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Pairs of Fan-type heavy subgraphs for pancyclicity of 2-connected graphs

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Abstract

A graph G on n vertices is Hamiltonian if it contains a spanning cycle, and pancyclic if it contains cycles of all lengths from 3 to n. In 1984, Fan presented a degree condition involving every pair of vertices at distance two for a 2-connected graph to be Hamiltonian. Motivated by Fan's result, we say that an induced subgraph H of G is f_1-heavy if for every pair of vertices $u, v \in V(H)$, $d_H(u, v) = 2$ implies $\max\{d(u), d(v)\} \geq (n+1)/2$. For a given graph R, G is called R-f_1-heavy if every induced subgraph of G isomorphic to R is f_1-heavy. In this paper we show that for a connected graph S with $S \neq P_3$ and a 2-connected claw-f_1-heavy graph G which is not a cycle, G being S-f_1-heavy implies G is pancyclic if $S = P_4, Z_1$ or Z_2, where claw is $K_{1,3}$ and Z_i is the path $a_1a_2a_3 \ldots a_{i+2}a_{i+3}$ plus the edge a_1a_3. Our result partially improves a previous theorem due to Bedrossian on pancyclicity of 2-connected graphs.

1 Introduction

We use Bondy and Murty [5] for terminology and notation not defined here, and we only consider simple graphs.

Let G be a graph, H a subgraph and v a vertex of G. We use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H, respectively. We call $d_H(v)$ the *degree* of v in H. For $x, y \in V(G)$, an (x, y)-*path* is a path P connecting x and y. If $x, y \in V(H)$, the *distance* between x and y in H, denoted by $d_H(x, y)$, is the length of a shortest (x, y)-path in H. When no confusion occurs, we use $N(v)$, $d(v)$ and $d(x, y)$ instead of $N_G(v)$, $d_G(v)$ and $d_G(x, y)$, respectively.

Let G be a graph on n vertices. For a given graph R, G is called *R-free* if G contains no induced subgraph isomorphic to R, and *R-f_i-heavy* if for every induced subgraph H of G isomorphic to R and every pair of vertices $u, v \in V(H)$, $d_H(u, v) = 2$ implies that $\max\{d(u), d(v)\} \geq (n+i)/2$. For a family $\mathcal{R}$ of graphs, G is called *$\mathcal{R}$-free* (*$\mathcal{R}$-f_i-heavy*) if G is R-free (R-f_i-heavy) for each $R \in \mathcal{R}$. In particular, similar

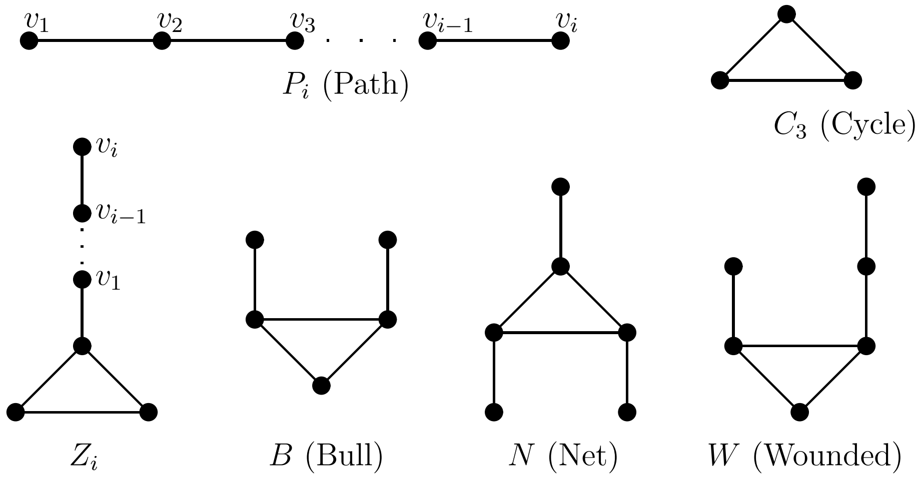

Figure 1: Graphs P_i, C_3, Z_i, B, N and W

as in [9], we use R-f-heavy ($\mathcal{R}$-f-heavy) instead of R-f_0-heavy ($\mathcal{R}$-f_0-heavy). Note that every $\mathcal{R}$-free graph is also $\mathcal{R}$-f_1-heavy ($\mathcal{R}$-f-heavy).

The bipartite graph $K_{1,3}$ is called the *claw*, its (only) vertex of degree 3 is called its *center* and the other vertices are called its *end vertices*. In this paper, we use claw-f_1-heavy instead of $K_{1,3}$-f_1-heavy.

A graph G on n vertices is said to be *Hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all vertices of G, and *pancyclic* if G contains cycles of all lengths from 3 to n. Bedrossian [1] completely characterized all the pairs of forbidden subgraphs for a 2-connected graph to be Hamiltonian and to be pancyclic.

Theorem 1 (Bedrossian [1]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Figure 1).*

Theorem 2 (Bedrossian [1]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

In 1984, Fan [6] presented a degree condition (so-called Fan's condition) involving every pair of vertices at distance two for a 2-connected graph to be Hamiltonian.

Theorem 3 (Fan [6]). *Let G be a 2-connected graph on n vertices. If $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v such that $d(u, v) = 2$, then G is Hamiltonian.*

Obviously, Fan's condition is equivalent to every 2-connected P_3-f-heavy graph is Hamiltonian. By restricting Fan's condition to some induced subgraphs of 2-connected graphs, Ning and Zhang [9] extended Theorem 1 as follows.

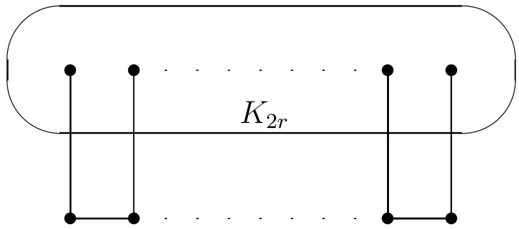

Figure 2: The Graph F_{4r}

Theorem 4 (Ning and Zhang [9]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-f-heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

Our aim in this paper is to consider the corresponding Fan-type heavy subgraph conditions for a 2-connected graph to be pancyclic. First, we notice that every 2-connected P_3-f_1-heavy graph is pancyclic. This can be easily deduced from the following result.

Theorem 5 (Benhocine and Wojda [3]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is P_3-f-heavy, then G is pancyclic unless $n = 4r$, $r > 2$, and $G = F_{4r}$ (see Figure 2), or n is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ is even and $G = K_{n/2,n/2} - e$.*

It is not difficult to see that P_3 is the only connected graph S such that every 2-connected S-f_1-heavy graph is pancyclic. For details, see [7, Theorem 13]. Furthermore, since every P_3-free graph is also P_3-f_1-heavy, P_3 is the only connected graph S such that every 2-connected S-f_1-heavy graph is pancyclic. So we have the following problem.

Problem 1. Which two connected graphs R and S other than P_3 imply that every 2-connected $\{R, S\}$-f_1-heavy graph is pancyclic?

By Theorem 2, we know that $R = K_{1,3}$ (up to symmetry) and S must be one of Z_1, Z_2, P_4 and P_5.

In this paper, we mainly prove the following result.

Theorem 6. *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, Z_2\}$-f_1-heavy, then G is pancyclic.*

As a corollary of Theorem 6, we have

Theorem 7. *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, P_4\}$-f_1-heavy, then G is pancyclic.*

In [2], Bedrossian et al. proved a theorem as follows.

Theorem 8 (Bedrossian, Chen and Schelp [2])**.** *Let G be a 2-connected graph on n vertices. If G is $\{K_{1,3}, Z_1\}$-f-heavy, then G is pancyclic unless $G = F_{4r}$ or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - e$ or else G is a cycle.*

By Theorem 8, we have

Theorem 9. *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, Z_1\}$-f_1-heavy, then G is pancyclic.*

Combining with Theorems 6, 7 and 9, we obtain Theorem 10, which partially answers Problem 1.

Theorem 10. *Let S be a connected graph with $S \neq P_3$ and let G be a 2-connected claw-f_1-heavy graph which is not a cycle. Then G being S-f_1-heavy implies G is pancyclic if $S = P_4, Z_1$ or Z_2.*

The rest of this paper is organized as follows. In Section 2, we will give additional terminology and list some useful lemmas. The proof of Theorem 6 will be postponed to Section 3.

2 Preliminaries

In this section, we first introduce some additional terminology and notation and then present four lemmas which will be used in our proof of Theorem 6.

Let G be a graph and S be a subset of of $V(G)$. We use $G[S]$ to denote the subgraph of G induced by S and $G - S$ to denote $G[V(G) \setminus S]$. In particular, if $S = \{u\}$, then we use $G - u$ instead of $G - \{u\}$. If $S = \{x_i : 1 \leq i \leq 5\}$ and $G[S]$ is isomorphic to Z_2, then we say that $\{x_1, x_2, x_3; x_4, x_5\}$ induces a Z_2, where $x_1x_2x_3x_1$ is a triangle and x_1 is the vertex of degree 3 in $G[S]$. If $S = \{x_i : 1 \leq i \leq 4\}$ and $G[S]$ is isomorphic to $K_{1,3}$, then we say that $\{x_1; x_2, x_3, x_4\}$ induces a claw, where x_1 is the center, and x_2, x_3, x_4 are the end vertices.

Let k, l $(k < l)$ be two integers. We say that G contains a *k-cycle* if G contains a cycle of length k, and G contains *$[k, l]$-cycles* if G contains cycles of all lengths from k to l. In particular, for a vertex $u \in V(G)$, we say that G contains a *u-triangle* if G contains the cycle $uxyu$, where $x, y \in V(G)$.

A vertex v of a graph G on n vertices is called *heavy* if $d(v) \geq n/2$, and *super-heavy* if $d(v) \geq (n+1)/2$. For two vertices u, v of G, $\{u, v\}$ is called a *heavy-pair* if $d(u) + d(v) \geq n$ and a *super-heavy pair* if $d(u) + d(v) \geq n + 1$.

Lemma 1 (Benhocine and Wojda [3])**.** *Let G be a graph on $n \geq 4$ vertices and C be a cycle of length $n - 1$ in G. If $d(x) \geq n/2$ for the vertex $x \in V(G) \backslash V(C)$, then G is pancyclic.*

Lemma 2 (Bondy [4]). *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d(x) + d(y) \geq n + 1$, then G is pancyclic.*

Lemma 3 (Hakimi and Schmeichel [10]). *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d(x) + d(y) \geq n$, then G is pancyclic unless G is bipartite or else G is missing only an $(n-1)$-cycle.*

Lemma 4 (Ferrara, Jacobson and Harris [8]). *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 2$ and $d(x) + d(y) \geq n + 1$, then G is pancyclic.*

3 Proof of Theorem 6

We prove Theorem 6 by contradiction. Suppose that G satisfies the condition of Theorem 6 but is not pancyclic. Since the result is easy to verify for $3 \leq n \leq 5$, we assume that $n \geq 6$.

If G is $\{K_{1,3}, Z_2\}$-free, then by Theorem 2, G is pancyclic. Thus we assume that G contains an induced claw or an induced Z_2. Therefore, there is a super-heavy vertex, say $u \in V(G)$. Set $G' = G - u$. Since G is $\{K_{1,3}, Z_2\}$-f_1-heavy, G' is $\{K_{1,3}, Z_2\}$-f-heavy. If G' is 2-connected, then by Theorem 4, G' is Hamiltonian. Hence G is pancyclic by Lemma 1. Now, it will be assumed that G' is not 2-connected. Then there exists a vertex $v \in V(G)$ $(v \neq u)$ such that $G - \{u, v\}$ is not connected. By Theorem 4, G is Hamiltonian. Hence $G - \{u, v\}$ consists of two components H_1 and H_2. Without loss of generality, we assume that $|V(H_1)| \leq |V(H_2)|$, where $V(H_1) = \{x_1, x_2, \ldots, x_{h_1}\}$ and $V(H_2) = \{y_1, y_2, \ldots, y_{h_2}\}$. Let $C = uy_1 \cdots y_{h_2}vx_{h_1} \cdots x_1u$ be a Hamilton cycle with the given orientation. In the following, for any two vertices $w_1, w_2 \in V(C)$, we use $C[w_1, w_2]$ to denote the segment of C from w_1 to w_2 along the orientation. Set $G_1 = G[V(H_1) \cup \{u\}]$ and $G_2 = G[V(H_2) \cup \{u\}]$.

Claim 1. There are no super-heavy vertices in H_1.

Proof. For any vertex $x \in V(H_1)$, x is adjacent to at most u, v and all the vertices in H_1 except for itself. Therefore, $d(x) \leq d_{H_1}(x) + 2 \leq h_1 - 1 + 2 \leq n/2$. Hence H_1 contains no super-heavy vertices. □

Claim 2. $N_{G_2}(u) \backslash \{y_1\} \subseteq N(y_1)$.

Proof. If there exists a vertex $y_i \in N_{G_2}(u) \backslash \{y_1\}$ such that $y_iy_1 \notin E(G)$, then $\{u; x_1, y_1, y_i\}$ induces a claw. By Claim 1, x_1 is not super-heavy. Since G is claw-f_1-heavy, y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$. By Lemma 2, G is pancyclic. □

Claim 3. There are no super-heavy pairs with distance one or two along the orientation of a Hamilton cycle in G.

Proof. Suppose not. By Lemma 2 or 4, G is pancyclic. □

Case 1. $h_1 = 1$.

Subcase 1.1. $uv \in E(G)$.

Note that G cannot be bipartite or missing an $(n-1)$ cycle. So if Lemma 3 applies to G then G is pancyclic. If u is adjacent to every vertex in C, then G is pancyclic. Now we can choose a vertex $y_i \in N_{G_2}(u)$ such that $uy_{i+1} \notin E(G)$. Let y_j be the first vertex on $C[y_i, y_{h_2}]$ such that $uy_{j+1} \in E(G)$, where assume that $y_{h_2+1} = v$. Obviously, $j \geq i+1$.

Claim 4. $i \geq 2$.

Proof. Assume there exists $y \in V(H_2)$ such that $y_1y \in E(G)$ and $uy \notin E(G)$. By Claim 2, we have $N_{G_2}(u)\backslash\{y_1\} \subset N(y_1)$. Since $d(u) \geq (n+1)/2$ and $u, y \in N(y_1)\backslash N(u)$, $d(y_1) \geq d(u) - 3 + 2 \geq (n-1)/2$. Therefore, $\{u, y_1\}$ is a heavy-pair such that $d_C(u, y_1) = 1$. By Lemma 3, G is pancyclic. Also, since $y_1y_2 \in E(G)$, then $uy_2 \in E(G)$ and $i \geq 2$. □

Next we assume that $i \leq h_2 - 2$. Note that $y_i, y_{i+1}, y_{i+2} \in C[y_2, y_{h_2}]$.

Claim 5. $j \geq i+2$.

Proof. Assume that $j = i+1$. First, we have $uy_i, uy_{i+2} \in E(G)$ and $uy_{i+1} \notin E(G)$.

If $y_iy_{i+2} \notin E(G)$, then $\{u; x_1, y_i, y_{i+2}\}$ induces a claw. Since $d(x_1) = 2 < (n+1)/2$ and G is claw-f_1-heavy, $\{y_i, y_{i+2}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+2}) = 2$, which contradicts to Claim 3.

Now assume that $y_iy_{i+2} \in E(G)$. If $y_1y_{i+1} \in E(G)$, then it follows $d(y_1) \geq (n-1)/2$ from Claim 2. Hence $\{u, y_1\}$ is a heavy pair with $d_C(u, y_1) = 1$, and G is pancyclic by Lemma 3. Therefore, $y_1y_{i+1} \notin E(G)$. We set $G' = G - y_i$. Clearly, $C' = C[y_{i+2}, y_i]y_iy_{i+2}$ is a Hamilton cycle in G'. Moreover, u, y_1 satisfy that $d_{G'}(u) + d_{G'}(y_1) = d(u) + d(y_1) \geq (n+1)/2 + (n-3)/2 = n-1$ and $d_{C'}(u, y_1) = 1$. By Lemma 3, G' is pancyclic. Together with the cycle C, G is pancyclic. □

By Claim 5, we obtain $uy_{i+2} \notin E(G)$.

Claim 6. $vy_{i+1} \in E(G)$.

Proof. Assume that $vy_{i+1} \notin E(G)$.

Claim 6.1. $vy_{i+2} \notin E(G)$.

Proof. Assume that $vy_{i+2} \in E(G)$. Then $\{v, x_1, u; y_{i+2}, y_{i+1}\}$ induces a Z_2. If v is a super-heavy vertex, then $\{u, v\}$ is a super-heavy pair such that $d_C(u, v) = 2$, contradicting Claim 3. Now assume that v is not super-heavy. Note that x_1 is not super-heavy. Since G is Z_2-f_1-heavy, $\{y_{i+1}, y_{i+2}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+1}) = 1$, contradicting Claim 3. □

Claim 6.2. $vy_i \notin E(G)$.

Proof. Assume that $vy_i \in E(G)$. By Claim 6.1, we have $vy_{i+2} \notin E(G)$. Note that $vy_{i+1} \notin E(G)$ by the initial hypothesis. If $y_iy_{i+2} \notin E(G)$, then $\{y_i, u, v; y_{i+1}, y_{i+2}\}$ induces a Z_2. Since v is not super-heavy, y_{i+1} is super-heavy. Hence either $\{y_{i+1}, y_{i+2}\}$ or $\{y_{i+1}, y_i\}$ is a super-heavy pair, a contradiction by Claim 3. If $y_iy_{i+2} \in E(G)$, then $\{y_i, y_{i+1}, y_{i+2}; v, x_1\}$ induces a Z_2. Since v is not super-heavy, $\{y_{i+1}, y_{i+2}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+1}) = 1$, a contradiction by Claim 3. □

Claim 6.3. y_i is super-heavy.

Proof. By Claims 6.2 and the initial hypothesis, we have $vy_i \notin E(G)$ and $vy_{i+1} \notin E(G)$. Since $\{u, v, x_1; y_i, y_{i+1}\}$ induces a Z_2 and x_1 is not super-heavy, y_i is super-heavy. □

By Claim 4, we have $i \geq 2$, and this implies y_{i-1} is well-defined.

Claim 6.4. $y_{i-1}y_{i+1} \notin E(G)$, $uy_{i-1} \in E(G)$, $y_iy_{i+2} \notin E(G)$ and $y_{i-1}y_{i+2} \notin E(G)$.

Proof. By Claim 6.3, y_i is super-heavy. If $y_{i-1}y_{i+1} \in E(G)$, then G is pancyclic by Lemma 1.

If $uy_{i-1} \notin E(G)$, then $\{y_i; y_{i-1}, y_{i+1}, u\}$ induces a claw. Hence either y_{i-1} or y_{i+1} is super-heavy. Therefore, either $\{y_{i-1}, y_i\}$ or $\{y_i, y_{i+1}\}$ is a super-heavy pair such that $d_C(y_{i-1}, y_i) = d_C(y_i, y_{i+1}) = 1$, a contradiction by Claim 3.

By Claim 2 and Lemma 3, $y_1y_{i+1} \notin E(G)$. If $y_iy_{i+2} \in E(G)$, then set $G' = G - y_{i+1}$. Now $C' = vx_1uy_1 \dots y_iy_{i+2} \dots y_{h_2}v$ is a Hamilton cycle in G', and $d_{G'}(u) + d_{G'}(y_1) \geq n - 1 = |G'|$ by Claim 2. By Lemma 3, G' is either pancyclic, bipartite, or missing only an $(n-2)$-cycle. Since $C' = vx_1uy_1 \cdots y_iy_{i+2} \cdots y_{h_2}v$ is an $(n-1)$-cycle and $C'' = vuy_1 \cdots y_iy_{i+2} \cdots y_{h_2}v$ is an $(n-2)$-cycle in G', G' is pancyclic. Therefore, G is pancyclic.

If $y_{i-1}y_{i+2} \in E(G)$, then set $G' = G - y_{i+1}$. Now $C' = uy_1 \dots y_{i-1}y_{i+2} \dots y_{h_2}vx_1u$ is a Hamilton cycle in $G'' = G' - y_i$ and $d_{G'}(y_i) \geq (n-1)/2 = |G'|/2$. By Lemma 1, G' is pancyclic. Together with the cycle C, G is pancyclic. □

By Claim 6.4, $\{y_i, u, y_{i-1}; y_{i+1}, y_{i+2}\}$ induces a Z_2. Since G is Z_2-f_1-heavy, either y_{i-1} or y_{i+1} is super-heavy. Then either $\{y_{i-1}, y_i\}$ or $\{y_{i+1}, y_i\}$ is a super-heavy pair such that $d_C(y_{i-1}, y_i) = d_C(y_{i+1}, y_i) = 1$. By Claim 3, a contradiction. □

Claim 7. For any $k \in \{i+1, \cdots, j\}$, $vy_k \in E(G)$.

Proof. By Claim 6, we have $vy_{i+1} \in E(G)$. Now we show that $vy_k \in E(G)$ for any $k \in \{i+2, \cdots, j\}$. Otherwise, assume that y_t is the first vertex on $C[y_{i+2}, y_j]$ such that $vy_t \notin E(G)$. Note that for any $k \in \{i+1, \cdots, j\}$, $uy_k \notin E(G)$. We have $uy_{t-1}, uy_t \notin E(G)$. Then $\{v, x_1, u; y_{t-1}, y_t\}$ induces a Z_2. Since x_1, v are not super-heavy, $\{y_{t-1}, y_t\}$ is a super-heavy pair such that $d_C(y_{t-1}, y_t) = 1$. By Claim 3, a contradiction, hence $vy_k \in E(G)$. □

Note that since $j \geq i+2$ and i could be selected to be $\leq h_2 - 2$, then if $(j+1) \leq h_2 - 2$, let $i = j+1$ and repeat the previous arguments to conclude that for any vertex $y \in \{y_2, y_3, \cdots, y_{h_2-2}\}$ such that $uy \notin E(G)$, we have $vy \in E(G)$. Hence

$d_{C[y_1,y_{h_2-2}]}(u) + d_{C[y_1,y_{h_2-2}]}(v) \geq h_2 - 2$. If $uy_{h_2-1} \in E(G)$ or $vy_{h_2-1} \in E(G)$ or $uy_{h_2} \in E(G)$, then $d_{G_2}(u)+d_{G_2}(v) \geq h_2 = n-3$. This implies that $d(u)+d(v) \geq n+1$. By Claim 3, a contradiction. Otherwise, assume that $uy_{h_2-1}, uy_{h_2} \notin E(G)$ and $vy_{h_2-1} \notin E(G)$. Then $\{v, x_1, u; y_{h_2}, y_{h_2-1}\}$ induces a Z_2. It follows that $\{y_{h_2}, y_{h_2-1}\}$ is a super-heavy pair such that $d_C(y_{h_2-1}, y_{h_2}) = 1$, contradicting Claim 3.

Subcase 1.2. $uv \notin E(G)$.

By Claim 2, $N_{G_2}(u)\backslash\{y_1\} \subseteq N(y_1)$. If $uy_2 \notin E(G)$, then since u is super-heavy and $u, y_2 \in N(y_1)\backslash N(u)$, y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, a contradiction by Claim 3. If $uy_2 \in E(G)$, then we have $d(y_1) \geq (n-1)/2$ and $\{u, y_1\}$ is a heavy-pair such that $d_C(u, y_1) = 1$. By Lemma 3, G is either pancyclic, bipartite, or missing only an $(n-1)$-cycle. The cycle uy_1y_2u (a triangle) is odd, so G is not bipartite. Since $C' = ux_1vy_{h_2}, \ldots, y_2u$ is an $(n-1)$-cycle, G is pancyclic.

Case 2. $h_1 \geq 2$.

Subcase 2.1. G_1 contains a u-triangle.

Without loss of generality, we denote a u-triangle in G_1 by $ux_kx_{k'}u$ where $k < k'$.

Subsubcase 2.1.1. u is not adjacent to every vertex of H_2.

Let $y_i \in V(H_2)$ be the vertex such that $uy_i \notin E(G)$ and i is as small as possible. Note that $\{u, x_k, x_{k'}; y_{i-1}, y_i\}$ induces a Z_2. By Claim 1, y_{i-1} is super heavy. So if $i = 2$ then $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, a contradiction by Claim 3. Therefore $i \geq 3$ and $uy_2 \in E(G)$.

If there exists $t \in \{1, 2, \ldots, h_1 - 1\}$ such that $ux_t \in E(G)$ and $ux_{t+1} \notin E(G)$, then $\{u, y_1, y_2; x_t, x_{t+1}\}$ induces a Z_2. Note that x_t is not super-heavy. Since G is Z_2-f_1-heavy, y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, contradicting Claim 3. Therefore, u is adjacent to every vertex of H_1. Note that $C' = ux_1 \cdots x_iu$ is an $(i+1)$-cycle, where $2 \leq i \leq h_1$, and G contains $[3, h_1+1]$-cycles. If $i = h_2$, then u is adjacent to every vertex of H_2 other than y_{h_2}. It follows G contains $[h_1+4, n]$-cycles. Furthermore, $C' = ux_2 \cdots x_{h_1}vy_{h_2}y_{h_2-1}u$ is an (h_1+3)-cycle. If $h_1 \geq 3$, then $C' = ux_3 \cdots x_{h_1}vy_{h_2}y_{h_2-1}u$ is an (h_1+2)-cycle, and G is pancyclic. If $h_1 = 2$ and $h_2 \geq 4$, then we can easily find a 4-cycle in G, and G is pancyclic. If $h_1 = 2$ and $h_2 = 2$ or 3, then $n = 6$ or 7. In these two cases, the result is easy to verify.

Now we suppose that $3 \leq i \leq h_2 - 1$ and try to get a contradiction. If there exists $y_k \in N_{G_2}(u)$ such that $y_ky_{i-2} \notin E(G)$ and $y_k \neq y_{i-2}$, then $\{u; x_1, y_k, y_{i-2}\}$ induces a claw. Since G is claw-f_1-heavy and x_1 is not super-heavy, y_{i-2} is super-heavy. Therefore, $\{y_{i-2}, y_{i-1}\}$ is a super-heavy pair such that $d_C(y_{i-2}, y_{i-1}) = 1$, a contradiction by Claim 3. So, $N_{G_2}(u)\backslash\{y_{i-2}\} \subseteq N(y_{i-2})$.

If $uv \in E(G)$, then we set $G' = G - V(H_1)$. Since $N(u) \cup \{u\}\backslash(V(H_1) \cup \{v, y_{i-2}\}) \subseteq N(y_{i-2})$, we have $d(y_{i-2}) \geq d(u) + 1 - h_1 - 2 \geq (n+1)/2 - h_1 - 1$. Furthermore, we obtain $d_{G'}(y_{i-2}) + d_{G'}(y_{i-1}) = d(y_{i-2}) + d(y_{i-1}) \geq n - h_1 = |G'|$.

Let $C' = uvy_{h_2}\cdots y_1u$. Then C' is a Hamilton cycle in G' and $d_{C'}(y_{i-2}, y_{i-1}) = 1$. By Lemma 3, G' is either pancyclic, bipartite, or missing only a $(|G'|-1)$-cycle. But G' contains the triangle uy_1y_2u, hence it is not bipartite. Note that G contains the cycle $C'' = uvy_{h_2}\cdots y_2u$ of length $|G'|-1$. Hence G' is pancyclic, and this implies that G contains $[3, |G'|]$-cycles. Since u is adjacent to every vertex of H_1, G contains $[|G'|+1, n]$-cycles. Hence G is pancyclic.

If $uv \notin E(G)$, then we set $G' = G - (V(H_1)\backslash\{x_{h_1}\})$. Now we have $d(y_{i-2}) \geq d(u) - h_1 - 1 + 1 \geq (n+1)/2 - h_1$. And we obtain $d_{G'}(y_{i-2}) + d_{G'}(y_{i-1}) \geq d(y_{i-2}) + d(y_{i-1}) \geq n + 1 - h_1 = |G'|$. Similarly, we can prove that G is pancyclic.

Subsubcase 2.1.2. u is adjacent to every vertex of H_2.

Note that uy_1y_2u is a u-triangle. If there exists a vertex $x_t \in V(H_1)$ such that $ux_t \in E(G)$ and $ux_{t+1} \notin E(G)$, then $\{u, y_1, y_2; x_t, x_{t+1}\}$ induces a Z_2. This implies that y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, a contradiction by Claim 3. If u is adjacent to every vertex in H_1, then u is adjacent to all vertices of $V(G)\backslash\{u, v\}$. This implies that $d(u) \geq n-2$, and $d(u) + d(y_1) \geq n$. By Lemma 3, G is either pancyclic, bipartite, or missing only an $(n-1)$-cycle. Since u is adjacent to every vertex in H_2, G is neither bipartite nor missing $(n-1)$-cycles. It follows that G is pancyclic.

Subcase 2.2. G_1 contains no u-triangles.

We first show that $N_{G_1}(u) = \{x_1\}$. Suppose not. If there is a vertex $x \in N_{G_1}(u)$ such that $x \neq x_1$, then since G_1 contains no u-triangles, we have $xx_1 \notin E(G)$. Now $\{u; x, x_1, y_1\}$ induces a claw. It follows that either x or x_1 is super-heavy, which contradicts to Claim 1.

If there exist two consecutive vertices, say $y_i, y_{i+1} \in V(H_2)$, such that $uy_i, uy_{i+1} \in E(G)$, then $\{u, y_i, y_{i+1}; x_1, x_2\}$ induce a Z_2. Hence $\{y_i, y_{i+1}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+1}) = 1$, a contradiction by Claim 3.

Therefore for any $y_i \in V(H_2)\backslash\{y_{h_2}\}$, $|\{uy_i, uy_{i+1}\} \cap E(G)| \leq 1$. This implies that u is adjacent to only one vertex x_1 in H_1 and at most $(h_1+1)/2$ vertices in H_2 and maybe adjacent to v or not. Hence we have $(n+1)/2 \leq d(u) \leq 1 + 1 + (h_2+1)/2$. This implies that $h_2 \geq n-4$. Noting that $h_2 = n - 2 - h_1 \leq n - 2 - 2 = n - 4$, we have $h_2 = n-4, h_1 = 2$, $uv \in E(G)$ and $N_{G_2}(u) = \{y_{2k+1} : k = 0, 1, \ldots, (n-5)/2\}$, where n is odd.

If $y_1y_3 \notin E(G)$, then $\{u; x_1, y_1, y_3\}$ induces a claw. Since G is claw-f_1-heavy, $\{y_1, y_3\}$ is a super-heavy pair such that $d_C(y_1, y_3) = 2$. By Claim 3, a contradiction.

If $y_1y_3 \in E(G)$, then $\{u, y_1, y_3; x_1, x_2\}$ induces a Z_2. Since G is Z_2-f_1-heavy, $\{y_1, y_3\}$ is a super-heavy pair such that $d_C(y_1, y_3) = 2$. By Claim 3, also a contradiction.

The proof is complete. □

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References

[1] P. Bedrossian, *Forbidden Subgraph and Minimum Degree Conditons for Hamiltonicity*, Ph.D. Thesis, Memphis State University, USA, 1991.

[2] P. Bedrossian, G. Chen and R.H. Schelp, A generalization of Fan's condition for Hamiltonicity, pancyclicity, and Hamiltonian connectedness, *Discrete Math.* **115** (1993), 39–50.

[3] A. Benhocine and A.P. Wojda, The Geng-Hua Fan conditions for pancyclic or Hamilton-connected graphs, *J. Combin. Theory Ser. B* **42** (1987), 167–180.

[4] J.A. Bondy, Pancyclic graphs I, *J. Combin. Theory Ser. B* **11** (1971), 80–84.

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan London and Elsevier, New York, 1976.

[6] G. Fan, New sufficient conditions for cycles in graphs, *J. Combin. Theory Ser. B* **37** (1984), 221–227.

[7] R.J. Faudree and R.J. Gould, Characterizing forbidden pairs for hamiltonian properties, *Discrete Math.* **173** (1997), 45–60.

[8] M. Ferrara, M.S. Jacobson and A. Harris, Cycle lengths in Hamiltonian graphs with a pair of vertices having large degree sum, *Graphs Combin.* **26** (2010), 215–223.

[9] B. Ning and S. Zhang, Ore- and Fan-type heavy subgraphs for Hamiltonicity of 2-connected graphs, *Discrete Math.* **313** (2013), 1715–1725.

[10] E.F. Schmeichel and S.L. Hakimi, A cycle structure theorem for hamiltonian graphs, *J. Combin. Theory Ser. B* **45** (1988), 99–107.

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