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Regular pinched maps

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Abstract

This paper concerns pinched surfaces, also known as pseudosurfaces. A map is a graph G embedded on an oriented pinched surface. An arc of a map is an edge of G with a fixed direction. A regular map is one with a group of orientation-preserving automorphisms that acts regularly on the arcs of a map, i.e., that acts both freely and transitively.

We study regular maps on pinched surfaces. We give a relation between a regular map on a pinched surface and a natural corresponding regular map on a surface with the pinch points pulled apart. We give several constructions for regular pinched maps and present a plethora of examples. These include strongly connected maps on pinched surfaces (those that do not have a finite set of disconnecting points), as well as examples formed by gluing other regular maps along a finite set of points.

1 Introduction

Let S be an orientable surface, not necessarily connected. Each component of S is a sphere with some number of handles attached. It is convenient to consider the surface together with a fixed orientation. A *map*, M, is the surface together with an embedded graph G. We require the embeddings to be *cellular*, that is, each component of $S - G$ is homeomorphic to an open disk. It follows that the number of components of S equals the number of components of G.

Let $X_1, X_2, \ldots, X_k$ be disjoint finite sets of points in S. Create $\check{S}$ by identifying each X_i into a single point x_i for $i = 1, \ldots, k$. Then $\check{S}$ is called a *pinched surface* and each x_i is called a *pinch point*. A small neighborhood of x_i in $\check{S}$ is homeomorphic to $|X_i|$ copies of the plane identified at their origins. Each of these planes is called a *sheet*. The surface $\check{S}$ can be turned into an *oriented* pinched surface by choosing an orientation of each connected component of S. We call the original S the *unpinched* surface corresponding to $\check{S}$. Observe that S is well-defined for a given pinched surface $\check{S}$. We say that S is formed by *pulling apart* the pinch points of $\check{S}$.

Figure 1 shows a cube embedded on a sphere, and then the related pinched surface with antipodal points $a = A$ identified. The resulting surface has a single pinch point with two sheets. Pulling apart the pinch point restores the original spherical cube.

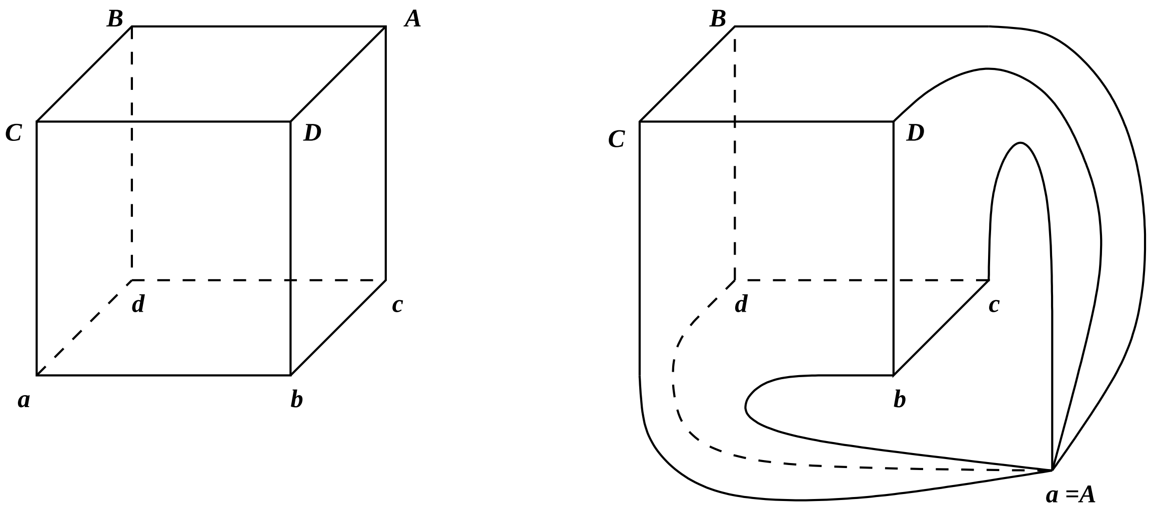

Figure 1: Identifying antipodes a and A on a cube

We assume throughout the paper that our pinched surfaces $\check{S}$ are connected. However, it is possible that pulling apart the pinch points disconnects $\check{S}$, that is, S is disconnected. Each component of S is called a *strong component* of $\check{S}$. If S has only one strong component, then $\check{S}$ is *strongly connected*. Equivalently, $\check{S}$ is strongly connected if and only if there does not exist a finite set of points disconnecting it.

As before, a *map* $\check{M}$ on $\check{S}$ is a pinched surface with an embedded graph $\check{G}$, but here we require that every pinch point is the image of a vertex. Following the model in [9] we still require maps to be cellular. Maps M on surfaces are a special case of maps $\check{M}$ on pinched surfaces: there are no pinch points. We will frequently make definitions for pinched surfaces with the understanding they apply to surfaces. Maps on surfaces are sometimes described in terms of *rotation systems*, each vertex is given a cyclic permutation of its incident edge-ends. Maps on pinched surfaces arise when you allow an arbitrary permutation of the edge-ends incident with a vertex.

An *automorphism* of $\check{M}$ is a self-homeomorphism of the oriented pinched surface that preserves both the orientation and the embedded graph. The set of all automorphisms form a group, called naturally enough the (full) *automorphism group*, denoted $Aut(\check{M})$.

Let Γ be a group of automorphisms of $\check{M}$, not necessarily the full automorphism group. We are interested on how Γ acts on the vertices, edges, and arcs of $\check{G}$ (an *arc* is an edge with a fixed direction). Recall that an action of Γ on a set X is *transitive* if for any pair $x, y \in X$ there is an element of Γ mapping x to y. The action is *free* if the only element of Γ that fixes a point of X is the identity. The action is *regular* if it is both transitive and free, i.e., for any $x, y \in X$ there is exactly one element of Γ mapping x to y.

Each automorphism of $\check{M}$ permutes the vertices. If Γ is regular on the arcs, then it is transitive on the vertices. Hence if one vertex is on a pinch point, then every vertex is on a pinch point. Moreover all pinch points have the same number of sheets and each sheet has the same number of arcs; in particular every vertex is of the same degree. A group of automorphisms acting regularly on the arcs has $deg(v)$ automorphisms fixing v, so it does not act freely on the vertices when this common degree exceeds one. It is convenient to have a name for all arcs pointed out from v and on the same sheet of v: call it a *cone*.

A *regular pinched map* is a pinched map with a group of automorphisms that act regularly on the arcs. Regular maps without pinch points have been widely studied. In this context orientation-reversing automorphisms are frequently allowed, so that the automorphism group is flag transitive, but we will not pursue this extension here. For more on this concept see the survey article [8], and for small examples see [2].

In this paper we study regular maps on pinched surfaces. In Section 2 we relate these to regular maps on the corresponding unpinched surface. In a sense this gives all regular pinched maps. However, in some circumstances there are more convenient ways to construct regular pinched maps. In Section 3 we give some strongly connected examples. In Section 4 we give examples that are not strongly connected. We then turn our attention in Section 5 to a pretty way of constructing pinched maps using powers of regular embedding and discuss some limitations of this method. We conclude in Section 6 with some open problems.

2 Group actions and the Main Theorem

Each pinched map $\check{M}$ with graph $\check{G}$ has a unique corresponding (possibly disconnected) unpinched map M with graph G. The arcs of $\check{M}$ are in direct correspondence with the arcs of M. Hence any permutation γ of the arcs of M corresponds to a permutation $\check{\gamma}$ of the arcs of $\check{M}$. The precise nature of this relationship will be made clear in our Main Theorem 2.1, where we show when the regularity of one group of automorphisms guarantees the regularity of the corresponding group.

We first need some notions from the theory of permutation groups [3]. Let Γ be a set of permutations acting on a set A. A *block* of this action is a subset $B \subset A$ such that for all $\gamma \in \Gamma$, either $\gamma(B) = B$ or $\gamma(B) \cap B = \emptyset$. If the action of Γ is transitive, as will always be the case in this paper, then any translate of a block is a block, all blocks are the same size, the blocks partition A, and every $\gamma \in \Gamma$ permutes the blocks.

Any action always has two *trivial* block partitions: (a) the partition where every block is a singleton, and (b) the partition with a single block $B = A$. If these are the only block partitions under a group action, then the action is *primitive*, otherwise the action is *imprimitive*. A block which is minimal in the set of all blocks of size > 1 is called a *minimal block*. Primitive and imprimitive actions of a group on a set are widely studied, again see [3].

Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of the vertex set of a graph G. Form the *quotient graph* $\check{G} = G/P$ whose vertices are the parts of P and with an edge joining V_iV_j for each edge with ends in these two parts. In other words, identify the vertices in each part to a single vertex keeping any edge multiplicities that arise. If G is embedded forming a map M, embed the *quotient map* $\check{M}$ by identifying the vertices on the surface. This $\check{M}$ has $\check{G}$ embedded in a pinched surface.

The following example illustrates these concepts.

Example 1 *Identifying antipodes in a cube:* Consider the 3-dimensional cube as a map M. Let a, A, b, B, c, C, d, D be the vertices of its graph G where x is antipode of X. The group Γ of (orientation-preserving) automorphisms of the cube is transitive on the 24 arcs, so M is a regular map. Each pair $\{x, X\}$ is a block as Γ acts on the vertices. Form a quotient map $\check{M} = M/P$ by identifying $a = A, b = B, c = C, d = D$, forming a sphere with four pinch points. Any automorphism of M is also an automorphism of $\check{M}$. The automorphisms are still transitive on the arcs of $\check{G}$, so $\check{M}$ is a regular pinched map.

Figure 1 shows a cube and the identification of two antipodal vertices, a and A. Figure 2 illustrates the simultaneous identification of all four pairs of antipodes. In Figure 2 the edge-ends are either dashed or solid depending on whether they are originally incident to a vertex or its antipode. The faces are found using the usual face-tracing algorthim (see [5]): walk along an edge and when encountering a vertex continue along the next edge anti-clockwise. But here since each vertex is a pinch point, continue along the next edge anti-clockwise *of the same nature*, dashed or solid. Thus at any vertex the three outward arcs with dashed ends form one cone and the three with solid ends form the other cone. This face-tracing algorithm yields the six quadrilateral faces of the pinched map corresponding to the six faces of the original unpinched cube.

With this example in mind, we present the Main Theorem.

Theorem 2.1 *Let M be a map formed by a graph G embedded on a (not necessarily connected) oriented surface S and let Γ be a group of automorphisms of the map*

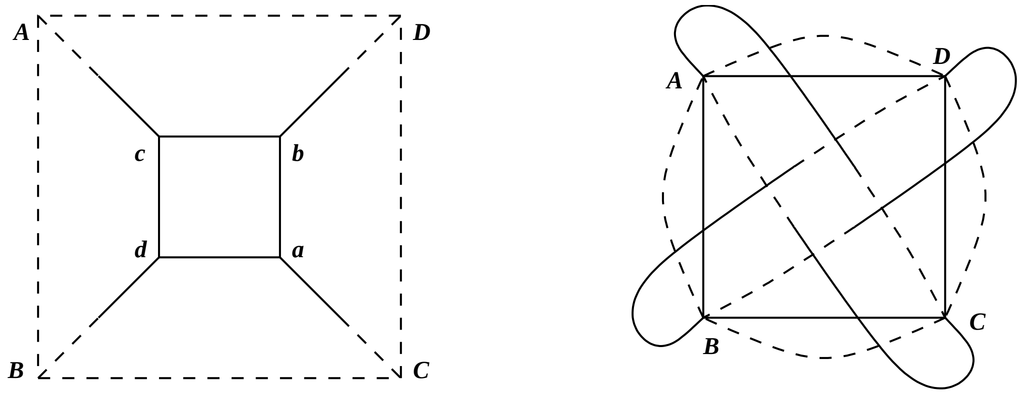

Figure 2: Identifying all antipodes on a cube

acting regularly on its arcs. Suppose that the action of Γ *on the vertices has a block structure* $P = \{V_1, \ldots, V_k\}$*. Then the corresponding group* $\check{\Gamma}$ *acts regularly as a group of automorphisms on the arcs of the pinched quotient map* $\check{M} = M/P$*.*

Conversely, if $\check{\Gamma}$ *acts regularly on the arcs of* $\check{M}$*, then the corresponding* Γ *acts regularly on the arcs of the unpinched map* M *formed by pulling apart the pinch points. The action of* Γ *on* $V(G)$ *has a block structure corresponding to the cones of the pinch points of* $\check{M}$*.*

Proof: An automorphism γ of M permutes the vertices of G. Since it has block structure P, it corresponds to an automorphism $\check{\gamma}$ of M/P. The group Γ acts regularly on the arcs of M, so $\check{\Gamma}$ acts regularly on $\check{M} = M/P$.

For the converse, each automorphism $\check{\gamma} \in \check{\Gamma}$ permutes the cones. Hence it also defines an automophism γ of M. This action of γ on the arcs of M is regular and has block structure P. ■

Let M be a regular unpinched map of degree d on a connected surface. Its automorphism group has a presentation in terms of two generators, one of order d generating the cyclic stabilizer of a vertex and the other of order 2 generating the stabilizer of an edge. More strongly, unpinched regular maps on connected surfaces may be identified with presentations of such two-generator groups. The automorphism group of a regular pinched map M/P does not always admit such a simple presentation since the vertex stabilizers need not be cyclic. The block structures of P correspond to subgroups of Γ containing a vertex stabilizer, and so do not correspond to two-generator groups.

We have considered a subgroup of automorphisms of the map acting regularly on the arcs. We examine when this is the full automorphism group.

Lemma 2.2 *Let* $\check{M}$ *be a strongly-connected oriented pinched embedding. Then any orientation-preserving automorphism acts freely on the arcs.*

Proof: Suppose that an automorphism γ fixes an arc a. We need to show that it fixes all arcs. Since γ fixes the embedding it fixes all corners. Since it is orientation preserving it fixes all arcs rooted at the tail of a. This γ also fixes the arc a^{-1}, and so fixes all arcs at the tail of a^{-1} which is the head of a. The map $\check{M}$ is strongly connected, so for any two arcs there sequence of these operations carry any arc to any other, so all arcs are fixed. (This can also be proven using Theorem 2.1 and the results in [5].) ■

The fact that the relations "share a tail" and "are inverse" extend transitively to the entire arc set can be considered as an alternate definition of strong connectivity.

Corollary 2.3 *Any group of automorphisms acting regularly on a strongly connected pinched map must be the full automorphism group.*

The situation when $\check{M}$ is not strongly connected is a bit more complicated. If $\check{M}$ is disconnected, then the full automorphism group does not necessarily act freely. Any automorphism of one component could be paired with the identity automorphism of a second component. In general, if there are n components of $\check{M}$ each isomorphic to M, then the automorphism group of $\check{M}$ is a wreath product of the automorphism group of M with the full symmetric group S_n.

3 Examples that are strongly-connected

Recall that a surface is strongly-connected if pulling apart the pinch points does not disconnect the surface. By Corollary 2.3 the only group that can act regularly on the arcs of a map on a strongly-connected surface is the full automorphism group. We give some examples.

Example 2 *Trivial quotients:* Every group action has two trivial block structures: that where all parts are of size 1, and that with only one part. If the action on the vertices has all parts of size 1, then the map M is the same as its quotient $\check{M}$. If there is only one vertex part, then $\check{M}$ has just a single point.

Example 3 *Bipartite graphs:* Let M be a connected regular map based on a bipartite graph with partite sets V_1, V_2. The action of the automorphism group on the vertices has $P = \{V_1, V_2\}$ as a block structure. Forming the quotient gives a regular pinched map with exactly two pinch points. The cube is bipartite and Example 1 gives a pinched quotient with four vertices. This shows that an action on the vertices can have more than one nontrivial block structure, and hence give rise to more than one regular pinched map.

Example 4 *Graphical properties:* Generalizing on the previous example, there can be other graphical properties that natually lead to block actions. For example, G could be a complete multipartite graph: the parts are blocks under any graph automorphism. Or G could be the octahedron, icosohedron, n-dimensional cube, or any other graph where there is a unique vertex of u of maximum distance from any given v. Any graph isomorphism must have u, v as a block. Regular embeddings of these graphs must respect these block structures, and so lead to regular pinched maps with each pinch point having two sheets.

Example 5 *The Heawood graph on the torus:* Consider the Heawood graph embedded as a regular map on the torus (this is the dual of the embedding of K_7 on the torus). The group of automorphisms of this map acts regularly on the arc set. It acts imprimitively on the vertex set: it is possible to partition the vertices into seven pairs which are blocks. Hence there is a quotient of this embedding yielding a regular pinched embedding with 7 pinch points each having 2 sheets.

Example 6 $C_4 \times C_4$ *on the torus:* Consider the regular embedding of $C_4 \times C_4$ on the torus. There are several block partitions on the vertices of this rotation. For the first, let a block be $\{(i,j),(i+2,j+2)\}$. This gives 8 blocks, each of size 2. By unioning together some of these blocks we get coarser block partitions. For example, if we union two of the old blocks to form $\{(i,j),(i+2,j),(i,j+2),(i+2,j+2)\}$ we again get a block partition with 4 blocks each of size four. We can again group these blocks in pairs to get the block partition corresponding to the bipartite vertex sets on $C_4 \times C_4$. Finally, we can union to a single block with a trivial block partition.

The underlying graph $C_4 \times C_4$ is the hypercube Q_4, so this construction is related to those with bipartite graphs and those with unique antipodal points.

4 Examples that are not strongly-connected

We now give examples that are formed, in a sense, by glueing together other regular maps.

Example 7 *Cloning:* Say that we have copies M_i, $i = 1, \ldots, n$, of a regular (possibly pinched) map M where M has automorphism group Γ. Let $\check{\Gamma}$ be a group of automorphisms acting regularly on arcs of the union of the M_i naturally extending the action of Γ on the individual maps. Such a group can be formed as say a direct product $\Gamma \times \mathbb{Z}_n$. Make a pinched map $\check{M}_n$ by identifying corresponding vertices in each of the M_i. Then $\check{M}_n$ is regular.

Example 8 *Inflating edges:* Let G be a connected graph admitting a regular action of a group Γ of automorphisms on arcs of the graph. Create $\check{M}$ by replacing each edge of G with a copy of the regular map K_2 on the sphere. Then $Aut(\check{M}) \cong \Gamma$ turns $\check{M}$ into a regular pinched map. The number of pinch components of $\check{M}$ is the number of edges in G.

Example 9 *Multiplying edges:* Again, let G be a connected graph admitting a regular action of a group Γ of automorphisms on arcs of the graph. Create $\check{M}$ by replacing every edge by a dipole with m parallel edges embedded on a sphere, and consider the induced natural arc-transitive action of the group $\mathbb{Z}_m \times \Gamma$ on the arcs of $\check{M}$. This turns $\check{M}$ into a regular pinched map. The number of pinch components of $\check{M}$ is the number of edges in G.

Example 10 *Doubling the Fano plane:* Replace each triple in the Fano plane with the regular map C_3 on the sphere. The resulting pinched surface has 7 pinch points each with 3 cones. Using the automorphism group of the Fano plane it can be seen that this group acts transitively on arcs of the pinched map. The action is not free, since we can switch the two triangles representing a fixed line in the Fano plane. However, there is again a subgroup of automorphisms that acts freely, giving rise to a regular pinched map. The map can be pictured by taking the embedding of K_7 on the torus, 2-coloring the faces, deleting the black faces, and duplicating the white faces.

Example 11 *The complement of the Fano plane:* For each triple in the Fano plane take its complement, a 4-set. Make 7 tetrahedra using each of these 4-sets and identify them along the vertices, forming a pinched surface. The underlying graph is $2K_7$, a complete graph with every edge replaced with two parallel edges. Be careful, when forming the quotient we identify only vertices, not edges, so as to preserve the set of arcs. The details are left to the reader.

5 Powers of maps

In this section we define the power of a map and examine the relation between regular pinched maps and their powers. First, we introduce some useful terminology. A permutation ρ on the arcs of a graph *respects feet* if for every arc, a and $\rho(a)$ eminate from the same vertex.

We start with a graph G underlying a map M on a possibly pinched oriented surface S. We consider the orientation as anti-clockwise as we view a vertex v. The arcs with their foot at v are permuted in this anti-clockwise manner by a permutation ρ_v. If v is not a pinch point, then ρ_v is cyclic. If v is a pinch point, then the number of orbits of ρ_v is the number of cones at v. Let ρ denote the permutation over all arcs of G. We call ρ the *rotation* of M (reference [5] gives a nice exposition on the relation between rotations and embeddings). Observe that ρ respects feet.

Suppose that we are given a graph G and an abstract permutation ρ on the arcs of G that respects feet. Then ρ determines an embedding of G on a pinched surface forming an oriented map. Because of this bijection between maps and rotations we refer to a map as a pair $M = (G, \rho)$.

We are most interested in regular maps, where the number of orbits in a rotation is independent of the vertex v and all of these orbits are of the same size. Denote the size of these orbits by $d_c(\rho)$, the *cone degree* of ρ, most conveniently thought of as the degree of v in each cone in a pinched embedding.

We begin by relating regular maps to the interaction of Γ and ρ. The following is commonly used for maps on surfaces and easily extends to maps on pinched surfaces.

Proposition 5.1 *Let $M = (G, \rho)$ be a pinched map. A permutation γ of arcs of G is an automorphism of M if and only if for all arcs a, $\gamma\rho(a) = \rho\gamma(a)$ and $\gamma(a^{-1}) = \gamma(a)^{-1}$.*

Proof: The permutation γ preserves the map if and only if it preserves the faces and inverse arcs. To preserve the faces you should preserve the rotation, that is, it is the same if you first rotate and then apply the automorphism, or apply the automorphism and then rotate. Hence ρ and γ commute as claimed. Similarly γ must preserve the involution swapping a and a^{-1}. ■

Our interest is to relate a map based on G with rotation ρ to the map on G with the power ρ^e of the rotation.

Corollary 5.2 *If $M = (G, \rho)$ is a regular pinched map, then $M^e = (G, \rho^e)$ is a regular pinched map.*

Proof: Let Γ be a group of automorphisms of M acting regularly on the arcs of the map. If $\gamma\rho = \rho\gamma$ for every $\gamma \in \Gamma$, then $\gamma\rho^e = \rho^e\gamma$ for every $\gamma \in \Gamma$. ■

We give some examples of regular pinched maps M^e derived from regular (sometimes pinched) maps M. We will consider $1 < e < d_c(\rho)$ to avoid redundancies. For convenience, let $d = d_c(\rho)$.

Example 12 *Reversing orientation:* Let $M = (G, \rho)$ be a regular pinched map and set $M^{-1} = (G, \rho^{d-1})$. The orbits of M and M^{-1} are the same size; in particular, if M has no pinch points, then M^{-1} has no pinch points. The pinched map M^{-1} corresponds to reversing the orientation on the surface of the map M.

Example 13 *Exponents of regular maps:* Let $M = (G, \rho)$ be a regular map without pinch points. If $M^e = (G, \rho^e)$ is isomorphic to M, then e is called an *exponent* of M. Any exponent e must be coprime with d. Exponents of maps have been widely studied [7, 1]. The definition of exponents extends to pinched maps for $1 < e < d_c(\rho)$. Again, any exponent e of a pinched map must be coprime to $d_c(\rho)$.

Example 14 *Non-coprime exponents—squaring K_7 on the torus:* Let M be a regular map with K_7 on the torus and let ρ be its rotation. Each orbit of ρ is of size 6. Form $M^2 = (K_7, \rho^2)$. The orbits of ρ^2 are all of size three, so M^2 is a pinched map with seven pinch points each having two cones of size three.

This square of K_7 is exactly the same pinched map formed by the seven parts with two cones arising from the imprimitive action on the vertices of the regular Heawood map on the torus.

In general, starting with a regular pinched map $M = (G, \rho)$ with each cone of size d, the cones in $M^e = (G, \rho^e)$ all have the same size, namely the order of e in $\mathbb{Z}_d$.

We now have several constructions of regular pinched maps. On occasion the same regular map can arise in two different ways, such as in the square of K_7 being the same as the quotient of the Heawood map under its imprimitive action on its vertices. This leads to the following question.

Question 5.3 *Suppose that we are given a strongly connected regular pinched map $M = (G, \rho)$. When is there a rotation ρ' and a power e such that $M' = (G, \rho')$ is regular and $M = (M')^e$?*

Replacing ρ' by $(\rho')^e$ can be viewed as wrapping the rotation e times around the vertex v. A stronger form of Question 5.3 asks *"Is every regular pinched map a wrapping of regular map?"* The answer to this stronger question is NO. Before giving a class of regular pinched maps that are not wrappings, we need the following.

Proposition 5.4 *A graph G underlies a regular map if and only if $Aut(G)$ has a subgroup H that acts regularly on the arcs and has cyclic vertex stabilizers.*

Proof: If G underlies a regular map, then the automorphism group of that map has the stated properties. The converse was proven in [4]. ■

Example 15 *Marušič's graphs:* Let $n \geq 5$ be odd. In [6] there is a construction of a Cayley graph of degree 4 using the alternating group A_n such that the full automorphism group of this Cayley graph is $G = S_n \times Z_2$. In this example $Aut(G)$ acts regularly on arcs in such a way that the stabilizer in $Aut(G)$ of every vertex is isomorphic to $Z_2 \times Z_2$.

These graphs can be regularly embedded on a pinched surface, with two cones at each vertex, each containing two edges. In the vertex stabilizer, the first and the second cone could correspond to, say, the elements 00, 10 and 01,11, while the elements 01 and 11 would swap the cones.

Since the vertex stabilizer is not cyclic, Proposition 5.4 shows that these graphs do not underlie a regular map. Hence these pinched maps are not wrappings.

6 Conclusion

In one sense the study of regular pinched maps is completed by our Main Theorem 2.1: all such maps are quotients of unpinched maps under a block structure of the group action on the arcs. But these maps come in a garden full of species, each beautiful in their own way. There are many areas left to explore. The first of these we have not yet examined.

Question 6.1 *What are the regular pinched non-orientable maps?*

A strongly-connected regular non-orientable map has an automorphism group of $4|E(G)|$ (not $2|E(G)|$) since there is no such concept of "orientation-preserving" maps. The description of these maps is more difficult, but it would be interesting to see how the techniques of this paper apply.

Back to the orientable case: we have a variety of construction techniques.

Question 6.2 *Can we take a combination of the constructions for regular maps, make regular pinched maps, modify them using our techniques, and create new regular maps?*

Finally, we were intrigued by Question 5.3, but could not answer it in any satisfactory manner. This can be phrased as "When can you unwrap a regular pinched map?"

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