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On the density of sumsets and product sets

Norbert Hegyvári*

ELTE TTK, Eötvös University, Institute of Mathematics, H-1117
Pázmány st. 1/c, Budapest
Hungary
hegyvari@elte.hu

François Hennecart

Univ. Jean-Monnet, Institut Camille Jordan CNRS 5208
23 rue Michelon, 42023 Saint-Étienne cedex 2
France
francois.hennecart@univ-st-etienne.fr

Péter Pál Pach†

Budapest University of Technology and Economics
1117 Budapest, Magyar tudósok körútja 2
Hungary
ppp@cs.bme.hu

Abstract

In this paper some links between the density of a set of integers and the density of its sumset, product set and set of subset sums are presented.

1 Introduction and notation

In the field of additive combinatorics a popular topic is to compare the densities of different sets (of, say, positive integers). The well-known theorem of Kneser gives a description of the sets A having lower density α such that the density of $A+A := \{a+b : a,b \in A\}$ is less than 2α (see for instance [9]). The analogous question with the product set $A^2 := \{ab : a,b \in A\}$ is apparently more complicated.

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For any set $A \subset \mathbb{N}$ of natural numbers, we define the lower asymptotic density $\underline{\mathbf{d}}A$ and the upper asymptotic density $\overline{\mathbf{d}}A$ in the natural way:

$$\underline{\mathbf{d}}A = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n}, \quad \overline{\mathbf{d}}A = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

If the two values coincide, then we denote by $\mathbf{d}A$ the common value and call it the *asymptotic density* of A.

Throughout the paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will use the notion $A(x) = \{n \in A : n \le x\}$ for $A \subseteq \mathbb{N}$ and $x \in \mathbb{R}$. For functions $f, g : \mathbb{N} \to \mathbb{R}_+$ we write $f = O(g)$ (or $f \ll g$), if there exists some $c > 0$ such that $f(n) \le cg(n)$ for large enough n.

In Section 2 we investigate the connection between the (upper-, lower-, and asymptotic) density of a set of integers and the density of its sumset. In Section 3 we give a partial answer to a question of Erdős by giving a necessary condition for the existence of the asymptotic density of the set of subset sums of a given set of integers. Finally, in Section 4 we consider analogous problems for product sets.

2 Density of sumsets

For subsets A, B of integers the sumset $A + B$ is defined to be the set of all sums $a + b$ with $a \in A$, $b \in B$. For $A \subseteq \mathbb{N}_0$ the following clearly hold:

$$\begin{aligned} \underline{\mathbf{d}}A &\le \overline{\mathbf{d}}A, \\ \underline{\mathbf{d}}A &\le \underline{\mathbf{d}}(A + A), \\ \overline{\mathbf{d}}A &\le \overline{\mathbf{d}}(A + A). \end{aligned}$$

We shall assume that our sets A are normalized in the sense that A contains 0 and $\gcd(A) = 1$.

First observe that there exists a set of integers A not having an asymptotic density such that its sumset $A + A$ has a density: for instance $A = \{0\} \cup \bigcup_{n\ge 0} [2^{2n}, 2^{2n+1}]$ has lower density $1/3$, upper density $2/3$ and its sumset $A + A$ has density 1, since it contains every nonnegative integer. For this kind of sets A, we denote respectively

$$\begin{aligned} \underline{\mathbf{d}}A &=: \alpha_A, \\ \overline{\mathbf{d}}A &=: \beta_A, \\ \mathbf{d}(A + A) &=: \gamma_A, \\ (\alpha_A, \beta_A, \gamma_A) &=: p_A, \end{aligned}$$

and we have

$$\alpha_A \le \beta_A \le \gamma_A.$$

The first question arising from this is to decide whether or not for any $p = (\alpha, \beta, \gamma)$ such that $0 \le \alpha \le \beta \le \gamma \le 1$ there exists a set A of integers such that $p = p_A$. This question has no positive answer in general, though the following weaker statement holds.

Proposition 2.1 *Let $0 \le \alpha \le 1$. There exists a normalized set $A \subset \mathbb{N}$ such that $\mathbf{d}A = \alpha$ and $\mathbf{d}(A+A) = 1$.*

Proof: Let $0 \in B$ be a thin additive basis (of order 2), that is, a basis containing 0 and satisfying $|B(x)| = o(x)$ as $x \to \infty$. For $\alpha = 0$ the choice $A = B$ is fine. For $\alpha > 0$ let $A = B \cup \{\lfloor n/\alpha \rfloor,\ n \ge 1\}$. Then A is a normalized set satisfying $A+A = \mathbb{N}_0$ and $\mathbf{d}A = \alpha$.

(Note that $B = \{0, 1, 2, \dots, \lfloor 1/\alpha \rfloor\}$ is also an appropriate choice for B in the case $\alpha > 0$.) ☐

Remark 1 We shall mention that Faisant et al. [1] proved the following related result: for any $0 \le \alpha \le 1$ and any positive integer k, there exists a sequence A such that $\mathbf{d}(jA) = j\alpha/k$, $j = 1, \dots, k$, where jA denotes the j-fold sumset $A + A + \cdots + A$ (j times). Well before that in [11, Theorem 2] the author established that for any positive real numbers $\alpha_1, \dots, \alpha_k, \beta$ satisfying $\sum_{i=1}^{k} \alpha_i \le \beta \le 1$ there exist sets $A_1, \dots, A_k$ such that $\mathbf{d}A_i = \alpha_i$ $(1 \le i \le k)$ and $\mathbf{d}(A_1 + \cdots + A_k) = \beta$.

After a conjecture stated by Pichorides, the related question about the characterisation of the two-dimensional domains $\{(\underline{\mathbf{d}}B, \overline{\mathbf{d}}B) : B \subset A\}$ has been solved (see [3] and [6]).

Note that if the density γ_A exists, then α_A, β_A and γ_A have to satisfy some strong conditions. For instance, by Kneser's theorem, we know that if for some set A we have $\gamma_A < 2\alpha_A$, then $A + A$ is, except possibly a finite number of elements, a union of arithmetic progressions in $\mathbb{N}$ with the same difference. This implies that γ_A must be a rational number. From the same theorem of Kneser, we also deduce that if $\gamma_A < 3\alpha_A/2$, then $A + A$ is an arithmetic progression from some point onward. It means that γ_A is a unit fraction, hence A contains any sufficiently large integer, if we assume that A is normalized.

Another strong connection between α_A and γ_A can be deduced from Freiman's theorem on the addition of sets (cf. [2]). Namely, every normalized set A satisfies

$$\gamma_A \ge \frac{\alpha_A}{2} + \min\left(\alpha_A, \frac{1}{2}\right).$$

A related but more surprising statement is the following:

Proposition 2.2 *There is a set of positive integers for which $\mathbf{d}(A)$ does exist and $\mathbf{d}(A+A)$ does not exist.*

Proof: Let us take $U = \{0, 2, 3\}$ and $V = \{0, 1, 2\}$, then observe that

$$U + (U \cup V) = \{0, 1, 2, 3, 4, 5, 6\} \quad V + (U \cup V) = \{0, 1, 2, 3, 4, 5\}.$$

Let $(N_k)_{k \ge 0}$ be a sufficiently quickly increasing sequence of integers with $N_0 = 0$, $N_1 = 1$, and define A by

$$A = (U \cup V) \cup \bigcup_{k \ge 1} \Big((U + 7\mathbb{Z}) \cap [7N_{2k}, 7N_{2k+1}] \cup (V + 7\mathbb{Z}) \cap [7N_{2k+1}, 7N_{2k+2}] \Big).$$

Then A has density $3/7$. Moreover, for any $k \geq 0$

$$[7N_{2k}, 7N_{2k+1}] \subset A + A,$$

thus $\overline{\mathbf{d}}(A + A) = 1$, if we assume $\lim_{k\to\infty} N_{k+1}/N_k = \infty$.
We also have

$$(A + A) \cap [14N_{2k-1}, 7N_{2k}] = (\{0, 1, 2, 3, 4, 5\} + 7\mathbb{N}) \cap [14N_{2k-1}, 7N_{2k}],$$

hence $\underline{\mathbf{d}}(A + A) = 6/7$ using again the assumption that $\lim_{k\to\infty} N_{k+1}/N_k = \infty$. □

For any set A having a density, let

$$\begin{aligned} \mathbf{d}A &=: \alpha_A, \\ \underline{\mathbf{d}}(A + A) &=: \underline{\gamma}_A, \\ \overline{\mathbf{d}}(A + A) &=: \overline{\gamma}_A, \\ (\alpha_A, \underline{\gamma}_A, \overline{\gamma}_A) &=: q_A; \end{aligned}$$

then we have

$$\alpha_A \leq \underline{\gamma}_A \leq \overline{\gamma}_A.$$

A question similar to the one asked for p_A can be stated as follows: given $q = (\alpha, \underline{\gamma}, \overline{\gamma})$ such that $0 \leq \alpha \leq \underline{\gamma} \leq \overline{\gamma} \leq 1$, does there exist a set A such that $q = q_A$?

We further mention an interesting question of Ruzsa: does there exist $0 < \nu < 1$ and a constant $c = c(\nu) > 0$ such that for any set A having a density,

$$\underline{\mathbf{d}}(A + A) \geq c \cdot (\overline{\mathbf{d}}(A + A))^{1-\nu}(\mathbf{d}A)^{\nu}\,?$$

Ruzsa proved (unpublished) that in case of an affirmative answer, we necessarily have $\nu \geq 1/2$.

3 Density of subset sums

Let $A = \{a_1 < a_2 < \cdots\}$ be a sequence of positive integers. Denote the set of all subset sums of A by

$$P(A) := \Big\{ \sum_{i=1}^{k} \varepsilon_i a_i \,:\, k \geq 0,\ \varepsilon_i \in \{0, 1\}\ (1 \leq i \leq k) \Big\}.$$

Zannier conjectured and Ruzsa proved that the condition $a_n \leq 2a_{n-1}$ implies that the density $\mathbf{d}(P(A))$ exists (see [8]). Ruzsa also asked the following questions:

i) Is it true that for every pair of real numbers $0 \leq \alpha \leq \beta \leq 1$, there exists a sequence of integers for which $\underline{\mathbf{d}}(P(A)) = \alpha$; $\overline{\mathbf{d}}(P(A)) = \beta$? This question was answered positively in [5].

ii) Is it true that the condition $a_n \le a_1 + a_2 + \cdots + a_{n-1} + c$ also implies that $\mathbf{d}(P(A))$ exists?

We shall prove the following statement.

Proposition 3.1 *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers. Assume that for some function θ satisfying $\theta(k) \ll \frac{k}{(\log k)^2}$ we have*

$$|a_n - s_{n-1}| = \theta(s_{n-1}) \text{ for every } n,$$

where $s_{n-1} := a_1 + a_2 + \cdots + a_{n-1}$.
Then $\mathbf{d}(P(A))$ exists.

Proof: We first prove that there exists a real number δ such that

$$|P(A)(s_n)| = \big(\delta + o(1)\big)s_n \quad \text{as } n \to \infty.$$

Let $n \ge 2$ be large enough. Then

$$P(A) \cap [1, s_n] = \Big(P(A) \cap [1, s_{n-1}]\Big) \cup \Big(P(A) \cap (s_{n-1}, s_n - \theta(s_{n-1}))\Big).$$

Since $a_n \ge s_{n-1} - \theta(s_{n-1})$, we have $P(A) \cap (s_{n-1}, s_n] \supseteq a_n + P(A) \cap (\theta(s_{n-1}), s_{n-1}]$, and thus

$$\Big|P(A) \cap [1, s_n]\Big| \ge 2\Big|P(A) \cap [1, s_{n-1}]\Big| - 2\theta(s_{n-1}) - 1. \tag{1}$$

On the other hand,

$$P(A) \cap [1, s_n] \subseteq (P(A) \cap [1, s_{n-1}]) \cup (a_n + P(A) \cap [1, s_{n-1}]) \cup [s_n - \theta(s_n), s_n],$$

since $a_{n+1} \ge s_n - \theta(s_n)$. Therefore,

$$\Big|P(A) \cap [1, s_n]\Big| \le 2\Big|P(A) \cap [1, s_{n-1}]\Big| + \theta(s_n) + 1. \tag{2}$$

Observe that $s_n = a_n + s_{n-1} \le 2s_{n-1} + \theta(s_{n-1})$; hence letting

$$\delta_n = \frac{\Big|P(A) \cap [1, s_n]\Big|}{s_n},$$

we obtain from (1) and (2) that

$$\delta_n - \delta_{n-1} = O\left(\frac{\theta(s_n)}{s_n}\right). \tag{3}$$

Now, we show that $s_n \gg 2^n$. Since

$$s_n = s_{n-1} + a_n \ge 2s_{n-1} - \theta(s_{n-1}) = s_{n-1}\left(2 - \frac{\theta(s_{n-1})}{s_{n-1}}\right), \tag{4}$$

the condition $\theta(k) \ll \frac{k}{(\log k)^2}$ implies that from (4) we obtain that $s_n \gg 1.5^n$. Therefore, in fact, for large enough n we have $s_n \geq s_{n-1}\left(2 - \frac{c}{n^2}\right)$ with some $c > 0$. Now, let $10c < K$ be a fixed integer. For $K < n$ we have

$$s_n \geq s_K \prod_{i=K+1}^{n} \left(2 - \frac{c}{i^2}\right) \geq s_K \left[2^{n-k} - 2^{n-k-1} \sum_{i=K+1}^{n} \frac{c}{i^2}\right] \gg 2^n,$$

since $\sum_{i=K+1}^{n} \frac{c}{i^2} < 1/10$. Hence, $s_n \gg 2^n$ indeed holds.

Therefore, using the assumption on θ we obtain that $\frac{\theta(s_n)}{s_n} \ll \frac{1}{n^2}$. So (3) yields that

$$\delta_n - \delta_{n-1} = O(n^{-2}).$$

Therefore, the sequence δ_n has a limit which we denote by δ. Furthermore, observe that

$$\delta_n = \delta + O(1/n). \tag{5}$$

The next step is to consider an arbitrary sufficiently large positive integer x and decompose it as

$$x = a_{n_1+1} + a_{n_2+1} + \cdots + a_{n_j+1} + z,$$

where $n_1 > n_2 > \cdots > n_j > k$ and $0 \leq z$ are defined in the following way. (Here k is a fixed, sufficiently large positive integer.) The index n_1 is chosen in such a way that $a_{n_1+1} \leq x < a_{n_1+2}$. If $x - a_{n_1+1} \geq a_{n_1}$, then $n_2 = n_1 - 1$, otherwise n_2 is the largest index for which $x - a_{n_1+1} \geq a_{n_2+1}$. The indices $n_3, n_4, \ldots$ are defined similarly. We stop at the point when the next index would be at most k and set $z := x - a_{n_1+1} - a_{n_2+1} - \cdots - a_{n_j+1}$. As $z \leq \theta(s_{n_1+1}) + s_k$, we have

$$z = o(x). \tag{6}$$

Furthermore, let

$$b_i = a_{n_1+1} + a_{n_2+1} + \cdots + a_{n_i+1}, \quad i = 0, 1, \ldots, j.$$

(The empty sum is $b_0 := 0$, as usual.)

Let $X_0 := (0, s_{n_1} - \theta(s_{n_1}))$ and for $1 \leq i \leq j-1$ let $X_i := (b_i + \theta(s_{n_i}), b_i + s_{n_{i+1}} - \theta(s_{n_{i+1}}))$ and consider

$$X := X_0 \cup X_1 \cup \cdots \cup X_{j-1} =$$
$$(0, s_{n_1} - \theta(s_{n_1})) \cup (b_1 + \theta(s_{n_1}), b_1 + s_{n_2} - \theta(s_{n_2})) \cup \cdots \cup (b_{j-1} + \theta(s_{n_{j-1}}), b_{j-1} + s_{n_j} - \theta(s_{n_j})).$$

Note that in this union each element appears at most once, since according to the definition of θ the sets $X_0, X_1, \ldots, X_{j-1}$ are pairwise disjoint as

$$b_i + s_{n_{i+1}} - \theta(s_{n_{i+1}}) \leq b_{i+1} = b_i + a_{n_{i+1}+1}$$

holds for every $0 \le i \le j-2$.

The set of those elements of $[1,x]$ that are not covered by X is:

$$\begin{aligned}[1,x]\setminus X = [s_{n_1}-\theta(s_{n_1}), b_1+\theta(s_{n_1})] \cup [b_1+s_{n_2}-\theta(s_{n_2}), b_2+\theta(s_{n_2})] \cup \dots \\ \cup [b_{j-2}+s_{n_{j-1}}-\theta(s_{n_{j-1}}), b_{j-1}+\theta(s_{n_{j-1}})] \cup [b_{j-1}+s_{n_j}-\theta(s_{n_j}), x].\end{aligned}$$

Therefore,

$$|[1,x]\setminus X| \le 3\sum_{i=1}^{j}\theta(s_{n_i}) + z.$$

Using $\sum\limits_{i=1}^{j}\theta(s_{n_i}) \ll \sum\limits_{i=1}^{j}\frac{s_{n_i}}{n_i^2} \ll \frac{x}{k^2}$ and (6), we obtain that $|[1,x]\setminus X| \le (\varepsilon_k + o(1))x$, where $\varepsilon_k \to 0$ (as $k\to\infty$). (Note that $\varepsilon_k \ll 1/k^2$.)

That is, the set X covers $[1,x]$ with the exception of a "small" portion of size $O(x/k^2)$. Therefore, by letting $k\to\infty$ the density of the uncovered part tends to 0.

Let us consider $P(A)\cap X_i$. If a sum is contained in $P(A)\cap X_i$, then the sum of the elements with indices larger than n_{i+1} is b_i. Otherwise, the sum is either at most $b_i+\theta(s_{n_i})$ or at least $b_i+s_{n_{i+1}}-\theta(s_{n_{i+1}})$.

Therefore $P(A)\cap X_i = (b_i + P(\{a_1,a_2,\dots,a_{n_{i+1}}\}))\cap X_i$.

Hence

$$\delta_{n_{i+1}}s_{n_{i+1}} - 2\theta(s_{n_{i+1}}) - 1 \le |P(A)\cap X_i| \le \delta_{n_{i+1}}s_{n_{i+1}}.$$

Therefore

$$\begin{aligned}&|P(A)\cap[x]| \ge \sum_{i=0}^{j-1}\big(\delta_{n_{i+1}}s_{n_{i+1}} - 2\theta(s_{n_{i+1}}) - 1\big)\\ &\ge \delta x - \delta z + \delta\sum_{i=0}^{j-1}(s_{n_{i+1}} - a_{n_{i+1}+1}) + \sum_{i=0}^{j-1}(\delta_{n_{i+1}}-\delta)s_{n_{i+1}} - 2\sum_{i=0}^{j-1}\big(\theta(s_{n_{i+1}})+1\big) \qquad (7)\end{aligned}$$

and

$$\begin{aligned}|P(A)\cap[x]| &\le \sum_{i=0}^{j-1}\delta_{n_{i+1}}s_{n_{i+1}}\\ &\le \delta x - \delta z + \delta\sum_{i=0}^{j-1}(s_{n_{i+1}} - a_{n_{i+1}+1}) + \sum_{i=0}^{j-1}(\delta_{n_{i+1}}-\delta)s_{n_{i+1}}.\end{aligned}$$

Now, observe that

- $|z| = o(x)$ by (6),
- $\sum\limits_{i=0}^{j-1}|s_{n_{i+1}} - a_{n_{i+1}+1}| = o(x)$, using $|s_{n_{i+1}} - a_{n_{i+1}+1}| = \theta(s_{n_{i+1}})$ and $\sum\limits_{i=0}^{j-1}a_{n_{i+1}+1} \le x$,

- $\sum_{i=0}^{j-1}(\delta_{n_{i+1}} - \delta)s_{n_{i+1}} \ll x/k$ by using (5). Letting $k \to \infty$ this term is also of size $o(x)$.

Hence we obtain from (7) and (8) that $|P(A) \cap [x]| = \delta x + o(x)$.

$\square$

4 Density of product sets

For any subsets $A, B \subseteq \mathbb{N}_0$, we denote by $A \cdot B$ the product set

$$AB = A \cdot B = \{ab \, : \, a \in A, b \in B\}.$$

For brevity, for $A = B$ we also write $A \cdot A = A^2$.

In this section we focus on the case $G = (\mathbb{N}, \cdot)$, the semigroup (for multiplication) of all positive integers. The restricted case $G = \mathbb{N} \setminus \{1\}$ is even more interesting, since $1 \in A$ implies $A \subset A^2$.

The sets of integers satisfying the small doubling hypothesis $\mathbf{d}(A + A) = \mathbf{d}A$ are well described through Kneser's theorem. The similar question for the product set does not plainly lead to a strong description. We can restrict our attention to sets A such that $\gcd(A) = 1$, since by setting $B := \frac{1}{\gcd A}A$ we have $\mathbf{d}A = \frac{1}{\gcd(A)}\mathbf{d}B$ and $\mathbf{d}A^2 = \frac{1}{(\gcd(A))^2}\mathbf{d}(B^2)$.

Examples 1 *i) Let A_{nsf} be the set of all non-squarefree integers. Letting $A = \{1\} \cup A_{\text{nsf}}$ we have $A^2 = A$ and*

$$\mathbf{d}A = 1 - \zeta(2)^{-1}.$$

ii) However, while $\gcd(A_{\text{nsf}}) = 1$*, we have*

$$\mathbf{d}A_{\text{nsf}}^2 < \mathbf{d}A_{\text{nsf}} = 1 - \zeta(2)^{-1}.$$

iii) Furthermore, the set A_{sf} of all squarefree integers satisfies

$$\mathbf{d}A_{\text{sf}} = \zeta(2)^{-1} \textit{ and } \mathbf{d}A_{\text{sf}}^2 = \zeta(3)^{-1},$$

since A_{sf}^2 consists of all cubefree integers.

iv) Given a positive integer k, the set $A_k = \big\{n \in \mathbb{N} \, : \, \gcd(n,k) = 1\big\}$ satisfies

$$A_k^2 = A_k \quad \textit{and} \quad \mathbf{d}A_k = \frac{\phi(k)}{k},$$

where ϕ is Euler's totient function.

We have the following result:

Proposition 4.1 *For any positive $\alpha < 1$ there exists a set $A \subset \mathbb{N}$ such that $\mathbf{d}A > \alpha$ and $\mathbf{d}A^2 < \alpha$.*

Proof: Assume first that $\alpha < 1/2$.
For $k \geq 1$ let $A_k = k\mathbb{N} = \{kn,\ n \geq 1\}$, then $A_k^2 = k^2\mathbb{N}$. Therefore, $\mathbf{d}A_k = 1/k$ and $\mathbf{d}(A_k^2) = 1/k^2$. If $1/(k+1) \leq \alpha < 1/k$, then A_k satisfies the requested condition. Since $\bigcup\limits_{k\geq 2} \left[\frac{1}{k+1}, \frac{1}{k}\right) = (0, 1/2)$, an appropriate k can be chosen for every $\alpha \in (0, 1/2)$.

Assume now that $1 > \alpha \geq 1/2$.
Let $p_1 < p_2 < \cdots$ be the increasing sequence of prime numbers and

$$B_r := \bigcup_{i=1}^{r} p_i\mathbb{N}.$$

The complement of the set B_r contains exactly those positive integers that are not divisible by any of $p_1, p_2, \ldots, p_r$, thus we have

$$\mathbf{d}(B_r) = 1 - \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) =: \gamma_r.$$

Similarly, the complement of the set B_r^2 contains exactly those positive integers that are not divisible by any of $p_1, \ldots, p_r$ or can be obtained by multiplying such a number by one of $p_1, \ldots, p_r$. Hence, we obtain that

$$\mathbf{d}(B_r^2) = 1 - \left(1 + \sum_{i=1}^{r} \frac{1}{p_i}\right) \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) =: \beta_r.$$

Note that

$$\beta_{r+1} = 1 - \left(1 + \sum_{i=1}^{r+1} \frac{1}{p_i}\right)\left(1 - \frac{1}{p_{r+1}}\right) \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) < 1 - \frac{3}{2} \cdot \frac{2}{3} \cdot \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) = \gamma_r. \tag{8}$$

As $(\beta_1, \gamma_1) = (1/4, 1/2)$, moreover $(\beta_r)_{r=1}^{\infty}$ and $(\gamma_r)_{r=1}^{\infty}$ are increasing sequences satisfying (8) and $\lim \gamma_r = 1$, we obtain that $[1/2, 1)$ is covered by $\bigcup\limits_{r=1}^{\infty} (\beta_r, \gamma_r)$. That is, for every $\alpha \in [1/2, 1)$ we have $\alpha \in (\beta_r, \gamma_r)$ for some r, and then $A = B_r$ is an appropriate choice.

□

We pose two questions about the densities of A and A^2.

Question 1 *If $1 \in A$ and $\mathbf{d}A = 1$, then $\mathbf{d}(A^2) = 1$, too. Given two integers k, ℓ, the set*

$$\{n \in \mathbb{N} \,:\, \gcd(n, k) = 1\} \cup k\ell\mathbb{N}$$

is multiplicatively stable. What are the sets A of positive integers such that $A^2 = A$ or less restrictively

$$1 \in A \text{ and } 1 > \mathbf{d}A^2 = \mathbf{d}A > 0\,?$$

Question 2 *It is clear that* $\mathbf{d}A > 0$ *implies* $\mathbf{d}A^2 > 0$*, since* $A^2 \supset (\min A)A$*.*
For any $\alpha \in (0,1)$ *we denote*

$$f(\alpha) := \inf_{A \subset \mathbb{N};\ \mathbf{d}A = \alpha} \mathbf{d}A^2.$$

Is it true that $f(\alpha) = 0$ *for any* α *or at least for* $\alpha < \alpha_0$ *?*

The next result shows that the product set of a set having density 1 and satisfying a technical condition must also have density 1.

Proposition 4.2 *Let* A*, with* $1 \notin A$*, be a set of positive integers with asymptotic density* $\mathbf{d}A = 1$*. Furthermore, assume that* A *contains an infinite subset of mutually coprime integers* $a_1 < a_2 < \cdots$ *such that*

$$\sum_{i \geq 1} \frac{1}{a_i} = \infty.$$

Then the product set A^2 *also has density* $\mathbf{d}(A^2) = 1$*.*

Proof: Let $\varepsilon > 0$ be arbitrary and choose a large enough k such that

$$\sum_{i=1}^{k} \frac{1}{a_i} > -\log \varepsilon. \tag{9}$$

Let x be a large integer. For any $i = 1, \ldots, k$, the set $A^2(x)$ contains all the products $a_i a$ with $a \leq x/a_i$. We shall use a sieve argument. Let A' be a finite subset of A and $X = [1, x] \cap \mathbb{N}$ for some $x > \max(A')$. For any $a \in A'$, let

$$X_a = \left\{ n \leq x \ : \ a \nmid n \text{ or } \frac{n}{a} \notin A \right\}.$$

Observe that

$$X \setminus X_a = (aA)(x).$$

Then

$$(A' \cdot A)(x) = \bigcup_{a \in A'} (X \setminus X_a).$$

By the inclusion-exclusion principle we obtain

$$|(A' \cdot A)(x)| = \sum_{k=1}^{|A'|} (-1)^{j-1} \sum_{\substack{B \subseteq A' \\ |B| = j}} \Big| \bigcap_{b \in B} (X \setminus X_b) \Big|,$$

whence

$$\Big| \bigcap_{a \in A'} X_a \Big| = \sum_{j=0}^{|A'|} (-1)^j \sum_{\substack{B \subseteq A' \\ |B| = j}} \Big| \bigcap_{b \in B} (X \setminus X_b) \Big|, \tag{10}$$

where the empty intersection $\bigcap_{b\in\emptyset}(X\setminus X_b)$ denotes the full set X.

For any finite set of integers B we denote by $\mathrm{lcm}(B)$ the least common multiple of the elements of B. Now, we consider

$$\bigcap_{b\in B}(X\setminus X_b)=\Big\{n\le x\,:\,\mathrm{lcm}(B)\mid n \text{ and } \frac{n}{b}\in A\ (\forall b\in B)\Big\}.$$

By the assumption $\mathbf{d}A=1$ we immediately get

$$\Big|\bigcap_{b\in B}(X\setminus X_b)\Big|=\frac{x}{\mathrm{lcm}(B)}(1+o(1)).$$

Plugging this into (10):

$$\Big|\bigcap_{a\in A'(x)}X_a\Big|=x\sum_{k=0}^{|A'|}(-1)^j\sum_{\substack{B\subseteq A'\\|B|=j}}\frac{1}{\mathrm{lcm}(B)}+o(x).$$

Since the elements of $A'=\{a_1,a_2,\dots,a_k\}$ are mutually coprime,

$$x-|A'\cdot A(x)|=x\sum_{j=0}^{k}(-1)^j\sum_{1\le a_{i_1}<\dots<a_{i_j}\le k}\frac{1}{a_{i_1}a_{i_2}\dots a_{i_j}}+o(x)=x\prod_{i=1}^{k}\Big(1-\frac{1}{a_i}\Big)+o(x).$$

(Note that for $j=0$ the empty product is defined to be 1, as usual.) Since $1-u\le\exp(-u)$ we get

$$x-|A'\cdot A(x)|\le x\exp\Big(-\sum_{i=1}^{k}\frac{1}{a_i}\Big)+o(x)<\varepsilon x+o(x)$$

by our assumption (9). Thus finally

$$|A^2(x)|\ge|A'\cdot A(x)|>x(1-\varepsilon-o(1)).$$

This ends the proof. □

Remark 2 *Specially, the preceding result applies when A contains a sequence of prime numbers $p_1<p_2<\cdots$ such that $\sum_{i\ge1}1/p_i=\infty$. For this it is enough to assume that*

$$\liminf_{i\to\infty}\frac{i\log i}{p_i}>0.$$

However, we do not know how to avoid the assumption on the mutually coprime integers having infinite reciprocal sum. We thus pose the following question:

Question 3 *Is it true that $\mathbf{d}A=1$ implies $\mathbf{d}(A^2)=1$?*

An example for a set A such that $\mathbf{d}(A) = 0$ and $\mathbf{d}(A^2) = 1$.

According to the fact that the multiplicative properties of the elements play an important role, one can build a set whose elements are characterized by their number of prime factors. Let

$$A = \left\{ n \in \mathbb{N} \, : \, \Omega(n) \leq 0.75 \log\log n + 1 \right\},$$

where $\Omega(n)$ denotes the number of prime factors (with multiplicity) of n. An appropriate generalisation of the Hardy-Ramanujan theorem (cf. [4] and [10]) shows that the normal order of $\Omega(n)$ is $\log\log n$ and the Erdős-Kac theorem asserts that

$$\mathbf{d} \left\{ n \in \mathbb{N} \, : \, \alpha < \frac{\Omega(n) - \log\log n}{\sqrt{\log\log n}} < \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt,$$

which implies $\mathbf{d}A = 0$. Now we prove that $\mathbf{d}A^2 = 1$. The principal feature in the definition of A is that A^2 must contain almost all integers n such that $\omega(n) \leq 1.2 \log\log n$.

For $n \in \mathbb{N}$ let

$$P_+(n) := \max \left\{ p : p \text{ is a prime divisor of } n \right\}.$$

Let us consider first the density of the integers n such that

$$P_+(n) > n \exp(-(\log n)^{4/5}). \tag{11}$$

Let x be a large number and write

$$\begin{aligned} &\left| \left\{ n \leq x \, : \, P_+(n) \leq n \exp(-(\log n)^{4/5}) \right\} \right| \\ &\qquad = \left| \left\{ n \leq x \, : \, P_+(n) \leq x \exp(-(\log x)^{4/5}) \right\} \right| + o(x). \end{aligned}$$

By a theorem of Hildebrand (cf. [7]) on the estimation of $\Psi(x, z)$, the number of z-friable integers up to x, we conclude that the above cardinality is $x + o(x)$. Hence, we may avoid the integers n satisfying (11). By the same estimation we may also avoid those integers n for which $P_+(n) < \exp((\log n)^{4/5})$.

Let n be an integer such that $\Omega(n) \leq 1.2 \log\log n$ and

$$\exp((\log n)^{4/5}) \leq P_+(n) \leq n \exp(-(\log n)^{4/5}).$$

Our goal is to find a decomposition $n = n_1 n_2$ with $\Omega(n_i) \leq 0.75 \log\log n_i + 1$, $i = 1, 2$.

Let

$$n = p_1 p_2 \dots p_{t-1} P_+(n),$$

where $t = \Omega(n)$. We also assume that $p_1 \leq p_2 \leq \dots \leq p_{t-1} \leq P_+(n)$. Let $m = \frac{n}{P_+(n)}$. Then

$$\exp((\log n)^{4/5}) \leq m \leq n \exp(-(\log n)^{4/5}).$$

Let

$$n_1 = p_1 p_2 \dots p_{u-1} P_+(n) \text{ and } n_2 = p_u \dots p_{t-1},$$

where $u = \lfloor (t-1)/2 \rfloor$. Then $n_2 \geq \sqrt{m}$, which yields

$$\log\log n_2 \geq \log\log m - \log 2 \geq 0.8 \log\log n - \log 2.$$

On the other hand,

$$\Omega(n_2) = t - u \leq \frac{t}{2} + 1 \leq 0.6 \log\log n + 1 \leq 0.75 \log\log n_2 + \frac{3\log 2}{4}.$$

Now $n_1 \geq P_+(n) \geq \exp((\log n)^{4/5})$, hence

$$\log\log n_1 \geq 0.8 \log\log n$$

and

$$\Omega(n_1) \leq \frac{t-1}{2} \leq 0.6 \log\log n \leq 0.75 \log\log n_1.$$

Therefore, the following statement is obtained:

Proposition 4.3 *The set*

$$A = \left\{ n \in \mathbb{N} : \Omega(n) \leq 0.75 \log\log n + 1 \right\}$$

has density 0 *and its product set* A^2 *has density* 1.

By a different approach we may extend the above result as follows.

Theorem 4.4 *For every* α *and* β *with* $0 \leq \alpha \leq \beta \leq 1$, *there exists a set* $A \subseteq \mathbb{N}$ *such that* $\mathbf{d}A = 0$, $\underline{\mathbf{d}}(A \cdot A) = \alpha$ *and* $\overline{\mathbf{d}}(A \cdot A) = \beta$.

Proof: We start with defining a set Q such that $\mathbf{d}(Q) = 0$ and $\mathbf{d}(Q \cdot Q) = \beta$. Let us choose a subset P_0 of the primes such that $\prod_{p \in P_0} (1 - 1/p) = \beta$. Such a subset can be chosen, since $\sum 1/p = \infty$. Now, let p_k denote the k-th prime and let

$$P_1 = \{p_i : i \text{ is odd}\} \setminus P_0,$$

$$P_2 = \{p_i : i \text{ is even}\} \setminus P_0.$$

Furthermore, let

$$Q_1 = \{n : \text{all prime divisors of } n \text{ belong to } P_1\}$$

and

$$Q_2 = \{n : \text{all prime divisors of } n \text{ belong to } P_2\}.$$

Let $Q = Q_1 \cup Q_2$. Clearly, $Q \cdot Q = Q_1 \cdot Q_2$ contains exactly those numbers that do not have any prime factor in P_0, so $\mathbf{d}(Q \cdot Q) = \beta$. For $i \in \{1, 2\}$ and $x \in \mathbb{R}$ the probability that an integer does not have any prime factor being less than x from P_i

is $\prod_{p<x, p\in P_i}(1-1/p) \le \frac{1}{\beta}\prod_{p<x, p\in P_i\cup P_0}(1-1/p) \le \frac{1}{\beta}\exp\left\{-\sum_{\substack{j:\ p_j<x,\\ j\equiv i \pmod 2}}\frac{1}{p_j}\right\} = O\left(\frac{1}{\beta\sqrt{\log x}}\right)$.

Therefore, $\mathbf{d}(Q_1) = \mathbf{d}(Q_2) = 0$, and consequently $\mathbf{d}(Q) = 0$ also holds. If $\alpha = \beta$, then $A = Q$ satisfies the conditions. From now on let us assume that $\alpha < \beta$.

Our aim is to define a subset $A \subseteq Q$ in such a way that $\underline{\mathbf{d}}(A\cdot A) = \alpha$ and $\overline{\mathbf{d}}(A\cdot A) = \beta$. As $A \subseteq Q$ we will have $\mathbf{d}(A) = 0$ and $\overline{\mathbf{d}}(A\cdot A) \le \beta$. The set A is defined recursively. We will define an increasing sequence of integers $(n_j)_{j=1}^{\infty}$ and sets A_j ($j \in \mathbb{N}$) satisfying the following conditions (and further conditions to be specified later):

(i) $A_j \subseteq A_{j-1}$,

(ii) $A_j \cap [1, n_{j-1}] = A_{j-1} \cap [1, n_{j-1}]$,

(iii) $A_j \cap [n_j + 1, \infty] = Q \cap [n_j + 1, \infty]$.

That is, A_j is obtained from A_{j-1} by dropping out some elements of A_{j-1} in the range $[n_{j-1} + 1, n_j]$. Finally, we set $A = \bigcap_{j=1}^{\infty} A_j$.

Let $n_1 = 1$ and $A_1 = Q$. We define the sets A_j in such a way that the following condition holds for every j with some n_0 depending only on Q:

$$(*) \qquad |(A_j \cdot A_j)(n)| \ge \alpha n \text{ for every } n \ge n_0.$$

Since $d(Q \cdot Q) = \beta > \alpha$, a threshold n_0 can be chosen in such a way that $(*)$ holds for $A_1 = Q$ with this choice of n_0. Now, assume that n_j and A_j are already defined for some j. We continue in the following way depending on the parity of j:

Case I: j is odd.

Let $n_j < s$ be the smallest integer such that

$$|(A_j \setminus [n_j + 1, s]) \cdot (A_j \setminus [n_j + 1, s])(n)| < \alpha n$$

for some $n \ge n_0$. We claim that such an s exists, indeed it is at most $\lfloor n_j^2/\alpha\rfloor + 1$. For $s' = \lfloor n_j^2/\alpha\rfloor + 1$ we have

$$|(A_j \setminus [n_j + 1, s']) \cdot (A_j \setminus [n_j + 1, s'])(s')| \le n_j^2 < \alpha s'.$$

Hence, s is well-defined (and $s \le s'$). Let $n_{j+1} := s - 1$ and $A_{j+1} := A_j \setminus [n_j + 1, s - 1]$. (Specially, it can happen that $n_{j+1} = n_j$ and $A_{j+1} = A_j$.) Note that A_{j+1} satisfies $(*)$.

Case II: j is even.

Now, let $n_j < s$ be the smallest index for which $|(A_j \cdot A_j)(s)| > (\beta - 1/j)s$.

We have $\mathbf{d}(Q \cdot Q) = \beta$ and A_j is obtained from Q by deleting finitely many elements of it: $A_j = Q \setminus R$, where $R \subseteq [n_j]$. As $\mathbf{d}(Q) = 0$, we have that

$$|((Q \cdot Q) \setminus (Q \setminus R) \cdot (Q \setminus R))(n)| \leq |R|^2 + \sum_{r \in R} |Q(n/r)| = o(n),$$

therefore, $\mathbf{d}(A_j \cdot A_j) = \beta$. So for some $n > n_j$ we have that $(A_j \cdot A_j)(n) > (\beta - 1/j)n$, that is, s is well-defined. Let $n_{j+1} := s$ and $A_{j+1} = A_j$. Clearly, A_{j+1} satisfies $(*)$.

This way an increasing sequence $(n_j)_{j=1}^{\infty}$ and sets $A_j (j \in \mathbb{N})$ are defined; these satisfy conditions (i)–(iii). Finally, let us set $A := \bigcap_{j=1}^{\infty} A_j$. Note that $A(n_j) = A_j(n_j)$.

We have already seen that $A \subseteq Q$ implies that $\mathbf{d}(A) = 0$ and $\overline{\mathbf{d}}(A \cdot A) \leq \beta$. At first we show that $\underline{\mathbf{d}}(A \cdot A) \geq \alpha$. Let $n \geq n_0$ be arbitrary. If j is large enough, then $n_j > n$. As A_j satisfies $(*)$ and $(A \cdot A)(n) = (A_j \cdot A_j)(n)$ we obtain that

$$|(A \cdot A)(n)| = |(A_j \cdot A_j)(n)| \geq \alpha n.$$

This holds for every $n \geq n_0$, therefore, $\underline{\mathbf{d}}(A \cdot A) \geq \alpha$.

As a next step, we show that $\underline{\mathbf{d}}(A \cdot A) = \alpha$. Let j be odd. According to the definition of n_{j+1} and A_{j+1} there exists some $n \geq n_0$ such that

$$|((A_j \setminus \{n_{j+1} + 1\}) \cdot (A_j \setminus \{n_{j+1} + 1\}))(n)| < \alpha n.$$

For brevity, let $s := n_{j+1} + 1$. As $A \subseteq A_j$ we get that $|(A \setminus \{s\}) \cdot (A \setminus \{s\})(n)| < \alpha n$. Also,

$$|(A \cdot A) \setminus ((A \setminus \{s\}) \cdot (A \setminus \{s\})(n))| \leq 1 + |A(n/s)| \leq 1 + |Q(n/s)|,$$

since $A \subseteq Q$. Thus $|(A \cdot A)(n)| \leq \alpha n + 1 + |Q(n/s)| \leq n(\alpha + 1/n + 1/s)$. Clearly $s = n_{j+1} + 1 \leq n$, and as $j \to \infty$ we have $n_{j+1} \to \infty$, therefore $\underline{\mathbf{d}}(A \cdot A) = \alpha$.

Finally, we prove that $\overline{\mathbf{d}}(A \cdot A) = \beta$. Let j be even. According to the definition of A_{j+1} and n_{j+1}, we have $|(A_{j+1} \cdot A_{j+1})(n_{j+1})| > (\beta - 1/j)n_{j+1}$. However, $(A \cdot A)(n_{j+1}) = (A_{j+1} \cdot A_{j+1})(n_{j+1})$, therefore $\overline{\mathbf{d}}(A \cdot A) \geq \lim(\beta - 1/j) = \beta$, and thus $\overline{\mathbf{d}}(A \cdot A) = \beta$, as was claimed.

□

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