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Holey Schröder designs of type $4^n u^1$

F. E. Bennett

Department of Mathematics
Mount Saint Vincent University
Halifax, Nova Scotia B3M 2J6
Canada
frank.bennett@msvu.ca

Hantao Zhang

Computer Science Department
The University of Iowa
Iowa City, IA 52242
U.S.A.
hantao-zhang@uiowa.edu

Abstract

A holey Schröder design of type $h_1^{n_1} h_2^{n_2} \dots h_k^{n_k}$ (HSD($h_1^{n_1} h_2^{n_2} \dots h_k^{n_k}$)) is equivalent to a frame idempotent Schröder quasigroup (FISQ($h_1^{n_1} h_2^{n_2} \dots h_k^{n_k}$)) of order n with n_i missing subquasigroups (holes) of order $h_i, 1 \leq i \leq k$, which are disjoint and spanning, that is, $\sum_{1\leq i\leq k} n_i h_i = n$. In this paper, we consider the existence of HSD($4^n u^1$) for $0 \leq u \leq 36$ and show that these HSDs exist if and only if $0 \leq u \leq 2n-2$ and $n \geq 4$ with just nine possible exceptions. We also investigate the existence of HSD($4^n u^1$) for general u and prove that there exists an HSD($4^n u^1$) for $u \geq 37$ and $n \geq \lceil 2u/3 \rceil + 7$.

1 Introduction

A *quasigroup* is an ordered pair $(Q, *)$, where Q is a set and $(*)$ is a binary operation on Q such that the equations

$$a * x = b \text{ and } y * a = b \tag{1}$$

are uniquely solvable for every pair of elements $a, b \in Q$. A quasigroup is called *idempotent* if the identity

$$x * x = x \tag{2}$$

is satisfied for all $x \in Q$. If the identity

$$(x * y) * (y * x) = x \tag{3}$$

holds for all $x, y \in Q$, then it is called a *Schröder quasigroup.* The *order* of the quasigroup is $|Q|$.

For a finite set Q, it is well known that the multiplication table of the quasigroup $(Q, *)$ defines a Latin square; that is, a Latin square can be viewed as the multiplication table of the quasigroup with the headline and sideline removed. Two quasigroups of the same order are *orthogonal* if when the two corresponding Latin squares are superposed, each symbol in the first square meets each symbol in the second square exactly once. A quasigroup (Latin square) is called *self-orthogonal* if it is orthogonal to its transpose. A pair of orthogonal Latin squares (quasigroups), say $(Q, *)$ and $(Q, \cdot)$, are said to have the *Weisner property* if $x * y = z$ and $x \cdot y = w$ whenever $z * w = x$ and $z \cdot w = y$ for all $x, y, z, w \in Q$.

If $(Q, *)$ is a Schröder quasigroup, then it is self-orthogonal. Moreover, if $(Q, \cdot)$ is the transpose of $(Q, *)$, then $z \cdot w = w * z$. Consequently, from $z * w = x$ and $z \cdot w = y$, we have $x * y = (z * w) * (w * z) = z$. Similarly, we also have $x \cdot y = w$. That is, $(Q, *)$ and $(Q, \cdot)$ satisfy the Weisner property. Idempotent Schröder quasigroups, or ISQs, are also associated with other combinatorial configurations (see [10], [2], [13], [14] and [15]). In particular, an ISQ(v) corresponds to an edge-colored design CBD$[G_6; v]$ which is investigated in [10]. An *edge-colored design* CBD$[G_6; v]$ on a v-set Q is a partition of the colored edges of a triplicate complete graph $3K_v$, each K_v receives one color for its edges from three different colors, into blocks $\{a, b, c, d\}$ each containing edges $\{a, b\}, \{c, d\}$ colored with color 1, edges $\{a, c\}, \{b, d\}$ with color 2, and edges $\{a, d\}, \{b, c\}$ with color 3. If we define a binary operation $(\cdot)$ as $a \cdot b = c, b \cdot a = d, c \cdot d = a$ and $d \cdot c = b$ from the block $\{a, b, c, d\}$ and define $x \cdot x = x$ for every $x \in Q$, an ISQ(v) is obtained on set Q. On the other hand, suppose Q is an ISQ. If $a \cdot b = c$, $b \cdot a = d$, then we must have $c \cdot d = (a \cdot b) \cdot (b \cdot a) = a$ and $d \cdot c = (b \cdot a) \cdot (a \cdot b) = b$. So the block $\{a, b, c, d\}$ is determined and a CBD$[G_6; v]$ can be obtained in this way. The following theorem states the known results on the existence of Schröder quasigroups:

Theorem 1.1 ([13], [6], [10]) (*a*) *A Schröder quasigroup of order v exists if and only if $v \equiv 0, 1 \pmod 4$ and $v \neq 5$.*

(*b*) *An idempotent Schröder quasigroup of order v exists if and only if $v \equiv 0, 1 \pmod 4$ and $v \neq 5, 9$.*

Let Q be a set and $\mathcal{H} = \{S_1, S_2, \ldots, S_k\}$ be a set of subsets of Q. A *holey idempotent Schröder quasigroup* having hole set $\mathcal{H}$ is a triple $(Q, \mathcal{H}, *)$, which satisfies the following properties:

1. $(*)$ is a binary operation defined on Q; however, when both points a and b belong to the same set S_i, there is no definition for $a * b$;

2. the equations (1) hold when a, b are not contained in the same set $S_i, 1 \leq i \leq k$;

3. the identity (2) holds for any $x \notin \cup_{1\leq i\leq k} S_i$;

4. the identity (3) holds when x and y are not contained in the same set $S_i, 1 \leq i \leq k$.

We denote the holey ISQ by HISQ$(v; s_1, s_2, \ldots, s_k)$, where $v = |Q|$ is the *order* and $s_i = |S_i|, 1 \leq i \leq k$. Each S_i is called a *hole*. When $\mathcal{H} = \emptyset$, we obtain an ISQ, and denote it by ISQ(v). When $\mathcal{H} = \{S_1\}$, we obtain an *incomplete ISQ*, and denote it by IISQ$(v, |S_1|)$.

From the definition of HISQ, we can obtain the definition of frame ISQ as follows. If $\mathcal{H} = \{S_1, S_2, \ldots, S_k\}$ is a partition of Q, then a holey ISQ is called *frame ISQ*. The *type* of the frame ISQ is defined to be the multiset $\{|S_i| : 1 \leq i \leq k\}$. We shall use an "exponential" notation $s_1^{n_1} s_2^{n_2} \ldots s_t^{n_t}$ to describe the type of n_i occurrences of s_i, $1 \leq i \leq t$, in the multiset. We briefly denote a frame ISQ of type $s_1^{n_1} s_2^{n_2} \ldots s_t^{n_t}$ by FISQ$(s_1^{n_1} s_2^{n_2} \ldots s_t^{n_t})$.

Now from an FISQ$(s_1^{n_1} s_2^{n_2} \ldots s_t^{n_t})$, we can use the same method to obtain an edge-colored design which is called *a holey Schröder design* and denoted by HSD$(s_1^{n_1} s_2^{n_2} \ldots s_t^{n_t})$. A *holey Schröder design* is a triple $(X, \mathcal{H}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{H}$ is a partition of X into subsets (called *holes*);

2. $\mathcal{B}$ is a family of 4-subsets of X (called *blocks*) such that a hole and a block contain at most one common point;

3. the pairs of points in a block $\{a, b, c, d\}$ are colored as $\{a, b\}$ and $\{c, d\}$ with color 1, $\{a, c\}$ and $\{b, d\}$ with color 2, and $\{a, d\}$ and $\{b, c\}$ with color 3;

4. every pair of points from distinct holes occurs in 3 blocks with different colors.

The *type* of the HSD is the multiset $\{|H| : H \in \mathcal{H}\}$ and it is also described by an exponential notation. In particular, we note that an IISQ(v, n) is equivalent to an HSD$(1^{v-n} n^1)$.

An HSD can be viewed as a generalization of CBD$[G_6; v]$. An HSD of type $\{s_1, s_2, \ldots, s_k\}$ is a partition of the colored edges of a triplicate graph $3K_{s_1, s_2, \ldots, s_k}$ into blocks $\{a, b, c, d\}$ each containing edges $\{a, b\}, \{c, d\}$ with color 1, edges $\{a, c\}, \{b, d\}$ with color 2, and edges $\{a, d\}, \{b, c\}$ with color 3, where each $K_{s_1, s_2, \ldots, s_k}$ receives one color for its edges from three different colors.

An HSD is equivalent to a frame self-orthogonal Latin square (FSOLS) with the Weisner property. For the existence of FSOLS of type $4^n u^1$, [18] gives the following theorem.

Theorem 1.2 *There exists an FSOLS$(4^n u^1)$ if and only if $n \geq 4$ and $0 \leq u \leq 2n-2$.*

In view of Theorem 1.2, we have the following lemma.

Lemma 1.3 *If an HSD*($4^n u^1$) *exists, then* $n \geq 4$ *and* $0 \leq u \leq 2n-2$.

Another class of designs related to HSDs is a *group divisible design* (GDD). A GDD is a 4-tuple $(X, \mathcal{G}, \mathcal{B}, \lambda)$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*;
2. $\mathcal{B}$ is a family of subsets of X (called *blocks*) such that a group and a block contain at most one common point;
3. every pair of points from distinct groups occurs in exactly λ blocks.

The *type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We also use the notation GD$(K, M; \lambda)$ or K-GDD$(M; \lambda)$, to denote the GDD when its block sizes belong to K and group sizes belong to M.

If $M = \{1\}$, then the GDD becomes a *pairwise balanced design* (PBD). If $K = \{k\}$, $M = \{n\}$ and with the type n^k, then the GDD becomes a *transversal design* TD(k, n). The following results are well known (see [1] and [5], for example).

Theorem 1.4 (a) *There exists a TD*$(4, m)$ *for every positive integer* $m \notin \{2, 6\}$.

(b) *There exists a TD*$(5, m)$ *for every positive integer* $m \notin \{2, 3, 6, 10\}$.

(c) *There exists a TD*$(6, m)$ *for* $m \geq 5$ *and* $m \notin \{6, 10, 14, 18, 22\}$,

(d) *There exists a TD*$(7, m)$ *for* $m \geq 7$ *and* $m \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$.

It is well-known that the existence of a TD(k, n) is equivalent to the existence of $k-2$ *mutually orthogonal Latin squares* of order n (MOLS(n)). It is easy to see that if we erase the colors in the blocks, the HSD becomes a GDD with block size 4 and $\lambda = 3$. But the converse may be not true. It is proved in [9] that a $\{4\}$-GDD with $\lambda = 3$ and of type h^u exists if and only if $h^2 u(u-1) \equiv 0 \pmod 4$, while in [4, 19], the following theorem is proved.

Theorem 1.5 *An HSD*(h^u) *exists if and only if* $h^2 u(u-1) \equiv 0 \pmod 4$ *with the exception of* $(h, u) \in \{(1, 5), (1, 9), (2, 4)\}$.

For the existence of HSD$(2^n u^1)$, the following two theorems come from [5, 19].

Theorem 1.6 *For* $0 \leq u \leq 15$, *an HSD*$(2^n u^1)$ *exists if and only if* $n \geq u+1$ *with the exception of* $(n, u) \in \{(2, 1), (3, 1), (3, 2), (4, 0)\}$, *and with the possible exception of* $(n, u) \in \{(7, 5), (7, 6), (11, 9), (11, 10)\}$.

Theorem 1.7 *There exists an HSD*$(2^n u^1)$ *when* $u \geq 16$ *and* $n \geq [5u/4] + 14$.

The following results have been useful for studying Schröder quasigroups with a specified number of idempotent elements [7].

Theorem 1.8 (*a*) *There exists an HSD*($4^n 1^1$) *if and only if* $n \geq 4$.

(*b*) *There exists an HSD*($4^n 9^1$) *if and only if* $n \geq 6$.

(*c*) *There exists an HSD*($4^n 12^1$) *if and only if* $n \geq 7$.

(*d*) *There exists an HSD*($4^n u^1$) *for* $n = 4, 5$ *and* $0 \leq u \leq 6$.

In this paper, we consider more generally the existence of HSDs of type $4^n u^1$. These designs represent holey Schröder quasigroups, where the holes of size 4 can be filled with a Schröder quasigroup of order 4 and the hole of size u can be filled with a Schröder quasigroup of order u whenever it exists, thereby producing a subquasigroup of order u in the resulting Schröder quasigroup of order $v = 4n + u$. The main result of this paper is the following theorem.

Theorem 1.9 (*a*) *For* $0 \leq u \leq 36$, *an HSD*($4^n u^1$) *exists if and only if* $n \geq 4$ *and* $0 \leq u \leq 2n - 2$, *with the possible exception of* $n = 19$ *and* $u \in \{29, 30, 31, 33, 34, 35\}$, *and* $n = 22$ *and* $u \in \{33, 34, 35\}$.

(*b*) *There exists an HSD*($4^n u^1$) *for all* $u \geq 37$ *and* $n \geq \lceil 2u/3 \rceil + 7$.

2 Constructions

For direct constructions of HSDs, we sometimes use *starter blocks*. Suppose the block set $\mathcal{B}$ of an HSD is closed under the action of some abelian group G; then we are able to list only part of the blocks (starter or base blocks) which determines the structure of the HSD. To check the starter blocks, we need only calculate whether the differences $\pm(x - y)$ from all pairs $\{x, y\}$ with color i in the starter blocks are precisely $G \setminus S$ for $1 \leq i \leq 3$, where S is the set of the differences of the holes. We can also attach some infinite points to an abelian group G. When the group acts on the blocks, the infinite points remain fixed. In the following example the x_is are infinite points.

Example 2.1 An HSD($4^6 8^1$)

points: $Z_{24} \cup \{x_i : 1 \leq i \leq 8\}$

holes: $\{\{i, i+6, i+12, i+18\} : 0 \leq i \leq 5\} \cup \{\{x_i : 1 \leq i \leq 8\}\}$

starter blocks: $\{0, 8, 17, 3\}$, $\{0, 4, 11, x_1\}$, $\{0, 11, 21, x_2\}$, $\{0, 21, 8, x_3\}$, $\{0, 9, 14, x_4\}$, $\{0, 17, 1, x_5\}$, $\{0, 5, 9, x_6\}$, $\{0, 2, 4, x_7\}$, $\{0, 1, 2, x_8\}$.

In this example, the entire set of blocks is developed from the starter blocks by adding 1 (mod 24) to each point of the starter blocks; the infinite points are unchanged for addition. The above idea of starter blocks can be also generalized: instead of adding 1 to each point of the starter blocks, we may add 2 or more to develop the block set; we refer this as the +2 method.

Example 2.2 An HSD($4^4 3^1$)

points: $Z_{16} \cup \{x, y, z\}$

holes: $\{\{i, i+4, i+8, i+12\} : 0 \le i \le 3\} \cup \{\{x, y, z\}\}$

starter blocks: $\{0, 2, 15, 9\}$, $\{0, 3, 2, z\}$, $\{0, 6, 5, 7\}$, $\{0, 7, 9, 10\}$, $\{0, 11, 6, y\}$, $\{0, 13, 3, x\}$, $\{1, 0, 3, z\}$, $\{1, 8, 6, x\}$, $\{1, 12, 7, y\}$

By adding 2 (mod 16) to the 9 starter blocks, we obtain a set of 72 blocks.

Example 2.3 An HSD($3^4 4^1$)

points: $Z_{12} \cup \{x_1, x_2, x_3, x_4\}$

holes: $\{\{i, i+4, i+8\} : 0 \le i \le 3\} \cup \{\{x_1, x_2, x_3, x_4\}\}$

starter blocks: $\{0, 2, 3, x_2\}$, $\{0, 3, 9, x_3\}$, $\{0, 6, 5, x_1\}$, $\{0, 7, 10, x_4\}$, $\{0, 11, 1, 6\}$, $\{1, 0, 3, x_4\}$, $\{1, 3, 10, x_2\}$, $\{1, 4, 6, x_3\}$, $\{1, 8, 7, x_1\}$, $\{2, 1, 3, x_3\}$, $\{2, 3, 8, x_1\}$, $\{2, 4, 1, x_2\}$, $\{2, 5, 0, x_4\}$, $\{3, 5, 4, x_2\}$, $\{3, 6, 8, x_3\}$, $\{3, 9, 6, x_1\}$, $\{3, 10, 5, x_4\}$.

By adding 4 (mod 12) to the 17 starter blocks, we obtain a set of 51 blocks.

Next, we state several recursive constructions of HSDs, which are commonly used in other block designs. The following construction comes from the weighting construction of GDDs [16].

Construction 2.4 (Weighting) *Suppose* $(X, \mathcal{H}, \mathcal{B})$ *is a GDD with* $\lambda = 1$ *and let* $w : X \mapsto Z^+ \cup \{0\}$. *Suppose there exist HSDs of type* $\{w(x) : x \in B\}$ *for every* $B \in \mathcal{B}$. *Then there exists an HSD of type* $\{\sum_{x \in H} w(x) : H \in \mathcal{H}\}$.

Using Theorem 1.4(a), if we give every point of an HSD weight m and input TD$(4, m)$ to each block of the HSD, we can obtain the following construction.

Construction 2.5 *Suppose there exists an HSD*($h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k}$), *then there exists an HSD of type* $((mh_1)^{n_1}(mh_2)^{n_2} \ldots (mh_k)^{n_k})$, *where* $m \ne 2, 6$.

The next construction is sometimes called "filling in holes". It is used commonly in constructing designs.

Construction 2.6 *Suppose there exists an HSD of type* $\{s_i : 1 \le i \le k\}$ *and HSDs of type* $\{h_{i_j} : 1 \le j \le n_i\} \cup \{a\}$, *where* $\sum_{j=1}^{n_i} h_{i_j} = s_i$ *and* $1 \le i \le k-1$, *then there exists an HSD of type* $\{h_{i_j} : 1 \le j \le n_i, 1 \le i \le k-1\} \cup \{s_k + a\}$.

The next three constructions are a special case of the above construction.

Construction 2.7 *If there exist an HSD*($4^m t^1$) *and an HSD*($4^s u^1$), *where* $4s + u = t$, *then there exists an HSD*($4^{m+s} u^1$).

Construction 2.8 *If there exist an HSD*$(h^m t^1)$ *and an HSD*$(4^s u^1)$*, where* $4s = h$*, then there exists an HSD*$(4^{sm}(t+u)^1)$*.*

Proof We adjoin u infinite points to the HSD$(h^m t^1)$ and fill the holes of size h with an HSD$(4^s u^1)$. □

Construction 2.9 *If there exist an HSD*$(h^m t^1)$ *and an HSD*(4^{s+1})*, where* $4s = h$*, then there exists an HSD*$(4^{sm}(t+4)^1)$*.*

The next construction comes from [10].

Construction 2.10 *Suppose there exists an FSOLS*$(h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k})$*, then there exists an HSD*$((4h_1)^{n_1}(4h_2)^{n_2} \ldots (4h_k)^{n_k})$*.*

Lemma 2.11 *For any* $m \geq 4$*, there exists an HSD*$(4^{4m} t^1)$ *for* $0 \leq t \leq 8m-2$*.*

Proof For all integers $m \geq 4$ and $0 \leq j \leq 2m-2$, we have an FSOLS$(4^m j^1)$ from Theorem 1.2. From this, we first get an HSD$(16^m (4j)^1)$ by Construction 2.10. To this HSD we apply Construction 2.8 with $h = 16$ and $t = k$, where $0 \leq k \leq 6$, using HSD$(4^4 k^1)$ from Theorem 1.8 to get the desired HSD$(4^{4m}(4j+k)^1)$. Let $t = 4j+k$, a simple calculation shows that $0 \leq t \leq 8m-2$. □

Lemma 2.12 *For any* $n \geq 6$*, there exists an HSD*$(4^n u^1)$ *whenever* $n \equiv 0 \pmod 3$*,* $0 \leq u \leq 2n-2$*, and* $u \equiv 1 \pmod 3$*.*

Proof There exist 4-GDDs of the same type (see, for example, [11]). So we can give all points of this GDD weight one to get the desired HSD$(4^n u^1)$. □

Lemma 2.13 *If there exist a TD*$(5, m)$ *and an HSD*$(1^m u^1)$*, then there is an HSD*$(4^m u^1)$*.*

Proof First adjoin an infinite point, say x, to the groups of the TD$(5, m)$, then delete a point from the TD which is different from x. The resulting design is a $\{5, m+1\}$-GDD of type $4^m m^1$, where each block of size 5 intersects the group of size m in a point other than x and the blocks of size $m+1$ intersect this group in the point x. In the group of size m we give x a weight of u and all the other points weight zero. Give all the other points of the GDD a weight of one. Since we have an HSD(1^4) and an HSD$(1^m u^1)$, we then obtain the desired HSD$(4^m u^1)$. □

The following result is proved in [6].

Theorem 2.14 *For any* $n \geq 5$*, there exists an HSD*$(1^n 2^1)$ *whenever* $n \equiv 0, 1 \pmod 4$ *and* $n \neq 8$*.*

As an immediate consequence of Theorems 2.14 and 1.4(b), we have the following useful results.

Lemma 2.15 *For any $n \geq 5$, there exists an HSD$(4^n 2^1)$ whenever $n \equiv 0, 1 \pmod 4$ and $n \neq 8$.*

Lemma 2.16 *There exist an HSD$(4^n 3^1)$ for $n \in \{7, 8, 11, 12, 15\}$, an HSD$(4^n 5^1)$ for $n \in \{11, 12, 15, 19\}$, an HSD$(4^n 6^1)$ for $n \in \{13, 17\}$, and an HSD$(4^{19} 7^1)$.*

Proof In [6], it is shown that there exist an HSD$(1^n 3^1)$ for $n \in \{7, 8, 11\}$ and an HSD$(1^{19} 7^1)$. An HSD$(1^{12} 3^1)$ and an HSD$(1^{12} 5^1)$ can be obtained from an HSD$(3^4 2^1)$ given in Appendix A16 and an HSD$(3^4 4^1)$ given in Example 2.3, by applying Construction 2.8 with an HSD$(1^3 1^1)$. Similarly, an HSD$(1^{15} 3^1)$ and an HSD$(1^{15} 5^1)$ can be obtained from an HSD$(3^5 2^1)$ and an HSD$(3^5 4^1)$. The following base blocks can generate an HSD$(3^5 2^1)$ by the +1 method (mod 15):

$$\{0, 1, 8, 4\}, \{0, 3, 2, 9\}, \{0, 6, 4, x_1\}, \{0, 13, 1, x_2\}.$$

The following base blocks can generate an HSD$(3^5 4^1)$ by the +1 method (mod 15):

$$\{0, 1, 2, x_4\}, \{0, 2, 6, 13\}, \{0, 4, 7, x_2\}, \{0, 9, 1, x_1\}, \{0, 12, 3, x_3\}.$$

The other types of HSD$(1^n u^1)$ are given in Appendix A16. The result then follows from Lemma 2.13 and Theorem 1.4(b). □

Lemma 2.17 *Suppose there exist a TD$(5, m)$ and an HSD$(1^{m-1}(k+1)^1)$. Then there exists an HSD$(4^{m-1}(4+k)^1)$.*

Proof From the TD$(5, m)$, we first get a resolvable TD$(4, m)$ (briefly RTD$(4, m)$) by deleting a group of size m. This RTD has m parallel classes of blocks, that is, the blocks of size 4 can be partitioned into m sets of disjoint blocks of size 4. We select one of these parallel classes, say P, and select a block of size 4 from this parallel class, say $B = \{a_1, a_2, a_3, a_4\}$. There are four groups of size m in the RTD, say G_1, G_2, G_3, G_4, and without loss of generality we can assume that the block B intersects G_i in a_i. We adjoin a set S of k infinite points to the groups of this RTD as follows. On the group G_i we form an HSD$(1^{m-1}(k+1)^1)$, where the hole of size $k+1$ consists of the set S plus a_i. For the rest of the RTD, we ignore the parallel class P and form an HSD(1^4) on all of the other blocks of size 4. Now for the holes of size 4 of the required HSD, we take the blocks of the parallel class P other than B and for the hole of size $4 + k$ we take the set $S \cup B$. It is an easy matter to check that the resulting design is indeed and HSD of type $4^{m-1}(4 + k)^1$. □

Using Lemma 2.17, we can prove the following result:

Lemma 2.18 *There exist an HSD$(4^{13} 5^1)$, an HSD$(4^n 6^1)$ for $n \in \{7, 8, 11, 12, 15\}$, and an HSD$(4^n 7^1)$ for $n \in \{13, 17\}$.*

Proof Since both a TD$(5, 14)$ and an HSD$(1^{13} 2^1)$ (Theorem 2.14) exist, we obtain an HSD$(4^{13} 5^1)$ by applying Lemma 2.17 with $m = 14, k = 1$. Similarly, for $n \in$

$\{7, 8, 11, 12, 15\}$, because both a TD$(5, n+1)$ and an HSD$(1^n 3^1)$ exist (see the proof of Lemma 2.16), we obtain an HSD$(4^n 6^1)$ by applying Lemma 2.17 with $m = n+1, k = 2$. For $n \in \{13, 17\}$, because both a TD$(5, n+1)$ and an HSD$(1^n 4^1)$ exist (they can be obtained from HSD$(4^k 1^1)$, $k = 4, 5$, by filling $k-1$ holes of size four with an HSD(1^4).), we obtain an HSD$(4^n 7^1)$ by applying Lemma 2.17 with $m = n+1, k = 3$. □

We need some small designs in the following lemma to make use of some constructions using TD$(6, m)$ and TD$(7, m)$.

Lemma 2.19 *There exist HSDs of type $4^n u^1$ for $n = 4, 5, 6, 7$ and $0 \leq u \leq 2n-2$.*

Proof For $n = 4, 5$ and $0 \leq u \leq 6$, the designs are provided by Theorem 1.8(d). For $n = 6, 7$ and $u = 1$, the designs come from Theorem 1.8(a). For $n = 6, 7$ and $u = 2, 3$, the design are given in Appendices A1 and A2.

For $n = 6, 7$ and $u = 4$, the designs come from Theorem 1.5. For $n = 6, 7$ and $u = 5, 6$, the designs are given in Appendices A3 and A4.

For $n = 5, 6, 7$ and $u = 7$, the designs are provided in Appendix A5 and Lemma 2.12. For $n = 5, 7$ and $u = 8$, we apply Theorem 2.10 with $m = 4$ on an FSOLS$(1^n 2^1)$. For $n = 6$ and $u = 8$, the construction is given in Example 2.1.

For $n = 6, 7$ and $u = 9, 10$, the designs come from Theorem 1.8(b), Lemma 2.12 and Appendix A6. For $n = 7$ and $u = 11, 12$, the designs come from Appendix A7 and Theorem 1.8(c). □

Lemma 2.20 *If there exists a TD$(6, m)$, then there exists an HSD$((4m)^4 (4k)^1 t^1)$, where $0 \leq k \leq m$ and $0 \leq t \leq 6m$.*

Proof Give weight 4 to each point of the first four groups of a TD$(6, m)$. Give a weight of 4 to k points of the fifth group and weight 0 to the remaining points of this group. Give a weight of 0, 1, 2, 3, 4, 5 or 6 to each point of the sixth group so that the total weight is t. By using HSDs of types 4^n for $n = 4, 5, 6$ and also HSD$(4^n k^1)$ for $n = 4, 5$ and $0 \leq k \leq 6$ by Lemma 2.19, we obtain the desired HSD by Construction 2.4. □

With a slight modification of the construction given in Lemma 2.20, we get the following:

Lemma 2.21 *If there exists a TD$(6, m)$, then there exists an HSD$(4^{4m+k} t^1)$, where $k = 0, 1, 4, 5, \ldots, m$ and $0 \leq t \leq 6m$.*

Proof Take an HSD$((4m)^4 (4k)^1 t^1)$, where $0 \leq k \leq m$ and $0 \leq t \leq 6m$, from Lemma 2.20, and fill the holes of sizes $4m$ and $4k$ with HSDs of types 4^m and 4^k, as these HSDs exist by Theorem 1.5, we obtain the desired HSD. □

Lemma 2.22 *If there exists a TD$(6, m)$, then there exists an HSD$((2m)^4 (2k)^1 t^1)$, where $0 \leq k \leq m$ and $m \leq t \leq 3m$.*

Proof Give weight 2 to each point of first four groups of a TD$(6, m)$. Give k points of the fifth group weight 2 and the remaining points weight 0. Give weight 1, 2 or 3 to the points of the sixth group. Since there exist HSDs of type $2^4 1^1, 2^5, 2^4 3^1, 2^5 1^1, 2^6, 2^5 3^1$ from Theorem 1.6, we obtain the desired HSD. □

Lemma 2.23 *If there exists a TD$(7, m)$, then there exists an HSD$(4^{4m+k} t^1)$, where $0 \le k \le 2$ or $4 \le k \le 2m$, and $0 \le t \le 6m$.*

Proof The proof is similar to that of Lemmas 2.20 and 2.21. Let $k = r + s$, where $r = 0, 1, 4, 5, \ldots, m$ and $s = 0, 1, 4, 5, \ldots, m$. Give weight 4 to each point of the first four groups of the TD$(7, m)$, give weight 4 to r points of the fifth group and weight 0 to the remaining points of this group. In the sixth group, give weight 4 to s points and weight 0 to the remaining points of this group. Finally, we give a weight of 0, 1, 2, 3, 4, 5 or 6 to the points of the seventh group such that the sum of these weights of the points is equal to t. By using HSDs of types 4^r, 4^s, and 4^n for $n = 4, 5, 6$ as well as HSD$(4^n k^1)$, where $0 \le k \le 6$, from Lemma 2.19, we first obtain an HSD$((4m)^4 (4r)^1 (4s)^1 t^1)$. Filling the holes of sizes $4m$, $4r$, $4s$ with HSDs of types 4^m, 4^r, 4^s, respectively, this gives an HSD$(4^{4m+r+s} t^1)$ and the desired result. □

In addition to the existence of HSD$(4^n u^1)$, we also have the following lemma regarding the existence of HSD$(12^4 u^1)$, which will be needed in Lemma 3.3.

Lemma 2.24 *There is an HSD$(12^4 u^1)$ for $2 \le u \le 18$.*

Proof We start with a TD$(5, 4)$. In the first four groups of the TD we give all the points weight 3. In the last group we give the points a weight of 0, 2, 3, or 4 for a total weight of u. The needed input HSDs of types 3^4 and 3^5, which come from Theorem 1.5, and the types $3^4 2^1$ and $3^4 4^1$, which are provided in Appendix A16 and Example 2.3. The resulting design is an HSD$(12^4 u^1)$ where $0 \le u \le 16$ and $u \ne 1$.

For $12 \le u \le 18$, we start with a TD$(4, 4)$ and add an infinite point, say x, to the groups and then delete a point other than x so as to form a $\{4, 5\}$-GDD of type $3^4 4^1$, where x appears only in blocks of size 5 and the group of size 4. In the group of size 4, we can then give x a weight of k, where $0 \le k \le 6$, and give all other points of the GDD a weight of 4. Using ingredient HSD$(4^4 k^1)$, this gives us an HSD of type $12^4 (12 + k)^1$ and the desired HSD$(12^4 u^1)$ for $12 \le u \le 18$. □

3 HSD$(4^n u^1)$ for some special n

As an application of the lemmas given in the previous section, we show at first the existence of HSD$(4^n u^1)$ for some special values of n and then a general result.

Lemma 3.1 *An HSD$(4^9 u^1)$ exists for $0 \le u \le 16$.*

Proof For $u = 0, 4$, they come from Theorem 1.5. For $u = 1, 9, 12$, they come from Theorem 1.8. For $u = 2$, it comes from Lemma 2.15. For $u = 3, 5, 6, 11, 14, 15$, they are given in Appendix A2, A3, A4, A7, A9, and A10. For $u = 7, 10, 13, 16$, they come from Lemma 2.12. Finally for $u = 8$, there exists an FSOLS($1^9 2^1$). Using Construction 2.10, we have an HSD($4^9 8^1$). □

Lemma 3.2 *An HSD*($4^{10}u^1$) *exists for* $0 \leq u \leq 18$.

Proof We have an HSD of type $4^{10}u^1$ from Theorems 1.5 and 1.8 and Appendices A1–A12. □

Lemma 3.3 *An HSD*($4^{12}u^1$) *exists for* $0 \leq u \leq 22$.

Proof For $u = 0, 1, 4$, the designs come from Theorems 1.5 and 1.8. For $u = 2, 3, 5$, the designs come from Lemmas 2.15 and 2.16. For $6 \leq u \leq 22$, we take an HSD($12^4 t^1$), where $2 \leq t \leq 18$, from Lemma 2.24 and then apply Construction 2.9 with $h = 12, m = 4, s = 3$ to get an HSD of type $4^{12}(t + 4)^1$. The result is an HSD($4^{12}u^1$) for $6 \leq u \leq 22$. □

Lemma 3.4 *An HSD*($4^{14}u^1$) *exists for* $0 \leq u \leq 26$.

Proof For $u = 0, 1, 4, 9, 12$, they are covered by Theorems 1.5 and 1.8. For $u = 2$, the design comes from an HSD of type ($4^{10}18^1$), given in Appendix A12, using Construction 2.7. For $u = 8, 16, 20, 24$, we apply Construction 2.10 on an FSOLS($1^{14}t^1$), where $t = u/4$. For the rest of the cases, we have an HSD of type $4^{14}u^1$ from Appendix A2 - A14. □

Lemma 3.5 *An HSD*($4^{15}u^1$) *exists for* $0 \leq u \leq 28$.

Proof For $u = 0, 1, 4, 9$, the designs come from Theorems 1.5 and 1.8. For $u = 2$, the design comes from an HSD of type ($4^{11}18^1$), given in Appendix A12, by using Construction 2.7. For $u = 3, 5$, the designs come from Lemma 2.15. For $u = 6, 7$, the designs come from Lemmas 2.12 and 2.18. For $u = 8$, we apply Construction 2.10 on an FSOLS($1^{15}2^1$). For $10 \leq u \leq 22$, we first form a $\{5, 6\}$-GDD of type 6^6 by deleting one block from a TD$(6, 7)$. In the first five groups of this GDD, we give all of the points weight 2. In the last group we give the points a weight of 1, 2, or 3 for a total weight of t where $6 \leq t \leq 18$. Since there are HSDs of types 2^n for $n = 5, 6$ and $2^n k^1$ for $n = 4, 5$ and $k = 1, 2, 3$, we get an HSD of type $12^5 t^1$ for $6 \leq t \leq 18$. To this HSD we apply Construction 2.9 with an HSD(4^4) and get an HSD of type $4^{15}(t + 4)^1$. The result is an HSD($4^{15}u^1$) for $10 \leq u \leq 22$. Finally, for $20 \leq u \leq 28$, from Theorem 1.5, we have an HSD of type 20^4 or equivalently, type $20^3 20^1$. To this we apply Construction 2.8 with $h = 20$, $m = 3$, and $u = k$, where $0 \leq k \leq 8$, by using an HSD of type $4^5 k^1$ from Lemma 2.19. The resulting design is an HSD of type $4^{15}(20 + k)^1$. Hence we obtain an HSD($4^{15}u^1$) for $0 \leq u \leq 28$ as stated, and this completes the proof of the lemma. □

Lemma 3.6 *An HSD*($4^{17}u^1$) *exists for* $0 \leq u \leq 32$.

Proof For $u = 0, 1, 4$, the designs come from Theorems 1.5 and 1.8. For $u = 2, 6$, the designs come from Lemmas 2.15 and 2.16. An HSD($4^{17}7^1$) coms from Lemma 2.18. For $u = 3, 5$, the designs come from HSDs of type $4^{13}t^1$ where $t = 19, 21$, respectively, given in Appendix A13 and [7]. Next, for $8 \le u \le 24$, we apply Lemma 2.22 with the parameters $m = 8$, $k = 2$, and $8 \le u \le 24$. The resulting design is an HSD($16^4 4^1 u^1$). This gives an HSD($4^{17}u^1$) by filling in the holes of size 16 with an HSD(4^4). For $u = 25, 26, 27, 29, 30, 31$, the designs are given in Appendix A15. For $u = 28, 32$, we apply Construction 2.10 on an FSOLS($1^{17}k^1$) with $k = 7, 8$. □

Lemma 3.7 *An HSD*($4^{18}u^1$) *exists for* $0 \le u \le 34$.

Proof The cases $u = 0, 1$, are covered by Theorems 1.5 and 1.8. For $u = 2, 3$, they are obtained from HSDs of type ($4^{14}u^1$), where $u = 18, 19$, which are given in Appendix A12 and A13, using Construction 2.7.

For $4 \le u \le 28$, we form a $\{6, 7\}$-GDD of type 6^7 by deleting one block from a TD$(7, 7)$. In the first six groups of this GDD, we give all of the points weight 2. In the last group we give the points a weight of 0, 1, 2, 3, or 4 for a total weight of t where $0 \le t \le 24$. Since there are HSDs of types 2^n for $n = 5, 6, 7$ and $2^n k^1$ for $n = 5, 6$ and $k = 0, 1, 2, 3, 4$ by Theorem 1.6, we get an HSD of type $12^6 t^1$ for $0 \le t \le 24$. To this HSD we apply Construction 2.9 to get an HSD($4^{18}(t+4)^1$) and hence an HSD($4^{18}u^1$) for $4 \le u \le 28$.

For $24 \le u \le 34$, from Theorem 1.5, we have an HSD of type 24^4 or equivalently, type $24^3 24^1$. To this we apply Construction 2.8 with $h = 24$, $m = 3$, and $u = k$, where $0 \le k \le 10$. Since we have an HSD of type $4^6 k^1$ from Lemma 2.19, the resulting design is an HSD of type $4^{18}(24+k)^1$. Hence we obtain an HSD($4^{18}u^1$) for $24 \le u \le 34$ as stated, and this completes the proof of the lemma. □

Lemma 3.8 *An HSD*($4^{19}u^1$) *exists for* $0 \le u \le 28$.

Proof For $u = 0, 1$, the designs come from Theorems 1.5 and 1.8. For $u = 2, 3$, the designs come from HSDs of type $4^{14}22^1$ and $4^{15}19^1$, given in Appendix A14 and Lemma 3.5, respectively, by applying Construction 2.7. For u even and $4 \le u \le 12$, we can construct the designs by first applying Lemma 2.22 with the parameters $m = 8$, $t = 12$, and $k = (u-4)/2$. The resulting design is an HSD($16^4 12^1 (u-4)^1$).

For $12 \le u \le 28$, we simply apply Lemma 2.22 with the parameters $m = 8$, $k = 6$, and $t = u - 4$, to get an HSD($16^4 12^1 (u-4)^1$).

Now we adjoin 4 infinite points to HSD($16^4 12^1 (u-4)^1$) and use types 4^4 and 4^5 to fill in the holes of sizes 12 and 16, respectively, leaving one hole of size u as desired. The resulting design is an HSD($4^{19}u^1$).

For $u = 5, 7$, the result comes from Lemma 2.16. For $u = 9, 11$, we prove a more general result for $8 \le u \le 24$ using a TD$(8, 8)$: In the first four groups of this TD we give all of the points a weight of two. In the fifth, sixth and seventh groups, we give two points weight two and the other points weight zero. In the last group, we give the points a weight of 1, 2, or 3, for a total weight of u. Since we have HSDs of types 2^n for $n = 5, 6, 7, 8$ and $2^n k^1$ for $n = 4, 5, 6, 7$ and $k = 1, 2, 3$, we get an HSD of type $16^4 4^3 u^1$ for $8 \le u \le 24$. By filling in the holes of size 16 with an HSD(4^4), the resulting design is an HSD($4^{19}u^1$) for $8 \le u \le 24$. □

Lemma 3.9 *An HSD*($4^{21}u^1$) *exists for* $0 \le u \le 40$.

Proof First of all, for $0 \le u \le 30$, we apply Lemma 2.21 with $m = 5$ and $k = 1$ to get an HSD($4^{21}u^1$).

Next, for $28 \le u \le 40$, we start with an HSD of type 28^4 and then apply Construction 2.8 with $h = 28$, $m = 3$, $s = 7$ and $u = k$, where $0 \le k \le 12$, since we have an HSD of type $4^7 k^1$ by Lemma 2.19. The resulting design is an HSD of type $4^{21}u^1$, where $28 \le u \le 40$, and this completes the proof of the lemma. □

Lemma 3.10 *An HSD*($4^{22}u^1$) *exists for* $0 \le u \le 32$.

Proof First of all, we start with a TD$(8, 8)$. In the first five groups of this TD we give all of the points a weight of two. In the sixth and seventh groups, we give two points weight two and the other points weight zero. In the last group, we give the points a weight of 0, 1, 2, 3, or 4 for a total weight of u. Since we have HSDs of type $2^n k^1$ for $n = 5, 6, 7$ and $k = 0, 1, 2, 3, 4$, we get an HSD of type $16^5 4^2 u^1$ for $0 \le u \le 32$. By filling in the holes of size 16 with an HSD(4^4), the resulting design is an HSD($4^{22}u^1$) for $0 \le u \le 32$. □

Lemma 3.11 *An HSD*($4^{23}u^1$) *exists for* $0 \le u \le 36$.

Proof For $0 \le u \le 6$, we first apply Lemma 2.22 with the parameters $m = 8$, $k = 6$, $t = 12 + u$ to get an HSD($16^4 12^1 (12 + u)^1$). Now we adjoin 4 infinite points to this HSD and use HSDs of types 4^4, 4^5, and $4^4 u^1$ to fill in the holes of sizes 12, 16, and $12 + u$, respectively. The resulting design is an HSD($4^{23}u^1$).

For $7 \le u \le 36$, we start with a TD$(7, 8)$. In the first five groups of this TD, we give all of the points a weight of two. In the sixth group, we give six points weight two and the other points weight zero. In the last group, we give the points a weight of 0, 1, 2, 3, or 4 for a total weight of $u - 4$. Since we have HSDs of type $2^n k^1$ for $n = 5, 6$ and $k = 0, 1, 2, 3, 4$, we get an HSD of type $16^5 12^1 (u-4)^1$ for $0 \le u-4 \le 32$. Finally, we adjoin 4 infinite points to this HSD and fill in the holes to get a resulting HSD($4^{23}u^1$) for $4 \le u \le 36$. This completes the proof. □

Lemma 3.12 *An HSD*($4^{26}u^1$) *exists for* $0 \le u \le 36$.

Proof The proof is similar to that of Lemma 3.10. Here we start with a TD$(9, 8)$ and in the first six groups of this TD we give all of the points a weight of two. In the seventh and eighth groups we give two points weight two and the other points weight zero. In the last group, we give the points a weight of 0, 1, 2, 3, 4, or 5 for a total weight of u. Since we have HSDs of type $2^n k^1$ for $n = 6, 7, 8$ and $k = 0, 1, 2, 3, 4$ by Theorem 1.6, we get an HSD of type $16^6 4^2 u^1$ for $0 \leq u \leq 32$. By filling in the holes of size 16 with an HSD(4^4), the resulting design is an HSD($4^{26}u^1$) for $0 \leq u \leq 32$. To complete the proof, for $28 \leq u \leq 36$, we can first apply Lemma 2.20 with $m = 7, k = 5, t = 0$ to get an HSD of type $28^4 20^1$. Since we have HSDs of type $4^n k^1$ for $n = 5, 7$ and $0 \leq k \leq 8$ from Lemma 2.19, we can fill in the holes to get an HSD of type $4^{26}(28 + k)^1$ for $0 \leq k \leq 8$. This completes the proof. □

Lemma 3.13 *An HSD*($4^{27}u^1$) *exists for* $0 \leq u \leq 52$.

Proof For $0 \leq u \leq 26$, we apply Lemma 2.20 with $m = 5$, $0 \leq k \leq 5$, and $t = 28$ to get an HSD($20^4 28^1 (4k)^1$). For $0 \leq w \leq 6$, we can adjoin w infinite points to this HSD and fill in the holes of sizes 20, 28 with HSDs of types $4^5 w^1$, $4^7 w^1$ from Lemma 2.19. The resulting design is an HSD of type $4^{27}(4k + w)^1$, that is, type $4^{27}u^1$ for $0 \leq u \leq 26$ as desired.

For $16 \leq u \leq 40$, we first apply Lemma 2.22 with $m = 12$, $k = 6$, and $t = u - 4$ to get an HSD($24^4 12^1 (u - 4)^1$). Now we adjoin 4 infinite points to this HSD and use HSDs of types 4^4, 4^7 to fill in the holes of sizes 12 and 24, respectively. The resulting design is an HSD($4^{27}u^1$).

Finally, for $40 \leq u \leq 52$, we start with an HSD of type 36^4 and then apply Construction 2.8 with $h = 36$, $m = 3$, $s = 9$ and $u = k$, where $0 \leq k \leq 16$, since we have an HSD of type $4^9 k^1$ by Lemma 3.1. The resulting design is an HSD of type $4^{27}u^1$, where $36 \leq u \leq 52$, and this completes the proof of the lemma. □

Lemma 3.14 *An HSD*($4^{31}u^1$) *exists for* $0 \leq u \leq 46$.

Proof For $0 \leq u \leq 6$, we apply Lemma 2.11 with $m = 6$ and $t = 28 + u$, to get an HSD of type $4^{24}(28 + u)^1$. By filling in the hole of size $28 + u$ with an HSD($4^7 u^1$), we get an HSD of type $4^{31}u^1$ as desired.

For $7 \leq u \leq 46$, we first apply Lemma 2.20 with the parameters $m = 7$, $k = 3$, and $t = u - 4$ to get an HSD($28^4 12^1 (u - 4)^1$). Now we adjoin 4 infinite points to this HSD and use types 4^4 and 4^8 to fill in the holes of sizes 12, 28, respectively, leaving one hole of size t. The resulting design is an HSD($4^{31}u^1$). □

Lemma 3.15 *For any* $m \geq 5$*, there exists an HSD*($4^{5m}u^1$) *for* $0 \leq u \leq 10m - 2$.

Proof For all odd integers $m \geq 5$ and $0 \leq j \leq (5m - 5)/2$, we have an FSOLS($5^m j^1$) ([18] Theorem 7.1). From this, we first get an HSD($20^m (4j)^1$) using Construction 2.10. To this HSD we apply Construction 2.8 with $h = 20$ and $u = k$, where $0 \leq k \leq 8$, using HSD($4^5 k^1$) from Theorem 1.8 to get the desired HSD($4^{5m}(4j + k)^1$). If $u = 4j + k$, then a simple calculation will show that $0 \leq u \leq 10m - 2$.

When $m = 4n$ for some $n \geq 1$, an HSD($4^{5m}u^1$) becomes an HSD($4^{4(5n)}u^1$) and we apply Lemma 2.11 (with $m = 5n$) to get the desired HSD.

When $m = 4n+2$ for some $n \geq 1$ and $0 \leq u \leq 10m-12$, we have an FSOLS$(5^m j^1)$ ([18] Theorem 7.1) for $m \geq 5$ and $0 \leq j \leq (5m-10)/2$. Applying Construction 2.10 to this FSOLS, we obtain an HSD of type $20^m(4j)^1$. To this HSD we apply Construction 2.8 with $h = 20$ and $u = k$, $0 \leq k \leq 8$, to get the desired HSD$(4^{5m}(4j+k)^1)$. If $u = 4j + k$, then a simple calculation will show that $0 \leq u \leq 10m - 12$. For the case when $10m - 12 < u \leq 10m - 2$, we note that $5m = 20t + 10 = 10(2t + 1)$ for some $t \geq 1$. Since there exists an FSOLS$(1^{2t+1}t^1)$ for every $t \geq 1$ by Theorem 7.1 of [18], we can first apply Construction 2.10 to obtain an HSD of type $4^{2t+1}(4t)^1$, and then apply Construction 2.5 to inflate by 10 and get an HSD of type $40^{2t+1}(40t)^1$. Finally, we add k infinite points to this HSD, where $0 \leq k \leq 18$, and fill in the holes, using HSDs of type $4^{10}k^1$ from Lemma 3.2, to get an HSD of type $4^{20t+10}(40t + k)^1$. That is, when $5m = 20t + 10$ and $u = 40t + k$, we can get an HSD of type $4^{5m}u^1$, where $10m - 20 \leq u \leq 10m - 2$. □

Lemma 3.16 *There exists an HSD*$(4^n u^1)$ *for* $0 \leq u \leq 42$ *and* $n \geq 32$.

Proof We make use of Lemma 2.23. The details of the parameters of m and $n = 4m + k$ are listed in Table 1. For the existence of TD$(7, m)$ see Theorem 1.4(d). □

n	32-42	36-48	40-54	48-66	56-78	72-102	80-114
m	7	8	9	11	13	17	19
n	96-138	128-186	176-258	248-366	≥ 252		
m	23	31	43	61	≥ 62		

Table 1: The proof of Lemma 3.16 (use Lemma 2.23)

Combining the lemmas in this section, we have the following result.

Theorem 3.17 (a) *There exists an HSD*$(4^n u^1)$ *for* $0 \leq u \leq 32$ *and* $n \geq 20$.

(b) *There exists an HSD*$(4^n u^1)$ *for* $0 \leq u \leq 36$ *and* $n \geq 23$.

(c) *There exists an HSD*$(4^n u^1)$ *for* $0 \leq u \leq 42$ *and* $n \geq 27$.

Proof (a): For $n = 20, 24, 28$, we apply Lemma 2.11. For $n = 21, 22, 23$, the result comes from Lemmas 3.9, 3.10 and 3.11. For $n = 25, 30$, we apply Lemma 3.15 with $m = 5, 6$. For $n = 26, 27, 31$, the result comes from Lemmas 3.12, 3.13, 3.14, respectively. For $n = 29$, we apply Lemma 2.21 with $m = 7$ and $k = 1$. For $n \geq 32$, the result comes from Lemma 3.16.

(b) and (c): The proof is identical to that of (a), except that when $n \geq 23(27)$, the value of u can be up to 36 (42). □

4 HSD$(4^n u^1)$ for $u \leq 36$

Lemma 4.1 *There exists an HSD*$(4^n u^1)$ *for any* $n \geq 4$ *and* $0 \leq u \leq 6$.

Proof The cases when $u = 0, 1, 4$ or $n \le 7$ are covered by Theorem 1.8 and Lemma 2.19. Theorem 3.17(a) covers the case when $n \ge 20$, Lemmas 3.1- 3.8 cover the cases when $n = 9, 10, 12, 14, 15, 17, 18$, and 19, and Lemma 2.11 covers the case when $n = 16$. So the remaining cases left to be covered are for $n = 8, 11, 13$.

For $n = 8, 11$ and $u = 2$, the designs are given in Appendix A1. For $n = 13$ and $u = 2$, the model is provided by Lemmas 2.15. For $n = 8, 11, 13$ and $u = 3$, the models are given in Appendix A2. For $n = 8, 11, 13$ and $u = 5$, the designs are given in Appendix A3, Lemmas 2.16, and 2.18, respectively. For $n = 8, 11, 13$ and $u = 6$, the designs are given in Lemmas 2.16 and 2.18, respectively. □

Lemma 4.2 *There exists an HSD*$(4^n u^1)$ *if* $7 \le u \le 13$ *and* $n \ge \lceil u/2 \rceil + 1$.

Proof The cases when $u = 9, 12$ or $n \le 7$ are covered by Theorem 1.8 and Lemma 2.19. When $u = 8$, for any $n \ge 8$, there exists an FSOLS$(1^n 2^1)$. Using Construction 2.10, we have an HSD$(4^n 8^1)$. Theorem 3.17(a) covers the case when $n \ge 20$, Lemmas 3.1- 3.8 cover the cases when $n = 9, 10, 12, 14, 15, 17, 18$, and 19, and Lemma 2.11 covers the case when $n = 16$. So the remaining cases are for $n = 8, 11, 13$.

For $n = 8, 11, 13$ and $u = 7$, the designs are given in Appendix A5 and Lemma 2.18. For $n = 8, 11, 13$ and $u = 10, 11, 13$, the models are given in Appendix A6, A7, and A8, respectively. □

Lemma 4.3 *There exists an HSD*$(4^n u^1)$ *if* $14 \le u \le 25$ *and* $n \ge \lceil u/2 \rceil + 1$.

Proof For $n \ge 20$, the lemma holds by Theorem 3.17(a). Lemmas 3.1- 3.8 cover the cases when $n = 9, 10, 12, 14, 15, 17, 18$, and 19, and Lemma 2.11 covers the case when $n = 16$, the remaining cases are for $n = 8, 11, 13$ and $n \ge \lceil u/2 \rceil$. For $u = 16, 20, 24$, let $u = 4k$, where $k = 4, 5, 6$, then for any $2k + 1 \le n \le 19$, there exists an FSOLS$(1^n k^1)$. Using Construction 2.10, we have an HSD$(4^n (4k)^1)$.

For $n = 8, 11, 13$ and $u = 14, 15, 17, 18, 19$, the designs are given in Appendix A9, A10, A11, A12, A13, respectively. For $n = 13$ and $u = 21, 22, 23$, the models are given in Appendix A14. □

Lemma 4.4 *There exists an HSD*$(4^n u^1)$ *if* $26 \le u \le 32$ *and* $n \ge \lceil u/2 \rceil + 1$*, with the possible exception of* $n = 19$ *and* $u \in \{29, 30, 31\}$.

Proof First of all, for $u = 28, 32$, the models can be generated by applying Lemma 2.10 on FSOLS$(1^n k^1)$ with $k = 7, 8$. For $n \ge 20$, the lemma holds by Theorem 3.17(a). Lemma 3.4 covers the case when $u = 26$ and $n = 14$. The model for $n = 15$ comes from Lemmas 3.5. For $n = 16, 17, 18$, the models are provided by Lemmas 2.11, 3.6, and 3.7, respectively. For $u = 26, 27$ and $n = 19$, the models come from Lemma 3.8. □

Lemma 4.5 *There exists an HSD*$(4^n u^1)$ *if* $33 \le u \le 36$ *and* $n \ge \lceil u/2 \rceil + 1$*, with the possible exception of* $n \in \{19, 22\}$ *and* $u \in \{33, 34, 35\}$.

Proof First of all, for $u = 36$, the models can be generated by applying Lemma 2.10 on FSOLS($1^n 9^1$). For $n \geq 23$, the lemma holds by Theorem 3.17(b). For $n = 18$ and $u = 33, 34$, the models are provided by Lemma 3.7. For $n = 20, 21$, the models are provided by Lemmas 2.11 and 3.9, respectively. □

In concluding this section, we have essentially proved the following theorem.

Theorem 4.6 *For* $0 \leq u \leq 36$, *an HSD*($4^n u^1$) *exists if and only if* $n \geq 4$ *and* $0 \leq u \leq 2n - 2$, *with the possible exception of* $n = 19$ *and* $u \in \{29, 30, 31, 33, 34, 35\}$, *and* $n = 22$ *and* $u \in \{33, 34, 35\}$.

Proof The necessary conditions come from Lemma 1.3. For the sufficiency, the conclusion comes directly from Lemmas 4.1–4.5. □

5 HSD($4^n u^1$) for general u

In this section, we present some constructions for the more general case of existence of HSD($4^n u^1$).

Lemma 5.1 *There exists an HSD*($4^n u^1$) *for* $37 \leq u \leq 42$ *and* $n \geq \lceil 2u/3 \rceil + 7$.

Proof We have $n \geq 27$ and Theorem 3.17(c) applies. □

From Theorem 1.4(d), we know that there exists a TD($7, m$) for $m \geq 61$. This fact will be used in proofs of the following lemmas.

Lemma 5.2 *There exists an HSD*($4^n u^1$) *for* $u \geq 366$ *and* $n \geq \lceil 2u/3 \rceil + 7$.

Proof From $u \geq 366$ and $n \geq \lceil 2u/3 \rceil + 7$, we know $n \geq 251$. Let $n = 4m + k$, where $4 \leq k \leq 7$ and $m \geq 61$. From $n \geq \lceil 2u/3 \rceil + 7$, we have $4m + 7 \geq 4m + k \geq \lceil 2u/3 \rceil + 7$, i.e., $4m \geq \lceil 2u/3 \rceil$ or $u \leq 6m$. That is, the maximal value of u for $n = 4m + k$ is bound by $6m$. Applying Lemma 2.23 with given values of m, k and u, we obtain the desired result. □

Lemma 5.3 *There exists an HSD*($4^n u^1$) *for* $42 \leq u \leq 366$ *and* $n \geq \lceil 2u/3 \rceil + 7$.

Proof From $u \geq 42$ and $n \geq \lceil 2u/3 \rceil + 7$, we know $n \geq 35$. Let $n = 4s + t$, where $4 \leq t \leq 7$ and $s \geq 7$. The range of u for this n is from 1 to $6s$. From the proof of Lemma 5.2, we know that if there exists a TD($7, s$), then using Lemma 2.23 with $m = s$, $k = t$, and $0 \leq u \leq 6s$, we have the desired HSD($4^n u^1$). So our focus is on such $n = 4s + t$, where a TD($7, s$) is missing.

According to Theorem 1.4(d), for $s \in M_7 = \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$, a TD($7, s$) is unknown. For those $s \in M_7$ except 14, since a TD($7, s + 1$) exists, we may apply Lemma 2.23 with $m = s + 1, k = t - 4$, and $0 \leq u \leq 6s$, for $t = 4, 5, 6$ to obtain the desired HSD($4^n u^1$).

For $s=14$ and $t=4,5$, we apply Lemma 2.21 with $m=15$, $k=t-4$, and $0\leq u\leq 6s$. For $s=15,20,26,30,34,38,46,60$ and $t=7$, we apply Lemma 2.21 with $m=s$, $k=7$, and $0\leq u\leq 6s$ to obtain the desired HSD($4^n u^1$). For $s=22$ and $t=7$, i.e., $n=95$, we apply Lemma 3.15 with $m=19$.

The remaining cases are for $s=10,14,18$ and $t=7$, and $s=14$ and $t=6$, or $n=47,63,79$ and 62.

For $n=47$, we at first apply Lemma 2.22 with $m=20, k=14$, and $20\leq u\leq 60$ to obtain an HSD($40^4(28)^1u^1$), and fill the holes of sizes 40 and 28 with an HSD(4^{10}) and an HSD(4^7), we obtain an HSD($4^{47}u^1$).

For $n=62,63,79$, we at first apply Lemma 2.20 with $m=13$ and $k=10,11$, and $m=17$ and $k=11$, and $0\leq t\leq 6(s-1)$ to obtain HSDs of types ($52^4 40^1 t^1$), ($52^4 44^1 t^1$), and ($68^4 44^1 t^1$). We then adjoin k infinite points, $0\leq k\leq 18$, to these HSDs and fill the holes of sizes 40, 44, 52, 68, with HSDs of types $4^r k^1$, for $r=10,11,13,17$, respectively, we obtain HSDs of types $4^n(t+k)^1$ for $n=62,63,79$, where $0\leq t+k\leq 6s+12$, as desired. □

Theorem 5.4 *There exists an HSD*($4^n u^1$) *when* $u\geq 37$ *and* $n\geq \lceil 2u/3\rceil+7$.

Proof The conclusion comes from Lemmas 5.1–5.3 directly. □

6 Conclusions

We have investigated the existence of HSD($4^n u^1$) and showed that for $0\leq u\leq 36$ the necessary conditions for existence are sufficient with just nine possible exceptions for (n,u). We have also established a general result for the existence of an HSD($4^n u^1$) for $n\geq\lceil 2u/3\rceil+7$, while the existence problem of HSD($4^n u^1$) for $u\geq 37$ and $\lceil u/2\rceil+1\leq n\leq\lceil 2u/3\rceil+6$ is still open and remains under investigation. In summary, by combining the results of Theorems 4.6 and 5.4, we have proved the main result in Theorem 1.9, which is restated below.

Theorem 6.1 (a) *For* $0\leq u\leq 36$, *an HSD*($4^n u^1$) *exists if and only if* $n\geq 4$ *and* $0\leq u\leq 2n-2$, *with the possible exception of* $n=19$ *and* $u\in\{29,30,31,33,34,35\}$, *and* $n=22$ *and* $u\in\{33,34,35\}$.

(b) *There exists an HSD*($4^n u^1$) *when* $u\geq 37$ *and* $n\geq\lceil 2u/3\rceil+7$.

Proof (a) This is a restatement of Theorem 4.6.

(b) This is a restatement of Theorem 5.4. □

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Appendix

Here we list some HSDs which are used in the previous sections. Most of them are obtained by computer. In the following list, the point set of an HSD($4^n u^1$) consists of Z_{4n} and u infinite points which are denoted by alphabet. For simplicity, we only list the starter blocks or the corresponding Latin square. We also use the +2 method to develop blocks, which means that we add 2 (mod $4n$) to each point of the starter blocks to obtain all blocks.

A1 HSD($4^n 2^1$) for $6 \le n \le 11$

$n = 6$ (+1 mod 24):
$\{0,1,20,3\},\{0,2,19,x_1\},\{0,3,23,13\},\{0,5,3,14\},\{0,8,7,16\},\{0,20,11,x_2\}$

$n = 7$ (+1 mod 28):
$\{0,1,17,11\},\{0,2,26,22\},\{0,3,1,9\},\{0,5,23,8\},\{0,9,24,x_1\},\{0,10,9,25\},\{0,17,12,x_2\}$,

$n = 8$ (+1 mod 32):
$\{0,1,23,3\},\{0,2,20,23\},\{0,4,17,7\},\{0,5,1,15\}$, $\{0,6,7,20\},\{0,7,13,2\},\{0,17,6,x_2\}$,
$\{0,23,28,x_1\}$

$n = 10$ (+1 mod 40):
$\{0,1,36,15\},\{0,2,3,14\},\{0,3,31,8\},\{0,4,17,2\}$, $\{0,5,8,24\},\{0,6,29,22\},\{0,8,1,x_2\}$,
$\{0,9,13,31\},\{0,12,33,6\},\{0,26,15,x_1\}$

$n = 11$ (+1 mod 44):
$\{0,1,9,41\},\{0,2,19,32\},\{0,3,10,4\},\{0,4,17,2\},\{0,7,6,30\},\{0,8,13,23\},\{0,9,28,x_1\}$,
$\{0,14,32,6\},\{0,16,7,34\},\{0,19,3,24\},\{0,39,24,x_2\}$

A2 HSD($4^n 3^1$) for $6 \le n \le 14$

$n = 6$ (+2 mod 24):
$\{0,1,16,x_1\},\{0,2,10,15\},\{0,3,22,17\},\{0,4,3,13\}$, $\{0,8,19,4\},\{0,9,11,x_3\},\{0,10,17,1\}$,
$\{0,13,23,21\},\{0,17,21,x_2\},\{0,23,7,3\},\{1,4,23,x_1\}$, $\{1,14,0,x_3\},\{1,18,20,x_2\}$

$n = 9$ (+2 mod 36):
$\{0,1,31,16\},\{0,3,17,11\},\{0,4,3,5\},\{0,5,28,6\}$, $\{0,6,30,17\},\{0,7,20,10\},\{0,8,22,34\}$,
$\{0,11,23,28\},\{0,16,12,x_2\},\{0,17,7,19\},\{0,19,15,29\},\{0,21,29,1\}$, $\{0,25,4,3\}$,
$\{0,29,13,33\}$, $\{0,33,2,x_1\},\{0,34,5,x_3\},\{1,24,13,x_1\},\{1,27,12,x_3\}$, $\{1,33,31,x_2\}$

$n = 10$ (+2 mod 40):
$\{0,1,12,x_2\},\{0,2,26,19\},\{0,3,29,x_1\},\{0,4,5,23\},\{0,5,38,12\},\{0,6,39,37\},\{0,7,1,15\}$,
$\{0,8,34,31\},\{0,9,36,21\},\{0,13,8,39\},\{0,15,37,9\}$, $\{0,16,3,11\},\{0,18,16,3\}$,
$\{0,19,15,32\}$, $\{0,21,9,25\},\{0,28,22,x_3\},\{0,29,21,22\}$, $\{0,35,11,17\},\{1,5,3,x_3\}$,
$\{1,24,28,x_1\},\{1,30,25,x_2\}$

$n = 13$ (+2 mod 52) :
$\{0,1,20,3\},\{0,2,42,24\},\{0,3,46,45\},\{0,4,50,20\},\{0,5,35,11\},\{0,6,31,27\},\{0,7,15,37\}$,
$\{0,8,44,41\},\{0,10,7,47\},\{0,11,23,38\},\{0,12,43,50\}$, $\{0,14,19,17\},\{0,15,48,x_3\}$,
$\{0,16,24,x_2\}$, $\{0,19,9,30\},\{0,20,11,21\},\{0,21,17,35\},\{0,23,5,x_1\}$, $\{0,24,34,23\}$,
$\{0,25,49,29\}$, $\{0,27,12,7\}$, $\{0,29,45,1\},\{0,33,47,4\},\{1,7,9,47\},\{1,36,35,x_3\}$,
$\{1,37,17,x_2\},\{1,44,26,x_1\}$

$n = 14$ (+2 mod 56):
$\{0,2,26,3\}, \{0,3,24,54\}, \{0,4,33,26\}, \{0,5,31,21\}, \{0,6,53,x_2\}, \{0,8,12,27\}, \{0,9,25,38\}, \{0,10,15,16\}, \{0,11,49,15\}, \{0,12,35,55\}, \{0,13,7,10\}, \{0,16,9,1\}, \{0,17,29,50\}, \{0,18,40,31\}, \{0,19,21,9\}, \{0,20,37,19\}, \{0,21,10,39\}, \{0,22,47,20\}, \{0,23,36,12\}, \{0,25,45,8\}, \{0,31,39,7\}, \{0,41,8,x_1\}, \{0,45,4,53\}, \{0,51,38,x_3\}, \{1,0,37,x_3\}, \{1,3,55,25\}, \{1,5,27,11\}, \{1,18,13,x_1\}, \{1,51,6,x_2\}$

A3 HSD(4^n5^1) for $6 \leq n \leq 14$

$n = 6$ (+2 mod 24):
$\{0,1,3,8\}, \{0,2,13,9\}, \{0,4,9,23\}, \{0,5,15,4\}, \{0,7,16,x_3\}, \{0,9,14,x_2\}, \{0,10,23,21\}, \{0,13,5,x_1\}, \{0,16,2,x_5\}, \{0,21,4,x_4\}, \{1,2,4,x_1\}, \{1,8,11,x_2\}, \{1,10,17,x_3\}, \{1,17,21,x_5\}, \{1,22,23,x_4\}$

$n = 7$ (+2 mod 28):
$\{0,1,18,20\}, \{0,3,4,15\}, \{0,5,11,23\}, \{0,8,17,x_3\}, \{0,10,16,1\}, \{0,15,2,19\}, \{0,16,6,x_4\}, \{0,19,3,4\}, \{0,22,20,x_5\}, \{0,23,5,25\}, \{0,24,1,x_2\}, \{0,25,27,x_1\}, \{1,19,16,x_2\}, \{1,20,4,x_1\}, \{1,23,6,x_3\}, \{1,25,21,x_4\}, \{1,27,7,x_5\}$

$n = 8$ (+2 mod 32):
$\{0,1,18,30\}, \{0,3,10,23\}, \{0,4,11,6\}, \{0,7,6,21\}, \{0,9,7,3\}, \{0,10,19,5\}, \{0,11,17,18\}, \{0,18,28,x_4\}, \{0,19,15,20\}, \{0,21,9,19\}, \{0,23,5,x_1\}, \{0,26,23,x_2\}, \{0,29,12,x_5\}, \{0,30,29,x_3\}, \{1,8,12,x_1\}, \{1,16,29,x_5\}, \{1,21,11,x_4\}, \{1,27,0,x_3\}, \{1,31,20,x_2\}$

$n = 9$ (+2 mod 36):
$\{0,1,35,29\}, \{0,2,33,x_3\}, \{0,3,4,1\}, \{0,4,17,19\}, \{0,5,8,20\}, \{0,6,30,25\}, \{0,7,13,21\}, \{0,8,10,x_5\}, \{0,10,16,x_1\}, \{0,13,34,17\}, \{0,14,22,3\}, \{0,15,23,x_4\}, \{0,16,12,11\}, \{0,23,11,7\}, \{0,25,5,x_2\}, \{0,29,3,23\}, \{1,13,8,x_3\}, \{1,15,11,x_5\}, \{1,16,30,x_2\}, \{1,26,0,x_4\}, \{1,27,13,x_1\}$

$n = 10$ (+2 mod 40):
$\{0,1,38,19\}, \{0,3,14,27\}, \{0,4,27,1\}, \{0,5,19,x_5\}, \{0,6,22,38\}, \{0,7,15,x_4\}, \{0,8,12,9\}, \{0,11,9,x_3\}, \{0,12,7,15\}, \{0,14,11,39\}, \{0,15,33,11\}, \{0,17,36,31\}, \{0,18,31,7\}, \{0,23,6,x_2\}, \{0,29,5,6\}, \{0,31,35,33\}, \{0,33,39,5\}, \{0,38,16,x_1\}, \{1,14,28,x_4\}, \{1,16,24,x_5\}, \{1,22,7,x_2\}, \{1,32,20,x_3\}, \{1,37,9,x_1\}$

$n = 14$ (+2 mod 56):
$\{0,1,53,31\}, \{0,2,3,29\}, \{0,3,9,49\}, \{0,4,38,21\}, \{0,6,51,53\}, \{0,7,39,52\}, \{0,8,49,20\}, \{0,9,2,25\}, \{0,11,29,44\}, \{0,12,10,46\}, \{0,13,50,47\}, \{0,15,37,27\}, \{0,16,31,x_2\}, \{0,17,40,16\}, \{0,18,55,23\}, \{0,19,36,11\}, \{0,21,4,3\}, \{0,22,30,55\}, \{0,26,13,x_5\}, \{0,27,7,51\}, \{0,33,24,13\}, \{0,35,45,50\}, \{0,37,21,x_1\}, \{0,46,8,x_3\}, \{0,49,19,x_4\}, \{1,5,53,3\}, \{1,10,42,x_4\}, \{1,37,32,x_2\}, \{1,39,51,x_3\}, \{1,49,14,x_5\}, \{1,52,22,x_1\}$

A4 HSD(4^n6^1) for $6 \leq n \leq 14$

$n = 6$ (+1 mod 24):
$\{0,5,21,14\}, \{0,8,19,9\}, \{0,13,8,x_2\}, \{0,15,11,x_3\}, \{0,20,17,x_4\}, \{0,21,14,x_1\}, \{0,22,20,x_5\}, \{0,23,22,x_6\}$

$n = 7$ (+1 mod 28):
$\{0,1,4,x_4\}, \{0,2,8,24\}, \{0,3,5,x_3\}, \{0,4,16,5\}, \{0,5,15,x_2\}, \{0,8,19,x_5\}, \{0,9,10,20\}, \{0,15,2,x_6\}, \{0,22,3,x_1\}$

$n = 9$ (+1 mod 36):
$\{0,1,5,29\}, \{0,2,13,17\}, \{0,3,29,35\}, \{0,7,20,6\}, \{0,8,30,5\}, \{0,15,12,x_3\}, \{0,19,11,x_1\}, \{0,20,22,x_2\}, \{0,23,2,x_5\}, \{0,26,10,x_6\}, \{0,31,19,x_4\}$

$n = 10$ (+1 mod 40):
$\{0,2,23,11\}, \{0,3,12,17\}, \{0,4,6,x_5\}, \{0,6,5,27\}, \{0,7,11,25\}, \{0,8,16,33\}, \{0,9,3,16\}, \{0,11,8,x_1\}, \{0,16,2,x_2\}, \{0,19,1,x_6\}, \{0,25,13,x_3\}, \{0,39,4,x_4\}$

$n = 14$ (+1 mod 56):
$\{0,1,31,51\}, \{0,2,45,29\}, \{0,3,1,16\}, \{0,4,48,x_5\}, \{0,5,37,20\}, \{0,6,16,9\}, \{0,8,26,15\}, \{0,12,20,x_2\}, \{0,13,9,31\}, \{0,18,24,1\}, \{0,19,2,23\}, \{0,24,5,34\}, \{0,26,33,x_6\}, \{0,31,34,x_1\}, \{0,46,35,x_4\}, \{0,47,12,x_3\}$

A5 HSD(4^n7^1) for $5 \le n \le 14$

$n = 5$ (+2 mod 20):
$\{0,1,7,x_6\}, \{0,2,13,x_7\}, \{0,3,2,x_1\}, \{0,8,6,x_5\}, \{0,13,1,4\}, \{0,14,11,x_2\}, \{0,16,4,x_3\}, \{0,19,12,x_4\}, \{1,3,4,x_7\}, \{1,5,2,x_2\}, \{1,9,13,x_5\}, \{1,10,19,x_4\}, \{1,12,18,x_6\}, \{1,14,7,x_1\}, \{1,15,17,x_3\}$

$n = 7$ (+2 mod 28):
$\{0,1,3,x_5\}, \{0,2,27,x_7\}, \{0,3,2,20\}, \{0,4,15,27\}, \{0,9,25,19\}, \{0,11,6,22\}, \{0,17,8,x_2\}, \{0,19,1,x_3\}, \{0,20,4,x_6\}, \{0,22,12,x_1\}, \{0,23,19,17\}, \{0,25,10,x_4\}, \{1,2,5,x_2\}, \{1,5,27,x_6\}, \{1,9,17,x_1\}, \{1,14,16,x_3\}, \{1,16,20,x_5\}, \{1,19,24,x_7\}, \{1,24,9,x_4\}$

$n = 8$ (+2 mod 32):
$\{0,1,3,29\}, \{0,2,6,15\}, \{0,4,29,19\}, \{0,5,9,10\}, \{0,7,12,x_6\}, \{0,10,19,x_5\}, \{0,12,10,1\}, \{0,13,31,x_1\}, \{0,14,2,x_7\}, \{0,15,25,13\}, \{0,17,18,11\}, \{0,19,7,x_3\}, \{0,26,11,x_2\}, \{0,27,4,x_4\}, \{1,3,28,x_2\}, \{1,4,10,x_1\}, \{1,5,0,x_5\}, \{1,12,15,x_4\}, \{1,15,21,x_7\}, \{1,22,11,x_6\}, \{1,30,16,x_3\}$

$n = 10$ (+2 mod 40):
$\{0,1,24,18\}, \{0,2,8,23\}, \{0,3,37,25\}, \{0,4,35,19\}, \{0,5,6,28\}, \{0,7,36,x_7\}, \{0,9,11,3\}, \{0,12,14,1\}, \{0,13,1,37\}, \{0,16,9,7\}, \{0,25,3,x_3\}, \{0,26,12,x_4\}, \{0,29,33,x_2\}, \{0,31,39,5\}, \{0,32,13,x_6\}, \{0,33,7,8\}, \{0,35,22,x_5\}, \{0,37,2,x_1\}, \{1,18,22,x_2\}, \{1,20,36,x_3\}, \{1,22,9,x_1\}, \{1,23,14,x_6\}, \{1,24,13,x_7\}, \{1,27,3,x_4\}, \{1,30,5,x_5\}$

$n = 11$ (+2 mod 44):
$\{0,1,30,18\}, \{0,2,42,1\}, \{0,4,35,42\}, \{0,5,37,x_3\}, \{0,6,23,38\}, \{0,7,15,x_1\}, \{0,8,7,5\}, \{0,9,28,41\}, \{0,10,29,23\}, \{0,14,4,39\}, \{0,15,21,x_7\}, \{0,16,24,x_2\}, \{0,17,8,x_6\}, \{0,18,31,x_4\}, \{0,19,34,14\}, \{0,21,17,7\}, \{0,31,3,27\}, \{0,39,18,x_5\}, \{1,2,26,x_3\}, \{1,4,36,x_7\}, \{1,5,7,21\}, \{1,13,27,11\}, \{1,18,21,x_6\}, \{1,19,37,x_2\}, \{1,20,11,x_5\}, \{1,22,6,x_1\}, \{1,37,0,x_4\}$

$n = 14$ (+2 mod 56):
$\{0,1,49,30\}, \{0,2,45,46\}, \{0,3,35,x_1\}, \{0,4,52,29\}, \{0,5,20,13\}, \{0,6,15,35\}, \{0,7,8,25\}, \{0,9,6,31\}, \{0,11,51,7\}, \{0,12,16,34\}, \{0,13,46,6\}, \{0,20,31,33\}, \{0,21,11,38\}, \{0,23,5,x_7\}, \{0,24,19,45\}, \{0,26,38,x_2\}, \{0,27,26,21\}, \{0,31,43,47\}, \{0,34,2,x_5\}, \{0,37,32,17\}, \{0,39,37,15\}, \{0,45,9,19\}, \{0,46,17,x_6\}, \{0,47,53,37\}, \{0,48,41,x_4\}, \{0,53,1,x_3\}, \{1,14,34,x_3\}, \{1,19,45,x_2\}, \{1,22,24,x_7\}, \{1,25,3,x_5\}, \{1,42,2,x_1\}, \{1,49,10,x_6\}, \{1,51,54,x_4\}$

A6 HSD(4^n10^1) for $7 \le n \le 14$

$n = 7$ (+1 mod 28):
$\{0,1,20,17\}, \{0,4,26,x_9\}, \{0,6,22,x_7\}, \{0,8,5,x_8\}, \{0,9,17,x_1\}, \{0,10,25,x_3\}, \{0,11,1,x_6\}, \{0,12,13,x_2\}, \{0,15,10,x_4\}, \{0,23,19,x_5\}, \{0,26,24,y_1\}$

$n = 8$ (+1 mod 32):
$\{0,1,11,2\}, \{0,3,10,x_4\}, \{0,6,17,29\}, \{0,14,18,x_2\}, \{0,15,27,x_3\}, \{0,19,2,x_5\}, \{0,21,26,x_6\}, \{0,22,28,x_1\}, \{0,25,12,x_8\}, \{0,27,13,x_9\}, \{0,28,29,x_7\}, \{0,30,7,y_1\}$

$n = 10$ (+1 mod 40):
$\{0,1,9,33\}, \{0,2,7,39\}, \{0,3,1,15\}, \{0,4,23,x_5\}, \{0,5,27,x_8\}, \{0,7,38,x_2\}, \{0,9,21,34\},$

$\{0,15,29,x_7\},\{0,18,14,x_1\},\{0,19,22,x_6\},\{0,23,6,x_9\},\{0,28,4,y_1\},\{0,29,16,x_4\},\{0,34,5,x_3\}$

$n=11$ (+1 mod 44):
$\{0,1,2,x_5\},\{0,2,30,37\},\{0,4,40,35\},\{0,6,36,x_4\},\{0,9,26,38\},\{0,10,28,x_9\},\{0,14,19,34\},$
$\{0,16,12,x_1\},\{0,17,5,24\},\{0,20,17,x_3\},\{0,23,10,y_1\},\{0,26,3,x_7\},\{0,31,6,x_8\},\{0,36,21,x_6\},$
$\{0,41,43,x_2\}$

$n=13$ (+1 mod 52):
$\{0,1,16,8\},\{0,2,50,29\},\{0,3,43,36\},\{0,4,34,25\},\{0,5,12,x_3\},\{0,11,30,14\},\{0,12,1,x_4\},$
$\{0,14,15,49\},\{0,15,46,x_7\},\{0,19,28,5\},\{0,27,47,x_9\},\{0,28,10,x_2\},\{0,30,20,x_1\},\{0,32,4,y_1\},$
$\{0,35,41,x_5\},\{0,42,44,x_8\},\{0,46,29,x_6\}$

$n=14$ (+1 mod 56):
$\{0,2,47,39\},\{0,3,35,1\},\{0,4,24,48\},\{0,6,31,49\},\{0,7,20,x_9\},\{0,9,11,50\},\{0,11,23,38\},$
$\{0,12,16,29\},\{0,16,49,19\},\{0,20,10,x_3\},\{0,29,51,y_1\},\{0,31,1,x_7\},\{0,33,38,x_5\},$
$\{0,35,26,x_6\},\{0,37,52,x_1\},\{0,46,6,x_4\},\{0,51,48,x_8\},\{0,55,34,x_2\}$

A7 HSD($4^n 11^1$) for $7 \le n \le 14$

$n=7$ (+2 mod 28):
$\{0,1,23,x_4\},\{0,4,1,x_8\},\{0,9,24,y_1\},\{0,11,19,x_9\},\{0,13,17,x_5\},\{0,16,11,y_2\},\{0,17,16,x_2\},$
$\{0,18,10,x_3\},\{0,19,8,3\},\{0,20,9,x_1\},\{0,22,6,x_7\},\{0,26,2,x_6\},\{1,2,4,x_9\},\{1,4,3,x_2\},$
$\{1,5,24,y_2\},\{1,7,23,x_6\},\{1,9,11,x_7\},\{1,13,26,x_8\},\{1,14,5,y_1\},\{1,19,14,x_1\},\{1,24,2,x_4\},$
$\{1,26,16,x_5\},\{1,27,9,x_3\}$

$n=8$ (+2 mod 32):
$\{0,2,20,x_6\},\{0,4,7,27\},\{0,5,3,x_2\},\{0,6,25,y_2\},\{0,7,17,26\},\{0,9,5,20\},\{0,12,22,x_9\},$
$\{0,15,9,x_5\},\{0,18,29,x_4\},\{0,19,30,x_3\},\{0,21,6,y_1\},\{0,22,19,x_8\},\{0,27,18,x_1\},$
$\{0,29,28,x_7\},\{1,0,31,y_1\},\{1,2,7,x_1\},\{1,3,15,x_9\},\{1,8,21,x_7\},\{1,19,5,x_6\},\{1,20,11,x_3\},$
$\{1,22,18,x_5\},\{1,23,6,y_2\},\{1,27,2,x_4\},\{1,29,22,x_8\},\{1,30,0,x_2\}$

$n=9$ (+2 mod 36):
$\{0,2,16,x_3\},\{0,5,31,x_2\},\{0,6,23,x_4\},\{0,7,28,x_7\},\{0,10,17,32\},\{0,11,3,33\},\{0,12,1,y_1\},$
$\{0,15,10,26\},\{0,17,29,x_5\},\{0,19,32,21\},\{0,22,6,x_1\},\{0,23,12,y_2\},\{0,28,5,x_9\},\{0,31,35,x_8\},$
$\{0,32,34,x_6\},\{0,35,19,3\},\{1,0,30,x_5\},\{1,4,16,x_8\},\{1,5,22,x_9\},\{1,8,7,x_7\},\{1,9,11,x_3\},$
$\{1,11,25,x_6\},\{1,23,29,x_1\},\{1,24,17,y_2\},\{1,25,24,x_4\},\{1,34,26,x_2\},\{1,35,4,y_1\}$

$n=10$ (+2 mod 40):
$\{0,2,14,x_6\},\{0,3,4,y_1\},\{0,4,19,32\},\{0,5,7,9\},\{0,6,17,21\},\{0,7,1,x_3\},\{0,8,27,14\},$
$\{0,14,18,37\},\{0,16,5,13\},\{0,17,31,x_8\},\{0,18,35,x_5\},\{0,25,33,x_4\},\{0,28,21,y_2\},$
$\{0,29,24,x_2\},\{0,33,29,x_9\},\{0,35,8,x_1\},\{0,39,38,x_7\},\{1,0,23,x_2\},\{1,4,2,x_3\},\{1,7,35,19\},$
$\{1,10,28,x_4\},\{1,15,30,x_5\},\{1,18,25,y_1\},\{1,19,3,x_6\},\{1,20,15,x_7\},\{1,26,32,x_8\},$
$\{1,29,38,y_2\},\{1,30,33,x_1\},\{1,32,16,x_9\}$

$n=11$ (+2 mod 44):
$\{0,2,37,x_3\},\{0,3,10,16\},\{0,7,23,x_1\},\{0,9,4,27\},\{0,10,30,2\},\{0,12,25,19\},\{0,13,43,29\},$
$\{0,14,6,x_8\},\{0,15,27,7\},\{0,17,21,x_9\},\{0,19,17,y_2\},\{0,21,3,15\},\{0,24,28,x_4\},\{0,26,20,x_7\},$
$\{0,27,18,3\},\{0,35,29,34\},\{0,36,13,x_2\},\{0,37,12,x_6\},\{0,40,19,y_1\},\{0,43,42,x_5\},\{1,0,41,x_5\},$
$\{1,3,4,y_1\},\{1,4,36,x_1\},\{1,14,40,y_2\},\{1,17,30,x_3\},\{1,20,6,x_9\},\{1,27,10,x_2\},\{1,35,15,x_4\},$
$\{1,37,3,x_8\},\{1,40,35,x_6\},\{1,41,33,x_7\}$

$n=13$ (+2 mod 52):
$\{0,2,27,5\},\{0,5,50,x_7\},\{0,6,35,x_8\},\{0,8,48,33\},\{0,9,24,x_2\},\{0,10,32,x_1\},\{0,11,29,43\},$
$\{0,12,47,x_5\},\{0,14,30,48\},\{0,15,31,x_4\},\{0,16,36,6\},\{0,17,14,45\},\{0,19,8,41\},\{0,21,6,x_3\},$
$\{0,24,19,17\},\{0,25,21,31\},\{0,27,17,18\},\{0,32,40,y_2\},\{0,35,5,23\},\{0,43,23,x_9\},\{0,45,51,2\},$
$\{0,47,7,x_6\},\{0,48,49,y_1\},\{1,0,49,x_7\},\{1,5,38,x_5\},\{1,7,35,15\},\{1,12,39,x_3\},\{1,24,0,x_9\},$
$\{1,25,17,x_1\},\{1,30,20,x_4\},\{1,37,16,y_1\},\{1,41,42,x_8\},\{1,45,43,y_2\},\{1,46,3,x_2\},\{1,50,12,x_6\}$

$n = 14$ (+2 mod 56):
$\{0,1,5,x_2\},\{0,2,45,8\},\{0,3,11,2\},\{0,4,49,29\},\{0,5,3,9\},\{0,6,22,4\},\{0,7,47,x_3\},$
$\{1,36,55,x_5\},\{0,10,35,20\},\{0,11,36,x_4\},\{0,12,29,y_2\},\{0,13,48,39\},\{0,17,7,22\},$
$\{0,19,55,11\},\{0,22,37,7\},\{0,23,30,x_9\},\{0,24,12,37\},\{0,26,27,y_1\},\{0,27,39,47\},$
$\{0,29,24,x_5\},\{0,36,31,x_6\},\{0,39,15,18\},\{0,40,16,x_7\},\{0,45,23,33\},\{0,48,38,x_1\},$
$\{0,51,33,35\},\{0,55,52,x_8\},\{1,5,35,x_1\},\{1,8,14,x_2\},\{1,14,40,x_3\},\{1,17,4,y_1\},$
$\{1,19,48,x_6\},\{1,22,7,x_9\},\{1,24,47,x_8\},\{1,26,25,x_4\},\{1,33,39,x_7\},\{1,35,38,y_2\}$

A8 HSD(4^n13^1) for $8 \le n \le 14$

$n = 8$ (+2 mod 32) :
$\{0,1,3,y_3\},\{0,2,5,y_2\},\{0,3,7,y_1\},\{0,4,9,y_0\},\{0,6,17,x_9\},\{0,7,12,x_8\},\{0,10,19,x_5\},$
$\{0,12,18,x_7\},\{0,13,22,x_4\},\{0,14,28,x_2\},\{0,15,26,13\},\{0,17,30,x_1\},\{0,21,11,x_3\},$
$\{0,27,15,x_6\},\ \{1,2,4,y_3\},\{1,4,8,y_1\},\{1,5,6,y_2\},\{1,7,13,x_7\},\{1,8,15,x_4\},\{1,10,0,x_3\},$
$\{1,11,10,x_9\},\{1,13,20,x_5\},\{1,15,29,x_2\},\{1,22,7,x_1\},\{1,24,12,x_6\},\{1,28,11,x_8\},$
$\{1,31,2,y_0\}$

$n = 10$ (+2 mod 40) :
$\{0,2,25,y_1\},\{0,4,35,38\},\{0,5,31,16\},\{0,7,23,x_3\},\{0,8,3,y_0\},\{0,9,16,x_1\},\{0,11,9,x_9\},$
$\{0,12,18,x_2\},\{0,13,36,12\},\{0,22,27,x_4\},\{0,23,19,7\},\{0,25,17,y_2\},\{0,26,13,x_8\},$
$\{0,27,15,21\},\{0,29,32,x_5\},\{0,31,12,y_3\},\{0,34,26,x_7\},\{0,39,38,x_6\},\{1,0,39,x_5\},$
$\{1,5,36,y_1\},\{1,6,19,x_1\},\{1,8,30,x_9\},\{1,15,4,y_0\},\{1,17,2,x_4\},\{1,20,5,x_6\},$
$\{1,22,8,x_3\},\{1,23,20,x_8\},\{1,24,13,y_3\},\{1,33,15,x_2\},\{1,38,34,y_2\},\{1,39,33,x_7\}$

$n = 11$ (+2 mod 44) :
$\{0,2,23,x_4\},\{0,3,1,x_5\},\{0,5,35,23\},\{0,6,41,x_1\},\{0,10,17,37\},\{0,12,7,24\},\{0,13,5,x_9\},$
$\{0,14,6,x_6\},\{0,15,31,13\},\{0,16,42,x_7\},\{0,17,30,y_0\},\{0,18,8,28\},\{0,23,40,y_3\},$
$\{0,29,28,x_8\},\{0,31,13,x_3\},\{0,35,39,29\},\{0,36,24,y_1\},\{0,37,18,17\},\{0,40,43,y_2\},$
$\{0,41,21,x_2\},\{1,0,42,x_5\},\{1,3,37,x_7\},\{1,6,20,x_3\},\{1,7,8,x_4\},\{1,17,36,y_2\},\{1,20,26,x_9\},$
$\{1,24,33,y_3\},\{1,26,30,x_2\},\{1,31,16,x_1\},\{1,36,41,x_8\},\{1,37,31,x_6\},\{1,38,35,y_0\},$
$\{1,41,29,y_1\}$

$n = 13$ (+2 mod 52) :
$\{0,2,31,y_1\},\{0,3,10,34\},\{0,4,41,9\},\{0,5,3,30\},\{0,7,12,x_7\},\{0,8,2,51\},\{0,9,44,23\},$
$\{0,12,36,x_2\},\{0,14,28,x_3\},\{0,15,4,y_2\},\{0,17,38,5\},\{0,18,9,33\},\{0,21,46,x_4\},\{0,22,18,7\},$
$\{0,27,45,16\},\{0,32,20,x_1\},\{0,33,17,x_8\},\{0,36,51,y_3\},\ \{0,42,22,y_0\},\{0,43,1,x_5\},$
$\{0,45,7,49\},\ \{0,46,35,x_6\},\{0,47,23,x_9\},\{1,0,35,y_2\},\{1,2,4,x_8\},\ \{1,9,6,x_6\},\{1,16,24,x_5\},$
$\{1,17,25,47\},\ \{1,18,43,x_4\},\{1,30,20,x_9\},\{1,35,34,y_1\},\{1,39,17,x_1\},\ \{1,41,21,x_3\},$
$\{1,42,9,x_7\},\{1,47,7,y_0\},\ \{1,49,28,y_3\},\{1,51,3,x_2\}$

$n = 14$ (+2 mod 56) :
$\{0,1,24,9\},\{0,2,20,y_1\},\{0,3,34,43\},\{0,4,33,51\},\{0,5,1,x_7\},\{0,6,26,8\},\{0,7,46,3\},$
$\{0,15,16,13\},\{0,16,50,11\},\{0,19,41,x_1\},\{0,21,37,2\},\{0,23,53,5\},\{0,25,13,32\},\{0,26,9,31\},$
$\{0,29,44,y_2\},\{0,32,3,x_4\},\{0,33,35,x_5\},\{0,34,38,x_6\},\{0,36,27,y_3\},\{0,43,8,y_0\},\{0,44,4,x_9\},$
$\{0,46,11,x_8\},\ \{0,47,29,53\},\{0,48,7,x_2\},\{0,51,15,19\},\{0,55,31,x_3\},\{1,3,36,x_4\},\{1,8,34,x_7\},$
$\{1,12,2,x_1\},\{1,13,12,x_2\},\{1,18,11,y_0\},\ \{1,26,32,x_5\},\{1,27,21,x_9\},\{1,30,18,x_3\},$
$\{1,37,45,x_6\},\ \{1,41,31,y_1\},\{1,46,35,y_2\},\{1,47,40,x_8\},\{1,51,14,y_3\}$

A9 HSD(4^n14^1) for $8 \le n \le 14$

$n = 8, u = 14$ (+1 mod 32) :
$\{0,4,31,y_0\},\{0,15,3,x_4\},\{0,18,5,x_3\},\{0,19,10,x_7\},\{0,20,13,x_8\},\{0,21,7,x_2\},\{0,22,12,x_6\},$
$\{0,23,17,x_9\},\{0,25,14,x_5\},\{0,26,11,x_1\},\ \{0,27,23,y_1\},\{0,29,26,y_2\},\{0,30,28,y_3\},$
$\{0,31,30,y_4\}$

$n = 9, u = 14$ (+1 mod 36) :
$\{0, 1, 17, x_8\}, \{0, 2, 6, 34\}, \{0, 3, 26, y_2\}, \{0, 4, 1, y_4\}, \{0, 5, 12, y_1\}, \{0, 6, 23, y_0\}, \{0, 11, 33, x_2\}$,
$\{0, 13, 34, x_1\}, \{0, 15, 7, x_3\}, \{0, 16, 21, x_7\}, \{0, 17, 5, x_4\}, \{0, 22, 11, x_5\}, \{0, 24, 14, x_6\}$,
$\{0, 26, 20, x_9\}, \{0, 29, 28, y_3\}$

$n = 10, u = 14$ (+1 mod 40) :
$\{0, 1, 28, 25\}, \{0, 4, 23, x_7\}, \{0, 5, 1, y_3\}, \{0, 6, 9, 31\}, \{0, 9, 8, x_3\}, \{0, 11, 27, y_2\}, \{0, 16, 22, x_6\}$,
$\{0, 17, 29, x_4\}, \{0, 21, 38, x_1\}, \{0, 25, 33, y_0\}, \{0, 26, 19, x_9\}, \{0, 27, 5, y_4\}, \{0, 28, 26, x_8\}$,
$\{0, 32, 6, x_2\}, \{0, 33, 4, y_1\}, \{0, 38, 3, x_5\}$

$n = 11, u = 14$ (+1 mod 44) :
$\{0, 1, 40, 43\}, \{0, 2, 5, 23\}, \{0, 4, 36, x_2\}, \{0, 12, 29, x_1\}, \{0, 14, 27, x_3\}, \{0, 17, 3, 24\}, \{0, 19, 28, x_7\}$,
$\{0, 24, 32, x_9\}, \{0, 28, 34, y_0\}, \{0, 29, 25, y_1\}, \{0, 31, 13, x_6\}, \{0, 34, 18, x_4\}, \{0, 35, 1, y_3\}$,
$\{0, 36, 38, y_2\}, \{0, 37, 30, x_8\}, \{0, 38, 9, y_4\}, \{0, 39, 20, x_5\}$

$n = 13, u = 14$ (+1 mod 52) :
$\{0, 1, 18, 23\}, \{0, 2, 12, 43\}, \{0, 8, 28, 11\}, \{0, 9, 33, x_7\}, \{0, 10, 4, x_5\}, \{0, 12, 10, 37\}, \{0, 14, 2, x_3\}$,
$\{0, 15, 31, 7\}, \{0, 16, 46, y_3\}, \{0, 18, 32, y_1\}, \{0, 29, 47, x_2\}, \{0, 30, 51, y_0\}, \{0, 32, 35, y_4\}$,
$\{0, 33, 29, x_6\}, \{0, 41, 14, x_9\}, \{0, 45, 37, x_1\}, \{0, 46, 45, y_2\}, \{0, 48, 43, x_8\}, \{0, 49, 16, x_4\}$

$n = 14, u = 14$ (+1 mod 56) :
$\{0, 2, 10, 21\}, \{0, 3, 22, 4\}, \{0, 5, 3, 29\}, \{0, 7, 41, y_3\}, \{0, 8, 49, x_4\}, \{0, 9, 45, x_8\}, \{0, 10, 26, 1\}$,
$\{0, 12, 18, 51\}, \{0, 15, 54, y_1\}, \{0, 16, 25, x_7\}, \{0, 17, 20, x_5\}, \{0, 19, 44, 24\}, \{0, 27, 16, y_2\}$,
$\{0, 32, 50, y_4\}, \{0, 34, 21, x_1\}, \{0, 35, 23, x_3\}, \{0, 43, 13, x_6\}, \{0, 50, 4, x_2\}, \{0, 52, 29, x_9\}$,
$\{0, 55, 48, y_0\}$

A10 HSD($4^n 15^1$) for $9 \le n \le 14$

$n = 9, u = 15$ (+2 mod 36):
$\{0, 2, 30, x_6\}, \{0, 4, 35, y_2\}, \{0, 5, 7, 26\}, \{0, 8, 1, y_5\}, \{0, 11, 34, y_1\}, \{0, 13, 14, y_4\}, \{0, 14, 26, x_1\}$,
$\{0, 16, 32, y_0\}, \{0, 19, 8, y_3\}, \{0, 21, 24, x_2\}, \{0, 23, 17, x_5\}, \{0, 24, 13, x_8\}, \{0, 26, 20, x_9\}$,
$\{0, 30, 31, x_4\}, \{0, 31, 23, x_7\}, \{0, 33, 11, x_3\}, \{1, 0, 23, y_4\}, \{1, 2, 17, y_3\}, \{1, 7, 4, y_2\}, \{1, 8, 13, x_2\}$,
$\{1, 12, 8, x_7\}, \{1, 17, 34, x_8\}, \{1, 22, 29, y_1\}, \{1, 23, 11, x_6\}, \{1, 25, 5, y_0\}, \{1, 27, 31, x_1\}$,
$\{1, 29, 3, x_9\}, \{1, 30, 16, x_3\}, \{1, 33, 12, y_5\}, \{1, 34, 32, x_5\}, \{1, 35, 18, x_4\}$

$n = 10, u = 15$ (+2 mod 40):
$\{0, 1, 28, x_8\}, \{0, 2, 31, y_3\}, \{0, 3, 4, x_5\}, \{0, 5, 27, x_4\}, \{0, 6, 38, x_1\}, \{0, 7, 15, y_2\}, \{0, 8, 26, y_1\}$,
$\{0, 11, 32, y_4\}, \{0, 13, 22, y_0\}, \{0, 15, 9, 16\}, \{0, 17, 13, x_6\}, \{0, 21, 33, 4\}, \{0, 22, 19, x_3\}$,
$\{0, 24, 7, x_2\}, \{0, 26, 21, x_7\}, \{0, 28, 16, x_9\}, \{0, 31, 6, 3\}, \{0, 36, 29, y_5\}, \{1, 2, 36, y_2\}, \{1, 3, 38, x_3\}$,
$\{1, 6, 15, x_8\}, \{1, 13, 6, y_5\}, \{1, 14, 25, y_0\}, \{1, 16, 33, y_4\}, \{1, 17, 3, x_1\}, \{1, 18, 4, x_6\}, \{1, 19, 18, x_2\}$,
$\{1, 22, 37, x_5\}, \{1, 27, 0, x_7\}, \{1, 32, 30, x_4\}, \{1, 33, 35, y_1\}$

$n = 11, u = 15$ (+2 mod 44):
$\{0, 2, 23, 8\}, \{0, 4, 2, y_4\}, \{0, 5, 26, x_2\}, \{0, 6, 10, x_6\}, \{0, 7, 36, x_9\}, \{0, 8, 5, y_3\}, \{0, 9, 4, y_1\}$,
$\{0, 14, 13, 34\}, \{0, 19, 27, 17\}, \{0, 20, 14, x_4\}, \{0, 21, 1, 18\}, \{0, 25, 37, y_0\}, \{0, 26, 29, x_1\}$,
$\{0, 28, 12, x_3\}, \{0, 29, 28, y_5\}, \{0, 31, 35, x_8\}, \{0, 32, 15, y_2\}, \{0, 34, 9, x_5\}, \{0, 35, 41, x_7\}, \{0, 37, 7, 9\}$,
$\{1, 0, 24, y_0\}, \{1, 2, 37, y_5\}, \{1, 4, 41, x_2\}, \{1, 6, 31, y_1\}, \{1, 9, 7, x_4\}, \{1, 15, 28, x_5\}, \{1, 19, 6, y_2\}$,
$\{1, 25, 35, x_3\}, \{1, 28, 14, x_8\}, \{1, 29, 13, x_6\}, \{1, 32, 20, x_7\}, \{1, 33, 26, y_3\}, \{1, 39, 21, y_4\}$,
$\{1, 41, 2, x_1\}, \{1, 42, 27, x_9\}$

$n = 13, u = 15$ (+2 mod 52):
$\{0, 2, 46, 30\}, \{0, 3, 47, 49\}, \{0, 5, 10, 19\}, \{0, 7, 4, 25\}, \{0, 11, 31, 35\}, \{0, 12, 49, x_3\}, \{0, 15, 5, x_1\}$,
$\{0, 17, 2, 29\}, \{0, 19, 41, 42\}, \{0, 22, 36, 21\}, \{0, 24, 20, x_7\}, \{0, 25, 18, x_8\}, \{0, 32, 12, y_2\}$,
$\{0, 33, 35, x_9\}, \{0, 34, 23, y_0\}, \{0, 35, 11, 51\}, \{0, 38, 3, y_4\}, \{0, 41, 1, x_4\}, \{0, 42, 30, x_6\}, \{0, 44, 45, y_1\}$,
$\{0, 46, 25, x_2\}, \{0, 47, 9, y_3\}, \{0, 48, 14, y_5\}, \{0, 49, 44, x_5\}, \{1, 0, 43, x_5\}, \{1, 8, 32, y_3\}, \{1, 10, 46, x_1\}$,
$\{1, 22, 20, x_4\}, \{1, 24, 5, x_8\}, \{1, 29, 45, y_5\}, \{1, 30, 36, x_9\}, \{1, 31, 38, y_0\}, \{1, 33, 10, y_4\}$,
$\{1, 35, 26, x_2\}, \{1, 37, 31, x_6\}, \{1, 39, 21, y_2\}, \{1, 43, 16, x_3\}, \{1, 45, 34, y_1\}, \{1, 47, 51, x_7\}$

$n = 14, u = 15$ (+2 mod 56):
$\{0,1,17,37\}, \{0,2,41,x_5\}, \{0,3,23,x_7\}, \{0,4,49,x_1\}, \{0,5,40,35\}, \{0,6,33,52\}, \{0,7,6,y_3\},$
$\{0,8,54,38\}, \{0,9,7,x_6\}, \{0,12,15,31\}, \{0,13,1,34\}, \{0,15,24,x_2\}, \{0,17,22,19\}, \{0,18,29,y_5\},$
$\{0,20,4,x_3\}, \{0,22,21,30\}, \{0,25,51,7\}, \{0,29,37,54\}, \{0,30,45,x_9\}, \{0,32,38,y_2\}, \{0,35,53,y_0\},$
$\{0,41,9,y_4\}, \{0,43,20,y_1\}, \{0,46,55,x_8\}, \{0,49,44,x_4\}, \{1,2,51,y_1\}, \{1,3,46,y_5\}, \{1,5,44,x_1\},$
$\{1,7,30,x_8\}, \{1,9,13,y_2\}, \{1,12,9,x_4\}, \{1,23,33,7\}, \{1,24,32,y_0\}, \{1,25,47,x_3\}, \{1,26,14,x_7\},$
$\{1,30,54,y_4\}, \{1,36,5,y_3\}, \{1,38,23,x_2\}, \{1,39,52,x_5\}, \{1,46,10,x_6\}, \{1,47,18,x_9\}$

A11 HSD($4^n 17^1$) for $10 \le n \le 14$

$n = 11, u = 17$ (+2 mod 44):
$\{0,1,2,y_3\}, \{0,2,19,y_5\}, \{0,3,1,x_2\}, \{0,5,26,1\}, \{0,7,35,y_7\}, \{0,9,3,y_4\}, \{0,10,31,x_8\},$
$\{0,14,39,y_2\}, \{0,15,20,x_5\}, \{0,16,40,x_6\}, \{0,17,10,y_0\}, \{0,18,32,x_3\}, \{0,20,30,x_1\},$
$\{0,21,38,29\}, \{0,25,5,12\}, \{0,32,16,y_1\}, \{0,36,27,x_4\}, \{0,38,9,y_6\}, \{0,40,36,x_9\},$
$\{0,43,29,x_7\}, \{1,3,7,x_3\}, \{1,4,22,x_7\}, \{1,5,2,x_4\}, \{1,6,15,y_3\}, \{1,9,4,y_6\}, \{1,13,25,x_9\},$
$\{1,14,33,y_0\}, \{1,15,28,x_8\}, \{1,16,24,x_2\}, \{1,17,27,x_6\}, \{1,18,11,x_5\}, \{1,21,8,y_2\},$
$\{1,22,20,y_7\}, \{1,27,30,y_5\}, \{1,32,38,y_4\}, \{1,35,17,y_1\}, \{1,39,3,x_1\}$

$n = 13, u = 17$ (+2 mod 52):
$\{0,1,21,x_8\}, \{0,2,3,y_5\}, \{0,4,34,49\}, \{0,5,42,y_3\}, \{0,6,23,x_5\}, \{0,7,47,x_1\}, \{0,8,36,x_7\},$
$\{0,9,40,y_6\}, \{0,10,4,27\}, \{0,11,1,y_7\}, \{0,16,27,35\}, \{0,17,32,x_3\}, \{0,19,38,29\}, \{0,22,30,3\},$
$\{0,28,24,x_6\}, \{0,31,8,y_2\}, \{0,32,46,y_0\}, \{0,33,25,y_1\}, \{0,34,2,x_9\}, \{0,38,37,x_2\}, \{0,40,49,x_4\},$
$\{0,47,43,y_4\}, \{0,51,5,45\}, \{1,3,41,37\}, \{1,4,44,y_7\}, \{1,7,18,x_5\}, \{1,8,15,y_6\}, \{1,12,46,y_4\},$
$\{1,15,45,x_9\}, \{1,16,21,y_2\}, \{1,18,2,x_8\}, \{1,19,24,y_5\}, \{1,21,23,x_6\}, \{1,24,22,y_1\}, \{1,26,5,x_3\},$
$\{1,29,38,x_4\}, \{1,31,3,x_7\}, \{1,32,42,x_1\}, \{1,37,12,x_2\}, \{1,43,25,y_0\}, \{1,50,17,y_3\}$

$n = 14, u = 17$ (+2 mod 56):
$\{0,1,21,x_4\}, \{0,2,9,y_6\}, \{0,3,51,30\}, \{0,4,13,52\}, \{0,5,18,55\}, \{0,7,11,16\}, \{0,9,3,34\},$
$\{0,11,43,y_3\}, \{0,13,2,y_2\}, \{0,16,5,17\}, \{0,18,6,x_7\}, \{0,19,37,y_1\}, \{0,20,55,24\}, \{0,22,19,x_2\},$
$\{0,24,34,y_7\}, \{0,29,41,49\}, \{0,30,12,x_6\}, \{0,33,10,y_4\}, \{0,39,24,y_5\}, \{0,44,45,x_8\},$
$\{0,45,20,x_3\}, \{0,46,17,x_1\}, \{0,47,7,13\}, \{0,48,40,x_9\}, \{0,49,30,y_0\}, \{0,50,15,x_5\}, \{1,2,22,x_4\},$
$\{1,3,18,x_2\}, \{1,4,55,y_2\}, \{1,11,30,x_8\}, \{1,14,8,y_1\}, \{1,16,49,x_3\}, \{1,19,17,x_9\}, \{1,22,13,y_0\},$
$\{1,27,53,x_7\}, \{1,30,32,y_3\}, \{1,33,11,x_6\}, \{1,34,39,y_5\}, \{1,35,4,x_5\}, \{1,37,34,y_6\}, \{1,41,31,y_7\},$
$\{1,42,25,y_4\}, \{1,53,24,x_1\}$

A12 HSD($4^n 18^1$) for $10 \le n \le 14$

$n = 10, u = 18$ (+1 mod 40):
$\{0,5,39,y_3\}, \{0,21,5,x_4\}, \{0,22,3,x_1\}, \{0,23,12,x_9\}, \{0,24,7,x_3\}, \{0,25,13,x_8\}, \{0,26,17,y_0\},$
$\{0,27,9,x_2\}, \{0,28,15,x_7\}, \{0,29,21,y_1\}, \{0,31,16,x_5\}, \{0,32,18,x_6\}, \{0,33,26,y_2\},$
$\{0,34,29,y_4\}, \{0,36,32,y_5\}, \{0,37,34,y_6\}, \{0,38,36,y_7\}, \{0,39,38,y_8\}$

$n = 11, u = 18$ (+1 mod 44):
$\{0,1,39,x_4\}, \{0,2,6,y_1\}, \{0,3,19,34\}, \{0,4,24,y_4\}, \{0,5,36,x_5\}, \{0,7,9,y_6\}, \{0,8,3,x_8\},$
$\{0,12,15,x_3\}, \{0,13,1,y_8\}, \{0,14,32,x_6\}, \{0,17,16,y_3\}, \{0,18,10,x_7\}, \{0,19,42,y_5\},$
$\{0,20,27,y_0\}, \{0,23,4,x_9\}, \{0,28,14,y_7\}, \{0,34,7,x_2\}, \{0,35,26,y_2\}, \{0,38,23,x_1\}$

$n = 13, u = 18$ (+1 mod 52):
$\{0,1,42,24\}, \{0,2,19,y_3\}, \{0,3,1,21\}, \{0,4,47,x_1\}, \{0,5,43,x_5\}, \{0,11,12,49\}, \{0,14,4,x_9\},$
$\{0,16,50,x_2\}, \{0,17,46,y_5\}, \{0,21,17,y_2\}, \{0,23,45,y_6\}, \{0,25,44,y_8\}, \{0,28,36,y_4\},$
$\{0,30,37,x_4\}, \{0,33,27,y_0\}, \{0,40,3,x_3\}, \{0,42,22,x_8\}, \{0,43,31,y_7\}, \{0,44,28,x_7\},$
$\{0,45,20,y_1\}, \{0,46,41,x_6\}$

$n = 14, u = 18$ (+1 mod 56):
$\{0,3,47,y_8\}, \{0,5,22,11\}, \{0,7,54,y_3\}, \{0,8,43,x_1\}, \{0,10,33,x_6\}, \{0,12,48,x_7\},$

$\{0,15,30,x_2\},\{0,17,55,22\},\{0,18,19,x_4\},\{0,19,15,50\},\{0,20,17,49\},\{0,22,32,y_4\},$
$\{0,25,12,y_6\},\{0,29,3,y_2\},\{0,30,49,x_9\},\{0,40,16,x_3\},\{0,43,18,x_8\},\{0,47,45,x_5\},$
$\{0,50,21,y_7\},\{0,52,36,y_0\},\{0,54,46,y_5\},\{0,55,4,y_1\}$

A13 HSD(4^n19^1) for $11 \le n \le 14$

$n=11, u=19$ (+2 mod 44):
$\{0,3,22,25\},\{0,4,9,y_6\},\{0,5,7,y_9\},\{0,6,13,y_4\},\{0,7,8,y_8\},\{0,9,15,y_5\},\{0,10,21,x_7\},$
$\{0,11,18,y_3\},\{0,14,28,x_6\},\{0,15,25,y_0\},\{0,17,31,x_9\},\{0,18,34,x_1\},\{0,19,41,22\},$
$\{0,21,6,31\},\{0,22,5,27\},\{0,24,1,x_4\},\{0,28,17,x_2\},\{0,29,11,x_3\},\{0,32,14,x_5\},\{0,35,12,y_1\},$
$\{0,36,4,y_2\},\{0,37,20,x_8\},\{0,42,2,y_7\},\{1,0,2,y_9\},\{1,1,1,1\},\{1,2,5,y_8\},\{1,4,10,y_5\},$
$\{1,5,8,y_6\},\{1,6,16,x_9\},\{1,7,12,y_4\},\{1,9,18,x_7\},\{1,12,25,x_8\},\{1,14,22,y_0\},\{1,17,37,x_1\},$
$\{1,18,42,x_3\},\{1,22,31,y_3\},\{1,25,33,y_2\},\{1,27,26,x_4\},\{1,31,19,x_6\},\{1,32,7,y_1\},$
$\{1,33,17,x_5\},\{1,35,6,x_2\},\{1,43,3,y_7\}$

$n=13, u=19$ (+2 mod 52):
$\{0,1,8,12\},\{0,3,19,x_8\},\{0,5,36,28\},\{0,7,3,y_2\},\{0,10,1,47\},\{0,11,6,y_0\},\{0,14,17,y_3\},$
$\{0,15,48,y_1\},\{0,18,2,y_9\},\{0,19,43,23\},\{0,22,28,x_2\},\{0,23,5,9\},\{0,24,9,x_1\},\{0,27,30,y_4\},$
$\{0,29,18,x_3\},\{0,32,12,y_8\},\{0,35,25,y_7\},\{0,36,11,y_6\},\{0,37,31,x_5\},\{0,40,10,x_7\},$
$\{0,43,14,y_5\},\{0,46,32,x_9\},\{0,49,27,x_6\},\{0,50,51,x_4\},\{1,2,6,x_6\},\{1,6,41,y_5\},\{1,8,25,x_3\},$
$\{1,11,32,x_4\},\{1,12,20,y_2\},\{1,17,3,y_9\},\{1,19,38,y_3\},\{1,20,30,x_5\},\{1,22,47,y_0\},\{1,23,8,y_6\},$
$\{1,28,35,y_4\},\{1,29,21,x_9\},\{1,32,31,y_1\},\{1,36,18,y_7\},\{1,39,37,y_8\},\{1,41,9,x_2\},$
$\{1,44,46,x_8\},\{1,45,4,x_1\},\{1,51,11,x_7\}$

$n=14, u=19$ (+2 mod 56):
$\{0,1,18,x_4\},\{0,2,21,y_0\},\{0,4,51,x_9\},\{0,5,43,21\},\{0,6,47,38\},\{0,7,55,15\},\{0,8,20,4\},$
$\{0,10,15,y_7\},\{0,13,33,50\},\{0,15,31,x_3\},\{0,17,46,x_7\},\{0,18,17,y_4\},\{0,19,7,22\},\{0,21,34,x_6\},$
$\{0,23,25,17\},\{0,26,6,y_1\},\{0,29,48,x_2\},\{0,32,2,y_3\},\{0,34,45,x_1\},\{0,36,12,y_2\},\{0,37,30,y_9\},$
$\{0,44,39,x_8\},\{0,47,37,y_8\},\{0,49,27,y_4\},\{0,51,16,y_6\},\{0,55,23,x_5\},\{1,4,33,x_6\},\{1,5,55,y_2\},$
$\{1,12,5,y_6\},\{1,14,23,x_7\},\{1,21,8,y_4\},\{1,22,47,x_2\},\{1,24,13,y_9\},\{1,26,28,x_3\},\{1,30,22,x_5\},$
$\{1,31,27,y_1\},\{1,32,48,y_4\},\{1,33,0,y_7\},\{1,39,16,x_9\},\{1,45,19,y_3\},\{1,46,21,x_4\},\{1,47,46,x_1\},$
$\{1,51,54,x_8\},\{1,54,44,y_8\},\{1,55,52,y_0\}$

A14 HSD(4^nu^1) for $n = 13, 14$ and $u = 21, 22, 23, 25, 26$

$n=13, u=21$ (+2 mod 52):
$\{0,3,50,y_5\},\{0,4,21,y_2\},\{0,5,6,29\},\{0,8,31,y_4\},\{0,9,16,7\},\{0,10,11,x_8\},\{0,11,41,46\},$
$\{0,14,22,x_5\},\{0,19,3,z_0\},\{0,25,27,y_0\},\{0,28,44,y_6\},\{0,29,23,y_7\},\{0,30,25,z_1\},\{0,32,4,x_1\},$
$\{0,33,24,y_3\},\{0,34,14,x_3\},\{0,35,49,y_9\},\{0,36,40,x_2\},\{0,37,19,x_6\},\{0,40,43,y_1\},$
$\{0,45,18,x_9\},\{0,46,9,y_8\},\{0,50,32,x_7\},\{0,51,10,x_4\},\{1,0,21,x_4\},\{1,4,18,y_7\},\{1,12,0,z_0\},$
$\{1,15,23,x_5\},\{1,22,24,x_6\},\{1,23,38,y_1\},\{1,26,7,y_3\},\{1,29,19,x_2\},\{1,32,2,y_9\},\{1,33,45,x_3\},$
$\{1,35,16,y_2\},\{1,36,46,y_0\},\{1,37,13,x_7\},\{1,38,11,y_5\},\{1,41,37,y_6\},\{1,43,8,y_4\},$
$\{1,45,48,x_8\},\{1,46,5,x_9\},\{1,47,15,x_1\},\{1,49,6,z_1\},\{1,51,20,y_8\}$

$n=13, u=22$ (+1 mod 52):
$\{0,1,48,y_7\},\{0,3,23,y_9\},\{0,4,38,z_2\},\{0,5,30,y_1\},\{0,8,10,y_8\},\{0,9,21,36\},\{0,10,11,y_0\},$
$\{0,11,18,x_2\},\{0,14,20,y_2\},\{0,16,19,x_6\},\{0,17,6,x_3\},\{0,19,15,y_5\},\{0,20,28,x_4\},$
$\{0,22,5,y_3\},\{0,24,3,y_6\},\{0,25,40,x_8\},\{0,29,51,x_1\},\{0,31,50,z_0\},\{0,34,43,x_7\},$
$\{0,40,16,y_4\},\{0,45,7,x_9\},\{0,46,17,z_1\},\{0,50,8,x_5\}$

$n=13, u=23$ (+2 mod 52):
$\{0,1,25,x_8\},\{0,3,36,x_1\},\{0,5,45,20\},\{0,7,9,x_9\},\{0,8,3,x_3\},\{0,14,42,y_3\},\{0,16,27,z_3\},$
$\{0,18,49,x_5\},\{0,19,12,y_6\},\{0,20,21,x_2\},\{0,22,20,x_7\},\{0,24,6,x_6\},\{0,31,14,y_1\},\{0,33,11,y_0\},$
$\{0,35,30,y_4\},\{0,37,8,z_2\},\{0,40,24,z_1\},\{0,42,33,y_5\},\{0,43,2,y_8\},\{0,46,34,y_7\},\{0,47,1,x_4\},$
$\{0,48,4,y_2\},\{0,49,35,y_9\},\{0,50,31,z_0\},\{1,2,6,y_9\},\{1,3,10,z_0\},\{1,5,39,x_6\},\{1,8,5,y_8\},$

$\{1,9,36,y_5\},\{1,12,34,x_9\},\{1,13,45,x_7\},\{1,24,38,y_0\},\{1,26,41,z_2\},\{1,29,25,y_3\},$
$\{1,30,24,x_8\},\{1,31,46,x_3\},\{1,32,3,y_4\},\{1,32,3,y_4\},\{1,33,12,z_3\},\{1,35,43,y_7\},\{1,36,35,y_6\},$
$\{1,37,2,x_2\},\{1,38,48,x_4\},\{1,39,30,x_5\},\{1,42,17,y_1\},\{1,43,33,y_2\},\{1,44,47,x_1\},\{1,47,31,z_1\}$

$n=14, u=21$ (+2 mod 56):
$\{0,1,36,x_7\},\{0,4,35,y_6\},\{0,6,30,x_9\},\{0,10,18,y_9\},\{0,13,37,y_0\},\{0,15,54,y_7\},\{0,16,31,11\},$
$\{0,17,12,x_1\},\{0,18,40,y_5\},\{0,19,39,12\},\{0,20,55,29\},\{0,21,23,30\},\{0,22,13,z_0\},$
$\{0,23,4,y_1\},\{0,24,19,13\},\{0,30,34,x_3\},\{0,33,50,x_4\},\{0,35,32,x_5\},\{0,37,11,z_1\},$
$\{0,39,29,x_2\},\{0,44,21,x_6\},\{0,45,46,x_8\},\{0,48,49,y_3\},\{0,51,17,y_8\},\{0,53,3,y_4\},\{0,54,8,y_2\},$
$\{1,2,0,z_1\},\{1,5,49,x_3\},\{1,9,16,z_0\},\{1,10,4,x_2\},\{1,13,42,x_6\},\{1,14,11,x_4\},\{1,16,5,y_7\},$
$\{1,19,27,y_9\},\{1,23,39,y_5\},\{1,26,33,x_7\},\{1,28,13,x_5\},\{1,32,14,y_0\},\{1,33,52,y_3\},$
$\{1,41,23,y_2\},\{1,46,55,x_1\},\{1,47,24,y_6\},\{1,48,32,y_4\},\{1,50,30,y_8\},\{1,52,21,x_8\},$
$\{1,54,41,y_1\},\{1,55,51,x_9\}$

$n=14, u=22$ (+1 mod 56):
$\{0,5,39,x_7\},\{0,6,25,z_2\},\{0,7,9,29\},\{0,8,15,y_4\},\{0,11,55,y_7\},\{0,13,3,y_0\},\{0,15,30,y_3\},$
$\{0,16,49,32\},\{0,18,13,x_8\},\{0,22,52,x_3\},\{0,23,32,x_4\},\{0,26,37,y_1\},\{0,27,11,x_5\},$
$\{0,31,35,y_8\},\{0,32,50,z_0\},\{0,35,36,x_2\},\{0,37,2,z_1\},\{0,44,8,y_6\},\{0,46,33,x_6\},\{0,47,44,y_2\},$
$\{0,52,46,x_1\},\{0,53,5,y_9\},\{0,54,29,y_4\},\{0,55,38,x_9\}$

$n=14, u=23$ (+2 mod 56):
$\{0,1,11,x_8\},\{0,4,33,y_5\},\{0,7,50,33\},\{0,9,17,24\},\{0,10,51,y_6\},\{0,13,19,z_3\},\{0,15,32,x_2\},$
$\{0,16,12,y_7\},\{0,17,18,x_5\},\{0,18,15,y_2\},\{0,19,16,x_6\},\{0,21,5,y_3\},\{0,24,8,y_1\},\{0,25,10,47\},$
$\{0,26,47,y_0\},\{0,29,22,y_8\},\{0,34,29,x_4\},\{0,36,7,z_0\},\{0,41,21,z_1\},\{0,44,55,y_9\},\{0,47,45,x_3\},$
$\{0,48,2,y_4\},\{0,50,52,x_1\},\{0,51,27,x_9\},\{0,54,20,z_2\},\{0,55,30,x_7\},\{1,4,9,x_5\},\{1,5,26,y_5\},$
$\{1,11,18,y_0\},\{1,12,0,z_3\},\{1,13,25,y_1\},\{1,14,22,z_1\},\{1,19,37,z_2\},\{1,21,54,y_9\},\{1,22,3,x_6\},$
$\{1,24,44,y_3\},\{1,25,47,y_4\},\{1,25,47,y_4\},\{1,26,13,y_8\},\{1,30,4,x_9\},\{1,31,14,y_2\},\{1,34,53,x_7\},$
$\{1,35,34,x_4\},\{1,41,32,z_0\},\{1,46,8,x_3\},\{1,49,19,y_7\},\{1,51,20,y_6\},\{1,52,41,x_2\},\{1,54,48,x_8\},$
$\{1,55,51,x_1\}$

$n=14, u=25$ (+2 mod 56):
$\{0,4,3,x_5\},\{0,7,4,y_9\},\{0,8,33,z_1\},\{0,10,27,53\},\{0,11,35,y_4\},\{0,13,51,x_4\},\{0,18,7,z_0\},$
$\{0,19,29,y_6\},\{0,24,48,x_1\},\{0,29,55,z_3\},\{0,30,53,z_5\},\{0,31,40,y_7\},\{0,33,25,y_5\},$
$\{0,34,30,x_6\},\{0,35,24,y_1\},\{0,36,10,z_2\},\{0,39,2,y_3\},\{0,40,18,x_8\},\{0,41,6,z_4\},\{0,44,49,y_0\},$
$\{0,45,41,x_7\},\{0,49,20,y_8\},\{0,50,9,y_2\},\{0,53,19,x_2\},\{0,54,34,x_9\},\{0,55,12,x_3\},\{1,0,18,y_6\},$
$\{1,6,0,z_3\},\{1,7,13,x_8\},\{1,9,25,x_6\},\{1,10,11,y_8\},\{1,14,7,y_1\},\{1,19,31,x_1\},\{1,20,53,y_9\},$
$\{1,21,46,z_1\},\{1,23,10,z_0\},\{1,30,20,x_2\},\{1,30,20,x_2\},\{1,32,3,y_7\},\{1,33,26,y_2\},\{1,34,42,y_5\},$
$\{1,36,21,z_4\},\{1,40,23,x_3\},\{1,41,39,z_2\},\{1,42,40,y_4\},\{1,45,9,x_9\},\{1,47,52,z_5\},\{1,48,36,x_4\},$
$\{1,52,12,x_7\},\{1,53,44,x_5\},\{1,54,17,y_3\},\{1,55,34,y_0\}$

$n=14, u=26$ (+1 mod 56):
$\{0,1,2,z_6\},\{0,2,4,z_5\},\{0,3,6,z_4\},\{0,4,8,z_3\},\{0,5,10,z_2\},\{0,6,12,z_1\},\{0,8,15,z_0\},$
$\{0,10,25,y_3\},\{0,12,24,y_4\},\{0,15,26,y_5\},\{0,17,27,y_6\},\{0,19,39,x_7\},\{0,21,47,x_8\},$
$\{0,22,45,x_4\},\{0,23,40,x_6\},\{0,24,3,x_3\},\{0,25,34,y_8\},\{0,26,51,x_2\},\{0,27,43,x_5\},$
$\{0,36,49,x_1\},\{0,38,20,x_9\},\{0,40,21,y_1\},\{0,43,19,y_0\},\{0,45,23,y_2\},\{0,47,18,y_7\},\{0,49,1,y_9\}$

A15 HSD($4^{17}u^1$) for $u=\{25,26,27,29,30,31\}$

$n=17, u=25$ (+2 mod 68):
$\{0,1,3,z_4\},\{0,2,7,z_2\},\{0,3,4,z_3\},\{0,4,10,z_1\},\{0,5,13,y_9\},\{0,6,2,z_5\},\{0,7,12,z_0\},$
$\{0,8,31,x_6\},\{0,9,16,y_8\},\{0,12,38,x_3\},\{0,13,25,y_5\},\{0,14,57,37\},\{0,15,24,y_4\},\{0,16,54,35\},$
$\{0,18,39,y_0\},\{0,19,6,y_6\},\{0,20,40,x_7\},\{0,23,36,y_2\},\{0,24,43,y_1\},\{0,25,35,y_7\},\{0,26,59,27\},$
$\{0,27,41,y_3\},\{0,28,29,50\},\{0,29,45,x_9\},\{0,30,11,39\},\{0,31,55,x_2\},\{0,32,48,x_8\},$
$\{0,33,5,47\},\{0,39,8,x_1\},\{0,46,18,x_4\},\{0,53,15,45\},\{0,58,26,x_5\},\{1,2,4,z_4\},\{1,3,6,z_2\},$
$\{1,4,7,z_3\},\{1,5,11,z_1\},\{1,6,13,z_0\},\{1,7,3,z_5\},\{1,8,16,y_9\},\{1,9,20,y_1\},\{1,10,19,y_8\},$
$\{1,11,29,x_8\},\{1,12,22,y_3\},\{1,14,36,x_9\},\{1,15,42,x_6\},\{1,17,32,y_0\},\{1,19,39,x_7\},$

$\{1,23,49,x_3\},\{1,24,53,x_1\},\{1,25,47,x_4\},\{1,26,37,y_6\},\{1,28,43,y_2\},\{1,32,8,x_2\},$
$\{1,34,48,y_5\},\{1,48,5,y_4\},\{1,57,25,x_5\},\{1,58,2,y_7\}$

$n=17,u=26$ (+1 mod 68):
$\{0,2,3,z_6\},\{0,5,12,y_9\},\{0,7,16,y_7\},\{0,8,23,x_8\},\{0,9,33,x_7\},\{0,10,50,19\},\{0,13,31,y_2\},$
$\{0,14,27,y_3\},\{0,15,21,z_1\},\{0,16,26,y_0\},\{0,19,57,27\},\{0,21,32,y_4\},\{0,22,61,37\},$
$\{0,25,28,z_3\},\{0,26,30,z_5\},\{0,27,5,x_3\},\{0,32,46,x_9\},\{0,33,49,y_1\},\{0,39,13,x_5\},$
$\{0,40,20,x_2\},\{0,45,10,x_1\},\{0,48,25,z_0\},\{0,50,29,y_5\},\{0,56,24,x_6\},\{0,57,14,x_4\},$
$\{0,62,6,y_8\},\{0,64,4,y_6\},\{0,65,2,z_2\},\{0,67,1,z_4\}$

$n=17,u=27$ (+2 mod 68):
$\{0,1,20,27\},\{0,2,44,y_9\},\{0,4,54,y_8\},\{0,5,11,z_6\},\{0,6,45,x_5\},\{0,9,61,z_1\},\{0,11,58,z_7\},$
$\{0,12,5,x_9\},\{0,14,66,z_3\},\{0,15,1,z_2\},\{0,18,38,x_7\},\{0,19,65,23\},\{0,22,8,37\},\{0,24,64,y_3\},$
$\{0,26,31,z_5\},\{0,28,7,22\},\{0,30,19,y_7\},\{0,31,27,41\},\{0,33,18,y_6\},\{0,36,12,y_2\},\{0,37,46,y_1\},$
$\{0,39,36,z_4\},\{0,43,41,x_6\},\{0,47,3,x_4\},\{0,48,47,y_5\},\{0,52,13,y_4\},\{0,55,37,x_3\},\{0,58,52,y_0\},$
$\{0,59,40,x_2\},\{0,60,35,x_1\},\{0,61,26,x_8\},\{0,67,57,z_0\},\{1,4,13,y_6\},\{1,5,61,y_9\},\{1,6,7,x_2\},$
$\{1,7,15,y_8\},\{1,12,20,x_6\},\{1,20,33,z_4\},\{1,21,44,y_7\},\{1,23,16,x_9\},\{1,24,21,y_1\},\{1,28,40,z_6\},$
$\{1,31,67,x_7\},\{1,33,53,y_2\},\{1,34,30,x_4\},\{1,41,10,x_1\},\{1,42,6,z_1\},\{1,44,14,z_0\},\{1,45,60,y_4\},$
$\{1,46,36,x_3\},\{1,48,46,z_2\},\{1,51,26,x_5\},\{1,53,25,y_3\},\{1,56,51,z_7\},\{1,57,2,y_5\},\{1,59,48,z_5\},$
$\{1,61,23,z_3\},\{1,66,31,x_8\},\{1,67,41,y_0\}$

$n=17,u=29$ (+2 mod 68):
$\{0,2,67,y_5\},\{0,3,66,x_5\},\{0,6,38,20\},\{0,7,36,y_9\},\{0,8,52,z_3\},\{0,12,39,y_6\},\{0,16,22,z_5\},$
$\{0,20,35,x_4\},\{0,21,61,z_8\},\{0,23,20,z_2\},\{0,25,21,y_7\},\{0,27,6,z_6\},\{0,30,44,y_1\},\{0,32,23,z_7\},$
$\{0,35,19,x_7\},\{0,39,3,26\},\{0,40,50,x_9\},\{0,42,29,x_6\},\{0,43,31,y_4\},\{0,44,56,y_0\},\{0,46,27,y_2\},$
$\{0,49,40,y_3\},\{0,54,4,x_8\},\{0,55,8,x_1\},\{0,57,47,z_9\},\{0,58,57,y_8\},\{0,59,26,x_3\},\{0,61,11,z_1\},$
$\{0,63,10,x_2\},\{0,64,25,z_0\},\{0,65,5,z_4\},\{1,0,61,z_6\},\{1,2,10,z_4\},\{1,5,60,y_5\},\{1,7,37,z_5\},$
$\{1,9,54,y_8\},\{1,15,14,y_2\},\{1,16,53,y_3\},\{1,21,7,x_9\},\{1,22,26,y_4\},\{1,23,25,49\},\{1,28,0,z_8\},$
$\{1,32,51,y_9\},\{1,36,31,x_5\},\{1,37,59,z_3\},\{1,38,13,x_2\},\{1,39,28,z_7\},\{1,40,3,z_2\},\{1,41,15,y_1\},$
$\{1,43,36,y_6\},\{1,50,20,x_7\},\{1,51,6,x_4\},\{1,53,29,y_0\},\{1,54,32,z_1\},\{1,56,4,y_7\},\{1,57,16,x_6\},$
$\{1,58,47,x_3\},\{1,59,24,z_0\},\{1,60,62,z_9\},\{1,64,21,x_1\},\{1,67,5,x_8\}$

$n=17,u=30$ (+1 mod 68):
$\{0,1,43,x_9\},\{0,2,48,z_8\},\{0,4,7,x_6\},\{0,7,35,y_7\},\{0,10,40,z_9\},\{0,11,23,y_5\},\{0,12,49,y_6\},$
$\{0,13,66,y_2\},\{0,16,6,z_4\},\{0,18,26,y_8\},\{0,19,39,x_1\},\{0,23,64,z_7\},\{0,26,5,35\},\{0,28,30,y_3\},$
$\{0,31,44,x_3\},\{0,32,27,y_4\},\{0,33,57,z_6\},\{0,39,32,z_0\},\{0,41,12,z_1\},\{0,43,52,z_2\},$
$\{0,44,55,x_4\},\{0,46,47,x_7\},\{0,47,22,z_5\},\{0,48,3,x_2\},\{0,53,37,z_3\},\{0,54,18,y_1\},\{0,59,10,x_0\},$
$\{0,60,54,y_9\},\{0,62,8,x_5\},\{0,63,67,y_0\},\{0,65,15,x_8\}$

$n=17,u=31$ (+2 mod 68):
$\{0,44,25,x_1\},\{1,59,16,x_1\},\{0,23,60,x_2\},\{1,20,23,x_2\},\{0,7,27,x_3\},\{1,44,8,x_3\},\{0,20,62,x_4\},$
$\{1,65,49,x_4\},\{0,27,41,x_5\},\{1,46,2,x_5\},\{0,38,48,x_6\},\{1,5,37,x_6\},\{0,56,3,x_7\},\{1,3,56,x_7\},$
$\{0,6,4,x_8\},\{1,41,59,x_8\},\{0,46,57,x_9\},\{1,25,48,x_9\},\{0,12,30,y_0\},\{1,23,39,y_0\},\{0,21,33,y_1\},$
$\{1,16,6,y_1\},\{0,26,45,y_2\},\{1,51,58,y_2\},\{0,8,9,y_3\},\{1,45,4,y_3\},\{0,9,56,y_4\},\{1,58,51,y_4\},$
$\{0,2,43,y_5\},\{1,57,54,y_5\},\{0,57,52,y_6\},\{1,42,5,y_6\},\{0,30,59,y_7\},\{1,61,32,y_7\},\{0,35,46,y_8\},$
$\{1,0,63,y_8\},\{0,65,18,y_9\},\{1,60,45,y_9\},\{0,48,32,z_0\},\{1,11,17,z_0\},\{0,37,24,z_1\},\{1,50,55,z_1\},$
$\{0,60,5,z_2\},\{1,29,38,z_2\},\{0,4,40,z_3\},\{1,39,13,z_3\},\{0,22,47,z_4\},\{1,9,14,z_4\},\{0,45,23,z_5\},$
$\{1,34,62,z_5\},\{0,39,42,z_6\},\{1,30,29,z_6\},\{0,33,19,z_7\},\{1,62,34,z_7\},\{0,47,14,z_8\},\{1,2,61,z_8\},$
$\{0,50,66,z_9\},\{1,63,43,z_9\},\{0,62,49,u_0\},\{1,67,30,u_0\},\{0,14,10,u_1\},\{1,37,67,u_1\},\{0,43,1,14\}$

A16 Miscellaneous HSD($h^n u^1$)

$h=1,n=11,u=5$ (+1 mod 11):
$\{0,1,2,x_5\},\{0,2,7,x_1\},\{0,4,6,x_4\},\{0,5,8,x_3\},\{0,8,1,x_2\}$

$h=1,n=19,u=5$ (+1 mod 19):
$\{0,1,2,x_5\},\{0,2,8,12\},\{0,5,14,8\},\{0,8,13,x_1\},\{0,10,12,x_4\},\{0,12,15,x_3\},\{0,16,1,x_2\}$

$h = 1, n = 13, u = 6$ (+1 mod 13):
$\{0,1,2,x_6\}, \{0,4,7,x_4\}, \{0,5,10,x_2\}, \{0,6,8,x_5\}, \{0,10,1,x_3\}, \{0,11,4,x_1\}$

$h = 1, n = 17, u = 6$ (+1 mod 17):
$\{0,1,2,x_6\}, \{0,2,8,x_1\}, \{0,4,13,7\}, \{0,5,10,x_2\}, \{0,8,11,x_4\}, \{0,10,12,x_5\}, \{0,14,1,x_3\}$

$h = 3, n = 4, u = 2$ (+6 mod 12):
$\{0,2,5,3\}, \{0,3,10,x_2\}, \{0,5,7,10\}, \{0,7,9,x_1\}, \{1,2,0,x_2\}, \{1,3,2,4\}, \{1,8,10,x_1\}, \{2,9,0,x_1\}$, $\{2,11,9,x_2\}, \{3,4,5,x_1\}, \{3,10,1,x_2\}, \{4,6,5,x_2\}, \{4,11,2,x_1\}, \{5,6,7,x_1\}, \{5,7,2,x_2\}, \{0,1,6,7\}$, $\{0,6,11,5\}, \{0,9,3,6\}, \{1,7,8,2\}, \{1,10,4,7\}, \{2,3,8,9\}, \{2,5,11,8\}, \{3,9,10,4\}, \{4,5,10,11\}$

Please note that the last nine starter blocks of HSD($3^4 2^1$) are invariant under the operation (+6 mod 12). So the total number of blocks for HSD($3^4 2^1$) is $2 * 15 + 9 = 39$.

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