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Stirling numbers of the second kind and Bell numbers for graphs

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Abstract

In this paper we summarize the known properties of Stirling numbers of the second kind and Bell numbers for graphs, and we also prove new results about them. These give us an alternative way to study r-Stirling numbers of the second kind and r-Bell numbers.

1 Introduction

Stirling numbers of the second kind and Bell numbers play a fundamental role in enumerative combinatorics, they count the number of partitions of a finite set. More precisely, for positive integers $k \le n$, the Stirling number of the second kind $\left\{ {n \atop k} \right\}$ is the number of partitions of an n-element set into k subsets, while the nth Bell number B_n is the number of partitions of a set with n elements (and as usual, let $\left\{ {n \atop 0} \right\} = 0$, $\left\{ {0 \atop 0} \right\} = 1$, $B_0 = 1$). Note that a partition of a non-empty finite set X means $\{A_1, \dots, A_k\}$ with $A_1, \dots, A_k$ being pairwise disjoint, non-empty subsets of X whose union is X. Here we notice that set partitions are the topic of a recent book [19].

Several variants and generalizations of Stirling numbers of the second kind and Bell numbers are known, we mention here only three of them. (For some other variants see, e.g., [20] and references therein.)

Let $n \ge 2$ and $k \le n$ be positive integers. A partition of $\{x_1, \dots, x_n\}$ is called a nonconsecutive partition if x_i and x_{i+1} cannot be in the same block ($i = 1, \dots, n-1$).

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Nonconsecutive partitions are also known under several other names in the literature, like Fibonacci, reduced or restricted partitions. Then $\left\{ {n \atop k} \right\}' = \left\{ {n-1 \atop k-1} \right\}$ is the number of nonconsecutive partitions of an n-element set into k subsets [24], [25] (see also a problem about banner colourings in the book [11]), and $B'_n = B_{n-1}$ gives the number of nonconsecutive partitions of a set with n elements [25], [27], [28], [31] (see also some problems in [1], [30]).

The notion of nonconsecutive partitions can be generalized as follows. Let n, k, m be positive integers with $n \geq m+1$ and $m \leq k \leq n$. A partition of $\{x_1, \ldots, x_n\}$ is an m-nonconsecutive partition if x_i and x_j cannot be in the same block for $1 \leq |i-j| \leq m$. m-nonconsecutive partitions also have some other names in the literature, for example m-Fibonacci, m-restricted, m-regular, m-separated or m-sparse partitions. Then $\left\{ {n \atop k} \right\}^{(m)} = \left\{ {n-m \atop k-m} \right\}$ counts the number of m-nonconsecutive partitions of an n-element set into k subsets [8], [9], [17], [24], [26] and $B_n^{(m)} = B_{n-m}$ is the number of m-nonconsecutive partitions of a set with n elements [2], [17], [18], [26], [27], [31].

Finally, let n, k, r be non-negative integers satisfying $n + r \geq 1$ and $k \leq n$. A partition of $\{x_1, \ldots, x_{n+r}\}$ is called an r-partition if $x_1, \ldots, x_r$ belong to distinct blocks. The r-Stirling number of the second kind $\left\{ {n \atop k} \right\}_r$ is the number of r-partitions of an $(n + r)$-element set into $k + r$ subsets. It was first defined by A. Z. Broder [7] and later redefined by R. Merris [21]. The nth r-Bell number $B_{n,r}$ is the number of r-partitions of a set with $n + r$ elements (and $B_{0,0} = 1$), which was recently defined and studied by I. Mező [23].

Stirling numbers of the second kind and Bell numbers for graphs were defined by B. Duncan and R. Peele [15], although they appeared implicitly earlier in several papers and books. In Section 2 of this paper we summarize the previously known results related to these numbers, but we provide new, simpler, alternative proofs if possible, additionally we show some new properties. In Section 3 we give the values of these numbers for particular graphs. Finally, by the help of the general theorems, properties of r-Stirling numbers of the second kind and r-Bell numbers will be proven in Section 4, some of them being unknown before.

2 Stirling numbers of the second kind and Bell numbers for graphs

Let G be a simple (finite) graph. A partition of $V(G)$ is called an independent partition if each block is an independent vertex set (i.e. adjacent vertices belong to distinct blocks). Then for a positive integer $k \leq |V(G)|$, let the Stirling number of the second kind $\left\{ {G \atop k} \right\}$ for graph G be the number of independent partitions of $V(G)$ into k subsets, moreover let $\left\{ {G \atop 0} \right\} = 0$, and define the Bell number B_G for graph G

as the number of independent partitions of $V(G)$. Then $B_G = \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\}$.

All the variants of Stirling numbers of the second kind and Bell numbers mentioned in the Introduction are special cases of these notions for certain graphs. For the empty graph $G = E_n$, the path graph $G = P_n$, the generalized m-path graph $G = P_n^{(m)}$ (a simple graph with $n \geq m+1$ vertices, in which two vertices are adjacent if and only if the difference of their indices is at most m), and $G = R_{n,r}$ (a graph that consists of a complete graph with r vertices and n isolated vertices), independent partitions of $V(G)$ are the ordinary partitions, the nonconsecutive partitions, the m-nonconsecutive partitions, and the r-partitions of the vertex set of G, respectively.

By simple observation we get for any simple graph G that $\left\{ {G \atop k} \right\} = 0$ if $0 \leq k \leq \chi(G) - 1$, where $\chi(G)$ is the chromatic number of G. This would allow us to start summation from $\chi(G)$ instead of 0 in some formulas below. Moreover $\left\{ {G \atop |V(G)|} \right\} = 1$ and $\left\{ {G \atop |V(G)|-1} \right\} = \binom{|V(G)|}{2} - |E(G)|$.

By a standard argument we can show that reduction relations hold for Stirling numbers of the second kind and Bell numbers for graphs, they can be found in [15].

Theorem 2.1. *If G is a simple graph, $e \in E(G)$ and $0 \leq k \leq |V(G)| - 1$, then*

$$\left\{ {G \atop k} \right\} = \left\{ {G-e \atop k} \right\} - \left\{ {G/e \atop k} \right\} \text{ and } B_G = B_{G-e} - B_{G/e},$$

where $G - e$ and G/e are the simplified graphs obtained by deleting and contracting edge e from G, respectively.

Since any proper vertex colouring of G with exactly k colours naturally induces an independent partition of $V(G)$ into k subsets, we can prove the following theorem. It appears relatively often in the literature about chromatic polynomials with different notation (see, e.g., [3], [4], [10], [13], [15], [29]).

Theorem 2.2. *If G is a simple graph and $p_G(x)$ is the chromatic polynomial of G, then*

$$p_G(x) = \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} x^{\underline{k}},$$

where $x^{\underline{k}}$ denotes the kth falling factorial of x.

C. Berceanu [2] showed that Bell numbers for graphs can be written as the linear combination of ordinary Bell numbers with the coefficients of the chromatic polynomial. He used some linear operators of the polynomial vector space in his proof. A similar equality appeared in matrix form for Stirling numbers of the second kind for graphs by B. Duncan and R. Peele [15]. Now we offer a different and simpler proof.

Theorem 2.3. *If G is a simple graph, $0 \le k \le |V(G)|$ and $p_G(x) = \sum_{j=0}^{|V(G)|} a_j x^j$, then*

$$\left\{ {G \atop k} \right\} = \sum_{j=k}^{|V(G)|} a_j \left\{ {j \atop k} \right\} \text{ and } B_G = \sum_{j=0}^{|V(G)|} a_j B_j.$$

Proof. For simplicity, denote by n the number of vertices of G. We prove the theorem by induction on $|E(G)|$.

It is true for $G = E_n$, since we have $p_G(x) = x^n$, $\left\{ {G \atop k} \right\} = \left\{ {n \atop k} \right\}$, $B_G = B_n$.

Let $|E(G)| \ge 1$ and suppose that our statement holds for graphs with fewer edges. Then $\left\{ {G \atop n} \right\} = 1 = \left\{ {n \atop n} \right\}$, which is the equation to be proved for $k = n$, because $a_n = 1$. It means that we need to prove the first formula only for $0 \le k \le n-1$.

Let $e \in E(G)$ and $p_{G-e}(x) = \sum_{j=0}^{n} b_j x^j$, $p_{G/e}(x) = \sum_{j=0}^{n-1} c_j x^j$. Since $p_G(x) = p_{G-e}(x) - p_{G/e}(x)$, we have

$$a_j = \begin{cases} b_j - c_j & \text{if } 0 \le j \le n-1 \\ b_j & \text{if } j = n. \end{cases}$$

Then Theorem 2.1 and the induction hypothesis for $G-e$ and G/e give

$$\left\{ {G \atop k} \right\} = \left\{ {G-e \atop k} \right\} - \left\{ {G/e \atop k} \right\} = \sum_{j=k}^{n} b_j \left\{ {j \atop k} \right\} - \sum_{j=k}^{n-1} c_j \left\{ {j \atop k} \right\} = \sum_{j=k}^{n} a_j \left\{ {j \atop k} \right\}$$

and

$$B_G = B_{G-e} - B_{G/e} = \sum_{j=0}^{n} b_j B_j - \sum_{j=0}^{n-1} c_j B_j = \sum_{j=0}^{n} a_j B_j.$$

□

The expected value of the chromatic polynomial value of a Poisson random variable can be given by a sum with Stirling numbers of the second kind for graphs.

Theorem 2.4. *If G is a simple graph, $\lambda > 0$ and $\xi \sim \mathrm{Poisson}(\lambda)$, then*

$$\mathsf{E} p_G(\xi) = \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} \lambda^k.$$

Proof. Using Theorem 2.2, we get

$$\mathsf{E} p_G(\xi) = \sum_{j=0}^{\infty} p_G(j) \frac{\lambda^j}{j!} e^{-\lambda} = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} e^{-\lambda} \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} j^{\underline{k}} =$$

$$\sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} j^{\underline{k}} = \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} e^{-\lambda} \lambda^k \sum_{j=k}^{\infty} \frac{\lambda^{j-k}}{(j-k)!} = \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} \lambda^k.$$

□

Now we are in position to prove a Dobiński type formula for Bell numbers for graphs, named after the author of [12].

Theorem 2.5. *If G is a simple graph, then*

$$B_G = \frac{1}{e} \sum_{j=0}^{\infty} \frac{p_G(j)}{j!}.$$

Proof. Let $\xi \sim \text{Poisson}(1)$. Then $\mathsf{E} p_G(\xi) = \sum_{j=0}^{\infty} p_G(j) \frac{1^j}{j!} e^{-1} = \frac{1}{e} \sum_{j=0}^{\infty} \frac{p_G(j)}{j!}$. On the other hand, by Theorem 2.4 we have $\mathsf{E} p_G(\xi) = \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} 1^k = B_G$. □

By the inclusion-exclusion principle, we can derive an explicit formula for Stirling numbers of the second kind for graphs. We should mention that a similar formula appeared in [24].

Theorem 2.6. *If G is a simple graph and $0 \le k \le |V(G)|$, then*

$$\left\{ {G \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} p_G(k-j).$$

Proof. It can be easily checked for $k = 0$, using $p_G(0) = 0$. For $1 \le k \le |V(G)|$, we count surjective proper vertex colourings of G with colour set $\{c_1, \ldots, c_k\}$ in two different ways.

If we consider a surjective proper vertex colouring of G, then the set of preimages of $c_1, \ldots, c_k$ is an independent partition of $V(G)$ into k subsets. We can assign $k!$ surjective proper vertex colourings for one such partition, hence their number is $k! \left\{ {G \atop k} \right\}$.

Denote by X the set of proper vertex colourings of G with the above colour set and $Y_i = \{f \in X \mid c_i \notin f(V(G))\}$ $(i = 1, \ldots, k)$. Then $|X| = p_G(k)$, and the cardinality of the intersection of any j sets of type Y_i is $p_G(k-j)$. By the inclusion-exclusion principle, the number of surjective proper vertex colourings with k colours is $|X \setminus (Y_1 \cup \ldots \cup Y_k)| = \sum_{j=0}^{k} (-1)^j \binom{k}{j} p_G(k-j)$. □

The following theorem is due to W. Yang [31] (for the definition of generalized m-trees, see that paper). This theorem immediately implies the results for the numbers of nonconsecutive and m-nonconsecutive partitions.

Theorem 2.7. *If $m \geq 1$, G is a generalized m-tree with $n \geq m+1$ vertices and $0 \leq k \leq n$, then*

$$\left\{ {G \atop k} \right\} = \begin{cases} 0 & \text{if } 0 \leq k \leq m-1 \\ \left\{ {n-m \atop k-m} \right\} & \text{if } m \leq k \leq n \end{cases} \quad \text{and } B_G = B_{n-m}.$$

Especially, if G is a tree with $n \geq 2$ vertices and $0 \leq k \leq n$, then

$$\left\{ {G \atop k} \right\} = \begin{cases} 0 & \text{if } k = 0 \\ \left\{ {n-1 \atop k-1} \right\} & \text{if } 1 \leq k \leq n \end{cases} \quad \text{and } B_G = B_{n-1}.$$

Proof. We give the proof for the values of Stirling numbers of the second kind for generalized m-trees, then we get the Bell numbers for them directly by summation. This could be done by induction on n, choosing the finally added vertex $v \in V(G)$, using the induction hypothesis for $G - v$ and applying the recurrence for ordinary Stirling numbers of the second kind, but our proof is based on Theorem 2.2.

The chromatic polynomial of a generalized m-tree G with n vertices is

$$p_G(x) = x^{\underline{m}}(x-m)^{n-m} = x^{\underline{m}} \sum_{k=0}^{n-m} \left\{ {n-m \atop k} \right\} (x-m)^{\underline{k}} = \sum_{k=m}^{n} \left\{ {n-m \atop k-m} \right\} x^{\underline{k}},$$

whence $\left\{ {G \atop k} \right\} = \left\{ {n-m \atop k-m} \right\}$ for $m \leq k \leq n$, and $\left\{ {G \atop k} \right\} = 0$ for $0 \leq k \leq m-1$ by Theorem 2.2. □

Remark. If G is a tree with $n+1$ vertices, then

$$p_G(x) = x(x-1)^n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} x^{j+1}.$$

Theorems 2.3 and 2.7 give $B_n = B_G = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{j+1}$ and $\left\{ {n \atop k} \right\} = \left\{ {G \atop k+1} \right\} = \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \left\{ {j+1 \atop k+1} \right\}$. By binomial transform we get the well-known formulas

$$B_{n+1} = \sum_{j=0}^{n} \binom{n}{j} B_j \text{ and } \left\{ {n+1 \atop k+1} \right\} = \sum_{j=k}^{n} \binom{n}{j} \left\{ {j \atop k} \right\}.$$

Finally, we should mention some further results. B. Duncan [14] described Stirling numbers of the second kind and Bell numbers for graphs having two components. The Bell polynomial (a variant of the so-called σ-polynomial) $B_G(x) = \sum_{k=0}^{|V(G)|} \left\{ {G \atop k} \right\} x^k$ of a graph G was studied by F. Brenti [5], and by F. Brenti, G. F. Royle, D. G. Wagner [6], and they deduced log-concavity and unimodality of $\left(\left\{ {G \atop k} \right\}\right)_{k=0}^{|V(G)|}$ for some families of graphs. After submission of the first version of the present paper, D. Galvin and D. T. Thanh [16] proved similar log-concavity results for Stirling numbers for forests and cycle graphs.

3 Examples

Now we summarize the values of Stirling numbers of the second kind and Bell numbers for some special graphs. This is an extended version of the table of [15], which contained only Bell numbers for fewer graphs. In our table E_n, K_n, S_n, P_n, $S_n^{(m)}$, $P_n^{(m)}$, C_n, W_n, $K_{m,n}$ denote the empty graph, the complete graph, the star graph, the path graph, the generalized m-star graph (the graph built up from a complete graph with m vertices by joining new vertices precisely to the original ones), the generalized m-path graph, the cycle graph, the wheel graph and the complete bipartite graph, respectively. Stirling numbers of the second kind for graphs with non-listed parameters k are equal to 0. (For $K_{m,n}$, we define $\left\{ {m \atop j} \right\}$ $(j > m)$ and $\left\{ {n \atop k-j} \right\}$ $(k-j > n)$ to be 0. While for the complements of P_n and C_n, f_{n+1} and ℓ_n denote the $(n+1)$th Fibonacci and the nth Lucas number.)

The values in our table can be derived by direct combinatorial arguments or using Theorems 2.1 and 2.7.

G	$\left\{ {G \atop k} \right\}$	B_G
E_n	$\left\{ {n \atop k} \right\}$ $(1 \le k \le n)$	B_n
K_n	1 $(k = n)$	1
S_n, P_n $(n \ge 2)$	$\left\{ {n-1 \atop k-1} \right\}$ $(1 \le k \le n)$	B_{n-1}
$S_n^{(m)}$, $P_n^{(m)}$ $(n \ge m+1)$	$\left\{ {n-m \atop k-m} \right\}$ $(m \le k \le n)$	B_{n-m}
C_n $(n \ge 3)$	$\sum_{j=k-1}^{n-1} (-1)^{n-1-j} \left\{ {j \atop k-1} \right\}$ $(2 \le k \le n)$	$\sum_{j=1}^{n-1} (-1)^{n-1-j} B_j$
W_n $(n \ge 4)$	$\sum_{j=k-2}^{n-2} (-1)^{n-2-j} \left\{ {j \atop k-2} \right\}$ $(3 \le k \le n)$	$\sum_{j=1}^{n-2} (-1)^{n-2-j} B_j$
$K_{m,n}$	$\sum_{j=0}^{k} \left\{ {m \atop j} \right\} \left\{ {n \atop k-j} \right\}$ $(1 \le k \le m+n)$	$B_m \cdot B_n$
$\overline{S}_n$ $(n \ge 2)$	$n-1$ $(k = n-1)$ and 1 $(k = n)$	n
$\overline{P}_n$ $(n \ge 2)$	$\binom{k}{n-k}$ $(\lceil \frac{n}{2} \rceil \le k \le n)$	f_{n+1}
$\overline{C}_n$ $(n \ge 4)$	$\frac{n}{k}\binom{k}{n-k}$ $(\lceil \frac{n}{2} \rceil \le k \le n)$	ℓ_n

4 r-Stirling numbers of the second kind and r-Bell numbers

As we said before, an r-partition of an $(n+r)$-element set is a partition where the first r elements of the set belong to distinct blocks. Or equivalently, it can be considered as an independent partition of the vertex set of the previously defined graph $R_{n,r}$. Then r-Stirling numbers of the second kind and r-Bell numbers count these partitions (see [7], [21], [22], [23]). In graph theoretical language, for non-negative integers n, k, r with $n + r \geq 1$ and $k \leq n$, $\left\{ {n \atop k} \right\}_r = \left\{ {R_{n,r} \atop k+r} \right\}$ and $B_{n,r} = B_{R_{n,r}}$. We notice that $\left\{ {R_{n,r} \atop l} \right\} = 0$ for $0 \leq l \leq r-1$, since $\chi(R_{n,r}) = r$.

For special parameters r and k, the following equalities hold (with the possible values of the other parameters):

$$\left\{ {n \atop k} \right\}_0 = \left\{ {n \atop k} \right\},\ \left\{ {n \atop k} \right\}_1 = \left\{ {n+1 \atop k+1} \right\},$$

$$\left\{ {n \atop 0} \right\}_r = r^n,\ \left\{ {n \atop 1} \right\}_r = (r+1)^n - r^n,\ \left\{ {n \atop n-1} \right\}_r = rn + \binom{n}{2},\ \left\{ {n \atop n} \right\}_r = 1,$$
$$B_{n,0} = B_n,\ B_{n,1} = B_{n+1},\ B_{0,r} = 1,\ B_{1,r} = r+1.$$

Now r-Stirling numbers of the second kind satisfy a recurrence relation. Namely, if $n, k \geq 1$, $r \geq 0$ and $k \leq n$, then

$$\left\{ {n+1 \atop k} \right\}_r = \left\{ {n \atop k-1} \right\}_r + (k+r)\left\{ {n \atop k} \right\}_r.$$

We prove the following theorems by using the results of Section 2 about Stirling numbers of the second kind and Bell numbers for graphs. The base of the proofs will be that the chromatic polynomial of the graph $R_{n,r}$ is $p_{R_{n,r}}(x) = x^n x^{\underline{r}}$.

Theorem 4.1. *If $n \geq 1$ and $r \geq 0$, then $(x+r)^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r x^{\underline{k}}$.*

Proof. By Theorem 2.2, we have

$$(x+r)^n (x+r)^{\underline{r}} = p_{R_{n,r}}(x+r) = \sum_{k=0}^{n+r} \left\{ {R_{n,r} \atop k} \right\} (x+r)^{\underline{k}} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r (x+r)^{\underline{k+r}},$$

which yields our statement. □

Theorem 4.2. *If $n, k, r \geq 0$, $n + r \geq 1$ and $k \leq n$, then*

$$\left\{ {n \atop k} \right\}_r = \sum_{j=\max\{k+r-n,0\}}^{r} (-1)^{r-j} \left[{r \atop j} \right] \left\{ {n+j \atop k+r} \right\} \text{ and } B_{n,r} = \sum_{j=0}^{r} (-1)^{r-j} \left[{r \atop j} \right] B_{n+j},$$

where $\left[{r \atop j} \right]$ is an (unsigned) Stirling number of the first kind.

Proof. The chromatic polynomial of $R_{n,r}$ is

$$p_{R_{n,r}}(x) = x^n x^{\underline{r}} = x^n \cdot \sum_{j=0}^{r} (-1)^{r-j} \begin{bmatrix} r \\ j \end{bmatrix} x^j = \sum_{j=0}^{r} (-1)^{r-j} \begin{bmatrix} r \\ j \end{bmatrix} x^{n+j}.$$

Then applying Theorem 2.3, we get the desired equations. $\square$

Corollary. *If $n, k, r \geq 0$, $n + r \geq 1$ and $k \leq n$, then*

$$\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\} = \sum_{j=\max\{k+r-n,0\}}^{r} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \left\{ \begin{matrix} n \\ k+r-j \end{matrix} \right\}_j \text{ and } B_{n+r} = \sum_{j=0}^{r} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} B_{n,j}.$$

Proof. The second statement follows from Theorem 4.2 by Stirling transforms. However, both formulas can be proved by a direct combinatorial argument; we give the details for Stirling numbers of the second kind.

We need to partition an $(n + r)$-element set into $k + r$ subsets. Suppose that the first r elements belong to j blocks ($j = \max\{k + r - n, 0\}, \ldots, r$). These blocks can be regarded as j new elements. Then $n + j$ elements have to be partitioned into $k + r$ subsets such that these new elements are in distinct blocks. It means that for a given j, the number of partitions is $\left\{ \begin{matrix} r \\ j \end{matrix} \right\} \left\{ \begin{matrix} n \\ k+r-j \end{matrix} \right\}_j$. $\square$

Theorem 4.3. *If ξ is a Poisson random variable with parameter $\lambda > 0$, $n \geq 1$ and $r \geq 0$, then*

$$\mathsf{E}(\xi + r)^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r \lambda^k.$$

Proof. First, it follows from Theorem 2.4 that

$$\mathsf{E} p_{R_{n,r}}(\xi) = \sum_{k=0}^{n+r} \left\{ \begin{matrix} R_{n,r} \\ k \end{matrix} \right\} \lambda^k = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r \lambda^{k+r} = \lambda^r \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r \lambda^k.$$

On the other hand

$$\mathsf{E} p_{R_{n,r}}(\xi) = \mathsf{E}(\xi^n \xi^{\underline{r}}) = \sum_{j=0}^{\infty} j^n j^{\underline{r}} \frac{\lambda^j}{j!} e^{-\lambda} = \sum_{j=r}^{\infty} j^n \frac{\lambda^j}{(j-r)!} e^{-\lambda} =$$

$$\lambda^r \sum_{j=0}^{\infty} (j+r)^n \frac{\lambda^j}{j!} e^{-\lambda} = \lambda^r \mathsf{E}(\xi + r)^n.$$

$\square$

Theorem 4.4. *If $n, r \geq 0$, then*

$$B_{n,r} = \frac{1}{e} \sum_{j=0}^{\infty} \frac{(j+r)^n}{j!}.$$

Proof. It is easy to check for $n = r = 0$. While applying Theorem 2.5, we get

$$B_{n,r} = B_{R_{n,r}} = \frac{1}{e}\sum_{j=0}^{\infty}\frac{j^n j^{\underline{r}}}{j!} = \frac{1}{e}\sum_{j=r}^{\infty}\frac{j^n}{(j-r)!} = \frac{1}{e}\sum_{j=0}^{\infty}\frac{(j+r)^n}{j!}.$$

□

Theorem 4.5. *If $n, k, r \geq 0$, $n + r \geq 1$ and $k \leq n$, then*

$$\left\{ {n \atop k} \right\}_r = \frac{1}{(k+r)!}\sum_{j=0}^{k+r}(-1)^j\binom{k+r}{j}(k+r-j)^n(k+r-j)^{\underline{r}}.$$

Proof. This immediately follows from Theorem 2.6 for the graph $R_{n,r}$. □

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