AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **58(1)** (2014), Pages 172–177

A result on fractional ID-$[a, b]$-factor-critical graphs*

Sizhong Zhou†

School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003
P. R. China
zsz_cumt@163.com

Jie Wu

Department of Science and Technology
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003
P. R. China

Quanru Pan

School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003
P. R. China

Abstract

A graph G is fractional ID-$[a, b]$-factor-critical if $G-I$ includes a fractional $[a, b]$-factor for every independent set I of G. In this paper, it is proved that if $\alpha(G) \leq \frac{4b(\delta(G)-a+1)}{(a+1)^2+4b}$, then G is fractional ID-$[a, b]$-factor-critical. Furthermore, it is shown that the result is best possible in some sense.

1 Introduction

We only consider finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote its vertex set and edge

* Supported by the National Natural Science Foundation of China (Grant No. 11371009) and the National Social Science Foundation of China (Grant No. 11BGL039).
† Corresponding author.

set, respectively. For $x \in V(G)$, the set of vertices adjacent to x in G is said to be the neighborhood of x, denoted by $N_G(x)$, and $|N_G(x)|$ is said to be the degree of x in G, denoted by $d_G(x)$. We write $N_G[x] = N_G(x) \cup \{x\}$. We use $\alpha(G)$ and $\delta(G)$ to denote the independence number and the minimum degree of G, respectively. For a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and $G - S = G[V(G) \setminus S]$. Let A and B be disjoint subsets of $V(G)$. Then we use $e_G(A, B)$ to denote the number of edges that join a vertex in A and a vertex in B. Let r be a real number. Recall that $\lfloor r \rfloor$ is the greatest integer such that $\lfloor r \rfloor \leq r$.

Let a and b be two integers such that $1 \leq a \leq b$. A spanning subgraph F of G with $a \leq d_F(x) \leq b$ for any $x \in V(G)$ is an $[a, b]$-factor of G. Suppose that $a = b$. Then F is called a k-factor of G. Let $h : E(G) \to [0, 1]$ be a function. Then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h if $a \leq \sum_{e \ni x} h(e) \leq b$ holds for every $x \in V(G)$, where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is fractional ID-$[a, b]$-factor-critical if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G. A fractional ID-$[k, k]$-factor-critical graph is a fractional ID-k-factor-critical graph. Notation and definitions not given here can be found in [1,2].

Graph factors and fractional factors have attracted a great deal of attention [3–7]. Sufficient conditions for a graph to be fractional ID-k-factor-critical can be found in [8–10]. The following result is a sufficient condition for a graph to be fractional ID-$[a, b]$-factor-critical.

Theorem 1 ([2]). *Let G be a graph of order n, and let a and b be two integers with $1 \leq a \leq b$. If $n \geq \frac{(a+2b)(a+b-2)+1}{b}$ and $\delta(G) \geq \frac{(a+b)n}{a+2b}$, then G is fractional ID-$[a, b]$-factor-critical.*

Now we proceed to investigate fractional ID-$[a, b]$-factor-critical graphs, and obtain an independence number and minimum degree condition on the existence of fractional ID-$[a, b]$-factor-critical graphs. The main result of the paper is the following theorem, which is a generalization of a result presented in [8].

Theorem 2 *Let G be a graph, and let $1 \leq a \leq b$ be two integers. If*

$$\alpha(G) \leq \frac{4b(\delta(G) - a + 1)}{(a+1)^2 + 4b},$$

then G is fractional ID-$[a, b]$-factor-critical.

If $a = b = k$ in Theorem 2, then we obtain the following corollary.

Corollary 1 ([8]). *Let G be a graph, and let k be an integer with $k \geq 1$. If*

$$\alpha(G) \leq \frac{4k(\delta(G) - k + 1)}{k^2 + 6k + 1},$$

then G is fractional ID-k-factor-critical.

2 The Proof of Theorem 2

In order to prove Theorem 2, we rely heavily on the following lemma.

Lemma 2.1 ([11]). *Let G be a graph. Then G has a fractional $[a, b]$-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a\}$ and $d_{G-S}(T) = \sum_{x\in T} d_{G-S}(x)$.

Proof of Theorem 2. Let X be an independent set of G and $H = G - X$. Obviously, $\delta(H) \geq \delta(G) - |X|$. Theorem 2 holds if and only if H has a fractional $[a, b]$-factor. Suppose, to the contrary, that H has no fractional $[a, b]$-factor. Then by using Lemma 2.1, there exists some subset $S \subseteq V(H)$ satisfying

$$\delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \leq -1, \quad (1)$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq a\}$. Clearly, $T \neq \emptyset$ by (1). Set

$$h = \min\{d_{H-S}(x) : x \in T\}.$$

From the definition of T, we obtain

$$0 \leq h \leq a.$$

Claim 1. $|S| \geq \delta(G) - \alpha(G) - h$.
Proof. We choose $x_1 \in T$ with $d_{H-S}(x_1) = h$. Thus, we have

$$\delta(H) \leq d_H(x_1) \leq d_{H-S}(x_1) + |S| = h + |S|,$$

that is,

$$|S| \geq \delta(H) - h. \quad (2)$$

Note that $\delta(H) \geq \delta(G) - |X|$. Combining this with (2), we have

$$|S| \geq \delta(G) - |X| - h. \quad (3)$$

Note that $|X| \leq \alpha(G)$. Then, using (3) we obtain

$$|S| \geq \delta(G) - \alpha(G) - h.$$

This completes the proof of Claim 1.

In the following, we consider the subgraph $H[T]$ of H induced by T. We write $T_1 = H[T]$. Assume $d_{T_1}(t_1)$ is the minimum value of $d_{T_1}(t)$ for any $t \in T_1$ and $M_1 = N_{T_1}[t_1]$. Let $T_i = H[T] - \bigcup_{1\leq j<i} M_j$. Moreover, for $i \geq 2$, suppose $d_{T_i}(t_i)$ is the minimum value of $d_{T_i}(t)$ for any $t \in T_i$ and $M_i = N_{T_i}[t_i]$. We denote the order of M_i by m_i. We continue these processing until we reach the situation in which $T_i = \emptyset$

for some i, say for $i = r + 1$. It is obvious that $\{t_1, t_2, \ldots, t_r\}$ is an independent set of H, and $r \geq 1$ by $T \neq \emptyset$.

We easily prove the following properties.

$$\alpha(H[T]) \;\geq\; r, \tag{4}$$

$$|T| = \sum_{1 \leq i \leq r} m_i. \tag{5}$$

Note that $\alpha(G) \geq \alpha(G[T]) = \alpha(H[T])$. Combining this with (4), we obtain

$$\alpha(G) \;\geq\; r. \tag{6}$$

Now, we prove the following claim.

Claim 2. $d_{H-S}(T) \geq \sum_{1 \leq i \leq r}(m_i^2 - m_i)$.

Proof. Since our choice of t_i implies that all vertices in M_i have degree at least $m_i - 1$ in T_i, we have

$$\sum_{1 \leq i \leq r}\Big(\sum_{x \in M_i} d_{T_i}(x)\Big) \;\geq\; \sum_{1 \leq i \leq r}(m_i^2 - m_i). \tag{7}$$

So (7) yields

$$d_{H-S}(T) \geq \sum_{1 \leq i \leq r}(m_i^2 - m_i) + \sum_{1 \leq i < j \leq r} e_H(M_i, M_j) \;\geq\; \sum_{1 \leq i \leq r}(m_i^2 - m_i).$$

This completes the proof of Claim 2.

In the following, we shall consider various cases for the value of h and derive a contradiction in each case.

Case 1. $0 \leq h \leq a - 1$.

It is easy to see that

$$m_i^2 - (a+1)m_i \;\geq\; -\frac{(a+1)^2}{4}. \tag{8}$$

According to Claim 1, Claim 2, (5), (6), (8), $0 \leq h \leq a - 1$ and the condition $\alpha(G) \leq \dfrac{4b(\delta(G) - a + 1)}{(a+1)^2 + 4b}$ of Theorem 2, we have

$$\begin{aligned}
\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\
&\geq b(\delta(G) - \alpha(G) - h) + \sum_{1 \leq i \leq r}(m_i^2 - m_i) - a\sum_{1 \leq i \leq r} m_i \\
&= b(\delta(G) - \alpha(G) - h) + \sum_{1 \leq i \leq r}(m_i^2 - (a+1)m_i)
\end{aligned}$$

$$
\begin{aligned}
&\geq b(\delta(G) - \alpha(G) - h) - \sum_{1\leq i\leq r} \frac{(a+1)^2}{4} \\
&= b(\delta(G) - \alpha(G) - h) - \frac{(a+1)^2}{4} r \\
&\geq b(\delta(G) - \alpha(G) - h) - \frac{(a+1)^2}{4} \alpha(G) \\
&= b(\delta(G) - h) - \frac{(a+1)^2 + 4b}{4} \alpha(G) \\
&\geq b(\delta(G) - a + 1) - \frac{(a+1)^2 + 4b}{4} \alpha(G) \\
&\geq b(\delta(G) - a + 1) - \frac{(a+1)^2 + 4b}{4} \cdot \frac{4b(\delta(G) - a + 1)}{(a+1)^2 + 4b} \\
&= 0,
\end{aligned}
$$

which contradicts (1).

Case 2. $h = a$.

By using (1), we obtain

$$
\begin{aligned}
-1 &\geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \\
&\geq b|S| + h|T| - a|T| = b|S| \geq 0,
\end{aligned}
$$

which is a contradiction. The proof of Theorem 2 is complete. It is obvious that

$$
\begin{aligned}
\frac{4b(\delta(G) - a + 1)}{(a+1)^2 + 4b} &< \alpha(G) \\
&= t + 1 \\
&= \lfloor \frac{4b(\delta(G) - a + 1)}{(a+1)^2 + 4b} \rfloor + 1 \\
&\leq \frac{4b(\delta(G) - a + 1)}{(a+1)^2 + 4b} + 1.
\end{aligned}
$$

We take a vertex x_i $(1 \leq i \leq t+1)$ in every K_{a+1}. Set $X = \{x_1, x_2, \ldots, x_{t+1}\}$. Apparently, X is an independent set of G. We write $H = G - X = K_t \vee (t+1)K_a$, $S = V(K_t)$ and $T = V((t+1)K_a)$. Then we obtain $|S| = t$, $|T| = (t+1)a$, $d_{H-S}(T) = a(a-1)(t+1)$. Note that $(b-a)t \leq a - 1$. Thus, we have

$$
\begin{aligned}
\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\
&= bt + a(a-1)(t+1) - (t+1)a^2 = (b-a)t - a \leq -1 < 0.
\end{aligned}
$$

In view of Lemma 2.1, H has no fractional $[a,b]$-factor, and so the result in Theorem 2 is sharp.

Acknowledgments

The authors would like to thank the anonymous referees and the editor for their helpful comments and valuable suggestions in improving the quality of this paper.

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(Received 15 Mar 2013)