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Broadcast domination and multipacking: bounds and the integrality gap

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Abstract

The dual concepts of coverings and packings are well studied in graph theory. Coverings of graphs with balls of radius one and packings of vertices with pairwise distances at least two are the well-known concepts of domination and independence, respectively. In 2001, Erwin introduced *broadcast domination* in graphs, a covering problem using balls of various radii where the cost of a ball is its radius. The minimum cost of a dominating broadcast in a graph G is denoted by $\gamma_b(G)$. The dual (in the sense of linear programming) of broadcast domination is *multipacking*: a multipacking is a set $P \subseteq V(G)$ such that for any vertex $v \in V(G)$ and any positive integer r, the ball of radius r around v contains at most r vertices of P. The maximum size of a multipacking in a graph G is denoted by $\mathrm{mp}(G)$. Naturally, $\mathrm{mp}(G) \leq \gamma_b(G)$. Hartnell and Mynhardt proved that $\gamma_b(G) \leq 3\mathrm{mp}(G) - 2$ (whenever $\mathrm{mp}(G) \geq 2$). In this paper, we show that $\gamma_b(G) \leq 2\mathrm{mp}(G) + 3$. Moreover, we conjecture that this can be improved to $\gamma_b(G) \leq 2\mathrm{mp}(G)$ (which would be sharp).

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1 Introduction

All graphs in this paper are undirected, finite, and simple. The reader is referred to [2] for standard definitions and notation.

The dual concepts of coverings and packings are well studied in graph theory, see [6]. Coverings of graphs with balls of radius one and packings of vertices with pairwise distances at least two are the well-known concepts of domination and independence respectively, see [13]. Typically we are interested in minimum (cost) coverings and maximum (weight) packings. Two natural questions to ask are the following: For what graphs do these dual problems have equal (integer) values? In the case of non-equality, can we bound the difference between the two values? The second question is the focus of this paper.

The particular covering problem we study is broadcast domination. Let $G = (V, E)$ be a graph. Define the *ball of radius r around v* by

$$N_r[v] = \{u : d(u, v) \leq r\}.$$

A *dominating broadcast* of G is a collection of balls $N_{r_1}[v_1], N_{r_2}[v_2], \ldots, N_{r_t}[v_t]$ (each $r_i > 0$) such that $\bigcup_{i=1}^{t} N_{r_i}[v_i] = V$. Alternatively, a dominating broadcast is a function $f : V \to \mathbb{N}$ such that for any vertex $u \in V$, there is a vertex $v \in V$ with $f(v) > 0$ and $\mathrm{dist}(u, v) \leq f(v)$. A vertex v with $f(v) > 0$ can be thought of as the site from which the broadcast is transmitted with power $f(v)$. The ball $N_{f(v)}[v]$ is the set of vertices that hear the broadcast from v. (The ball $N_{f(v)}[v]$ belongs to the covering.) Vertices u for which $f(u) = 0$ do not broadcast and the trivial ball $N_0[u]$ is not included in the cover.

The *cost* of a dominating broadcast f is $\sum_{v \in V} f(v)$. The minimum cost of a dominating broadcast in G (taken over all dominating broadcasts) is the *broadcast domination number* of G, denoted by $\gamma_b(G)$.[1]

The broadcast domination number can be defined as an integer linear program:

$$\gamma_b(G) = \min\{cx \mid x_{(i,k)} \in \{0, 1\}, Ax \geq \mathbf{1}\}$$

where the vectors c, x and the columns of A are indexed by (i, k) for $i \in V(G)$ and $1 \leq k \leq \mathrm{diam}(G)$. The entry $c_{(i,k)} = k$. The entry $x_{(i,k)} = 1$ if and only if $N_k[i]$ is in the dominating broadcast, i.e. $f(i) = k$. Finally, the matrix $A = [a_{j,(i,k)}]$ is defined by $a_{j,(i,k)} = 1$ if vertex j belongs to $N_k[i]$ in G and is zero otherwise.

The dual to this problem is the *maximum multipacking problem* [3, 18]. A *multipacking* in a graph G is a subset $P \subseteq V(G)$ such that for any positive integer r and any vertex v in V, the ball of radius r centered at v contains at most r vertices of P. The maximum size of a multipacking of G, its *multipacking number*, is denoted by $\mathrm{mp}(G)$. That is,

$$\mathrm{mp}(G) = \max\{y\mathbf{1} \mid y_j \in \{0, 1\}, yA \leq c\}.$$

[1]One may consider the cost to be any function of the powers (for example the sum of the squares), see e.g. [14]. We shall stick to the classical convention of linear cost.

We observe that these two parameters are well defined for disconnected graphs; however, in the work below our bounds are based on the diameter of the graph in question. Hence, we shall restrict our attention to connected graphs with the observation that disconnected graphs can be studied component-wise.

Broadcast domination was introduced by Erwin [9, 10] in his doctoral thesis in 2001. Multipacking was then defined in Teshima's Master's Thesis [18] in 2012, see also [3] (and [4, 12, 19] for subsequent studies). This work fits into the general study of coverings and packings, which has a rich history in Graph Theory: Cornuéjols wrote a monograph on the topic [6].

In early work, Meir and Moon [17] studied various coverings and packings in trees, providing several inequalities relating the size of a minimum covering and a maximum packing. Giving such inequalities connecting the parameters γ_b and mp is the focus of our work. Since broadcast domination and multipacking are dual problems, we know that for any graph G,

$$\mathrm{mp}(G) \leq \gamma_b(G).$$

This bound is tight, in particular for strongly chordal graphs, see [11, 16, 18]. (In a recent companion work we prove equality for grids [1].) A natural question comes to mind. How far apart can these two parameters be? Hartnell and Mynhardt [12] gave a family of connected graphs $(G_k)_{k\in\mathbb{N}}$ for which the difference between both parameters is k. In other words, the difference can be arbitrarily large. Nonetheless, they proved that for any connected graph G with $\mathrm{mp}(G) \geq 2$,

$$\gamma_b(G) \leq 3\mathrm{mp}(G) - 2$$

and asked [12, Section 5] whether the factor 3 can be improved. Answering their question in the affirmative, our main result is the following.

Theorem 1.1. *Let G be a connected graph. Then,*

$$\gamma_b(G) \leq 2\mathrm{mp}(G) + 3.$$

Moreover, we conjecture that the additive constant in the bound of Theorem 1.1 can be removed.

Conjecture 1.2. *For any connected graph G, $\gamma_b(G) \leq 2\mathrm{mp}(G)$.*

In Section 2, we prove Theorem 1.1. In Section 3, we show that Conjecture 1.2 holds for all graphs with multipacking number at most 4. We conclude the paper with some discussions in Section 4.

2 Proof of Theorem 1.1

We want to bound the domination broadcast number of a graph by a function of its multipacking number. We first state a key counting result which is used throughout the remainder of this paper.

For any two integers a and b such that $a \leq b$, $[\![a, b]\!]$ denotes the set $\mathbb{Z} \cap [a, b]$. A path P in a graph G is *isometric* if for any two vertices in P, $\mathrm{dist}_P(x, y) = \mathrm{dist}_G(x)$. That is, P is a geodesic in G.

Lemma 2.1. *Let G be a graph, k be a positive integer and $(u_0, \ldots, u_{3k})$ be an isometric path of length $3k$ in G. Let $P = \{u_{3i} | i \in [\![0, k]\!]\}$ be the set of every third vertex on this path. Then, for any positive integer r and any ball B of radius r in G,*

$$|B \cap P| \leq \left\lceil \frac{2r+1}{3} \right\rceil.$$

Proof. Let B be a ball of radius r in G, then any two vertices in B are at distance at most $2r$. Since the path $(u_0, \ldots, u_{3k})$ is isometric the intersection of the path and B is included in a subpath of length $2r$. This subpath contains at most $2r+1$ vertices and only one third of those vertices can be in P. □

Note for any positive integer r, we have $r \geq \lceil \frac{2r+1}{3} \rceil$. Thus, Lemma 2.1 ensures that P is a valid multipacking of size $k+1$. We have the following (see also [8]):

Proposition 2.2. *For any connected graph G,*

$$\mathrm{mp}(G) \geq \left\lceil \frac{diam(G)+1}{3} \right\rceil.$$

Building on this idea, we have the following result.

Theorem 2.3. *Given a graph G and two positive integers k and k' such that $k' \leq k$, if there are four vertices x, y, u and v in G such that*

$$d_G(x,u) = d_G(x,v) = 3k,\ d_G(u,v) = 6k \text{ and } d_G(x,y) = 3k + 3k',$$

then

$$\mathrm{mp}(G) \geq 2k + k'.$$

Proof. Let $(u_{-3k}, \ldots, u_0, \ldots, u_{3k})$ be the vertices of an isometric path from u to v going through x. Note that $u = u_{-3k}$, $x = u_0$ and $v = u_{3k}$. We shall select every third vertex of this isometric path and let P_1 be the set $\{u_{3i} | i \in [\![-k, k]\!]\}$.

We thus have already selected $2k+1$ vertices. If $k' \leq 1$, then these vertices give us the desired multipacking. Thus assume $k' \geq 2$. In order to complete our goal, we need $k'-1$ additional vertices. Let $(x_0, \ldots, x_{3k+3k'})$ be the vertices of an isometric path from x to y. Note that $x = x_0$ and $y = x_{3k+3k'}$. We shall select every third vertex on this isometric path starting at x_{3k+6}. Formally, we let P_2 be the set $\{x_{3k+3(i+2)} | i \in [\![0, k'-2]\!]\}$. Finally, we let P be the union of P_1 and P_2. An illustration of this is displayed in Figure 1.

Since every vertex of P_2 is at distance at least $3k+6$ from x, while every vertex of P_1 is at distance at most $3k$ from x, we infer that P_1 and P_2 are disjoint. Thus $|P| = 2k + k'$. We shall now prove that P is a valid multipacking.

Let r be an integer between 1 and $|P|-1$, and let B be a ball of radius r in G (with a fixed but arbitrary centre). If this ball B intersects only P_1 or only P_2, then we know by Lemma 2.1 that it cannot contain more than r vertices of P. We may then consider that the ball B intersects both P_1 and P_2. Let l denote the greatest

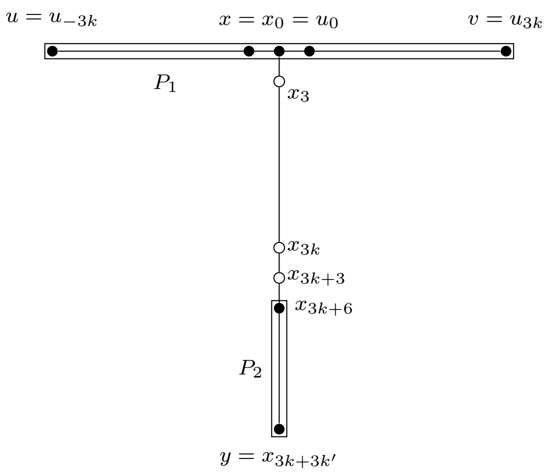

Figure 1: Building of P.

integer i such that $x_{3k+3(i+2)}$ is in B and in P_2. Let us name this vertex z. From this, we may say that

$$|B \cap P_2| \leq l + 1 \tag{1}$$

Before ending this preamble, we state an easy inequality. For every integer n,

$$\left\lceil \frac{n}{3} \right\rceil \leq \frac{n}{3} + \frac{2}{3} \tag{2}$$

We now split the remainder of the proof into two cases.

Case 1: $3(l+2) \leq r$**.** In this case, we just use Lemma 2.1 for P_1. We have

$$|B \cap P_1| \leq \left\lceil \frac{2r+1}{3} \right\rceil,$$

and by Inequality (2), this quantity is bounded above by $\frac{2r+1}{3} + \frac{2}{3}$. We obtain with Inequality (1),

$$\begin{aligned} |B \cap P| &\leq l + 1 + \frac{2r+1}{3} + \frac{2}{3} \\ &\leq l + 2 + \frac{2r}{3} \\ &\leq \frac{r}{3} + \frac{2r}{3} && \text{(by our case hypothesis)} \\ &\leq r. \end{aligned}$$

Therefore, the ball B contains at most r vertices of P, as required.

Case 2: $3(l+2) > r$. Here we need some more insight. Recall that $l+2$ cannot exceed k' and that $k' \leq k$. Thus

$$\begin{aligned} r &< 3(l+2) \\ &< 2k' + l + 2 \\ &< 2k + l + 2, \end{aligned}$$

and since r is an integer, we get

$$r \leq 2k + l + 1. \tag{3}$$

We also note that any vertex u_i for $|i| \leq 3k + 3(l+2) - (2r+1)$ is at distance at least $2r+1$ from z. By the triangle inequality $d(z, u_i) \geq d(z, x) - d(u_i, x)$, where $d(z, x) = 3k + 3(l+2)$, and $d(u_i, x) = |i|$. Since the ball B has radius r, no such vertex can be in B. Since we assumed that B intersects P_1, not all the vertices of the uv-path are excluded from B. This means that

$$3k > 3k + 3(l+2) - (2r+1). \tag{4}$$

We partition the vertices of P_1 into three sets: U_L, U_M, U_R. The vertex u_i belongs to: U_L if $i < -3k - 3(l+2) + 2(r+1)$; U_M if $|i| \leq 3k + 3(l+2) - (2r+1)$; and U_R if $i > 3k + 3(l+2) - (2r+1)$. See Figure 2(a). The distance from $u = u_{-3k}$ to the first vertex (smallest positive index) in U_R is then $6k + 3(l+2) - (2r+1) + 1$. We compare this distance with $2r+1$ resulting in the following two cases.

Case 2.1: $6k + 3(l+2) - (2r+1) + 1 \geq 2r+1$. We match U_L with U_R so that each pair is at distance at least $2r+1$ (match u_{-3k} with the first vertex in U_R and so on, as pictured in Figure 2(a)). Therefore the ball B contains at most one vertex of each matched pair. In other words, B contains at most $\lceil |U_R|/3 \rceil$ vertices from $U_L \cup U_R$, and so

$$|B \cap P_1| \leq \left\lceil \frac{3k - (3k + 3(l+2) - 2r) + 1}{3} \right\rceil.$$

By using Inequality (1) again,

$$\begin{aligned} |B \cap P| &\leq l + 1 + \left\lceil \frac{2r+1}{3} \right\rceil - (l+2) \\ &\leq r. \end{aligned}$$

Therefore, the ball B contains at most r vertices of P, as required.

Case 2.2: $6k + 3(l+2) - (2r+1) + 1 < 2r+1$. We partition each of U_L and U_R as shown in Figure 2(b). The vertices that are distance at least $2r+1$ from a vertex in $U_L \cup U_R$ are the sets U'_L and U'_R, and those that are close to all other vertices are U''_L and U''_R. We can match pairs of vertices $U'_L \cup U'_R$. This allows us to say that

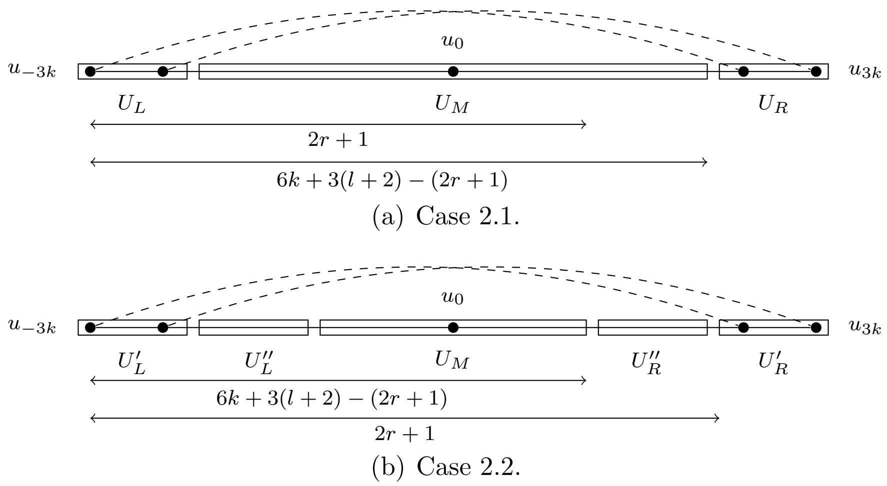

Figure 2: Illustrations for Case 2.

the extremities of P_1 will contribute at most $\left\lceil \frac{6k-(2r+1)+1}{3} \right\rceil$ which equals $2k + \lceil \frac{-2r}{3} \rceil$. Using again Inequality (2), this is bounded above by $2k - \frac{2r}{3} + \frac{2}{3}$.

For any integer i between $6k+3(l+2)-(2r+1)+1$ and $2r$, vertices u_{-i} and u_i belong to U''_L and U''_R respectively. Such vertices may be in B. Since P_1 contains every third vertex on these two subpaths, this amounts to at most

$$2\left\lceil \frac{2r-6k-3(l+2)+(2r+1)}{3} \right\rceil$$

such vertices. This quantity is equal to

$$2\left\lceil \frac{4r+1}{3} \right\rceil - 4k - 2(l+2),$$

which in turn, using Inequality (2) is bounded above by

$$\frac{8r}{3} + 2 - 4k - 2(l+2).$$

By putting everything together, we derive that

$$\begin{aligned} |B \cap P| &\leq (l+1) + \left(2k - \frac{2r}{3} + \frac{2}{3}\right) + \left(\frac{8r}{3} + 2 - 4k - 2(l+2)\right) \\ &\leq 2r - 2k - l - \frac{1}{3}. \end{aligned}$$

But since $|B \cap P|$ is an integer, we may rewrite this last inequality as

$$\begin{aligned} |B \cap P| &\leq r + (r - 2k - l - 1) \\ &\leq r. && \text{(by Inequality (3))} \end{aligned}$$

Thus, $|B \cap P|$ cannot exceed r and the ball B contains at most r vertices of P, as required. This concludes the proof of Theorem 2.3. □

Theorem 2.3 allows us to give a lower bound on the size of a maximum multi-packing in a graph in terms of its diameter and radius.

Corollary 2.4. *For any connected graph G of diameter d and radius r,*

$$\mathrm{mp}(G) \geq \frac{d}{6} + \frac{r}{3} - \frac{3}{2}.$$

Proof. We just pick the integer k such that d can be expressed as $6k + \alpha$ where α is in $[\![0, 5]\!]$ and the integer k' such that r can be expressed as $3k + 3k' + \beta$ where β is in $[\![0, 2]\!]$.

We must have two vertices at distance $6k$ in G. On a shortest path of length $6k$, the middle vertex has some vertex at distance $3k + 3k'$. We can then apply Theorem 2.3.

$$\begin{aligned} \mathrm{mp}(G) &\geq 2k + k' \\ &\geq \frac{1}{3}(d-\alpha) + \frac{1}{3}\left(r - \beta - \frac{1}{2}(d-\alpha)\right) \\ &\geq \frac{d}{6} + \frac{r}{3} - \frac{9}{6}. \end{aligned}$$

□

We can now finalize the proof of our main theorem.

Proof of Theorem 1.1. Since the diameter of a graph is always greater than or equal to its radius, we conclude from Corollary 2.4 that

$$\frac{\mathrm{rad}(G) - 3}{2} \leq \mathrm{mp}(G) \leq \gamma_b(G) \leq \mathrm{rad}(G).$$

Hence, for any connected graph G,

$$\gamma_b(G) \leq 2\mathrm{mp}(G) + 3,$$

proving Theorem 1.1. □

Note that in our proof, we chose the length of the long path to be a multiple of 6 for the reading to be smooth. We think that the same ideas implemented with more care would work for multiples of 3. This might slightly improve the additive constant in our bound, but we believe that it would not be enough to prove Conjecture 1.2 (while adding too much complexity to the proof).

3 Proving Conjecture 1.2 when $\mathrm{mp}(G) \leq 4$

The following collection of results shows that Conjecture 1.2 holds for graphs G when $\mathrm{mp}(G) \leq 4$.

Lemma 3.1. *Let G be a graph and P a subset of vertices of G. If, for every subset U of at least two vertices of P, there exist two vertices of U that are at distance at least $2|U| - 1$, then P is a multipacking of G.*

Proof. We prove the contrapositive. Let G be a graph and P a subset of its vertices which is not a multipacking. Then there is a ball B of radius r which contains $r+1$ vertices of P.

Let U be the set $B \cap P$, then U has size at least $r+1$. Moreover, any two vertices in U are at distance at most $2r$ which is stricly smaller than $2|U|-1$. ☐

Proposition 3.2. *Let G be a connected graph. If* $\text{mp}(G) = 3$*, then* $\gamma_b(G) \leq 6$.

Proof. We prove the contrapositive again. Let G be a graph with domination broadcast number at least 7. Then, the eccentricity of any vertex is at least 7 (otherwise we could cover the whole graph by broadcasting with power 6 from a single vertex).

Let x be any vertex of G. There must be a vertex y at distance 7 from x. Let u be any vertex at distance 3 from x and on a shortest path from x to y. Then u is at distance 4 from y. But u has also eccentricity at least 7. So there is a vertex v at distance 7 from u. By the triangle inequality, v is at distance at least 4 from x and at least 3 from y. Therefore the set $\{u, v, x, y\}$ satisfies the condition of Lemma 3.1 giving $\text{mp}(G) \geq 4$ (and so $\text{mp}(G) \neq 3$). ☐

The following proposition improves Theorem 1.1 for graphs G with $\text{mp}(G) \leq 6$ and shows that Conjecture 1.2 holds when $\text{mp}(G) = 4$.

Proposition 3.3. *Let G be a connected graph. If* $\text{mp}(G) \geq 4$*, then* $\gamma_b(G) \leq 3\text{mp}(G) - 4$.

Proof. For a contradiction, let G be a counterexample, that is a graph with multipacking number p at least 4 while $\gamma_b(G) \geq 3p-3$. Then, the eccentricity of any vertex of G is at least $3p-3$ (otherwise we could broadcast at distance $3p-4$ from a single vertex). Let x be a vertex of G and let V_i denote the set of vertices at distance exactly i of x. By our previous remark, V_{3p-3} is non-empty. Let y be a vertex in V_{3p-3} and consider a shortest path P_{xy} from x to y in G. Let $v_0 = x$, and for $1 \leq i \leq p-1$, let v_i be the vertex on P_{xy} belonging to V_{3i} (thus $v_{p-1} = y$).

Now, since $\gamma_b(G) \geq 3p-3$, there must be a vertex u at distance at least $3p-3$ of v_{p-2} (otherwise we could broadcast from that single vertex). Note that the triangle inequality ensures that the distance between u and v_i is at least $3+3i$ for i between 0 and $p-2$. The distance from u to v_{p-1} is at least $3p-6$ which is at least 6 since p is at least 4. Consider the set $P = \{u, v_0, \ldots, v_{p-1}\}$. We claim that P is a multipacking of G of size $p+1$, which is a contradiction.

Let B be a ball of radius r. Since P_{xy} is an isometric path, Lemma 2.1 ensures us that B contains at most

$$\left\lceil \frac{2r+1}{3} \right\rceil$$

vertices from $P \cap P_{xy}$ which is smaller than r. When B does not include u, the ball is satisfied. For balls that contain vertex u, the maximum size of $P \cap B$ is

$$\left\lceil \frac{2r+1}{3} \right\rceil + 1.$$

Whenever r is 4 or more, this quantity does not exceed r. So every ball with radius 4 or more is satisfied. We still need to check balls of radius 1,2, and 3 which contain u.

- Balls of radius 1 are easy to check since every vertex of P_{xy} is at distance at least 3 from u.
- For balls of radius 2, it is enough to check that there is only one vertex at distance 4 or less from u in $P \cap P_{xy}$.
- For balls of radius 3, there is only one way to select u and three vertices in $P \cap P_{xy}$ within distance 6 from u. We should take v_0, v_1 and v_{p-1}. But since v_0 and v_{p-1} are at distance $3p-3$ from each other, they cannot appear simultaneously in a ball of radius 3 (since p is at least 4, $3p-3$ is at least 9).

Therefore P is a multipacking of size $p+1$, which is a contradiction. □

Corollary 3.4. *Let G be a connected graph. If* $\mathrm{mp}(G) \leq 4$*, then* $\gamma_b(G) \leq 2\mathrm{mp}(G)$.

Proof. When $\mathrm{mp}(G) \leq 2$, this is shown in [12]. The case $\mathrm{mp}(G) = 3$ is implied by Proposition 3.2, and the case $\mathrm{mp}(G) = 4$ follows from Proposition 3.3. □

4 Concluding remarks

We conclude the paper with some remarks.

4.1 The optimality of Conjecture 1.2

We know a few examples of connected graphs G which achieve the conjectured bound, that is, $\gamma_b(G) = 2\mathrm{mp}(G)$. For example, one can easily check that C_4 and C_5 have multipacking number 1 and broadcast number 2. In Figure 3, we depict three examples having multipacking number 2 and domination broadcast number 4. By making disjoint unions of these graphs, we can build further extremal graphs with arbitrary multipacking number. However, if we only consider connected graphs, we do not even know an example with multipacking number 3 and domination broadcast number 6. Hartnell and Mynhardt [12] constructed an infinite family of connected graphs G with $\gamma_b(G) = \frac{4}{3}\mathrm{mp}(G)$, but we do not know any construction with a higher ratio. Are there arbitrarily large connected graphs that reach the bound of Conjecture 1.2?

4.2 An approximation algorithm

The computational complexity of broadcast domination has been extensively studied, see for example [7, 14] and references of [3, 18, 19]. It is particularly interesting to note that, unlike most other natural covering problems, broadcast domination is solvable in polynomial ($O(n^6)$) time [14]. It is not known whether this is also the case for multipacking, but a cubic-time algorithm exists for strongly chordal graphs [4, 19], as well as a linear-time algorithm for trees [3, 4, 19]. (The results on trees

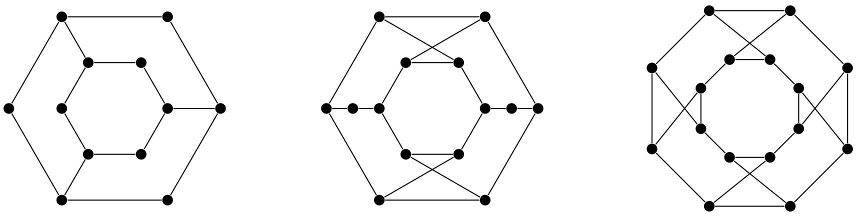

Figure 3: Graphs with multipacking number 2 and domination broadcast number 4. Graph (b) comes from L. Teshima's Master's Thesis [18] and (c) was found by C. R. Dougherty (private communication).

build on structural work in [5, 15].) We note that our proof of Theorem 1.1, being constructive, implies the existence of a $(2+o(1))$-factor approximation algorithm for the multipacking problem.

Corollary 4.1. *There is a polynomial-time algorithm that, given a graph G, constructs a multipacking of G of size at least $\frac{\mathrm{mp}(G)-3}{2}$.*

Proof. To construct the multipacking, one first needs to compute the radius r and diameter d of the graph G. Then, as described in the proof of Corollary 2.4, we compute α and k, and find the four vertices x, y, u, v and the two isometric paths P_1 and P_2 described in Theorem 2.3. Finally, we proceed as in the proof of Theorem 2.3, that is, we essentially select every third vertex of these two paths to obtain the multipacking P. All distances and paths can be computed in polynomial time using classic methods. By Corollary 2.4, P has size at least $\frac{\mathrm{rad}(G)-3}{2}$. Since $\mathrm{mp}(G) \leq \mathrm{rad}(G)$, the approximation factor follows. □

References

[1] L. Beaudou and R. C. Brewster, On the multipacking number of grid graphs, submitted. arXiv e-prints:1803.09639.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Grad. Texts in Math. 244, Springer, New York, 2008.

[3] R. C. Brewster, C. M. Mynhardt and L. Teshima, New bounds for the broadcast domination number of a graph, *Cent. Eur. J. Math.* **11** (2013), 1334–1343.

[4] R. C. Brewster, G. MacGillivray and F. Yang, Broadcast domination and multipacking in strongly chordal graphs, *Discrete Appl. Math.* (to appear).

[5] E. J. Cockayne, S. Herke and C. M. Mynhardt, Broadcasts and domination in trees, *Discrete Math.* **311** (2011), 1235–1246.

[6] G. Cornuéjols, *Combinatorial Optimization: packing and covering*, CBMS-NSF regional conference series in applied mathematics, vol. 74, SIAM, Philadelphia, 2001.

[7] J. Dabney, B. C. Dean and S. T. Hedetniemi, A linear-time algorithm for broadcast domination in a tree, *Networks* **53** (2009), 160–169.

[8] J. E. Dunbar, D. J. Erwin, T. W. Haynes, S. M. Hedetniemi and S. T. Hedetniemi, Broadcasts in graphs, *Discrete Appl. Math.* **154** (2006), 59–75.

[9] D. J. Erwin, *Cost domination in graphs*, PhD Thesis, Department of Mathematics, Western Michigan University, 2001.

[10] D. J. Erwin, Dominating broadcasts in graphs, *Bull. Inst. Combin. Appl.* **42** (2004), 89–105.

[11] M. Farber, Domination, Independent Domination, and Duality in Strongly Chordal Graphs, *Discrete Appl. Math.* **7** (1984), 115–130.

[12] B. L. Hartnell and C. M. Mynhardt, On the difference between broadcast and multipacking numbers of graphs, *Util. Math.* **94** (2014), 19–29.

[13] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Monographs and Textbooks in Pure and Applied Math., 208. Marcel Dekker, Inc., New York, 1998.

[14] P. Heggernes and D. Lokshtanov, Optimal broadcast domination in polynomial time, *Discrete Math.* **306** (2006), 3267–3280.

[15] S. Herke and C. M. Mynhardt, Radial trees, *Discrete Math.* **309** (2009), 5950–5962.

[16] A. Lubiw, Doubly Lexical Orderings of Matrices, *SIAM J. Comput.* **16** (1987), 854–879.

[17] A. Meir and J. W. Moon, Relations between packing and covering numbers of a tree, *Pacific J. Math.* **61** (1975), 225–233.

[18] L. Teshima, *Broadcasts and multipackings in graphs*, Master's Thesis, Department of Mathematics and Statistics, University of Victoria, 2012.

[19] F. Yang, *New results on broadcast domination and multipacking*, Master's Thesis, Department of Mathematics and Statistics, University of Victoria, 2015.

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