2$. We construct a graph $H$ of order $n$ by starting with the complete bipartite graph $K_{\frac{n}{2}-1,\frac{n}{2}+1}$ and adding the matching $e_1, \ldots, e_{k+1}$ to the partite set of size $\frac{n}{2} + 1$. Removing any $k = \frac{n-2}{4}$ of the edges $e_i$ yields a non-hamiltonian graph. Thus, if $G$ is $E'$-avoiding hamiltonian, $|E'| < \frac{n-2}{4}$. □

## 3 An Extension of Theorem 2.3

If we relax the degree condition in Theorem 2.3 slightly, it becomes possible that $G - E(H)$ is no longer hamiltonian. We can show however, that if $G - E(H)$ is not hamiltonian then it must fall into one of two exceptional classes. The following is the main result of this paper.

**Theorem 3.1.** *Let $k \geq 0$ be an integer and let $G$ be a graph on $n \geq 2k + 3$ vertices with $\sigma_2(G) \geq n + 2k - 1$. If $F$ is a graph with maximum degree at most $k$, then $G$ is $F$-avoiding hamiltonian, or there is some subgraph $H$ of $G$ such that $H \cong F$ and either $G - E(H)$ is a butterfly, or $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G - E(H) \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$.*

Again, we note that $\sigma_2(G - E(H))$ may be significantly smaller than $\sigma_2(G)$, as the deletion of $E(H)$ may introduce new pairs of nonadjacent vertices. Consequently, it would be incorrect to conclude that Theorem 3.1 follows immediately from the characterization of $\sigma_2$-maximal nonhamiltonian graphs mentioned above.

Before we begin we will give some useful notation and lemmas. Define $G'$ to be the graph $G - E(H)$ and observe that $\sigma_2(G) \geq n + 2k - 1$ and $n \geq 2k + 3$ implies that the minimum degree of $G$ is at least $2k + 1$ and hence that the minimum degree of $G'$ is at least $k + 1$. Moreover, for any vertices $x, y \in V(G)$ such that $xy$ is not an edge in $G$, $d_{G'}(x) + d_{G'}(y) \geq n - 1$.

The following two lemmas will be used to prove Theorem 3.1.

**Lemma 3.2.** *Let $G$ be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$. Then for any subgraph $H \subseteq G$ with $\Delta(H) \leq k$, either $G'$ is a butterfly or $G'$ is 2-connected.*

*Proof.* Let $H$ be a subgraph of $G$ with $\Delta(H) \leq k$. We will show by way of contradiction that $G'$ is a butterfly or is 2-connected.

Suppose that $G'$ contains a vertex $v$ such that $G' - v$ is disconnected, and let $S_i$ and $S_j$ be two components of $G' - v$ with $|V(S_i)| = s_i$ and $|V(S_j)| = s_j$. Since the minimum degree of $G'$ is at least $k+1$, the minimum degree of $G' - v$ is at least $k$. Hence each component of $G' - v$ has at least $k+1$ vertices. Let $x$ be a vertex in $S_i$. Then there exists $y \in S_j$ such that $xy$ is not an edge in $G$, and therefore $d_{G'}(x) + d_{G'}(y) \geq n-1$. (Note that this is true for every vertex of $G' - v$.) We also have that $d_{G'}(x) \leq s_i$ and $d_{G'}(y) \leq s_j$. Combining the inequalities we get $n-1 \leq d_{G'}(x) + d_{G'}(y) \leq s_i + s_j \leq n-1$, which implies that $s_i + s_j = n-1$ (and also that $G' - v$ has exactly two components), $d_{G'}(x) = s_i$, $d_{G'}(y) = s_j$ and both $x$ and $y$ are adjacent $v$. This is true for all $x \in V(S_i)$, $y \in V(S_j)$, so $G'$ is a butterfly. $\square$

**Lemma 3.3.** *Let $G$ be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$ and let $H$ be any subgraph of $G$ with $\Delta(H) \leq k$. Let $C$ be a longest cycle in $G'$ such that $|C| = t$. For any component $S$ of $G' - C$ with $|N_C(S)| \geq 2$ we have the following:*

1. *For all $x \in V(S)$ and for all $x_i, x_j \in N_C(S), x_i^+ x_j^+ \notin E(G')$ and $x_i^+ x \notin E(G')$. (For all $x \in V(S)$ and for all $x_i, x_j \in N_C(S), x_i^- x_j^- \notin E(G')$ and $x_i^- x \notin E(G')$.) Furthermore, $|N_C(S)| \leq \frac{t}{2}$.*
2. *For all $x \in V(S)$ and for all $x_i \in N_C(S)$ such that $xx_i^+ \notin E(G)$, and for all $y \in V(G') - N_C^+(S) - V(S)$, $x_i^+ y \in E(G')$. (For all $x \in V(S)$ and $x_i \in N_C(S)$ such that $xx_i^- \notin E(G)$, and for all $y \in V(G') - N_C^-(S) - V(S)$, $x_i^- y \in E(G')$.) Furthermore, $x$ is adjacent to every vertex of $V(S) - x$.*

*Proof.* For convenience let $|N_C(S)| = \ell$ and take note of the fact that we are considering $N_C(S)$ in the graph $G'$. The proof of *(1)* is straightforward, and standard throughout the literature, so we leave the proof to the reader. We do note here that the degree sum condition is not necessary to prove *(1)*.

*(2)* Let $x \in V(S)$ and $x_i \in N_C(S)$ such that $xx_i^+$ is not in $E(G)$. Then we know that $d_{G'}(x) + d_{G'}(x_i^+) \geq n-1$. Since there are no edges between components of $G' - C$, $d_{G'}(x) \leq |V(S)| - 1 + \ell$. Recall from (*1*) that $x_i^+$ is not adjacent to any vertex of $N_C^+(S)$ nor $V(S)$, so $d_{G'}(x_i^+) \leq t - \ell + n - t - |V(S)| = n - |V(S)| - \ell$. Combining the inequalities yields $n-1 \leq d_{G'}(x) + d_{G'}(x_i^+) \leq n-1$. Therefore equality must hold, so $x$ must be adjacent to every vertex in $V(S) - x$ and $x_i^+$ must be adjacent to every vertex in $G' - N_C^+(S) - V(S)$. The argument for $x_i^-$ is similar. $\square$

*Proof.* (of Theorem 3.1) Let $G$ be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$ and let $H$ be any subgraph of $G$ with $\Delta(H) \leq k$. Let $C$ be a longest cycle in $G'$ and let $t = |V(C)|$. If $C$ is a hamiltonian cycle, we are done. Suppose then that $C$ is not a hamiltonian cycle.

We begin by showing that $G' - C$ is connected. Suppose otherwise and let $S_1, \ldots, S_h$ be the components of $G' - C$, with $|V(S_i)| = s_i$ for $1 \leq i \leq h$. Without loss of generality we will assume that $s_i \leq s_{i+1}$ for $1 \leq i \leq h-1$.

Let $x \in S_i$ and $y \in S_j$ for some distinct $i$ and $j$. Then by part (*1*) of Lemma 3.3, $d_{G'}(x) \leq s_i - 1 + \frac{t}{2}$ and $d_{G'}(y) \leq s_j - 1 + \frac{t}{2}$ which implies that $d_{G'}(x) + d_{G'}(y) \leq s_i + s_j + t - 2 \leq n - 2$. Consequently, as $x$ and $y$ are nonadjacent in $G'$, $xy$ must be an edge in $H$. At most $k$ edges of $H$ were incident with each vertex in each $S_i$, so $s_i \leq k$ for all $1 \leq i \leq h$.

Assume without loss of generality that $x \in S_1$ and consider the neighborhood of $x$ on $C$. In $G$

$$d_C(x) \geq 2k + 1 - (s_i - 1) - \sum_{\ell \neq 1} s_\ell \geq 2k + 2 - hs_h.$$

Since at most $k$ edges of $H$ were incident with each vertex in $G$, at most $(h-1)s_h$ edges between $S_1$ and $S_j$ are in $H$ for all $j \neq 1$, so in $G'$

$$d_C(x) \geq 2k + 2 - hs_h - (k - (h-1)s_h) = k + 2 - s_h.$$

At most $k - s_h$ of the non-neighbors of $x$ on $C$ in $G'$ were neighbors of $x$ on $C$ in $G$. Therefore, there exist $x_i, x_j \in N_C(x)$ such that $xx_i^+, xx_j^+$ is neither an edge in $G'$ nor $G$. By part (*2*) of Lemma 3.3 both $x_i^+$ and $x_j^+$ are adjacent to every vertex in $V(G') - N_C^+(x) - V(S_1)$. Recall that $y \in S_j$, where $j \neq 1$. Then the cycle $xx_iC^-x_j^+yx_i^+C^+x_jx$ is longer than $C$, which contradicts that $C$ is a longest cycle of $G'$. Therefore, $G' - C$ is connected.

Let $S$ be the graph $G' - C$ and define the neighborhood of $S$ in $C$ to be $N_C(S)$. Suppose that $|N_C(S)| = \ell$. Note that if $G$ is not 2-connected, then by Lemma 3.2 $G'$ is a butterfly. We will assume that $G'$ is 2-connected.

Suppose that the order of $S$ is at least $k+1$ and let $x_i$ be in $N_C(S)$. Since $S$ is connected, no vertex in $S$ can be adjacent to $x_i^+$, but at most $k$ edges incident with $x_i^+$ were in $H$. Consequently, as there are at least $k+1$ vertices in $S$, there exists $v \in V(S)$ such that $x_i^+v$ is neither an edge in $G'$ nor $G$, which implies $d_{G'}(x_i^+) + d_{G'}(v) \geq n-1$. Then by part (*2*) of Lemma 3.3 we know that for every $x_i \in N_C(S)$, $N_{G'}(x_i^+) = V(C) - N_C^+(S)$. This means that $d_{G'}(x_i^+) = t - \ell$ for all $x_i \in N_C(S)$. Since $d_{G'}(v) \leq |V(S)| - 1 + \ell = n - t - 1 + \ell$, we also have that $d_{G'}(x_i^+) + d_{G'}(v) \leq n - 1$, so equality must hold. Consequently $v$ must be adjacent to every vertex in $N_C(S)$ and $V(S) - v$.

Suppose there is $x_j \in N_C(S)$ such that $x_j^- \notin N_C^+(S)$. We have shown above that $x_j^+x_j^-$ and $x_{j-1}^+x_j \in E(G')$, where $x_{j-1}$ denotes the predecessor of $x_j$ in $N_C(S)$. Then the cycle $vx_{j-1}C^-x_j^+x_j^-C^-x_{j-1}^+x_jv$ has length $t+1$, which contradicts that $C$ is a longest cycle of $G'$. So for each $x_j$, $(x_j^-)^-$ is in $N_C(S)$, implying that $|N_C(S)| = \frac{t}{2}$. Since $G'$ is 2-connected, there exists $u \in V(S)$ with $u \neq v$ and $x_i \in N_C(S)$ such that

$ux_i \in E(G')$. Then the cycle $vx_{i-1}C^-x_iuv$ has length $t+1$, which contradicts that $C$ is a longest cycle of $G'$. Therefore, we will assume that $S$ has order at most $k$.

Suppose then that $|V(S)| = r$, where $2 \leq r \leq k$. Since the minimum degree of $G'$ is at least $k+1$, every vertex in $S$ has at least $k+1-(r-1) = k-r+2 \geq 2$ neighbors on $C$. Let $u, v$ be in $V(S)$ and $x_i, x_j$ be in $N_C(S)$, such that $ux_i, vx_j \in E(G')$, and let $P$ be any $u-v$ path in $S$. For every vertex $y \in V(C)$ such that $x_i^+y \in E(G')$ either $x_j^+y^- \notin E(G')$ or $x_j^+y^+ \notin E(G')$. Indeed, if $y$ were to lie between $x_i^+$ and $x_j^+$ on $C$, the cycle $ux_iC^-x_j^+y^-C^-x_i^+yC^+x_jvPu$ would be longer than $C$ and were $y$ to lie between $x_j^+$ and $x_i^+$ on $C$ the cycle $ux_iC^-y^+x_j^+C^+yx_i^+C^+x_jvPu$ would be longer than $C$. Thus $t \leq n-2$ implies that $d_{G'}(x_i^+) + d_{G'}(x_j^+) = d_C(x_i^+) + d_C(x_j^+) \leq n-2$ and therefore that $x_i^+x_j^+ \in E(G)$ for all $x_i, x_j \in N_C(S)$. Since $x_i^+x_j^+$ is not in $E(G')$ for any $x_i, x_j \in N_C(S)$, these edges must be in $E(H)$. But the minimum degree of $G'$ is at least $k+1$, so $|N_C(S)| \geq k-r+2$ implies that there are at least $k-r+1$ such edges for each $x_i^+$. Then $r \leq k$ implies that there is at least one vertex $w \in V(S)$ for each $x_i^+$ such that $wx_i^+$ is neither an edge in $G'$ nor $G$. (These $w$ are not necessarily distinct.) By the same argument used above we see that $|N(S)| = \frac{t}{2}$, so we can find a cycle longer than $C$, which is a contradiction.

Hence we may assume that $|V(S)| = 1$. Let $x$ be the vertex in $S$ and suppose that $d_{G'}(x) < \frac{n-1}{2}$. Then there is a vertex $x_i^+ \in N_C^+(x)$ such that $xx_i^+$ is neither an edge in $G'$ nor $G$ and a vertex $x_j^- \in N_C^-(x)$ such that $xx_j^-$ is neither an edge in $G'$ nor $G$. Then by part (*2*) of Lemma 3.3, $x_i^+$ is adjacent to every vertex in $V(C) - N_C^+(x)$ and $x_j^-$ is adjacent to every vertex in $V(C) - N_C^-(x)$ . First suppose that $x_i^+ = x_{i+1}^-$; that is, $x_i^+$ is the only vertex between $x_i$ and $x_{i+1}$ on $C$. Then the cycle $xx_iC^-x_jx_i^+x_j^-C^-x_{i+1}x$ is hamiltonian, which contradicts that $C$ is a longest cycle in $G'$. So $x_i^+ \neq x_{i+1}^-$. By a similar argument we find that $x_j^- \neq x_{j-1}^+$. Hence by part (*2*) of Lemma 3.3 we know that $x_i^+x_j$ and $x_j^-x_j^+$ are edges in $G'$. Then the cycle $xx_iC^-x_j^+x_j^-C^-x_i^+x_jx$ is a hamiltonian cycle, which contradicts that $C$ is a longest cycle in $G'$.

Therefore we may assume that $d_{G'}(x) = \frac{n-1}{2}$. Observe that $N_C^+(x) \cup x$ is an independent set of order $\frac{n+1}{2}$. Since $n \geq 2k+3$, $\frac{n+1}{2} \geq k+2$, so for every vertex $y \in N_C^+(x) \cup x$ there is a vertex $z \in N_C^+(x)$ such that $yz$ is neither an edge in $G'$ nor $G$. Then every vertex in $N_C^+(x) \cup x$ is adjacent to exactly $N_C(x)$. It follows that $K_{\frac{n+1}{2},\frac{n-1}{2}} \subseteq G' \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$. □

The conclusion that there is a subgraph $H \cong F$ such that $G - E(H)$ either falls into the class of butterflies or is a supergraph of $K_{\frac{n-1}{2},\frac{n+1}{2}}$ is only feasible for certain choices of $F$. The following two corollaries reflect this.

**Corollary 3.4.** *Let $k \geq 0$ be an integer and let $G$ be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$. If $F$ is a graph of order $n$ with minimum degree at least 1 and maximum degree at most $k$, then either $G$ is $F$-avoiding hamiltonian or there is some subgraph $H$ of $G$ such that $H \cong F$ and $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G - E(H) \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$.*

**Corollary 3.5.** *Let $k \geq 0$ be an integer and let $G$ be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$. If $F$ is a connected graph with maximum degree at most $k$ and order at least $\frac{n}{2}+1$, then either $G$ is $F$-avoiding hamiltonian or $F$ is bipartite, has order at most $n-1$, and there is some subgraph $H$ of $G$ such that $H \cong F$ and $G-E(H)$ is a butterfly.*

## 4 Applications of Theorem 3.1

We now apply Theorem 3.1 to problems involving not only hamiltonian cycles, but also hamiltonian paths and perfect matchings. We say that $G$ is $F$-avoiding traceable if for any subgraph $H$ of $G$ such that $H \cong F$, $G-E(H)$ has a hamiltonian path. The following is an immediate corollary of Theorem 3.1.

**Corollary 4.1.** *Let $k \geq 0$ be an integer and let $G$ be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$. If $F$ is a graph with maximum degree at most $k$, then $G$ is $F$-avoiding traceable.*

We now return our attention to hamiltonian cycles. The problem of determining when a graph contains $k$ edge-disjoint hamiltonian cycles has long been of interest. In [7], it was shown that a graph $G$ of sufficiently large order $n$ with $\sigma_2(G) \geq n+2k-2$ contains $k$ edge-disjoint hamiltonian cycles. The problem of finding disjoint hamiltonian cycles in bipartite graphs has also been examined [8]. Other results focus on finding $k$ edge-disjoint hamiltonian cycles in graphs that satisfy the Ore condition. In [6], it is shown that if $G$ is a graph of sufficiently large order $n$ with $\sigma_2(G) \geq n$ and $\delta(G) \geq 4k-2$ then $G$ contains $k$ edge-disjoint hamiltonian cycles.

In light of these results, we present the following variation. Let $H$ be a family of $k \geq 1$ edge-disjoint hamiltonian cycles in a graph $G$. If $G-E(H)$ is hamiltonian, then $G-E(H)$ contains a hamiltonian cycle $C$ which, together with $H$, would comprise a family of $k+1$ edge-disjoint hamiltonian cycles in $G$. In fact, if $G$ is $F$-avoiding hamiltonian for graph $F$ isomorphic to $k$ edge-disjoint hamiltonian cycles, then we are not only finding disjoint families of hamiltonian cycles, but in fact we are able to *extend* any family of $k$ edge-disjoint hamiltonian cycles to a family of $k+1$ edge-disjoint hamiltonian cycles. Taking into account Corollaries 3.4 and 3.5, the following is an immediate consequence of Theorem 3.1.

**Corollary 4.2.** *Let $k > 0$ be an integer and let $G$ be a graph on $n \geq 4k+3$ vertices with $\sigma_2(G) \geq n+4k-1$ and let $H$ be any collection of $k$ edge-disjoint hamiltonian cycles in $G$. Then $H$ can be extended to a family of $k+1$ edge-disjoint hamiltonian cycles. This result is sharp.*

Corollary 4.2 complements the results mentioned above pertaining to the existence of $k$ edge-disjoint hamiltonian cycles. To see that Corollary 4.2 is sharp, consider a graph $G$ of even order $n \geq 4k+4$ which is comprised of two disjoint cliques of order $\frac{n}{2}$, denoted $G_1$ and $G_2$, and a family $H$ of $k$ edge-disjoint hamiltonian cycles with the property that $H$ is bipartite with partite sets $V(G_1)$ and $V(G_2)$. Then $\sigma_2(G) = n + 4k - 2$, but $G - E(H)$ is isomorphic to $2K_{\frac{n}{2}}$ which is not hamiltonian.

Since a hamiltonian cycle of even order can be viewed as the union of two disjoint perfect matchings, we also obtain the following result pertaining to extending families of perfect matchings.

**Corollary 4.3.** *Let $k > 0$ be an integer and let $G$ be a graph of even order $n \geq 2k+3$ with $\sigma_2(G) \geq n+2k-1$ and let $H$ be any collection of $k$ edge-disjoint perfect matchings in $G$. Then $H$ can be extended to a family of $k+2$ edge-disjoint perfect matchings in $G$. This result is sharp.*

To see that Corollary 4.3 is sharp, let $t$ be an odd integer such that $2t \geq 2k - 1$ consider a graph $G$ of order $2t$ which is comprised of two disjoint cliques of order $t$, denoted $G_1$ and $G_2$, and a family $H$ of $k$ edge-disjoint perfect matchings with the property that $H$ is bipartite with partite sets $V(G_1)$ and $V(G_2)$. Then $\sigma_2(G) = n + 2k - 2$, but $G - E(H)$ is isomorphic to $2K_t$ which does not contain a perfect matching as $t$ is odd.

As mentioned above, certain supergraphs of $K_{\frac{n-1}{2},\frac{n+1}{2}}$ and the class of butterflies serve to establish the sharpness of Ore's Theorem. That is, they are examples of nonhamiltonian graphs of with $\sigma_2 = n - 1$. If we let $k = 0$ in Theorem 3.1 we can see that these are in fact the only such graphs. As mentioned above, this fact was also noted in [1], [11] and [12].

**Corollary 4.4.** *Let $G$ be a nonhamiltonian graph of order $n$ with $\sigma_2(n) = n - 1$. Then either $G$ is a butterfly or $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$.*

## 5 F-avoiding Pancyclicity

A graph $G$ is *pancyclic* if $G$ contains a cycle of each length from 3 up to $|G|$. The study of pancyclic graphs is a natural extension of the hamiltonian problem. Having developed necessary conditions for a graph $G$ to be $F$-avoiding hamiltonian, we turn our attention to the analogous notion for pancyclic graphs. Let $F$ and $G$ be graphs. If $G - E(H)$ is pancyclic for every subgraph $H$ of $G$ such that $H \cong F$, then we say that $G$ is *$F$-avoiding pancyclic*. In this section we will give several conditions on $G$ and $F$ which assure that $G$ is $F$-avoiding pancyclic. In addition to Theorem 3.1, the following two theorems from [10] will be useful.

**Theorem 5.1.** *Let $G$ be a graph of order $n$ with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and hamiltonian cycle $v_0, \ldots, v_{n-1}, v_0$. If $d(v_0)+d(v_{n-1}) \geq n$ then $G$ is either pancyclic, bipartite or missing only an $(n-1)$-cycle.*

**Theorem 5.2.** *Let $G$ be a graph of order $n$ with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and hamiltonian cycle $v_0, \ldots, v_{n-1}, v_0$. If $d(v_0) + d(v_{n-1}) \geq n + 1$ then $G$ is pancyclic.*

We begin with an Ore-type condition for $H$ avoiding pancyclicity that leaves us with no exception graphs.

**Theorem 5.3.** *Let $G$ be a graph of order $n$ and let $F$ be a graph with maximum degree $k$. If $\sigma_2(G) \geq n + 2k + 1$ then $G$ is $F$-avoiding pancyclic. This result is sharp for all values of $k$.*

*Proof.* By Theorem 2.3 we know that $G' = G - E(F)$ is hamiltonian. Let $x$ be a vertex of $G$ with $d(x) = \delta(G)$. Then there is a vertex $y$ of $G$ with $d(y) \geq n + 2k + 1 - \delta(G)$. Let $C$ be a hamiltonian cycle in $G'$. Then $d_{G'}(y) + d_{G'}(y^+) \geq n + k + 1 - \delta(G) + \delta(G) - k = n + 1$, so $G'$ is pancyclic by Theorem 5.2.

To see that this result is best possible, let $n \geq 2k + 2$ be an even integer and let $H$ be any $k$-regular graph on $\frac{n}{2}$ vertices. We create the graph $G$ by taking the join of two copies of $H$. Then $\sigma_2(G) = n + 2k$ and the removal of the edges of each copy of $H$ leaves us with $K_{\frac{n}{2},\frac{n}{2}}$, which is not pancyclic since it contains no odd cycles. □

The following is a well-known result of Bondy [3].

**Theorem 5.4.** *Let $G$ be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$ then either $G$ is pancyclic or $G$ is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$.*

If we relax the conditions on $\sigma_2(G)$ given in Theorem 5.3 slightly we obtain a similar result.

**Theorem 5.5.** *Let $k \geq 0$ be an integer and let $G$ be a graph on $n \geq 6k + 4$ vertices with $\sigma_2(G) \geq n + 2k$. If $F$ is a graph with maximum degree at most $k$, then $G$ is $F$-avoiding pancyclic or there is some subgraph $H$ of $G$ such that $H \cong F$ and $G - E(H)$ is $K_{\frac{n}{2},\frac{n}{2}}$. This result is sharp for all values of $k$.*

*Proof.* For simplicity we let $G' = G - E(H)$. By Theorem 2.3 we know that $G'$ contains a hamiltonian cycle $C$. If $\sigma_2(G') \geq n$ the result follows by Theorem 5.4. Suppose that $\sigma_2(G') < n$ and that $G'$ is not pancyclic. Let $v$ be a vertex with degree $\delta(G') < \frac{n}{2}$. Then, as $\sigma_2(G) \geq n + 2k$ and $d_H(v) \leq k$ there are at least $\frac{n-1}{2} - k$ vertices of degree at least $n - \delta(G') \geq \frac{n+1}{2}$. Since $n \geq 6k + 4$, $\frac{n-1}{2} - k > \frac{n}{3}$, and we can find two vertices $x$ and $y$ on $C$ such that both $x$ and $y$ have degree at least $\frac{n+1}{2}$ and $1 \leq dist_C(x, y) \leq 2$.

If $dist_C(x, y) = 1$ then $G'$ is pancyclic by Theorem 5.2. Therefore we may assume that $dist_C(x, y) = 2$.

We assume without loss of generality that $x = y^{++}$ on $C$ and let $x = v_0, v_1, \ldots, v_{n-2} = y, v_{n-1}, v_0$ be the vertices of $C$ in order following a clockwise direction. By Theorem 5.1, since $d_{G'}(x) + d_{G'}(v_{n-1}) \geq (n - \delta(G')) + \delta(G') = n$, we need only show that $G'$ contains an $(n-1)$-cycle. In $G" = G' - v_{n-1}$, consider the hamiltonian path $v_0, \ldots, v_{n-2}$. We have $d_{G"}(v_0) + d_{G"}(v_{n-2}) \geq n + 1 - 2 = n - 1$, hence $G"$ is hamiltonian and therefore $G$ contains an $n-1$-cycle. The result follows.

To see that the result is sharp, let $n \equiv 3 (\mathrm{mod}\ 4)$ and let $H$ be any $k$-regular graph on $\frac{n+1}{2}$ vertices. If we let $G$ denote the join of $H$ and $\overline{K_{\frac{n-1}{2}}}$ then $\sigma_2(G) = n + 2k - 1$ but $G - E(H)$ is isomorphic to $K_{\frac{n-1}{2},\frac{n+1}{2}}$. □

## 6 Conclusion

Given an arbitrary graph $F$ and a graph $G$ of order less than $2|F|$, we would like to determine sharp bounds on $\sigma_2(G)$ that determine when $G$ is $F$-avoiding hamiltonian. This would allow us to strengthen Theorem 2.3 in some sense.

Currently, we are investigating other notions similar to those introduced in this paper. In particular, we are developing conditions under which a bipartite graph is $F$-avoiding hamiltonian.

More generally, we pose the following problem. Let $P$ be a graph property and let $G$ be a graph containing some $H$ as a subgraph. It would be interesting to find meaningful conditions on $G$ (and possibly $H$) that assure $G - E(H)$ has property $P$.

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