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# The nonexistence of projective planes of order 12 with a collineation group of order 9

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*In memory of Professor Yutaka Hiramine*

**Abstract**

In this paper, we prove that there are no projective planes of order 12 admitting a collineation group of order 9.

## 1 Introduction

A finite projective plane is one of the most fundamental concepts in finite geometry. For every prime power $q$ there exists a projective plane of order $q$, because the desarguesian plane $\mathrm{PG}(2, q)$ gives an example of a projective plane of order $q$. But the order of any known finite projective plane is always a prime power. Is the order of any finite projective plane a prime power? For this question, Bruck and Ryser proved the following remarkable theorem in 1949 [8].

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**The Bruck-Ryser Theorem** *If $n \equiv 1$ or $2$ $(mod\, 4)$, there does not exist a projective plane of order $n$ unless $n$ can be expressed a sum of two integral squares.*

For example, this theorem yields that there does not exist a projective plane of order $n$, where $n \leq 25$, if $n = 6$, 14, 21, or 22. Therefore, the smallest composite integer not covered by the Bruck-Ryser Theorem is 10.

In [26] there is an interesting description of the search for a projective plane of order 10. There exists a projective plane of order $n$ if and only if there exists a complete set of $n-1$ mutually orthogonal Latin squares of order $n$. Euler conjectured that there is no pair of orthogonal Latin squares of order $n$ if $n \equiv 2 \pmod 4$. It was proved that this conjecture is false for all orders greater than six (see [9, 10, 27, 28]). This raised the hope for the existence of a projective plane of order 10. Many mathematicians were interested in a projective plane of order 10. At first it was proved that the projective plane has a trivial collineation group [2, 17, 31]. Lam and his colleagues started the research of this problem in 1980 and after a huge effort, finally proved the non-existence of a projective plane of order 10. They examined the weight enumerator of the vector space generated by the rows of the incidence matrix of a putative projective plane of order 10. They used computers for the exhaustive research and the computer time was about 2,000 hours on a CRAY.

The next composite order not covered by the Bruck-Ryser theorem is 12. Actually it is still unknown whether or not a projective plane of order 12 exists. The study of projective planes of order 12 was begun by Janko and van Trung in 1980. Now let $G$ be be a collineation group of a projective plane of order 12. Janko and van Trung proved in their articles [15, 16, 18, 19, 20, 21, 22, 23] that $G$ has the following four properties.

(i) $G$ is a $\{2, 3\}$-group.

(ii) If $|G| = 6$, then $G$ is an abelian group.

(iii) If $|G| = 4$, then $G$ is a cyclic group.

(iv) If $|G| = 3$ or 4, then G is not an elation group.

Horvatic-Baldasar, Kramer, and Matulic-Bedenic [6, 7] showed that $|G|$ divides 16 or 9. Suetake [30], Akiyama and Suetake [3] showed that $|G|$ divides 4 or 9. Morover Akiyama and Suetake [4] proved that if $|G| = 9$, then $G$ is an elementary abelian group and is not planar.

Projective planes of order 15 were studied in [1, 13, 29].

Kang and Ju-Hyun Lee [25] studied an explicit formula and its fast computational algorithm for projective planes of prime order. The GAP System for Computational Discrete Algebra [12] is very useful (however we did not use the system). Casiello, Indaco, and Nagy [11] , on the computational approach to the problem of the existence of a projective plane of order 10, quite recently implemented a new enumerative procedure using the GAP System in order to considerably reduce the computational time of some essential parts.

This paper is a sequel of [4] and we prove the following theorem.

**Theorem** *There are no projective planes of order* 12 *admitting a collineation group of order* 9*.*

Any finite projective plane of order $n$ contains a symmetric transversal design $\mathrm{STD}_1[n,n]$ as a substructure. Conversely any symmetric transversal design $\mathrm{STD}_1$ $[n,n]$ can be uniquely extended to a projective plane of order $n$, up to isomorphism.

Let $\pi = (\mathcal{Q}, \mathcal{L}, J)$ be a projective plane of order 12 with a collineation group $G$ of order 9 and $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be the symmetric transversal design $\mathrm{STD}_1[12,12]$ contained in $\pi$ having the automorphism group $G^{\mathcal{P}\cup\mathcal{B}}$. Then we determine explicitly all types of the action on $\mathcal{P}$ and $\mathcal{B}$ of $G$ in Sections 4 and 5. If $G$ contains a nontrivial planar element, we prove that the subplane of order 3 fixed point wise by the collineation does not exist in Section 6. Otherwise, we prove the nonexistence of $\pi$ by availing the groupring $\mathbb{Z}[G]$ in Section 7. We used a computer for both cases. We also have the following result from the theorem.

**Corollary** *If $G$ is a collineation group of a projective plane $\pi$ of order* 12*, then $G$ is cyclic and $|G|$ divides* 3 *or* 4*.*

Throughout this paper all sets are assumed to be finite. Most definitions and notation are standard and are taken from [5, 14, 24].

## 2 Preliminaries

In this section we state some basic definitions and results about a projective plane and a symmetric transversal design, which will be needed to prove our result.

**Notation 2.1** Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an incidence structure, where $\mathcal{P}$ is a point set, $\mathcal{B}$ is a block set and $I$ is an incidence relation, that is, $I$ is a subset of $\mathcal{P}\times\mathcal{B}$. Then for $p \in \mathcal{P}$ and $B \in \mathcal{B}$, $pIB$ denotes $(p,B) \in I$. For $p \in \mathcal{P}$ set $(p) = \{X \in \mathcal{B} |\ pIX\}$ and for $B \in \mathcal{B}$ set $(B) = \{x \in \mathcal{P} |\ xIB\}$. If $\mathcal{D}$ is a projective plane, since $\mathcal{B} \ni B \longmapsto (B) \in 2^{\mathcal{P}}$ is a one-to-one mapping, we identify $B$ with $(B)$ for $B \in \mathcal{B}$.

**Notation 2.2** Let $(G, \Lambda)$ be a permutation group acting on the set $\Lambda$, which is not always faithful, and $H$ a non empty subset of $G$. Then set $F_\Lambda(H) = \{x \in \Lambda |\ x^\mu = x \text{ for all } \mu \in H\}$ and $\theta_\Lambda(H) = |F_\Lambda(H)|$. If $H = \{\varphi\}$, especially set $F_\Lambda(\{\varphi\}) = F_\Lambda(\varphi)$ and $\theta_\Lambda(\{\varphi\}) = \theta_\Lambda(\varphi)$. $t_\Lambda(G) = t_\Lambda$ denotes the number of orbits of the permutation group $(G, \Lambda)$.

**Lemma 2.3 (Burnside-Frobenius)** *Let $G$ be a permutation group acting on a set $\Lambda$ and $t$ the number of orbits of $(G, \Lambda)$. Then*

$$t|G| = \sum_{\alpha\in G} \theta_\Lambda(\alpha).$$

**Lemma 2.4** *Let $\pi = (\mathcal{Q}, \mathcal{L}, J)$ be a projective plane. Let $\varphi$ be a collineation and $G$ a collineation group of $\pi$. Then*

$$\theta_{\mathcal{Q}}(\varphi) = \theta_{\mathcal{L}}(\varphi) \text{ and } t_{\mathcal{Q}}(G) = t_{\mathcal{L}}(G).$$

**Lemma 2.5** *Let $\pi = (\mathcal{Q}, \mathcal{L}, J)$ be a projective plane. Let $\varphi$ be a collineation of $\pi$ with $\theta_{\mathcal{Q}}(\varphi) \neq 0$. Then one of the following statements holds:*

(i) *$\varphi$ is a generalized elation. That is, there exist $L \in F_{\mathcal{L}}(\varphi)$ and $p \in F_{\mathcal{Q}}(\varphi)$ such that $F_{\mathcal{Q}}(\varphi) \subseteq (L)$, $F_{\mathcal{L}}(\varphi) \subseteq (p)$, $p \in (L)$, where $L$, $p$ are called an axis, a center of $\varphi$ respectively. In this case, since the axis and the center of $\varphi$ are unique for $\pi$ respectively, $\varphi$ is called a $(p, L)$-generalized elation.*

(ii) *$\varphi$ is a generalized homology. That is, there exist $L \in F_{\mathcal{L}}(\varphi)$ and $p \in F_{\mathcal{Q}}(\varphi)$ such that $F_{\mathcal{Q}}(\varphi) \subseteq (L) \cup \{p\}$, $F_{\mathcal{L}}(\varphi) \subseteq (p) \cup \{L\}$, $p \notin (L)$, where $L$, $p$ are called an axis, a center of $\varphi$ respectively. In this case, since the axis and the center of $\varphi$ are unique for $\pi$ respectively, $\varphi$ is called a $(p, L)$-generalized homology.*

(iii) *$\varphi$ is planar. That is, the substructure $(F_{\mathcal{Q}}(\varphi), F_{\mathcal{L}}(\varphi))$ of $\pi$ is a projective plane (a subplane of $\pi$).*

**Lemma 2.6** *Let $\pi = (\mathcal{Q}, \mathcal{L}, J)$ be a projective plane. Let $\varphi$, $\tau \in$Aut $\pi$ such that $\varphi\tau = \tau\varphi$. Then $F_{\mathcal{Q}}(\varphi)^{\tau} = F_{\mathcal{Q}}(\varphi)$ and $F_{\mathcal{L}}(\varphi)^{\tau} = F_{\mathcal{L}}(\varphi)$.*

**Definition 2.7** Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an incidence structure. Then $\mathcal{D}$ is called a *symmetric transversal design* $\mathrm{STD}_{\lambda}[k, u]$, if the following axioms are satisfied, where $\lambda$, $k$, $u$ are positive integers and $k \geq 2$:

(i) For $B \in \mathcal{B}$, $|(B)| = k$.

(ii) There exists a partition of $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{k-1}$ such that for any $0 \leq i \leq k-1$ $|\mathcal{P}_i| = u$ and for distinct $p, q \in \mathcal{P}$

$$|(p) \cap (q)| = \begin{cases} 0 & \text{if } p,\ q \in \mathcal{P}_i \text{ for some } i, \\ \lambda & \text{otherwise} \end{cases} .$$

($\mathcal{P}_0, \ldots, \mathcal{P}_{k-1}$ are called *point classes* of $\mathcal{D}$. We denote the set of point classes by $\Omega(\mathcal{D})$.)

(iii) The dual structure $\mathcal{D}^d$ of $\mathcal{D}$ also satisfies (i) and (ii).

(The point classes of $\mathcal{D}^d$ $\mathcal{B}_0, \ldots, \mathcal{B}_{k-1}$ are called *block classes of* $\mathcal{D}$. We denote the set of block classes by $\Delta(\mathcal{D})$.)

In this definition we give some remarks. From the definition it follows that $k = u\lambda$ and $|\mathcal{P}| = |\mathcal{B}| = uk$. Since $\mathcal{B} \ni B \longmapsto (B) \in 2^{\mathcal{P}}$ is a one-to-one mapping, we identify $B$ with $(B)$ for $B \in \mathcal{B}$.

**Lemma 2.8** *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an $\mathrm{STD}_{\lambda}[k, u]$ with a set of point classes $\Omega(\mathcal{D}) = \{\mathcal{P}_0, \ldots, \mathcal{P}_{k-1}\}$ and a set of block classes $\Delta(\mathcal{D}) = \{\mathcal{B}_0, \ldots, \mathcal{B}_{k-1}\}$. Let $\mathcal{P}_i = \{p_{ui}, p_{ui+1}, \ldots, p_{ui+(u-1)}\}$ and $\mathcal{B}_j = \{B_{uj}, B_{uj+1}, \ldots, B_{uj+(u-1)}\}$ $(0 \leq i, j \leq k-1)$. Let*

$$N = (n_{r,s})_{0 \leq r,s \leq ku-1} = \begin{pmatrix} N_{0,0} & \ldots & N_{0,k-1} \\ \vdots & & \vdots \\ N_{k-1,0} & \ldots & N_{k-1,k-1} \end{pmatrix}$$

*be the incidence matrix of $\mathcal{D}$ corresponding to these numberings of the points and the blocks, that is*

$$n_{r,s} = \begin{cases} 1 & \text{if } p_r I B_s \\ 0 & \text{otherwise} \end{cases},$$

*where each $N_{i,j}$ $(0 \le i,j \le k-1)$ is a $u \times u$ matrix. Then the following statements hold.*

*(i) Each $N_{i,j}$ $(0 \le i,j \le k-1)$ is a permutation matrix of degree $u$ and*

$$NN^T = N^T N = \begin{pmatrix} kE & \lambda J & \dots & \lambda J \\ \lambda J & kE & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda J \\ \lambda J & \dots & \lambda J & kE \end{pmatrix},$$

*where $E$ is the identity matrix of degree $u$ and $J$ is the $u \times u$ all one matrix.*

*(ii) Let $\varphi \in \mathrm{Sym}\, \mathcal{P} \cup \mathcal{B}$ such that $\mathcal{P}^\varphi = \mathcal{P}$ and $\mathcal{B}^\varphi = \mathcal{B}$. We define $\varphi_f$, $\varphi_g \in \mathrm{Sym}\,\{0,1,\dots,ku-1\}$ by $\varphi : p_r \longmapsto p_{r^{\varphi_f}}$, $B_s \longmapsto B_{s^{\varphi_g}}$ $(0 \le r,s \le ku-1)$. Then the following hold.*
- *$\varphi \in \mathrm{Aut}\ \mathcal{D} \iff pIB$ if and only if $p^\varphi I B^\varphi$ $(p \in \mathcal{P}, B \in \mathcal{B}) \iff n_{r,s} = n_{r^{\varphi_f}, s^{\varphi_g}}$ $(0 \le r,s \le ku-1)$.*
- *If $\varphi \in \mathrm{Aut}\ \mathcal{D}$, then from the definition of STD, it follows that $\varphi$ induces permutations on both $\Omega(\mathcal{D})$ and $\Delta(\mathcal{D})$. Let these permutations be $\widetilde{\varphi}$ and $\widetilde{\widetilde{\varphi}}$ respectively.*

**Lemma 2.9** [3] *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an $\mathrm{STD}_\lambda[k,u]$ with the set of point classes $\Omega = \Omega(\mathcal{D})$ and the set of block classes $\Delta = \Delta(\mathcal{D})$. Let $\varphi \in$Aut $\mathcal{D}$ and let $G$ an automorphism group of $\mathcal{D}$. Then*

$$\theta_{\mathcal{P}}(\varphi) + \theta_\Delta(\varphi) = \theta_{\mathcal{B}}(\varphi) + \theta_\Omega(\varphi) \text{ and } \theta_{\mathcal{P}}(G) + \theta_\Delta(G) = \theta_{\mathcal{B}}(G) + \theta_\Omega(G).$$

The following result is well-known (see Proposition 7.19 in [5]).

**Lemma 2.10** *Let $\pi = (\mathcal{Q}, \mathcal{L}, J)$ be a projective plane of order $n$. Choose $r_\infty \in \mathcal{Q}$ and $L_\infty \in \mathcal{L}$ such that $r_\infty \in (L_\infty)$. Set $\mathcal{P} = \mathcal{Q} \backslash (L_\infty)$ and $\mathcal{B} = \mathcal{L} \backslash (r_\infty)$. Let $(r_\infty)\backslash\{L_\infty\} = \{L_0, L_1, \dots, L_{n-1}\}$ and $(L_\infty)\backslash\{r_\infty\} = \{r_0, r_1, \dots, r_{n-1}\}$. Set $\mathcal{P}_i = (L_i)\backslash\{r_\infty\}$, $\mathcal{B}_j = (r_j)\backslash\{L_\infty\}$ $(0 \le i,j \le n-1)$, $\Omega = \{\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{n-1}\}$ and $\Delta = \{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{n-1}\}$. Then the substructure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ $(I = J \cap (\mathcal{P} \times \mathcal{B}))$ of $\pi$ is an $\mathrm{STD}_1[n,n]$ having the set of point classes $\Omega$ and the set of block classes $\Delta$. In this case we say that $\mathcal{D}$ is the $\mathrm{STD}_1[n,n]$ with respect to a point $r_\infty$ and a line $L_\infty$.*

**Lemma 2.11** *Let $\pi = (\mathcal{Q}, \mathcal{L}, J)$ be a projective plane of order $n$. Choose $r_\infty \in \mathcal{Q}$ and $L_\infty \in \mathcal{L}$ such that $r_\infty \in (L_\infty)$. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be the $\mathrm{STD}_1[n,n]$ with respect to $r_\infty$ and $L_\infty$. Set $\Omega = \Omega(\mathcal{D})$ and $\Delta = \Delta(\mathcal{D})$. Let $G$ be a collineation group of $\pi$ such that ${L_\infty}^\mu = L_\infty$ and ${r_\infty}^\mu = r_\infty$ for all $\mu \in G$. Then the following statements hold.*

*(i)* *For all* $\mu \in G$, $\mu|_{\mathcal{P}\cup\mathcal{B}} \in \mathrm{Aut}\ \mathcal{D}$.

*(ii)* $G \ni \mu \longmapsto \mu|_{\mathcal{P}\cup\mathcal{B}} \in \mathrm{Aut}\ \mathcal{D}$ *is a monomorphism. (In the rest of the paper, we identify* $\mu|_{\mathcal{P}\cup\mathcal{B}}$ *with* $\mu$.*)*

*(iii)* *Both* $G \ni \mu \longmapsto \widetilde{\mu} \in \mathrm{Sym}\ \Omega$ *and* $G \ni \mu \longmapsto \widetilde{\widetilde{\mu}} \in \mathrm{Sym}\ \Delta$ *are homomorphisms.*

## 3 Projective planes of order 12 admitting a collineation group of order 9

We assume the following in this section.

**Hypothesis 3.1** $\pi = (\mathcal{Q}, \mathcal{L}, J)$ is a projective plane of order 12 admitting a collineation group $G$ of order 9.

**Lemma 3.2** [18] $\pi$ *does not have an elation of order* 3.

**Lemma 3.3** [4] $G$ *is an elementary abelian group of order* 9 *and the substructure* $(F_{\mathcal{Q}}(G), F_{\mathcal{L}}(G))$ *of* $\pi$ *is not a subplane of* $\pi$.

**Lemma 3.4** [4] *Let* $\mu \in G\backslash\{1\}$. *If* $\pi_1 = (F_{\mathcal{Q}}(\mu), F_{\mathcal{L}}(\mu))$ *is a subplane of* $\pi$, *then the order of* $\pi_1$ *is* 3.

**Lemma 3.5** *Let* $\mu \in G$, $L \in \mathcal{L}$ *and* $r \in (L)$. *If* $\mu$ *is a* $(r, L)$*-generalized elation, then* $r \in F_{\mathcal{Q}}(G)$ *and* $L \in F_{\mathcal{L}}(G)$.

Proof. Let $\xi \in G$. Now $\xi^{-1}\mu\xi = \mu$ is a $(r^\xi, L^\xi)$-generalized elation. Since the center $r$ and the axis $L$ of $\mu$ are unique for $\mu$, respectively, $r^\xi = r$ and $L^\xi = L$. □

**Lemma 3.6** *If* $\mu \in G\backslash\{1\}$, *then one of the following* (1) *to* (5) *holds:*

| | $\mu$ | $\theta_\Omega(\mu)$ | $\theta_\mathcal{B}(\mu)$ | $\theta_\Delta(\mu)$ | $\theta_\mathcal{P}(\mu)$ |
|---|---|---|---|---|---|
| (1) | *planar* | 3 | 9 | 3 | 9 |
| (2) | $(r_\infty, L)$*–g.e.* | $n_2$ | 0 | 0 | $n_2$ |
| (3) | $(r_\infty, L_\infty)$*-g.e.* | $n_3$ | 0 | $n_3$ | 0 |
| (4) | $(r, L_\infty)$*-g.e.* | 0 | $n_4$ | $n_4$ | 0 |
| (5) | $(r_\infty, L_\infty)$*-g.e.* | 0 | 0 | 0 | 0 |

*where* $n_2, n_3, n_4 \in \{3, 6, 9\}$, $r \in (L_\infty)\backslash\{r_\infty\}$ *and* $L \in (r_\infty)\backslash\{L_\infty\}$.

Proof. If $\mu$ is planar, (1) holds by Lemma 3.4. Suppose that $\mu$ is not planar. Then $\mu$ is a generalized elation. The axis of $\mu$ is a line through $r_\infty$ and the center of $\mu$ is a point on $L_\infty$. If $L_\infty$ is the axis of $\mu$, then (3), (4) or (5) holds. If $L_\infty$ is not the axis of $\mu$, then there exists a line $L \in (r_\infty)\backslash\{L_\infty\}$ such that $L$ is the axis of $\mu$. This yields that the center of $\mu$ is $r_\infty$. Therefore (2) holds. □

**Lemma 3.7** *$G\backslash\{1\}$ contains a planar collineation, if and only if $G$ is not semiregular on $\mathcal{P} = \mathcal{Q}\backslash(L_\infty)$ and also on $\mathcal{B} = \mathcal{L}\backslash(r_\infty)$.*

Proof. Suppose that $G$ is not semiregular on $\mathcal{P}$ and also on $\mathcal{B}$. Then there exist $\varphi \in G\backslash\{1\}$, $M \in \mathcal{L}$ such that $M \notin (r_\infty)$, $M^\varphi = M$. There also exist $\tau \in G\backslash\{1\}$, $p \in \mathcal{P}$ such that $p^\tau = p$. Set $L = pr_\infty \in \mathcal{L}$. Suppose that $G\backslash\{1\}$ does not have a planar collineation. Then $\tau$ is a $(r_\infty, L)$-generalized elation and $L \in F_{\mathcal{L}}(G)$ by Lemma 3.5. Set $M \cap L_\infty = r$ and $M \cap L = s$. Thus $r, s, r_\infty$ are not collinear and these points are fixed by $\varphi$. This yields that $\varphi$ is planar, which is a contradiction. Therefore $G\backslash\{1\}$ contains a planar collineation.

The converse is clear. Thus we have the lemma. □

Since $|G| = 9$, $G$ fixes a point $r_\infty$ and a line $L_\infty$ with $r_\infty \in (L_\infty)$. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be the $\mathrm{STD}_1[12, 12]$ with respect to $r_\infty$ and $L_\infty$. Actually, $\mathcal{P} = \mathcal{Q}\backslash(L_\infty)$, $\mathcal{B} = \mathcal{L}\backslash(r_\infty)$ and $\Omega = \{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{11}\}$, $\Delta = \{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{11}\}$ are point classes and block classes of $\mathcal{D}$ respectively, where $(r_\infty)\backslash\{L_\infty\} = \{L_0, L_1, \ldots, L_{11}\}$, $(L_\infty)\backslash\{r_\infty\} = \{r_0, r_1, \ldots, r_{11}\}$, $\mathcal{P}_i = (L_i)\backslash\{r_\infty\}$ and $\mathcal{B}_j = (r_j)\backslash\{L_\infty\}$ $(0 \le i, j \le 11)$.

**Lemma 3.8** *The sizes of $G$-orbits on $L_\infty$ are as follows*:
**Case 1** $(1, 1, 1, 1, 1, 1, 1, 3, 3)$;
**Case 2** $(1, 1, 1, 1, 3, 3, 3)$;
**Case 3** $(1, 1, 1, 1, 9)$;
**Case 4** $(1, 3, 3, 3, 3)$;
**Case 5** $(1, 3, 9)$.

Proof. If $G$ has $G$-orbits on $L_\infty$ different from Cases 1 to 5, then the sizes of $G$-orbits on $L_\infty$ is $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3)$. Then there exists $\mu \in G\backslash\{1\}$ such that $|F_{(L_\infty)}(\mu)| = 13$. This is contrary to Lemma 3.2. □

## 4 The case that $G\backslash\{1\}$ contains a planar collineation

In this section we consider the case that $G\backslash\{1\}$ contains a planar collineation. We assume Hypothesis 3.1 and also the following in this section.

**Hypothesis 4.1** $G\backslash\{1\}$ contains a planar collineation.

Then, by Lemma 3.7, $G$ does not act semiregularly on $\mathcal{P}$, nor on $\mathcal{B}$. In the rest of this section, for each of Cases 1 to 5 obtained in Section 3, if that case occurs, we determine the actions on $\Omega \cup \Delta$ of $\varphi$ and $\tau$, where $G = \langle \varphi, \tau \rangle$. Moreover, if $\varphi(\tau)$ fixes a class $X \in \Omega \cup \Delta$, we also determine the action on $X$ of $\varphi(\tau)$. We will show in Section 6 that actions on $\Omega \cup \Delta$ of $\varphi$ and $\tau$ yield explicitly the actions on $\mathcal{P} \cup \mathcal{B}$ of $\varphi(\tau)$.

**Lemma 4.2** *Case 1 does not occur.*

Proof. Let $\varphi$ be a planar collineation in $G\backslash\{1\}$. Then $\theta_\Delta(\varphi) = 3$. This is contrary to the assumption of Case 1. □

**Lemma 4.3** *If Case 2 occurs, then one of the following two types holds.*
**Type 1** (i) $G = \langle\varphi, \tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_5, \mathcal{P}_4)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_4)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$.
(ii) *$\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \le i \le 2$ and three blocks of $\mathcal{B}_j$ for $0 \le j \le 2$. Also $G$ acts semiregularly on both $\mathcal{P}\backslash F_\mathcal{P}(\varphi)$ and $\mathcal{B}\backslash F_\mathcal{B}(\varphi)$, while $\langle\tau\rangle$ acts semiregularly on both $F_\mathcal{P}(\varphi)$ and $F_\mathcal{B}(\varphi)$.*

**Type 2** (i) $G = \langle\varphi, \tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_8, \mathcal{P}_7)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_8, \mathcal{B}_7)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$.
(ii) *$\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \le i \le 2$ and three blocks of $\mathcal{B}_j$ for $0 \le j \le 2$. Also $G$ acts semiregularly on both $\mathcal{P}\backslash F_\mathcal{P}(\varphi)$ and $\mathcal{B}\backslash F_\mathcal{B}(\varphi)$, while $\langle\tau\rangle$ acts semiregularly on both $F_\mathcal{P}(\varphi)$ and $F_\mathcal{B}(\varphi)$.*

Proof. Let $\varphi$ be a planar collineation in $G\backslash\{1\}$. Then we can assume that $\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$ and $\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$, where $\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \le i \le 2$ and three blocks of $\mathcal{B}_j$ for $0 \le j \le 2$.

($\alpha$) Assume that there exists $\tau \in G\backslash\langle\varphi\rangle$ with $F_\mathcal{P}(\tau) \ne \emptyset$. Since $\tau$ is planar by Lemma 3.6, $\theta_\Omega(\tau) = \theta_\Delta(\tau) = 3$. Applying the Burnside-Frobenius theorem to the permutation group $(G, \Delta)$, we have $\theta_\Delta(\varphi) + \theta_\Delta(\tau) + \theta_\Delta(\varphi\tau) + \theta_\Delta(\varphi^2\tau) = 21$. This yields $\theta_\Delta(\varphi\tau) + \theta_\Delta(\varphi^2\tau) = 15$. Since $\theta_\Delta(\varphi\tau) \ne 12$ and $\theta_\Delta(\varphi^2\tau) \ne 12$, by Lemma 3.2, $(\theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (6, 9)$ or $(9, 6)$. Considering $\varphi^2$ instead of $\varphi$ if necessary, we may assume that $(\theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (6, 9)$. Now $\varphi\tau$ and $\varphi^2\tau$ are generalized elations having $L_\infty$ as an axis. Therefore $\theta_\mathcal{P}(\varphi\tau) = \theta_\mathcal{P}(\varphi^2\tau) = 0$. From this we have $\theta_\Omega(\varphi\tau) + \theta_\mathcal{B}(\varphi\tau) = \theta_\Delta(\varphi\tau) + \theta_\mathcal{P}(\varphi\tau) = 6 + 0 = 6$. Similarly we have $\theta_\Omega(\varphi^2\tau) + \theta_\mathcal{B}(\varphi^2\tau) = 9$.

Suppose that $F_\Omega(\varphi) \cap F_\Omega(\tau) \ne \emptyset$. Then $F_\Omega(\varphi) = F_\Omega(\tau) = \{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2\}$. If there exists $p \in \mathcal{P}_0$ such that $p^\varphi = p^\tau = p$, then $(F_\mathcal{Q}(G), F_\mathcal{L}(G))$ is a subplane of $\pi$ of order 3. This is contrary to Lemma 3.3. Therefore $F_{\mathcal{P}_0}(\varphi) \cap F_{\mathcal{P}_0}(\tau) = \emptyset$.

Since $\theta_\mathcal{P}(\varphi\tau) = \theta_\mathcal{P}(\varphi^2\tau) = 0$, $G$ acts semiregularly on $\mathcal{P}_0\backslash(F_{\mathcal{P}_0}(\varphi) \cup F_{\mathcal{P}_0}(\tau))$. Therefore $9 = |G| \,|\, |\mathcal{P}_0\backslash(F_{\mathcal{P}_0}(\varphi) \cup F_{\mathcal{P}_0}(\tau))| = 6$. This is a contradiction. Thus $F_\Omega(\varphi) \cap F_\Omega(\tau) = \emptyset$. Therefore $(\theta_\Omega(\varphi^2\tau), \theta_\mathcal{B}(\varphi^2\tau)) = (0, 9)$ by Lemma 3.6. Let $r_0 (\ne r_\infty)$ be the center of $\varphi^2\tau$. Then $r_0 \in F_\mathcal{Q}(G)$ by Lemma 3.5. Set $\mathcal{B}_0 = (r_0)\backslash\{L_\infty\} \in \Delta$. By a similar argument to that the above, $F_{\mathcal{B}_0}(\varphi) \cap F_{\mathcal{B}_0}(\tau) = \emptyset$. There exists $L \in (r_0)$ such that $L^{\varphi^2\tau} = L$ and $L$ is fixed by $\varphi$ or $\tau$. Therefore $L$ is fixed by $\varphi$ and $\tau$. This is also a contradiction.

($\beta$) Assume that $F_{\mathcal{P}}(\mu) = \emptyset$ for all $\mu \in G\backslash\langle\varphi\rangle$. Let $\tau \in G\backslash\langle\varphi\rangle$. We may assume that $\theta_\Delta(\tau) \le \theta_\Delta(\varphi\tau) \le \theta_\Delta(\varphi^2\tau)$. Since $\theta_\Delta(\tau) + \theta_\Delta(\varphi\tau) + \theta_\Delta(\varphi^2\tau) = 18$, $(\theta_\Delta(\tau), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (3, 6, 9)$ or $(6, 6, 6)$. $\tau, \varphi\tau$ and $\varphi^2\tau$ are generalized elations having $L_\infty$ as an axis by Lemma 3.6. The center of each collineation of $\tau$, $\varphi\tau$, and $\varphi^2\tau$ is an element of $F_{(L_\infty)}(\varphi)$. Set $\pi_S = (F_{\mathcal{Q}}(\varphi), F_{\mathcal{L}}(\varphi))$. Then $\pi_S$ is a subplane of $\pi$ of order 3. Now $\tau|_{\pi_S} = \varphi\tau|_{\pi_S} = \varphi^2\tau|_{\pi_S}$ and this is an elation of $\pi_S$ having $L_\infty$ as an axis. We may assume that the center of $\tau|_{\pi_S}$ is $r_\infty$. Therefore $\tau|_{\pi_S}$ fixes all lines through the point $r_\infty$. Let $M_0, M_1, M_2$ be these lines except $L_\infty$. Since $M_0, M_1, M_2$ are fixed by $\varphi$ and $\tau$, these three lines are fixed by any collineation in $G$.

Assume that $(\theta_\Delta(\tau), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (3, 6, 9)$. Then $F_{(r_\infty)}(\varphi) = F_{(r_\infty)}(\tau)$ and $F_{(r_\infty)}(\varphi) \subseteq F_{(r_\infty)}(\varphi\tau) \cap F_{(r_\infty)}(\varphi^2\tau)$. The center of each collineation of $\tau$, $\varphi\tau$, and $\varphi^2\tau$ is $r_\infty$. If there exists $M \in (r_\infty)$ such that $M^\varphi \ne M$, $M^{\varphi\tau} = M$ and $M^{\varphi^2\tau} = M$, then $M = M^\varphi$, because $M^{\varphi\tau} = M = M^{\varphi^2\tau}$ yields $M = M^\varphi$. This is a contradiction. Therefore $F_{(r_\infty)}(\varphi\tau) \cap F_{(r_\infty)}(\varphi^2\tau) = \{L_\infty, M_0, M_1, M_2\} = F_{(r_\infty)}(\varphi) = F_{(r_\infty)}(\tau)$. In this case we have Type 1.

Assume that $(\theta_\Delta(\tau), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (6, 6, 6)$. Then $F_{(r_\infty)}(\varphi) \subseteq F_{(r_\infty)}(\tau) \cap F_{(r_\infty)}(\varphi\tau) \cap F_{(r_\infty)}(\varphi^2\tau)$. The center of each collineation of $\tau$, $\varphi\tau$, and $\varphi^2\tau$ is $r_\infty$. If there exists $M \in (r_\infty)$ such that $M^\varphi \ne M$, $M^\tau = M$ and $M^{\varphi\tau} = M$, then $M = M^\varphi$, because $M^\tau = M = M^{\varphi\tau}$ yields $M = M^\varphi$. This is a contradiction. Therefore $F_{(r_\infty)}(\tau) \cap F_{(r_\infty)}(\varphi\tau) = F_{(r_\infty)}(\varphi)$. By a similar argument, $F_{(r_\infty)}(\tau) \cap F_{(r_\infty)}(\varphi^2\tau) = F_{(r_\infty)}(\varphi\tau) \cap F_{(r_\infty)}(\varphi^2\tau) = F_{(r_\infty)}(\varphi)$. In this case we have Type 2. □

**Lemma 4.4** *If Case* 3 *occurs, then one of the following three types holds.*

**Type 3** (i) $G = \langle\varphi, \tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_9)(\mathcal{B}_4, \mathcal{B}_7, \mathcal{B}_{10})(\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_{11})$.
(ii) *Each of* $\varphi$, $\tau$, $\varphi\tau$, $\varphi^2\tau$ *fixes three points of* $\mathcal{P}_i$ *for* $0 \le i \le 2$ *and three blocks of* $\mathcal{B}_j$ *for* $0 \le j \le 2$. *Any two point sets of* $F_{\mathcal{P}}(\varphi)$, $F_{\mathcal{P}}(\tau)$, $F_{\mathcal{P}}(\varphi\tau)$, *and* $F_{\mathcal{P}}(\varphi^2\tau)$ *are disjoint from each other. Any two block sets of* $F_{\mathcal{B}}(\varphi)$, $F_{\mathcal{B}}(\tau)$, $F_{\mathcal{B}}(\varphi\tau)$, *and* $F_{\mathcal{B}}(\varphi^2\tau)$ *are disjoint from each other.*

**Type 4** (i) $G = \langle\varphi, \tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_9)(\mathcal{B}_4, \mathcal{B}_7, \mathcal{B}_{10})(\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_{11})$.
(ii) $\varphi$ *fixes three points of* $\mathcal{P}_i$ *for* $0 \le i \le 2$ *and three blocks of* $\mathcal{B}_j$ *for* $0 \le j \le 2$. *Also* $G$ *fixes any block of* $F_{\mathcal{B}_0}(\varphi)$, *and* $G$ *acts semiregularly on the each block set of* $\mathcal{B}_0\backslash F_{\mathcal{B}_0}(\varphi)$, $\mathcal{B}_1\backslash F_{\mathcal{B}_1}(\varphi)$, *and* $\mathcal{B}_2\backslash F_{\mathcal{B}_2}(\varphi)$. *Moreover,* $\langle\tau\rangle$ *acts regularly on the both block sets* $F_{\mathcal{B}_1}(\varphi)$ *and* $F_{\mathcal{B}_2}(\varphi)$.

**Type 5** (i) $G = \langle \varphi, \tau \rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_9)(\mathcal{B}_4, \mathcal{B}_7, \mathcal{B}_{10})(\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_{11})$.
(ii) *$\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \leq i \leq 2$ and three blocks of $\mathcal{B}_j$ for $0 \leq j \leq 2$. Also $\langle \tau \rangle$ acts regularly on $F_{\mathcal{P}_i}(\varphi)$ for $0 \leq i \leq 2$, and $G$ acts regularly on $\mathcal{P}_i \backslash F_{\mathcal{P}_i}(\varphi)$ for $0 \leq i \leq 2$. Moreover, $\langle \tau \rangle$ acts regularly on $F_{\mathcal{B}_j}(\varphi)$ for $0 \leq j \leq 2$, and $G$ acts regularly on $\mathcal{B}_j \backslash F_{\mathcal{B}_j}(\varphi)$ for $0 \leq j \leq 2$.*

Proof. Suppose that Case 3 occurs. Let $\varphi$ be a planar collineation in $G \backslash \{1\}$. Then we may assume that $\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)$ $(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$ and $\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$, where $\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \leq i \leq 2$ and three blocks of $\mathcal{B}_j$ for $0 \leq j \leq 2$. Let $\varphi^{\mathcal{P}_0} = (p_0)(p_1)(p_2)(p_3, p_4, p_5)$ $(p_6, p_7, p_8)(p_9, p_{10}, p_{11})$, $\varphi^{\mathcal{P}_1} = (p_{12})(p_{13})(p_{14})(p_{15}, p_{16}, p_{17})(p_{18}, p_{19}, p_{20})(p_{21}, p_{22}, p_{23})$, $\varphi^{\mathcal{P}_2} = (p_{24})(p_{25})(p_{26})(p_{27}, p_{28}, p_{29})(p_{30}, p_{31}, p_{32})(p_{33}, p_{34}, p_{35})$ and $F_{(L_\infty)}(\varphi) = \{r_\infty, r_0, r_1, r_2\}$. We distinguish two cases.

Case I. Suppose that there exists $\tau \in G \backslash \langle \varphi \rangle$ with $F_{\mathcal{P}}(\tau) \neq \emptyset$. Then $\tau$ is planar and $F_{(L_\infty)}(\tau) = \{r_\infty, r_0, r_1, r_2\}$. Since $(F_{\mathcal{Q}}(G), F_{\mathcal{L}}(G))$ is not a subplane of $(\mathcal{Q}, \mathcal{L}, J)$ by Lemma 3.3, $F_{\mathcal{P}}(\varphi) \cap F_{\mathcal{P}}(\tau) = \emptyset$.

($\alpha$) Suppose that ${\mathcal{P}_0}^\tau = \mathcal{P}_0$. Since $\tau$ induces a permutation on $\{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2\}$, ${\mathcal{P}_1}^\tau = \mathcal{P}_1$ and ${\mathcal{P}_2}^\tau = \mathcal{P}_2$. Let $\tau^{\mathcal{P}_0} = (p_0, p_1, p_2)(p_3)(p_4)(p_5)(p_6, p_8, p_7)(p_9, p_{10}, p_{11})$,
$\tau^{\mathcal{P}_1} = (p_{12}, p_{13}, p_{14})(p_{15})(p_{16})(p_{17})$ $(p_{18}, p_{20}, p_{19})(p_{21}, p_{22}, p_{23})$ and
$\tau^{\mathcal{P}_2} = (p_{24}, p_{25}, p_{26})(p_{27})(p_{28})(p_{29})(p_{30}, p_{32}, p_{31})(p_{33}, p_{34}, p_{35})$. Therefore
$\varphi\tau^{\mathcal{P}_0} = (p_0, p_1, p_2)(p_3, p_4, p_5)(p_6)(p_7)(p_8)(p_9, p_{11}, p_{10})$,
$\varphi\tau^{\mathcal{P}_1} = (p_{12}, p_{13}, p_{14})$ $(p_{15}, p_{16}, p_{17})(p_{18})(p_{19})(p_{20})(p_{21}, p_{23}, p_{22})$,
$\varphi\tau^{\mathcal{P}_2} = (p_{24}, p_{25}, p_{26})(p_{27}, p_{28}, p_{29})(p_{30})(p_{31})(p_{32})(p_{33}, p_{35}, p_{34})$,
$\varphi^2\tau^{\mathcal{P}_0} = (p_0, p_1, p_2)(p_3, p_5, p_4)(p_6, p_7, p_8)(p_9)(p_{10})(p_{11})$,
$\varphi^2\tau^{\mathcal{P}_1} = (p_{12}, p_{13}, p_{14})(p_{15}, p_{17}, p_{16})(p_{18}, p_{19}, p_{20})(p_{21})(p_{22})(p_{23})$ and
$\varphi^2\tau^{\mathcal{P}_2} = (p_{24}, p_{25}, p_{26})(p_{27}, p_{29}, p_{28})(p_{30}, p_{31}, p_{32})(p_{33})(p_{34})(p_{35})$.

Thus any collineation of $\varphi$, $\tau$, $\varphi\tau$, $\varphi^2\tau$ is planar. Therefore $\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)$ $(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$. By the assumption, $\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)$ $(\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_9)(\mathcal{B}_4, \mathcal{B}_7, \mathcal{B}_{10})(\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_{11})$. Thus we have Type 3.

($\beta$) Suppose that ${\mathcal{P}_0}^\tau \neq \mathcal{P}_0$. Then we may assume that $\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)$ $(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$ or $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_8, \mathcal{P}_7)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. If the former occurs, $\widetilde{\varphi^2\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_5, \mathcal{P}_4)(\mathcal{P}_6)(\mathcal{P}_7)(\mathcal{P}_8)(\mathcal{P}_9)(\mathcal{P}_{10})(\mathcal{P}_{11})$ and therefore $\varphi^2\tau$ is neither a generalized elation nor a planar collineation. This is a contradiction. Therefore $\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_8, \mathcal{P}_7)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. Set $\mathcal{S} = (F_{\mathcal{Q}}(\varphi), F_{\mathcal{L}}(\varphi))$. Then $\mathcal{S}$ is a subplane of $\pi$ of order 3. And also $\tau|_{\mathcal{S}}$ is a $(r_i, L_\infty)$-elation of $\mathcal{S}$ for some $0 \leq i \leq 2$ and $\tau$ fixes all lines of $F_{(r_i)}(\varphi)$ through $r_i$. In this case we can reduce to case ($\alpha$) by considering $r_i$ instead of $r_\infty$.

Case II. Suppose that for all $\mu \in G\backslash\langle\varphi\rangle$, $F_{\mathcal{P}}(\mu) = \emptyset$. Then $\theta_\Delta(\mu) = 3$, $\theta_{\mathcal{P}}(\mu) = 0$, $\theta_\Omega(\mu) + \theta_{\mathcal{B}}(\mu) = \theta_\Delta(\mu) + \theta_{\mathcal{P}}(\mu) = 3$ and $(\theta_\Omega(\mu), \theta_{\mathcal{B}}(\mu)) = (0, 3)$ or $(3, 0)$. Let $G = \langle\varphi, \tau\rangle$. Then we may assume that $\theta_\Omega(\tau) \leq \theta_\Omega(\varphi\tau) \leq \theta_\Omega(\varphi^2\tau)$. In this case $(\theta_\Omega(\tau), \theta_\Omega(\varphi\tau), \theta_\Omega(\varphi^2\tau)) = (0, 0, 0)$, $(0, 0, 3)$, $(0, 3, 3)$ or $(3, 3, 3)$.

($\gamma$) Suppose that $(\theta_\Omega(\tau), \theta_\Omega(\varphi\tau), \theta_\Omega(\varphi^2\tau)) = (0, 0, 0)$. Then $\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ $(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$ and $(\theta_{\mathcal{B}}(\tau), \theta_{\mathcal{B}}(\varphi\tau), \theta_{\mathcal{B}}(\varphi^2\tau)) = (3, 3, 3)$. Any collineation of $\tau$, $\varphi\tau$, or $\varphi^2\tau$ is a generalized elation having the axis $L_\infty$. We may assume that the center of $\tau$ is $r_0$. We distinguish three cases.

- Suppose that both $\varphi\tau$ and $\varphi^2\tau$ have the center $r_0$. Then ($\gamma$-1) $F_{\mathcal{B}_0}(\tau) = F_{\mathcal{B}_0}(\varphi)$ or ($\gamma$-2) $|F_{\mathcal{B}_0}(\tau)| = |F_{\mathcal{B}_0}(\varphi\tau)| = |F_{\mathcal{B}_0}(\varphi^2\tau)| = |F_{\mathcal{B}_0}(\varphi)| = 3$ and $\mathcal{B}_0 = F_{\mathcal{B}_0}(\tau) \cup F_{\mathcal{B}_0}(\varphi\tau) \cup F_{\mathcal{B}_0}(\varphi^2\tau) \cup F_{\mathcal{B}_0}(\varphi)$ is a disjoint union.
- Suppose that the center of $\varphi\tau$ is $r_0$ and $r_0$ is not the center of $\varphi^2\tau$. In this case we may assume that the center of $\varphi^2\tau$ is $r_1$. Therefore ($\gamma$-3) $|F_{\mathcal{B}_0}(\tau)| = |F_{\mathcal{B}_0}(\varphi\tau)| = |F_{\mathcal{B}_0}(\varphi)| = 3$, $|F_{\mathcal{B}_1}(\varphi^2\tau)| = |F_{\mathcal{B}_1}(\varphi)| = 3$ and $F_{\mathcal{B}_0}(\tau)$, $F_{\mathcal{B}_0}(\varphi\tau)$, $F_{\mathcal{B}_0}(\varphi)$ do not intersect each other. Moreover $F_{\mathcal{B}_1}(\varphi^2\tau) \cap F_{\mathcal{B}_1}(\varphi) = \emptyset$.
- Suppose that the centers of $\tau$, $\varphi\tau$, $\varphi^2\tau$ are different each other. Then we may assume that the center of $\varphi\tau$ is $r_1$ and the center of $\varphi^2\tau$ is $r_2$. Therefore ($\gamma$-4) $|F_{\mathcal{B}_0}(\tau)| = |F_{\mathcal{B}_0}(\varphi)| = 3$, $|F_{\mathcal{B}_1}(\varphi\tau)| = |F_{\mathcal{B}_1}(\varphi)| = 3$, $|F_{\mathcal{B}_2}(\varphi^2\tau)| = |F_{\mathcal{B}_2}(\varphi)| = 3$, $F_{\mathcal{B}_0}(\tau) \cap F_{\mathcal{B}_0}(\varphi) = \emptyset$, $F_{\mathcal{B}_1}(\varphi\tau) \cap F_{\mathcal{B}_1}(\varphi) = \emptyset$ and $F_{\mathcal{B}_2}(\varphi^2\tau) \cap F_{\mathcal{B}_2}(\varphi) = \emptyset$.

($\gamma$-1) yields Type 4.

Assume that ($\gamma$-2) occurs. Let $p \in F_{\mathcal{P}_0}(\varphi)$. Then $p^\tau \in F_{\mathcal{P}_1}(\varphi)$. Let $B$ be the block through $p$ and $p^\tau$. Then $B \in F_{\mathcal{B}}(\varphi)$. Since $p$, $p^\tau \in (B)$, we have $p^\tau$, $p^{\tau^2} \in (B^\tau)$. Therefore $B$ and $B^\tau$ are through the point $p^\tau$. But $B$, $B^\tau \in \mathcal{B}_i$ for some $0 \leq i \leq 2$. This is a contradiction. Thus ($\gamma$-2) does not occur.

Assume that ($\gamma$-3) occurs. Since $G$ acts semiregularly on $\mathcal{B}_1\backslash(F_{\mathcal{B}_1}(\varphi^2\tau) \cup F_{\mathcal{B}_1}(\varphi))$, $9||\mathcal{B}_1\backslash(F_{\mathcal{B}_1}(\varphi^2\tau) \cup F_{\mathcal{B}_1}(\varphi))| = 6$. This is a contradiction. Thus ($\gamma$-3) does not occur.

Assume that ($\gamma$-4) occurs. Since $G$ acts semiregularly on $\mathcal{B}_0\backslash(F_{\mathcal{B}_0}(\tau) \cup F_{\mathcal{B}_0}(\varphi))$, we have $9||\mathcal{B}_0\backslash(F_{\mathcal{B}_0}(\tau) \cup F_{\mathcal{B}_0}(\varphi))| = 6$. This is a contradiction. Thus ($\gamma$-4) does not occur.

($\delta$) Suppose that $(\theta_\Omega(\tau), \theta_\Omega(\varphi\tau), \theta_\Omega(\varphi^2\tau)) = (0, 0, 3)$. Since $\theta_\Omega(\tau) = \theta_\Omega(\varphi\tau) = 0$, we may assume that $\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. Therefore $\widetilde{\varphi^2\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ $(\mathcal{P}_3)(\mathcal{P}_4) \ldots (\mathcal{P}_{11})$. This is contrary to $\theta_\Omega(\varphi^2\tau) = 3$. Thus ($\delta$) does not occur.

($\epsilon$) Suppose that $(\theta_\Omega(\tau), \theta_\Omega(\varphi\tau), \theta_\Omega(\varphi^2\tau)) = (0, 3, 3)$. Since $\theta_\Omega(\tau) = 0$, $\theta_\Omega(\varphi\tau) = 3$, we may assume that $\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_5, \mathcal{P}_4)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. Therefore $\widetilde{\varphi^2\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ $(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)$ $(\mathcal{P}_6)(\mathcal{P}_7) \ldots (\mathcal{P}_{11})$. This is contrary to $\theta_\Omega(\varphi^2\tau) = 3$. Thus ($\epsilon$) does not occur.

($\zeta$) Suppose that $(\theta_\Omega(\tau), \theta_\Omega(\varphi\tau), \theta_\Omega(\varphi^2\tau)) = (3, 3, 3)$. Then since

$$(\theta_{\mathcal{B}}(\tau), \theta_{\mathcal{B}}(\varphi\tau), \theta_{\mathcal{B}}(\varphi^2\tau)) = (0, 0, 0) \text{and } \theta_\Omega(\tau) = \theta_\Omega(\varphi\tau) = 3,$$

we may assume that

$$\begin{aligned} \widetilde{\tau} &= (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11}), \\ & \quad (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_6,\mathcal{P}_9)(\mathcal{P}_4,\mathcal{P}_7,\mathcal{P}_{10})(\mathcal{P}_5,\mathcal{P}_8,\mathcal{P}_{11}) \\ \text{or} & \quad (\mathcal{P}_0,\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_8,\mathcal{P}_7)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11}). \end{aligned}$$

If the first case on $\widetilde{\tau}$ occurs, then $\widetilde{\varphi^2\tau} = (\mathcal{P}_0)(\mathcal{P}_1)\dots(\mathcal{P}_{11})$. This is a contradiction. The second case on $\widetilde{\tau}$ yields Type 5. If the third case on $\widetilde{\tau}$ occurs, we have a contradiction by the same argument as in ($\gamma$-2). □

**Lemma 4.5** *Let $G = \langle\varphi,\tau\rangle$. In Case 4, if both $\varphi$ and $\tau$ are planar and $F_\Omega(\varphi) \cap F_\Omega(\tau) = \emptyset$, then $F_{(L_\infty)}(\varphi) = F_{(L_\infty)}(\tau)$.*

Proof. Suppose that both $\varphi$ and $\tau$ are planar and $F_\Omega(\varphi) \cap F_\Omega(\tau) = \emptyset$, $F_{(L_\infty)}(\varphi) \neq F_{(L_\infty)}(\tau)$. Then $F_{(L_\infty)}(\varphi) \cap F_{(L_\infty)}(\tau) = \{r_\infty\}$. Let $x \in F_{\mathcal{P}}(\varphi)$ and $y \in F_{\mathcal{P}}(\tau)$. Since $x, y$ are not contained in the same point class, there exists $B \in \mathcal{B}$ such that $x \in (B)$ and $y \in (B)$.

Assume that there exists $x_1(\neq x) \in (B)$ such that $x_1 \in F_{\mathcal{P}}(\varphi)$. Then $|(B) \cap F_{\mathcal{P}}(\varphi)| = 3$, $B \in F_{\mathcal{B}}(\varphi)$ and therefore $(B) = (B^\varphi) \ni y^\varphi$. Moreover $y^\varphi \neq y$ and $y^\varphi \in F_{\mathcal{P}}(\tau)$. Let $L$ be the extension to a line in $\mathcal{L}$ of $B$. Then $(L) \cap (L_\infty)$ is fixed by both $\varphi$ and $\tau$. This is a contradiction. Therefore $\{B\} \cap F_{\mathcal{P}}(\varphi) = \{x\}$, $(B) \cap F_{\mathcal{P}}(\tau) = \{y\}$.

Moreover $(B) \cap (L_\infty) \notin F_{(L_\infty)}(\varphi) \cup F_{(L_\infty)}(\tau)$. If we move points $x \in F_{\mathcal{P}}(\varphi)$ and points $y \in F_{\mathcal{P}}(\tau)$, the number of these lines $L$ (the extensions to lines in $\mathcal{L}$ of the blocks $B$) is 81. Therefore these lines $L$ intersect with $L_\infty$ in the points except $F_{(L_\infty)}(\varphi) \cup F_{(L_\infty)}(\tau)$. But $|\{X \in \mathcal{L} \mid X \neq L_\infty,\ (X) \cap (L_\infty) \notin F_{(L_\infty)}(\varphi) \cup F_{(L_\infty)}(\tau)\}| = 6 \times 12 = 72$. This is a contradiction. Thus we have the lemma. □

**Lemma 4.6** *If Case 4 occurs, then one of the following three types holds.*

**Type 6** (i) $G = \langle\varphi,\tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)(\mathcal{B}_6,\mathcal{B}_7,\mathcal{B}_8)(\mathcal{B}_9,\mathcal{B}_{10},\mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0,\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0,\mathcal{B}_1,\mathcal{B}_2)(\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)(\mathcal{B}_6,\mathcal{B}_7,\mathcal{B}_8)(\mathcal{B}_9,\mathcal{B}_{10},\mathcal{B}_{11})$.
(ii) *$\varphi$ fixes three points on $\mathcal{P}_i$ for $0 \le i \le 2$ and three blocks of $\mathcal{B}_j$ for $0 \le j \le 2$. $\langle\varphi^2\tau\rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ for $3 \le i, j \le 11$.*

**Type 7** (i) $G = \langle\varphi,\tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)(\mathcal{B}_6,\mathcal{B}_7,\mathcal{B}_8)(\mathcal{B}_9,\mathcal{B}_{10},\mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0,\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_5,\mathcal{P}_4)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0,\mathcal{B}_1,\mathcal{B}_2)(\mathcal{B}_3,\mathcal{B}_5,\mathcal{B}_4)(\mathcal{B}_6,\mathcal{B}_7,\mathcal{B}_8)(\mathcal{B}_9,\mathcal{B}_{10},\mathcal{B}_{11})$.
(ii) *$\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \le i \le 2$ and three blocks of $\mathcal{B}_j$ for $0 \le j \le 2$. $\langle\varphi\tau\rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ for $3 \le i, j \le 5$. $\langle\varphi^2\tau\rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ for $6 \le i, j \le 11$.*

**Type 8** (i) $G = \langle \varphi, \tau \rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_8, \mathcal{P}_7)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_8, \mathcal{B}_7)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$.
(ii) *$\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \leq i \leq 2$ and three blocks of $\mathcal{B}_j$ for $0 \leq j \leq 2$. $\langle \tau \rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ for $3 \leq i, j \leq 5$. $\langle \varphi\tau \rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ for $6 \leq i, j \leq 8$. $\langle \varphi^2\tau \rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ for $9 \leq i, j \leq 11$.*

Proof. Suppose that Case 4 occurs. Let $\varphi$ be a planar collineation in $G \backslash \{1\}$. Let $G = \langle \varphi, \tau \rangle$ and $F_{(L_\infty)}(\varphi) = \{r_\infty, r_0, r_1, r_2\}$. Then $\langle \tau \rangle$ acts regularly on $\{r_0, r_1, r_2\}$. We may assume that $\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$ and $\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$, where $\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \leq i \leq 2$ and three blocks of $\mathcal{B}_j$ for $0 \leq j \leq 2$. Applying the Burnside-Frobenius theorem to the permutation group $(G, \Delta)$, we have $\theta_\Delta(\tau) + \theta_\Delta(\varphi\tau) + \theta_\Delta(\varphi^2\tau) = 9$. Then, since we may assume that $\theta_\Delta(\tau) \leq \theta_\Delta(\varphi\tau) \leq \theta_\Delta(\varphi^2\tau)$, we find that $(\theta_\Delta(\tau), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (0, 0, 9)$, $(0, 3, 6)$ or $(3, 3, 3)$ holds.

($\alpha$) Suppose that $(\theta_\Delta(\tau), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (0, 0, 9)$. Since $\theta_\Delta(\tau) = 0$, $\theta_{\mathcal{B}}(\tau) = 0$ and $\theta_\Omega(\tau) = \theta_{\mathcal{P}}(\tau)$.

Assume that $\theta_\Omega(\tau) \neq 0$. Now $\tau$ is a $(r_\infty, L)$-generalized elation for some $L \in (r_\infty) \backslash \{L_\infty\}$ by Lemma 3.6. Since $L^\varphi = L$ by Lemma 3.5, $L \in F_{\mathcal{L}}(G)$. Let $L_i$ be the line of $\pi$ through $r_\infty$ corresponding to $\mathcal{P}_i$ $(0 \leq i \leq 11)$. Then since $\{L_0, L_1, L_2\}^\tau = \{L_0, L_1, L_2\}$, $L_0 \in F_{\mathcal{L}}(G)$. This is a contradiction. Therefore $\theta_\Omega(\tau) = \theta_{\mathcal{P}}(\tau) = 0$ and $\theta_\Delta(\tau) = \theta_{\mathcal{B}}(\tau) = 0$.

Since $\theta_\Delta(\varphi\tau) = 0$, the similar argument yields $\theta_\Omega(\varphi\tau) = \theta_{\mathcal{P}}(\varphi\tau) = 0$ and $\theta_\Delta(\varphi\tau) = \theta_{\mathcal{B}}(\varphi\tau) = 0$. Since $\varphi^2\tau$ is a $(r_\infty, L_\infty)$-generalized elation by Lemma 3.5, $\theta_\Omega(\varphi^2\tau) = 9$. Therefore $\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. It also follows that $\langle \varphi^2\tau \rangle$ acts semiregulary on both $\mathcal{P}_i$ and $\mathcal{B}_j$ for $3 \leq i, j \leq 11$. Thus we have Type 6.

($\beta$) Suppose that $(\theta_\Delta(\tau), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (0, 3, 6)$. Then, $\theta_\Omega(\tau) = \theta_{\mathcal{P}}(\tau) = 0$ and $\theta_\Delta(\tau) = \theta_{\mathcal{B}}(\tau) = 0$ hold by the same argument as in ($\alpha$), because $\theta_\Delta(\tau) = 0$. Since $\theta_\Delta(\varphi\tau) = 3$, by Lemma 4.5 $\varphi\tau$ is a generalized elation. Let $F_{(L_\infty)}(\varphi\tau) = \{r_3, r_4, r_5, r_\infty\}$. From the assumption of Case 4, it follows that $\{r_0, r_1, r_2\} \cap \{r_3, r_4, r_5\} = \emptyset$. $\varphi\tau$ is a $(r_\infty, L_\infty)$-generalized elation by Lemma 3.5. Therefore

$$\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_5, \mathcal{P}_4)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$$

and

$$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_4)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11}).$$

It also follows that $\langle \varphi\tau \rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ $(3 \leq i, j \leq 5)$ and $\langle \varphi^2\tau \rangle$ acts semiregularly on both $\mathcal{P}_i$ and $\mathcal{B}_j$ $(6 \leq i, j \leq 11)$. Thus we have Type 7.

($\gamma$) Suppose that $(\theta_\Delta(\tau), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau)) = (3,3,3)$. Then all $\tau$, $\varphi\tau$, $\varphi^2\tau$ are generalized elations by Lemmas 3.5 and 4.5. For $\mu \neq \xi \in \{\varphi, \tau, \varphi\tau, \varphi^2\tau\}$, $F_\Delta(\mu) \cap F_\Delta(\xi) = \emptyset$ and $F_\Omega(\mu) \cap F_\Omega(\xi) = \emptyset$. In this case we have Type 8. □

**Lemma 4.7** *If Case 5 occurs, then the following holds.*
**Type 9** (i) $G = \langle \varphi, \tau \rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_9)(\mathcal{B}_4, \mathcal{B}_7, \mathcal{B}_{10})(\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_{11})$.
(ii) *$\varphi$ fixes three points of $\mathcal{P}_i$ for $0 \leq i \leq 2$ and three blocks of $\mathcal{B}_j$ for $0 \leq j \leq 2$.*

Proof. Suppose that Case 5 occurs. Let $\varphi$ be a planar collineation in $G\backslash\{1\}$. Let $G = \langle \varphi, \tau \rangle$. Then $\theta_\Omega(\tau) = \theta_{\mathcal{P}}(\tau) = 0$ and $\theta_\Delta(\tau) = \theta_{\mathcal{B}}(\tau) = 0$. By considering the assumption of Case 5, we have Type 9. □

## 5 The case that $G\backslash\{1\}$ does not contain a planar collineation

If $G\backslash\{1\}$ does not contain a planar collineation, then $G$ is semiregular on $\mathcal{P} = \mathcal{Q}\backslash(L_\infty)$ or $G$ is semiregular on $\mathcal{B} = \mathcal{L}\backslash(r_\infty)$ by Lemma 3.7. In this section we assume Hypothesis 3.1 and the following.

**Hypothesis 5.1** $G\backslash\{1\}$ does not contain a planar collineation and $G$ is semiregular on $\mathcal{Q}\backslash(L_\infty)$.

Then every $\mu \in G$ is a generalized elation of $\pi$ with $L_\infty$ as an axis.

In the rest of this section, we investigate the actions on both $\Omega \cup \Delta$ and $\mathcal{P} \cup \mathcal{B}$ of $\varphi$ and $\tau$, where $G = \langle \varphi, \tau \rangle$, as in Section 4 under these assumptions. The extensions of $\varphi$ and $\tau$ on $\mathcal{P} \cup \mathcal{B}$ will be determined in Section 7.

**Lemma 5.2** *Case 1 does not occur.*

Proof. Suppose that Case 1 occurs. Let $G = \langle \varphi, \tau \rangle$ and $F_{(L_\infty)}(G) = \{r_\infty, r_0, r_1, r_2, r_3, r_4, r_5\}$. Since $|\{r_i |\ r_i$ is the center of $\mu$ for some $\mu \in G\backslash\{1\}\ \}| \leq 4$, there exists $1 \leq j \leq 5$ such that $r_j$ is not a center of any collineation of $\varphi, \tau, \varphi\tau, \varphi^2\tau$. Therefore $G$ acts semiregularly on $(r_j)\backslash\{L_\infty\}$ and therefore $9 = |G| | |(r_j)\backslash\{L_\infty\}| = 12$. This is a contradiction. □

**Lemma 5.3** *Case 2 does not occur.*

Proof. Suppose that Case 2 occurs. Let $G = \langle \varphi, \tau \rangle$ and $F_{(L_\infty)}(G) = \{r_\infty, r_0, r_1, r_2\}$.

If there exists $i \in \{\infty, 0, 1, 2\}$ such that $r_i$ is not the center of any collineation $\in G\backslash\{1\}$, then $G$ acts semiregularly on $(r_i)\backslash\{L_\infty\}$ and therefore $9 = |G| | |(r_i)\backslash\{L_\infty\}| = 12$. This is a contradiction. Thus the centers of $\varphi, \varphi\tau, \varphi^2\tau, \tau$ are different each other.

The Burnside-Frobenius Theorem yields $\theta_\Delta(\varphi)+\theta_\Delta(\varphi\tau)+\theta_\Delta(\varphi^2\tau)+\theta_\Delta(\tau) = 21$. Since we may assume that $\theta_\Delta(\varphi) \le \theta_\Delta(\varphi\tau) \le \theta_\Delta(\varphi^2\tau) \le \theta_\Delta(\tau)$, we find that

$$(\theta_\Delta(\varphi), \theta_\Delta(\varphi\tau), \theta_\Delta(\varphi^2\tau), \theta_\Delta(\tau)) = (3, 3, 6, 9) \text{ or } (3, 6, 6, 6),$$

here we may also assume that the center of $\tau$ is $r_\infty$. Set $\Phi_1 = \{L \in (r_\infty)\backslash\{L_\infty\} | \ L^\tau = L\}$ and $\Phi_2 = \{L \in (r_\infty)\backslash\{L_\infty\} | \ L^\tau \neq L\}$. We remark that $|\Phi_2| = 3$ or 6, because $\theta_\Delta(\tau) = |\Phi_1| = 9$ or 6. Then $G$ induces a permutation group on $\Phi_i$ $(i = 1, 2)$. Since $G$ acts semiregularly on $\Phi_2$, we have $9 = |G|||\Phi_2|$. This is a contradiction. □

**Lemma 5.4** *If Case* 3 *occurs, then the following hold.*
**Type 10** (i) $G = \langle\varphi, \tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_9)(\mathcal{B}_4, \mathcal{B}_7, \mathcal{B}_{10})(\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_{11})$.
(ii) *$G$ acts semiregularly on $\mathcal{P}$ and $|F_{\mathcal{B}_0}(\tau)| = |F_{\mathcal{B}_1}(\varphi\tau)| = |F_{\mathcal{B}_2}(\varphi^2\tau)| = 3$.*

Proof. Let $G = \langle\varphi, \tau\rangle$. Then we may assume that

$$\begin{aligned} \widetilde{\widetilde{\varphi}} &= (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11}), \\ \text{and} \quad \widetilde{\widetilde{\tau}} &= (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_9)(\mathcal{B}_4, \mathcal{B}_7, \mathcal{B}_{10})(\mathcal{B}_5, \mathcal{B}_8, \mathcal{B}_{11}). \end{aligned}$$

Let $F_{(L_\infty)}(G) = \{r_\infty, r_0, r_1, r_2\}$. A similar argument as in Lemma 5.2 yields that centers of $\varphi, \tau, \varphi\tau, \varphi^2\tau$ are different from each other. Therefore we may assume that the center of $\varphi$ is $r_\infty$. Since $\theta_\Omega(\varphi) = 3$ by Lemma 3.6, we may assume that $\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. Since $\theta_\Omega(\mu) = 0$ for all $\mu \in G\backslash\langle\varphi\rangle$, $\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_6, \mathcal{P}_9)(\mathcal{P}_4, \mathcal{P}_7, \mathcal{P}_{10})(\mathcal{P}_5, \mathcal{P}_8, \mathcal{P}_{11})$. Since the centers of $\varphi, \tau, \varphi\tau, \varphi^2\tau$ are different from each other, we may assume that $|F_{\mathcal{B}_0}(\tau)| = |F_{\mathcal{B}_1}(\varphi\tau)| = |F_{\mathcal{B}_2}(\varphi^2\tau)| = 3$. □

**Lemma 5.5** *If Case* 4 *occurs, then one of the following four types holds.*

**Type 11** (i) $G = \langle\varphi, \tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$.
(ii) *$G$ acts semiregularly on both $\mathcal{P}$ and $\mathcal{B}$.*

**Type 12** (i) $G = \langle\varphi, \tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{11}, \mathcal{P}_{10})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{11}, \mathcal{B}_{10})$.

(ii) *$G$ acts semiregularly on both $\mathcal{P}$ and $\mathcal{B}$.*

**Type 13** (i) $G = \langle \varphi, \tau \rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6)(\mathcal{P}_7)(\mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6)(\mathcal{B}_7)(\mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$.
(ii) *$G$ acts semiregularly on both $\mathcal{P}$ and $\mathcal{B}$.*

**Type 14** (i) $G = \langle \varphi, \tau \rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{11}, \mathcal{P}_{10})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{11}, \mathcal{B}_{10})$.
(ii) *$G$ acts semiregularly on both $\mathcal{P}$ and $\mathcal{B}$.*

Proof. Let $G = \langle \varphi, \tau \rangle$. By the assumption of Case 4, any collineation in $G$ is a $(r_\infty, L_\infty)$-generalized elation. Therefore $G$ acts semiregularly on $\mathcal{B}$. We may assume that $\theta_\Delta(\varphi) \leq \theta_\Delta(\mu) \leq \theta_\Delta(\tau)$ for all $\mu \in G \backslash \{1\}$. The Burnside-Frobenius theorem yields $\theta_\Delta(\varphi) + \theta_\Delta(\varphi\tau) + \theta_\Delta(\varphi^2\tau) + \theta_\Delta(\tau) = 12$ and therefore $\theta_\Delta(\varphi) = 0, 3$.

Suppose that $\theta_\Delta(\varphi) = 0$. Then we may assume that
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$ and
$\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. Since $\theta_\Delta(\tau) = 6, 9$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$,
$(\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{11}, \mathcal{B}_{10})$ or
$(\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6)(\mathcal{B}_7)(\mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$.

($\alpha$) Suppose that $\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$. Then $\theta_\Omega(\tau) = 6$. Since $\widetilde{\widetilde{\varphi\tau}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_8, \mathcal{B}_7)(\mathcal{B}_9, \mathcal{B}_{11}, \mathcal{B}_{10})$, $\theta_\Omega(\varphi\tau) = 0$. Therefore $\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. In this case we have Type 11.

($\beta$) Suppose that $\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{11}, \mathcal{B}_{10})$. Then $\theta_\Omega(\tau) = 6$. Since $\widetilde{\widetilde{\varphi\tau}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_8, \mathcal{B}_7)(\mathcal{B}_9)(\mathcal{B}_{11})(\mathcal{B}_{10})$, $\theta_\Omega(\varphi\tau) = 3$. Therefore $\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{11}, \mathcal{P}_{10})$. In this case we have Type 12.

($\gamma$) Suppose that $\widetilde{\widetilde{\tau}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6)(\mathcal{B}_7)(\mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$. Then $\theta_\Omega(\tau) = 9$. Since

$$\widetilde{\widetilde{\varphi\tau}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{11}, \mathcal{B}_{10}),$$

$\theta_\Omega(\varphi\tau) = 0$. Therefore $\widetilde{\tau} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6)(\mathcal{P}_7)(\mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$. In this case we have Type 13.

Suppose that $\theta_\Omega(\varphi) = 3$. Then $\theta_\Omega(\varphi) = \theta_\Omega(\varphi\tau) = \theta_\Omega(\varphi^2\tau) = \theta_\Omega(\tau) = 3$. Since $\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$, $\theta_\Omega(\varphi) = 3$. Therefore $\widetilde{\varphi} =$

$(\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11})$ and $\widetilde{\widetilde{\tau}} = (\mathcal{B}_0,\mathcal{B}_1,\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}_4)(\mathcal{B}_5)(\mathcal{B}_6,\mathcal{B}_7,\mathcal{B}_8)(\mathcal{B}_9,\mathcal{B}_{11},\mathcal{B}_{10})$. Since $\theta_\Omega(\tau) = \theta_\Omega(\varphi\tau) = 3$, $\widetilde{\tau} = (\mathcal{P}_0,\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}_3)(\mathcal{P}_4)(\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{11},\mathcal{P}_{10})$. In this case we have Type 14. □

**Lemma 5.6** *If Case* 5 *occurs, then the following hold.*

**Type 15** (i) $G = \langle\varphi,\tau\rangle$,
$\widetilde{\varphi} = (\mathcal{P}_0)(\mathcal{P}_1)(\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5)(\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)(\mathcal{P}_9,\mathcal{P}_{10},\mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0)(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)(\mathcal{B}_6,\mathcal{B}_7,\mathcal{B}_8)(\mathcal{B}_9,\mathcal{B}_{10},\mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0,\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}_3,\mathcal{P}_6,\mathcal{P}_9)(\mathcal{P}_4,\mathcal{P}_7,\mathcal{P}_{10})(\mathcal{P}_5,\mathcal{P}_8,\mathcal{P}_{11})$,
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0,\mathcal{B}_1,\mathcal{B}_2)(\mathcal{B}_3,\mathcal{B}_6,\mathcal{B}_9)(\mathcal{B}_4,\mathcal{B}_7,\mathcal{B}_{10})(\mathcal{B}_5,\mathcal{B}_8,\mathcal{B}_{11})$.
(ii) *$G$ acts semiregularly on both $\mathcal{P}$ and $\mathcal{B}$.*

Proof. There exists $\varphi \in G\backslash\{1\}$ such that $\theta_\Delta(\varphi) = 3$ by the assumption of Case 5. Since $\theta_\mathcal{P}(\varphi) + \theta_\Delta(\varphi) = \theta_\mathcal{B}(\varphi) + \theta_\Omega(\varphi)$ and $\theta_\mathcal{P}(\varphi) = 0$, $\theta_\mathcal{B}(\varphi) + \theta_\Omega(\varphi) = 3$. Since $\varphi$ is a $(r_\infty, L_\infty)$-generalized elation, $\theta_\Omega(\varphi) = 3$ and therefore $\theta_\mathcal{B}(\varphi) = 0$. There exists $\tau \in G\backslash\langle\varphi\rangle$ such that $\theta_\Delta(\tau) = 0$ by the assumption of Case 5. Then $\theta_\Delta(\varphi\tau) = \theta_\Delta(\varphi^2\tau) = 0$. Therefore $\tau, \varphi\tau, \varphi^2\tau$ are $(r_\infty, L_\infty)$-generalized elations. Hence $\theta_\Omega(\tau) = \theta_\Omega(\varphi\tau) = \theta_\Omega(\varphi^2\tau) = 0$ and $\theta_\mathcal{B}(\tau) = \theta_\mathcal{B}(\varphi\tau) = \theta_\mathcal{B}(\varphi^2\tau) = 0$. Thus we have Type 15.□

**Lemma 5.7** *Let $G$ be a collineation group of order* 9 *of $\pi = (\mathcal{Q},\mathcal{L},J)$. If $G\backslash\{1\}$ does not contain a planar collineation, then one of Types* 10 *to* 15 *occurs, up to duality of $\pi$.*

Proof. From Lemmas 5.2 to 5.6, and Lemma 3.7, the lemma holds. □

## 6 Types 1 to 9

In this section we consider Types 1 to 9 in Section 4 and we show that none of these types occurs, by considering the first 36 rows of the incidence matrix of $\mathcal{D}$, which corresponds to the subplane of order 3.

Let $\mathcal{D} = (\mathcal{P},\mathcal{B},I)$ be the $\mathrm{STD}_1[12,12]$ with the set of point classes $\Omega = \{\mathcal{P}_0,\ldots,\mathcal{P}_{11}\}$ $(0 \le i \le 11)$ and the set of block classes $\Delta = \{\mathcal{B}_0,\ldots,\mathcal{B}_{11}\}$ $(0 \le j \le 11)$. Let $\mathcal{P}_i = \{p_{12i}, p_{12i+1},\ldots,p_{12i+11}\}$ $(0 \le i \le 11)$ and $\mathcal{B}_j = \{B_{12j}, B_{12j+1},\ldots,B_{12j+11}\}$ $(0 \le j \le 11)$. Let $H = (h_{i,j})_{0\le i,j\le 143}$ be the incidence matrix corresponding to the numberings $p_0,\ldots,p_{143}$ and $B_0,\ldots,B_{143}$ of points and blocks of $\mathcal{D}$ and set $H_{r,s} = (h_{12r+i,12s+j})_{0\le i,j\le 11}$ for $0 \le r,s \le 11$. Then $H_{r,s}$ $(0 \le r,s \le 11)$ is a permutation matrix and $H = (H_{r,s})_{0\le r,s\le 11}$. Moreover set $H_1 = (h_{i,j})_{0\le i\le 35, 0\le j\le 143}$. Then $H_1 = (H_{r,s})_{0\le r\le 2, 0\le s\le 11}$. At first we determine the form of $H_1$ for each type of Types 1 to 9. We need several symbols for that.

**Notation 6.1** (i) Let $\Lambda_1$ be the set of $12 \times 12$ permutation matrices

$$\left(\begin{array}{c|ccc} C_0 & O_3 & O_3 & O_3 \\ \hline O_3 & C_1 & C_2 & C_3 \\ O_3 & C_3 & C_1 & C_2 \\ O_3 & C_2 & C_3 & C_1 \end{array}\right),$$

where $C_i$ $(0 \le i \le 3)$ are $3 \times 3$ cyclic matrices.

Let $\Lambda_2$ be the set of $12 \times 12$ permutation matrices

$$S = \begin{pmatrix} P & O_3 & O_3 & O_3 \\ O_3 & C_0 & C_1 & C_2 \\ O_3 & C_3 & C_4 & C_5 \\ O_3 & C_6 & C_7 & C_8 \end{pmatrix},$$

where $P$ is a $3 \times 3$ permutation matrix and $C_i$ $(0 \le i \le 8)$ are $3 \times 3$ cyclic matrices.

Let $\Lambda_3$ be the set of $12 \times 12$ permutation matrices $\begin{pmatrix} A & O_3 & O_3 & O_3 \\ O_3 & B & O_3 & O_3 \\ O_3 & O_3 & C & O_3 \\ O_3 & O_3 & O_3 & D \end{pmatrix}$, where $A, B, C, D$ are $3 \times 3$ permutation matrices.

(ii) For a $3 \times 3$ matrix $X = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{pmatrix} = (x_{i,j})_{0 \le i,j \le 2}$ with entries from $\{0, 1\}$ and $f, g \in$Sym$\{0, 1, 2\}$, we define $X^{(f,g)} = (y_{i,j})_{0 \le i,j \le 2}$ by $y_{i,j} = x_{i^f, j^g}$ $(0 \le i, j \le 2)$. In particular, for $r, s \in \{1, 2\}$, set $X^{(f^r, f^s)} = X^{(r,s)}$ where $f = (0, 1, 2)$.

Then, let $\Phi_1$ be the set of $12 \times 12$ permutation matrices

$$\begin{pmatrix} C_0 & C_1 & C_2 & C_3 \\ X_0 & X_1 & X_2 & X_3 \\ X_0{}^{(1,1)} & X_1{}^{(1,1)} & X_2{}^{(1,1)} & X_3{}^{(1,1)} \\ X_0{}^{(2,2)} & X_1{}^{(2,2)} & X_2{}^{(2,2)} & X_3{}^{(2,2)} \end{pmatrix},$$

$\Phi_2$ the set of $12 \times 12$ permutation matrices

$$\begin{pmatrix} C_0 & C_1 & C_2 & C_3 \\ X_0 & X_1 & X_2 & X_3 \\ X_0{}^{(2,1)} & X_1{}^{(2,1)} & X_2{}^{(2,1)} & X_3{}^{(2,1)} \\ X_0{}^{(1,2)} & X_1{}^{(1,2)} & X_2{}^{(1,2)} & X_3{}^{(1,2)} \end{pmatrix}$$

and $\Phi_3$ the set of $12 \times 12$ permutation matrices

$$\begin{pmatrix} C_0 & C_1 & C_2 & C_3 \\ X_0 & X_1 & X_2 & X_3 \\ X_0{}^{(0,1)} & X_1{}^{(0,1)} & X_2{}^{(0,1)} & X_3{}^{(0,1)} \\ X_0{}^{(0,2)} & X_1{}^{(0,2)} & X_2{}^{(0,2)} & X_3{}^{(0,2)} \end{pmatrix},$$

where $C_i$ $(0 \le i \le 3)$ are cyclic matrices and $X_i$ $(0 \le i \le 3)$ are $3 \times 3$ matrices.

We remark that $|\Lambda_i|$ and $|\Phi_i|$ $(1 \le i \le 3)$ are not big. Actually, $|\Lambda_1| = 3^4 = 81$, $|\Lambda_2| = 6^2 \times 3^2 = 972$, $|\Lambda_3| = 6^4 = 1296$ and $|\Phi_1| = |\Phi_2| = |\Phi_3| = 4 \times 3 \times 9 \times 6 \times 3 = 1944$.

(iii) We define a $12 \times 12$ permutation matrix $X^{(f,g)} = (y_{i,j})_{0 \le i,j \le 11}$ by $y_{i,j} = x_{i^f, j^g}$ $(0 \le i, j \le 11)$ for a $12 \times 12$ permutation matrix $X = (x_{i,j})_{0 \le i,j \le 11}$ and $f \in$ Sym$\{0, 1, \ldots, 11\}$. In particular, we set $X^{(f,1)} = X^f$.

It follows that the actions of $\varphi$ and $\tau$ on both $\mathcal{P}$ and $\mathcal{B}$ in Types 1 to 9 are determined explicitly from Section 4.

### Type 1

**(6.1.1)** $\varphi = (x_0)(x_1)(x_2)(x_3,x_4,x_5)(x_6,x_7,x_8)(x_9,x_{10},x_{11})(x_{12})(x_{13})(x_{14})(x_{15},x_{16},x_{17})(x_{18},x_{19},x_{20})$ $(x_{21},x_{22},x_{23})(x_{24})(x_{25})(x_{26})(x_{27},x_{28},x_{29})(x_{30},x_{31},x_{32})(x_{33},x_{34},x_{35})(x_{36},x_{48},x_{60})(x_{37},x_{49},x_{61})(x_{38},x_{50},x_{62})$ $(x_{39},x_{51},x_{63})(x_{40},x_{52},x_{64})(x_{41},x_{53},x_{65})(x_{42},x_{54},x_{66})(x_{43},x_{55},x_{67})(x_{44},x_{56},x_{68})(x_{45},x_{57},x_{69})(x_{46},x_{58},x_{70})$ $(x_{47},x_{59},x_{71})(x_{72},x_{84},x_{96})(x_{73},x_{85},x_{97})(x_{74},x_{86},x_{98})(x_{75},x_{87},x_{99})(x_{76},x_{88},x_{100})(x_{77},x_{89},x_{101})(x_{78},x_{90},x_{102})$ $(x_{79},x_{91},x_{103})(x_{80},x_{92},x_{104})(x_{81},x_{93},x_{105})(x_{82},x_{94},x_{106})(x_{83},x_{95},x_{107})(x_{108},x_{120},x_{132})(x_{109},x_{121},x_{133})$ $(x_{110},x_{122},x_{134})(x_{111},x_{123},x_{135})(x_{112},x_{124},x_{136})(x_{113},x_{125},x_{137})$ $(x_{114},x_{126},x_{138})(x_{115},x_{127},x_{139})$ $(x_{116},x_{128},x_{140})(x_{117},x_{129},x_{141})$ $(x_{118},x_{130},x_{142})(x_{119},x_{131},x_{143})$ and

$\tau = (x_0,x_1,x_2)(x_3,x_6,x_9)(x_4,x_7,x_{10})(x_5,x_8,x_{11})(x_{12},x_{13},x_{14})(x_{15},x_{18},x_{21})(x_{16},x_{19},x_{22})(x_{17},x_{20},x_{23})$ $(x_{24},x_{25},x_{26})(x_{27},x_{30},x_{33})(x_{28},x_{31},x_{34})(x_{29},x_{32},x_{35})(x_{36},x_{61},x_{50})(x_{37},x_{62},x_{48})(x_{38},x_{60},x_{49})(x_{39},x_{64},x_{53})$ $(x_{40},x_{65},x_{51})(x_{41},x_{63},x_{52})(x_{42},x_{67},x_{56})(x_{43},x_{68},x_{54})(x_{44},x_{66},x_{55})(x_{45},x_{70},x_{59})(x_{46},x_{71},x_{57})(x_{47},x_{69},x_{58})$ $(x_{72},x_{85},x_{98})(x_{73},x_{86},x_{96})(x_{74},x_{84},x_{97})(x_{75},x_{88},x_{101})(x_{76},x_{89},x_{99})(x_{77},x_{87},x_{100})(x_{78},x_{91},x_{104})$ $(x_{79},x_{92},x_{102})(x_{80},x_{90},x_{103})(x_{81},x_{94},x_{107})(x_{82},x_{95},x_{105})(x_{83},x_{93},x_{106})(x_{108},x_{121},x_{134})(x_{109},x_{122},x_{132})$ $(x_{110},x_{120},x_{133})(x_{111},x_{124},x_{137})(x_{112},x_{125},x_{135})(x_{113},x_{123},x_{136})(x_{114},x_{127},x_{140})(x_{115},x_{128},x_{138})$ $(x_{116},x_{126},x_{139})(x_{117},x_{130},x_{143})(x_{118},x_{131},x_{141})(x_{119},x_{129},x_{142})$, where $x \in \{p, B\}$.

Proof. Since $|F_{\mathcal{P}_i}(\varphi)| = 3$ $(0 \leq i \leq 2)$, let $F_{\mathcal{P}_0}(\varphi) = \{p_0,p_1,p_2\}$, $F_{\mathcal{P}_1}(\varphi) = \{p_{12},p_{13},p_{14}\}$ and $F_{\mathcal{P}_2}(\varphi) = \{p_{24},p_{25},p_{26}\}$. Since $\langle\varphi\rangle$ acts semiregularly on $\mathcal{P}_0 \backslash F_{\mathcal{P}_0}(\varphi)$, let $\varphi^{\mathcal{P}_0} = (p_0)(p_1)(p_2)$ $(p_3,p_4,p_5)(p_6,p_7,p_8)(p_9,p_{10},p_{11})$. Since $\langle\tau\rangle$ acts semiregularly on $\mathcal{P}_0$, we may assume that $\tau^{\mathcal{P}_0} = (p_0,p_1,p_2)(p_3,p_6,p_9)\dots$. From this, we have ${p_3}^{\tau} = p_6$ and therefore ${p_3}^{\varphi\tau} = {p_3}^{\tau\varphi} = {p_6}^{\varphi}$. This yields ${p_4}^{\tau} = p_7$. By a similar argument, it follows that

$\tau^{\mathcal{P}_0} = (p_0,p_1,p_2)(p_3,p_6,p_9)$ $(p_4,p_7,p_{10})(p_5,p_8,p_{11})$. Similarly, we have
$\varphi^{\mathcal{P}_1} = (p_{12})(p_{13})(p_{14})(p_{15},p_{16},p_{17})$ $(p_{18},p_{19},p_{20})(p_{21},p_{22},p_{23})$,
$\tau^{\mathcal{P}_1} = (p_{12},p_{13},p_{14})(p_{15},p_{18},p_{21})(p_{16},p_{19},p_{22})(p_{17},p_{20},p_{23})$,
$\varphi^{\mathcal{P}_2} = (p_{24})(p_{25})(p_{26})(p_{27},p_{28},p_{29})(p_{30},p_{31},p_{32})(p_{33},p_{34},p_{35})$ and
$\tau^{\mathcal{P}_2} = (p_{24},p_{25},p_{26})(p_{27},p_{30},p_{33})(p_{28},p_{31},p_{34})(p_{29},p_{32},p_{35})$.

Since ${\mathcal{P}_i}^{\varphi\tau} = \mathcal{P}_i$ $(3 \leq i \leq 5)$, we may assume that
$\varphi\tau^{\mathcal{P}_3} = (p_{36},p_{37},p_{38})(p_{39},p_{40},p_{41})(p_{42},p_{43},p_{44})(p_{45},p_{46},p_{47})$,
$\varphi\tau^{\mathcal{P}_4} = (p_{48},p_{49},p_{50})(p_{51},p_{52},p_{53})(p_{54},p_{55},p_{56})(p_{57},p_{58},p_{59})$ and
$\varphi\tau^{\mathcal{P}_5} = (p_{60},p_{61},p_{62})(p_{63},p_{64},p_{65})(p_{66},p_{67},p_{68})(p_{69},p_{70},p_{71})$.
Since $\widetilde{\varphi} = \dots(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5)\dots$, we may assume that $\varphi^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = (p_{36},p_{48},p_{60})\dots$. From this, we have ${p_{36}}^{\varphi} = p_{48}$ and therefore ${p_{36}}^{\varphi\tau\varphi} = {p_{36}}^{\varphi\varphi\tau} = {p_{48}}^{\varphi\tau} = p_{49}$. This yields ${p_{37}}^{\varphi} = p_{49}$. By a similar argument, it follows that
$\varphi^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = (p_{36},p_{48},p_{60})(p_{37},p_{49},p_{61})(p_{38},p_{50},p_{62})\dots$. Similarly, we have
$\varphi^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = \dots(p_{39},p_{51},p_{63})(p_{40},p_{52},p_{64})(p_{41},p_{53},p_{65})\dots$,
$\varphi^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = \dots(p_{42},p_{54},p_{66})(p_{43},p_{55},p_{67})(p_{44},p_{56},p_{68})\dots$ and
$\varphi^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = \dots(p_{45},p_{57},p_{69})(p_{46},p_{58},p_{70})(p_{47},p_{59},p_{71})\dots$. Thus
$\varphi^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = (p_{36},p_{48},p_{60})(p_{37},p_{49},p_{61})(p_{38},p_{50},p_{62})(p_{39},p_{51},p_{63})$ $(p_{40},p_{52},p_{64})$ $(p_{41},p_{53},p_{65})$ $(p_{42},p_{54},p_{66})(p_{43},p_{55},p_{67})$ $(p_{44},p_{56},p_{68})(p_{45},p_{57},p_{69})(p_{46},p_{58},p_{70})$ $(p_{47},p_{59},p_{71})$. Since

$\varphi\tau^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = (p_{36}, p_{37}, p_{38})(p_{39}, p_{40}, p_{41})(p_{42}, p_{43}, p_{44})(p_{45}, p_{46}, p_{47})\ (p_{48}, p_{49}, p_{50})$ $(p_{51}, p_{52}, p_{53})(p_{54}, p_{55}, p_{56})(p_{57}, p_{58}, p_{59})\ (p_{60}, p_{61}, p_{62})(p_{63}, p_{64}, p_{65})(p_{66}, p_{67}, p_{68})$ $(p_{69}, p_{70}, p_{71})$, from $\tau = \varphi^2(\varphi\tau)$, it follows that
$\tau^{\mathcal{P}_3\cup\mathcal{P}_4\cup\mathcal{P}_5} = (p_{36}, p_{61}, p_{50})(p_{37}, p_{62}, p_{48})(p_{38}, p_{60}, p_{49})(p_{39}, p_{64}, p_{53})\ (p_{40}, p_{65}, p_{51})$ $(p_{41}, p_{63}, p_{52})(p_{42}, p_{67}, p_{56})(p_{43}, p_{68}, p_{54})\ (p_{44}, p_{66}, p_{55})(p_{45}, p_{70}, p_{59})(p_{46}, p_{71}, p_{57})$ $(p_{47}, p_{69}, p_{58})$.

Since ${\mathcal{P}_i}^{\varphi^2\tau} = \mathcal{P}_i$ $(6 \le i \le 8)$, we may assume that
$\varphi^2\tau^{\mathcal{P}_6} = (p_{72}, p_{73}, p_{74})(p_{75}, p_{76}, p_{77})(p_{78}, p_{79}, p_{80})(p_{81}, p_{82}, p_{83})$,
$\varphi^2\tau^{\mathcal{P}_7} = (p_{84}, p_{85}, p_{86})(p_{87}, p_{88}, p_{89})(p_{90}, p_{91}, p_{92})(p_{93}, p_{94}, p_{95})$ and
$\varphi^2\tau^{\mathcal{P}_8} = (p_{96}, p_{97}, p_{98})(p_{99}, p_{100}, p_{101})(p_{102}, p_{103}, p_{104})(p_{105}, p_{106}, p_{107})$.
Since $\widetilde{\varphi} = \dots(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)\dots$, we may assume that $\varphi^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = (p_{72}, p_{84}, p_{96})$ $\dots$. From this, we have ${p_{72}}^{\varphi} = p_{84}$ and therefore ${p_{72}}^{\varphi^2\tau\varphi} = {p_{72}}^{\varphi\varphi^2\tau} = {p_{84}}^{\varphi^2\tau}$ $= p_{85}$. This yields ${p_{73}}^{\varphi} = p_{85}$. By a similar argument, it follows that
$\varphi^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = (p_{72}, p_{84}, p_{96})(p_{73}, p_{85}, p_{97})(p_{74}, p_{86}, p_{98})\dots$. Similarly, we have
$\varphi^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = \dots(p_{75}, p_{87}, p_{99})(p_{76}, p_{88}, p_{100})(p_{77}, p_{89}, p_{101})\dots$,
$\varphi^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = \dots(p_{78}, p_{90}, p_{102})(p_{79}, p_{91}, p_{103})(p_{80}, p_{92}, p_{104})\dots$ and
$\varphi^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = \dots(p_{81}, p_{93}, p_{105})(p_{82}, p_{94}, p_{106})(p_{83}, p_{95}, p_{107})\dots$. Thus
$\varphi^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = (p_{72}, p_{84}, p_{96})(p_{73}, p_{85}, p_{97})(p_{74}, p_{86}, p_{98})(p_{75}, p_{87}, p_{99})\ (p_{76}, p_{88}, p_{100})$ $(p_{77}, p_{89}, p_{101})(p_{78}, p_{90}, p_{102})(p_{79}, p_{91}, p_{103})\ (p_{80}, p_{92}, p_{104})(p_{81}, p_{93}, p_{105})(p_{82}, p_{94}, p_{106})$ $(p_{83}, p_{95}, p_{107})$. Since
$\varphi^2\tau^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = (p_{72}, p_{73}, p_{74})(p_{75}, p_{76}, p_{77})(p_{78}, p_{79}, p_{80})(p_{81}, p_{82}, p_{83})\ (p_{84}, p_{85}, p_{86})$ $(p_{87}, p_{88}, p_{89})(p_{90}, p_{91}, p_{92})(p_{93}, p_{94}, p_{95})\ (p_{96}, p_{97}, p_{98})(p_{99}, p_{100}, p_{101})(p_{102}, p_{103}, p_{104})$ $(p_{105}, p_{106}, p_{107})$, from $\tau = \varphi(\varphi^2\tau)$, it follows that
$\tau^{\mathcal{P}_6\cup\mathcal{P}_7\cup\mathcal{P}_8} = (p_{72}, p_{85}, p_{98})(p_{73}, p_{86}, p_{96})\ (p_{74}, p_{84}, p_{97})(p_{75}, p_{88}, p_{101})\ (p_{76}, p_{89}, p_{99})$ $(p_{77}, p_{87}, p_{100})\ (p_{78}, p_{91}, p_{104})(p_{79}, p_{92}, p_{102})\ (p_{80}, p_{90}, p_{103})(p_{81}, p_{94}, p_{107})\ (p_{82}, p_{95}, p_{105})$ $(p_{83}, p_{93}, p_{106})$.

The actions of $\varphi$ and $\tau$ on $\mathcal{P}_9 \cup \mathcal{P}_{10} \cup \mathcal{P}_{11}$ are obtained by the same argument as the above, because ${\mathcal{P}_i}^{\varphi^2\tau} = \mathcal{P}_i$ $(9 \le i \le 11)$. That is
$\varphi^{\mathcal{P}_9\cup\mathcal{P}_{10}\cup\mathcal{P}_{11}} = (p_{108}, p_{120}, p_{132})(p_{109}, p_{121}, p_{133})\ (p_{110}, p_{122}, p_{134})(p_{111}, p_{123}, p_{135})$ $(p_{112}, p_{124}, p_{136})(p_{113}, p_{125}, p_{137})\ (p_{114}, p_{126}, p_{138})(p_{115}, p_{127}, p_{139})\ (p_{116}, p_{128}, p_{140})$ $(p_{117}, p_{129}, p_{141})\ (p_{118}, p_{130}, p_{142})(p_{119}, p_{131}, p_{143})$ and
$\tau^{\mathcal{P}_9\cup\mathcal{P}_{10}\cup\mathcal{P}_{11}} = (p_{108}, p_{121}, p_{134})(p_{109}, p_{122}, p_{132})\ (p_{110}, p_{120}, p_{133})(p_{111}, p_{124}, p_{137})$ $(p_{112}, p_{125}, p_{135})(p_{113}, p_{123}, p_{136})(p_{114}, p_{127}, p_{140})(p_{115}, p_{128}, p_{138})\ (p_{116}, p_{126}, p_{139})$ $(p_{117}, p_{130}, p_{143})\ (p_{118}, p_{131}, p_{141})(p_{119}, p_{129}, p_{142})$.

Therefore we have the actions of $\varphi$ and $\tau$ on $\mathcal{P}$ described in (6.1.1). Since the permutation group $(G, \mathcal{P})$ is isomorphic to the permutation group $(G, \mathcal{B})$, we may assume that the numbering of the actions of $\varphi$ and $\tau$ on $\mathcal{B}$ are the same as these on the points. □

**(6.1.2)** Let $f = (0)(1)(2)(3,4,5)(6,7,8)(9,10,11) \in \mathrm{Sym}\{0, 1, \dots, 11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & {A_0}^f & {A_0}^{f^2} & A_1 & {A_1}^f & {A_1}^{f^2} & A_2 & {A_2}^f & {A_2}^{f^2} \\ S_3 & S_4 & S_5 & B_0 & {B_0}^f & {B_0}^{f^2} & B_1 & {B_1}^f & {B_1}^{f^2} & B_2 & {B_2}^f & {B_2}^{f^2} \\ S_6 & S_7 & S_8 & C_0 & {C_0}^f & {C_0}^{f^2} & C_1 & {C_1}^f & {C_1}^{f^2} & C_2 & {C_2}^f & {C_2}^{f^2} \end{array}\right),$$

where $S_0, \dots, S_8 \in \Lambda_1$, $A_0, B_0, C_0 \in \Phi_1$, $A_i, B_i, C_i \in \Phi_2$ $(i = 1, 2)$.

Proof. We remark that $h_{i,j} = 1 \iff p_i I B_j \iff {p_i}^\mu I {B_j}^\mu \iff h_{i',j'} = 1$ and $h_{i,j} = 0 \iff p_i \not\!I B_j \iff {p_i}^\mu \not\!I {B_j}^\mu \iff h_{i',j'} = 0$, where ${p_i}^\mu = p_{i'}$ and ${B_j}^\mu = B_{j'}$, for $0 \le i, j \le 143$, $\mu \in G$.

We define an action on $\mathcal{P} \times \mathcal{B}$ of $G$ by $(p, B)^\mu = (p^\mu, B^\mu)$ for $(p, B) \in \mathcal{P} \times \mathcal{B}$. Then, if $A \subseteq \mathcal{P} \times \mathcal{B}$ is a $G$-orbit, $h_{i,j} = h_{i',j'}$ for $(p_i, B_j)$, $(p_{i'}, B_{j'}) \in A$.

Since $(p_3, B_3)^G = \{(p_3, B_3), (p_4, B_4), \dots, (p_{11}, B_{11})\}$,

$(p_3, B_4)^G = \{(p_3, B_4), (p_4, B_5), (p_5, B_3), (p_6, B_7), (p_7, B_8), (p_8, B_6), (p_9, B_{10}), (p_{10}, B_{11}), (p_{11}, B_9)\}$,

$(p_3, B_5)^G = \{(p_3, B_5), (p_4, B_3), (p_5, B_4), (p_6, B_8), (p_7, B_6), (p_8, B_7), (p_9, B_{11}), (p_{10}, B_9), (p_{11}, B_{10})\}$,

$(p_3, B_6)^G = \{(p_3, B_6), (p_4, B_7), (p_5, B_8), (p_6, B_9), (p_7, B_{10}), (p_8, B_{11}), (p_9, B_3), (p_{10}, B_4), (p_{11}, B_5)\}$,

$(p_3, B_7)^G = \{(p_3, B_7), (p_4, B_8), (p_5, B_6), (p_6, B_{10}), (p_7, B_{11}), (p_8, B_9), (p_9, B_4), (p_{10}, B_5), (p_{11}, B_3)\}$,

$(p_3, B_8)^G = \{(p_3, B_8), (p_4, B_6), (p_5, B_7), (p_6, B_{11}), (p_7, B_9), (p_8, B_{10}), (p_9, B_5), (p_{10}, B_3), (p_{11}, B_4)\}$,

$(p_3, B_9)^G = \{(p_3, B_9), (p_4, B_{10}), (p_5, B_{11}), (p_6, B_3), (p_7, B_4), (p_8, B_5), (p_9, B_6), (p_{10}, B_7), (p_{11}, B_8)\}$,

$(p_3, B_{10})^G = \{(p_3, B_{10}), (p_4, B_{11}), (p_5, B_9), (p_6, B_4), (p_7, B_5), (p_8, B_3), (p_9, B_7), (p_{10}, B_8), (p_{11}, B_6)\}$, and

$(p_3, B_{11})^G = \{(p_3, B_{11}), (p_4, B_9), (p_5, B_{10}), (p_6, B_5), (p_7, B_3), (p_8, B_4), (p_9, B_8), (p_{10}, B_6), (p_{11}, B_7)\}$,

if we set $h_0 = h_{0,0}, h_1 = h_{0,1}, h_2 = h_{0,2}$ and $h_3 = h_{3,3}, h_4 = h_{3,4}, \dots, h_{11} = h_{3,11}$, then $H_{0,0} = \left(\begin{array}{c|ccc} C_0 & O_3 & O_3 & O_3 \\ \hline O_3 & C_1 & C_2 & C_3 \\ O_3 & C_3 & C_1 & C_2 \\ O_3 & C_2 & C_3 & C_1 \end{array}\right)$, where $C_0 = \begin{pmatrix} h_0 & h_1 & h_2 \\ h_2 & h_0 & h_1 \\ h_1 & h_2 & h_0 \end{pmatrix}$, $C_1 = \begin{pmatrix} h_3 & h_4 & h_5 \\ h_5 & h_3 & h_4 \\ h_4 & h_5 & h_3 \end{pmatrix}$, $C_2 = \begin{pmatrix} h_6 & h_7 & h_8 \\ h_8 & h_6 & h_7 \\ h_7 & h_8 & h_6 \end{pmatrix}$ and $C_3 = \begin{pmatrix} h_9 & h_{10} & h_{11} \\ h_{11} & h_9 & h_{10} \\ h_{10} & h_{11} & h_9 \end{pmatrix}$. Set $S_0 = H_{0,0} \in \Lambda_1$.

By repeating the argument similarly, we obtain $H_1 = \left(\begin{array}{ccc|c|c|c} S_0 & S_1 & S_2 & & & \\ S_3 & S_4 & S_5 & * & ** & *** \\ S_6 & S_7 & S_8 & & & \end{array}\right)$, where $S_0, S_1, \dots, S_8 \in \Lambda_1$. By a similar argument as above, we can find the remaining submatrices of $H_1$. Note that $G$ acts semiregularly on $\bigcup_{0 \le i \le 2} \mathcal{P}_i \times \bigcup_{3 \le j \le 11} \mathcal{B}_j$. For example, since $(p_3, B_{36})^G = \{(p_3, B_{36}), (p_7, B_{37}), (p_{11}, B_{38}), (p_4, B_{48}), (p_8, B_{49}), (p_9, B_{50}), (p_5, B_{60}), (p_6, B_{61}), (p_{10}, B_{62})\}$, we have $h_{3,36} = h_{7,37} = h_{11,38} = h_{4,48} = h_{8,49} = h_{9,50} = h_{5,60} = h_{6,61} = h_{10,62}$. □

The proof of statements which will appear in the remaining types are omitted, because we can prove these by arguments similar to those used in Type 1.

**Type 2**

**(6.2.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_1, x_2)(x_3, x_6, x_9)(x_4, x_7, x_{10})(x_5, x_8, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{18}, x_{21})(x_{16}, x_{19}, x_{22})(x_{17}, x_{20}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27}, x_{30}, x_{33})(x_{28}, x_{31}, x_{34})(x_{29}, x_{32}, x_{35})$
$(x_{36}, x_{37}, x_{38})(x_{39}, x_{40}, x_{41})(x_{42}, x_{43}, x_{44})(x_{45}, x_{46}, x_{47})$
$(x_{48}, x_{49}, x_{50})(x_{51}, x_{52}, x_{53})(x_{54}, x_{55}, x_{56})(x_{57}, x_{58}, x_{59})$
$(x_{60}, x_{61}, x_{62})(x_{63}, x_{64}, x_{65})(x_{66}, x_{67}, x_{68})(x_{69}, x_{70}, x_{71})$
$(x_{72}, x_{97}, x_{86})(x_{73}, x_{98}, x_{84})(x_{74}, x_{96}, x_{85})(x_{75}, x_{100}, x_{89})$
$(x_{76}, x_{101}, x_{87})(x_{77}, x_{99}, x_{88})(x_{78}, x_{103}, x_{92})(x_{79}, x_{104}, x_{90})$
$(x_{80}, x_{102}, x_{91})(x_{81}, x_{106}, x_{95})(x_{82}, x_{107}, x_{93})(x_{83}, x_{105}, x_{94})$
$(x_{108}, x_{121}, x_{134})(x_{109}, x_{122}, x_{132})(x_{110}, x_{120}, x_{133})(x_{111}, x_{124}, x_{137})$
$(x_{112}, x_{125}, x_{135})(x_{113}, x_{123}, x_{136})(x_{114}, x_{127}, x_{140})(x_{115}, x_{128}, x_{138})$
$(x_{116}, x_{126}, x_{139})(x_{117}, x_{130}, x_{143})(x_{118}, x_{131}, x_{141})(x_{119}, x_{129}, x_{142})$, where $x \in \{p, B\}$.

**(6.2.2)** Let $f = (0)(1)(2)(3, 4, 5)(6, 7, 8)(9, 10, 11) \in \mathrm{Sym}\{0, 1, \ldots, 11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_0 & S_2 & A_0 & {A_0}^f & {A_0}^{f^2} & A_1 & {A_1}^f & {A_1}^{f^2} & A_2 & {A_2}^f & {A_2}^{f^2} \\ S_3 & S_4 & S_5 & B_0 & {B_0}^f & {B_0}^{f^2} & B_1 & {B_1}^f & {B_1}^{f^2} & B_2 & {B_2}^f & {B_2}^{f^2} \\ S_6 & S_7 & S_8 & C_0 & {C_0}^f & {C_0}^{f^2} & C_1 & {C_1}^f & {C_1}^{f^2} & C_2 & {C_2}^f & {C_2}^{f^2} \end{array}\right),$$

where $S_0, \ldots, S_8 \in \Lambda_1,\ A_0, B_0, C_0 \in \Phi_3,\ A_1, B_1, C_1 \in \Phi_1,\ A_2, B_2, C_2 \in \Phi_2$.

**Type 3**

**(6.3.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$

$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_1, x_2)(x_3)(x_4)(x_5)(x_6, x_8, x_7)(x_9, x_{10}, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15})(x_{16})(x_{17})(x_{18}, x_{20}, x_{19})(x_{21}, x_{22}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27})(x_{28})(x_{29})(x_{30}, x_{32}, x_{31})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{72}, x_{108})(x_{48}, x_{84}, x_{120})(x_{60}, x_{96}, x_{132})(x_{37}, x_{73}, x_{109})$
$(x_{49}, x_{85}, x_{121})(x_{61}, x_{97}, x_{133})(x_{38}, x_{74}, x_{110})(x_{50}, x_{86}, x_{122})$
$(x_{62}, x_{98}, x_{134})(x_{39}, x_{75}, x_{111})(x_{51}, x_{87}, x_{123})(x_{63}, x_{99}, x_{135})$
$(x_{40}, x_{76}, x_{112})(x_{52}, x_{88}, x_{124})(x_{64}, x_{100}, x_{136})(x_{41}, x_{77}, x_{113})$
$(x_{53}, x_{89}, x_{125})(x_{65}, x_{101}, x_{137})(x_{42}, x_{78}, x_{114})(x_{54}, x_{90}, x_{126})$
$(x_{66}, x_{102}, x_{138})(x_{43}, x_{79}, x_{115})(x_{55}, x_{91}, x_{127})(x_{67}, x_{103}, x_{139})$
$(x_{44}, x_{80}, x_{116})(x_{56}, x_{92}, x_{128})(x_{68}, x_{104}, x_{140})(x_{45}, x_{81}, x_{117})$
$(x_{57}, x_{93}, x_{129})(x_{69}, x_{105}, x_{141})(x_{46}, x_{82}, x_{118})(x_{58}, x_{94}, x_{130})$
$(x_{70}, x_{106}, x_{142})(x_{47}, x_{83}, x_{119})(x_{59}, x_{95}, x_{131})(x_{71}, x_{107}, x_{143})$, where $x \in \{p, B\}$.

**(6.3.2)** Let $f = (0)(1)(2)(3,5,4)(6,8,7)(9,11,10)$, $g = (0,2,1)(3)(4)(5)(6,7,8)(9,11,10)$, $h = (0,2,1)(3,5,4)(6)(7)(8)(9,10,11)$, $k = (0,2,1)(3,4,5)(6,8,7)(9)(10)(11) \in \mathrm{Sym}\{0,1,\ldots,11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & {A_0}^f & {A_0}^{f^2} & {A_0}^g & {A_0}^h & {A_0}^k & {A_0}^{g^2} & {A_0}^{k^2} & {A_0}^{h^2} \\ S_3 & S_4 & S_5 & A_1 & {A_1}^f & {A_1}^{f^2} & {A_1}^g & {A_1}^h & {A_1}^k & {A_1}^{g^2} & {A_1}^{k^2} & {A_1}^{h^2} \\ S_6 & S_7 & S_8 & A_2 & {A_2}^f & {A_2}^{f^2} & {A_2}^g & {A_2}^h & {A_2}^k & {A_2}^{g^2} & {A_2}^{k^2} & {A_2}^{h^2} \end{array}\right),$$

where $S_0, S_1, \ldots, S_8 \in \Lambda_3$ and $A_0, A_1, A_2$ are $12 \times 12$ permutation matrices.

## Type 4

**(6.4.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$, where $x \in \{p, B\}$.

$\tau = (p_0, p_{12}, p_{24})(p_1, p_{13}, p_{25})(p_2, p_{14}, p_{26})(p_3, p_{15}, p_{27})$
$(p_4, p_{16}, p_{28})(p_5, p_{17}, p_{29})(p_6, p_{18}, p_{30})(p_7, p_{19}, p_{31})$
$(p_8, p_{20}, p_{32})(p_9, p_{21}, p_{33})(p_{10}, p_{22}, p_{34})(p_{11}, p_{23}, p_{35})$
$(p_{36}, p_{72}, p_{108})(p_{48}, p_{84}, p_{120})(p_{60}, p_{96}, p_{132})(p_{37}, p_{73}, p_{109})$
$(p_{49}, p_{85}, p_{121})(p_{61}, p_{97}, p_{133})(p_{38}, p_{74}, p_{110})(p_{50}, p_{86}, p_{122})$
$(p_{62}, p_{98}, p_{134})(p_{39}, p_{75}, p_{111})(p_{51}, p_{87}, p_{123})(p_{63}, p_{99}, p_{135})$
$(p_{40}, p_{76}, p_{112})(p_{52}, p_{88}, p_{124})(p_{64}, p_{100}, p_{136})(p_{41}, p_{77}, p_{113})$
$(p_{53}, p_{89}, p_{125})(p_{65}, p_{101}, p_{137})(p_{42}, p_{78}, p_{114})(p_{54}, p_{90}, p_{126})$
$(p_{66}, p_{102}, p_{138})(p_{43}, p_{79}, p_{115})(p_{55}, p_{91}, p_{127})(p_{67}, p_{103}, p_{139})$

$(p_{44}, p_{80}, p_{116})(p_{56}, p_{92}, p_{128})(p_{68}, p_{104}, p_{140})(p_{45}, p_{81}, p_{117})$
$(p_{57}, p_{93}, p_{129})(p_{69}, p_{105}, p_{141})(p_{46}, p_{82}, p_{118})(p_{58}, p_{94}, p_{130})$
$(p_{70}, p_{106}, p_{142})(p_{47}, p_{83}, p_{119})(p_{59}, p_{95}, p_{131})(p_{71}, p_{107}, p_{143})$ and
$\tau = (B_0)(B_1)(B_2)(B_3, B_6, B_9)(B_4, B_7, B_{10})(B_5, B_8, B_{11})$
$(B_{12}, B_{13}, B_{14})(B_{15}, B_{18}, B_{21})(B_{16}, B_{19}, B_{22})(B_{17}, B_{20}, B_{23})$
$(B_{24}, B_{25}, B_{26})(B_{27}, B_{30}, B_{33})(B_{28}, B_{31}, B_{34})(B_{29}, B_{32}, B_{35})$
$(B_{36}, B_{72}, B_{108})(B_{48}, B_{84}, B_{120})(B_{60}, B_{96}, B_{132})(B_{37}, B_{73}, B_{109})$
$(B_{49}, B_{85}, B_{121})(B_{61}, B_{97}, B_{133})(B_{38}, B_{74}, B_{110})(B_{50}, B_{86}, B_{122})$
$(B_{62}, B_{98}, B_{134})(B_{39}, B_{75}, B_{111})(B_{51}, B_{87}, B_{123})(B_{63}, B_{99}, B_{135})$
$(B_{40}, B_{76}, B_{112})(B_{52}, B_{88}, B_{124})(B_{64}, B_{100}, B_{136})(B_{41}, B_{77}, B_{113})$
$(B_{53}, B_{89}, B_{125})(B_{65}, B_{101}, B_{137})(B_{42}, B_{78}, B_{114})(B_{54}, B_{90}, B_{126})$
$(B_{66}, B_{102}, B_{138})(B_{43}, B_{79}, B_{115})(B_{55}, B_{91}, B_{127})(B_{67}, B_{103}, B_{139})$
$(B_{44}, B_{80}, B_{116})(B_{56}, B_{92}, B_{128})(B_{68}, B_{104}, B_{140})(B_{45}, B_{81}, B_{117})$
$(B_{57}, B_{93}, B_{129})(B_{69}, B_{105}, B_{141})(B_{46}, B_{82}, B_{118})(B_{58}, B_{94}, B_{130})$
$(B_{70}, B_{106}, B_{142})(B_{47}, B_{83}, B_{119})(B_{59}, B_{95}, B_{131})(B_{71}, B_{107}, B_{143})$.

**(6.4.2)** (i) For a $3 \times 3$ matrix $P = (p_{i,j})_{0 \le i,j \le 2}$, set

$$P^{[1]} = \begin{pmatrix} p_{0,2} & p_{0,0} & p_{0,1} \\ p_{1,2} & p_{1,0} & p_{1,1} \\ p_{2,2} & p_{2,0} & p_{2,1} \end{pmatrix} \text{ and } P^{[2]} = \begin{pmatrix} p_{0,1} & p_{0,2} & p_{0,0} \\ p_{1,1} & p_{1,2} & p_{1,0} \\ p_{2,1} & p_{2,2} & p_{2,0} \end{pmatrix}.$$

(ii) For $S = \begin{pmatrix} P & O_3 & O_3 & O_3 \\ O_3 & C_0 & C_1 & C_2 \\ O_3 & C_3 & C_4 & C_5 \\ O_3 & C_6 & C_7 & C_8 \end{pmatrix} \in \Phi_1$ set

$$S^{(*0)} = \begin{pmatrix} P & O_3 & O_3 & O_3 \\ O_3 & C_2 & C_0 & C_1 \\ O_3 & C_5 & C_3 & C_4 \\ O_3 & C_8 & C_6 & C_7 \end{pmatrix}, \quad S^{(*1)} = \begin{pmatrix} P^{[1]} & O_3 & O_3 & O_3 \\ O_3 & C_2 & C_0 & C_1 \\ O_3 & C_5 & C_3 & C_4 \\ O_3 & C_8 & C_6 & C_7 \end{pmatrix},$$

$$S^{(**0)} = \begin{pmatrix} P & O_3 & O_3 & O_3 \\ O_3 & C_1 & C_2 & C_0 \\ O_3 & C_4 & C_5 & C_3 \\ O_3 & C_7 & C_8 & C_6 \end{pmatrix} \text{ and } S^{(**1)} = \begin{pmatrix} P^{[2]} & O_3 & O_3 & O_3 \\ O_3 & C_1 & C_2 & C_0 \\ O_3 & C_4 & C_5 & C_3 \\ O_3 & C_7 & C_8 & C_6 \end{pmatrix}.$$

**(6.4.3)** Let $f = (0)(1)(2)(3,5,4)(6,8,7)(9,11,10) \in \mathrm{Sym}\{0,1,\ldots,11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & A_0{}^f & A_0{}^{f^2} & A_2 & A_2{}^f & A_2{}^{f^2} & A_1 & A_1{}^f & A_1{}^{f^2} \\ S_0{}^{(*0)} & S_1{}^{(*1)} & S_2{}^{(*1)} & A_1 & A_1{}^f & A_1{}^{f^2} & A_0 & A_0{}^f & A_0{}^{f^2} & A_2 & A_2{}^f & A_2{}^{f^2} \\ S_0{}^{(**0)} & S_1{}^{(**1)} & S_2{}^{(**1)} & A_2 & A_2{}^f & A_2{}^{f^2} & A_1 & A_1{}^f & A_1{}^{f^2} & A_0 & A_0{}^f & A_0{}^{f^2} \end{array}\right),$$

where $S_0, S_1, S_2 \in \Lambda_2$ and $A_0, A_1, A_2$ are $12 \times 12$ permutation matrices.

**Type 5**

**(6.5.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_1, x_2)(x_3, x_6, x_9)(x_4, x_7, x_{10})(x_5, x_8, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{18}, x_{21})(x_{16}, x_{19}, x_{22})(x_{17}, x_{20}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27}, x_{30}, x_{33})(x_{28}, x_{31}, x_{34})(x_{29}, x_{32}, x_{35})$
$(x_{36}, x_{72}, x_{108})(x_{48}, x_{84}, x_{120})(x_{60}, x_{96}, x_{132})(x_{37}, x_{73}, x_{109})$
$(x_{49}, x_{85}, x_{121})(x_{61}, x_{97}, x_{133})(x_{38}, x_{74}, x_{110})(x_{50}, x_{86}, x_{122})$
$(x_{62}, x_{98}, x_{134})(x_{39}, x_{75}, x_{111})(x_{51}, x_{87}, x_{123})(x_{63}, x_{99}, x_{135})$
$(x_{40}, x_{76}, x_{112})(x_{52}, x_{88}, x_{124})(x_{64}, x_{100}, x_{136})(x_{41}, x_{77}, x_{113})$
$(x_{53}, x_{89}, x_{125})(x_{65}, x_{101}, x_{137})(x_{42}, x_{78}, x_{114})(x_{54}, x_{90}, x_{126})$
$(x_{66}, x_{102}, x_{138})(x_{43}, x_{79}, x_{115})(x_{55}, x_{91}, x_{127})(x_{67}, x_{103}, x_{139})$
$(x_{44}, x_{80}, x_{116})(x_{56}, x_{92}, x_{128})(x_{68}, x_{104}, x_{140})(x_{45}, x_{81}, x_{117})$
$(x_{57}, x_{93}, x_{129})(x_{69}, x_{105}, x_{141})(x_{46}, x_{82}, x_{118})(x_{58}, x_{94}, x_{130})$
$(x_{70}, x_{106}, x_{142})(x_{47}, x_{83}, x_{119})(x_{59}, x_{95}, x_{131})(x_{71}, x_{107}, x_{143})$, where $x \in \{p, B\}$.

**(6.5.2)** Let $f = (0)(1)(2)(3, 5, 4)(6, 8, 7)(9, 11, 10)$,
$g = (0, 2, 1)(3, 9, 6)(4, 10, 7)(5, 11, 8)$, $h = (0, 2, 1)(3, 11, 7)(4, 9, 8)(5, 10, 6)$,
$k = (0, 2, 1)(3, 10, 8)(4, 11, 6)(5, 9, 7) \in \mathrm{Sym}\{0, 1, \ldots, 11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & {A_0}^f & {A_0}^{f^2} & {A_0}^g & {A_0}^h & {A_0}^k & {A_0}^{g^2} & {A_0}^{k^2} & {A_0}^{h^2} \\ S_3 & S_4 & S_5 & A_1 & {A_1}^f & {A_1}^{f^2} & {A_1}^g & {A_1}^h & {A_1}^k & {A_1}^{g^2} & {A_1}^{k^2} & {A_1}^{h^2} \\ S_6 & S_7 & S_8 & A_2 & {A_2}^f & {A_2}^{f^2} & {A_2}^g & {A_2}^h & {A_2}^k & {A_2}^{g^2} & {A_2}^{k^2} & {A_2}^{h^2} \end{array}\right),$$

where $S_0, \ldots, S_8 \in \Lambda_1$ and $A_0,\ A_1,\ A_2$ are $12 \times 12$ permutation matrices.

**Type 6**

**(6.6.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$

$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{16}, x_{29})$
$(x_4, x_{17}, x_{27})(x_5, x_{15}, x_{28})(x_6, x_{19}, x_{32})(x_7, x_{20}, x_{30})$
$(x_8, x_{18}, x_{31})(x_9, x_{22}, x_{35})(x_{10}, x_{23}, x_{33})(x_{11}, x_{21}, x_{34})$
$(x_{36}, x_{49}, x_{62})(x_{37}, x_{50}, x_{60})(x_{38}, x_{48}, x_{61})(x_{39}, x_{52}, x_{65})$
$(x_{40}, x_{53}, x_{63})(x_{41}, x_{51}, x_{64})(x_{42}, x_{55}, x_{68})(x_{43}, x_{56}, x_{66})$
$(x_{44}, x_{54}, x_{67})(x_{45}, x_{58}, x_{71})(x_{46}, x_{59}, x_{69})(x_{47}, x_{57}, x_{70})$
$(x_{72}, x_{85}, x_{98})(x_{73}, x_{86}, x_{96})(x_{74}, x_{84}, x_{97})(x_{75}, x_{88}, x_{101})$
$(x_{76}, x_{89}, x_{99})(x_{77}, x_{87}, x_{100})(x_{78}, x_{91}, x_{104})(x_{79}, x_{92}, x_{102})$
$(x_{80}, x_{90}, x_{103})(x_{81}, x_{94}, x_{107})(x_{82}, x_{95}, x_{105})(x_{83}, x_{93}, x_{106})$
$(x_{108}, x_{121}, x_{134})(x_{109}, x_{122}, x_{132})(x_{110}, x_{120}, x_{133})(x_{111}, x_{124}, x_{137})$
$(x_{112}, x_{125}, x_{135})(x_{113}, x_{123}, x_{136})(x_{114}, x_{127}, x_{140})(x_{115}, x_{128}, x_{138})$
$(x_{116}, x_{126}, x_{139})(x_{117}, x_{130}, x_{143})(x_{118}, x_{131}, x_{141})(x_{119}, x_{129}, x_{142})$, where $x \in \{p, B\}$.

**(6.6.2)** Let $f = (0)(1)(2)(3,4,5)(6,7,8)(9,10,11)$, $g = (0,1,2)(3,4,5)(6,7,8)(9,10,11) \in \mathrm{Sym}\{0,1,\ldots,11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & {A_0}^{(f^2,1)} & {A_0}^{(f,1)} & A_1 & {A_1}^{(f^2,1)} & {A_1}^{(f,1)} & A_2 & {A_2}^{(f^2,1)} & {A_2}^{(f,1)} \\ S_2 & S_0 & S_1 & {A_0}^{(1,g^2)} & {A_0}^{(f^2,g^2)} & {A_0}^{(f,g^2)} & {A_1}^{(1,g^2)} & {A_1}^{(f^2,g^2)} & {A_1}^{(f,g^2)} & {A_2}^{(1,g^2)} & {A_2}^{(f^2,g^2)} & {A_2}^{(f,g^2)} \\ S_1 & S_2 & S_0 & {A_0}^{(1,g)} & {A_0}^{(f^2,g)} & {A_0}^{(f,g)} & {A_1}^{(1,g)} & {A_1}^{(f^2,g)} & {A_1}^{(f,g)} & {A_2}^{(1,g)} & {A_2}^{(f^2,g)} & {A_2}^{(f,g)} \end{array}\right),$$

where $S_0, S_1, S_2 \in \Lambda_2$ and $A_0$, $A_1$ and $A_2$ are $12 \times 12$ permutation matrices.

## Type 7

**(6.7.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{16}, x_{29})$
$(x_4, x_{17}, x_{27})(x_5, x_{15}, x_{28})(x_6, x_{19}, x_{32})(x_7, x_{20}, x_{30})$
$(x_8, x_{18}, x_{31})(x_9, x_{22}, x_{35})(x_{10}, x_{23}, x_{33})(x_{11}, x_{21}, x_{34})$
$(x_{36}, x_{61}, x_{50})(x_{37}, x_{62}, x_{48})(x_{38}, x_{60}, x_{49})(x_{39}, x_{64}, x_{53})$

$(x_{40}, x_{65}, x_{51})(x_{41}, x_{63}, x_{52})(x_{42}, x_{67}, x_{56})(x_{43}, x_{68}, x_{54})$
$(x_{44}, x_{66}, x_{55})(x_{45}, x_{70}, x_{59})(x_{46}, x_{71}, x_{57})(x_{47}, x_{69}, x_{58})$
$(x_{72}, x_{85}, x_{98})(x_{73}, x_{86}, x_{96})(x_{74}, x_{84}, x_{97})(x_{75}, x_{88}, x_{101})$
$(x_{76}, x_{89}, x_{99})(x_{77}, x_{87}, x_{100})(x_{78}, x_{91}, x_{104})(x_{79}, x_{92}, x_{102})$
$(x_{80}, x_{90}, x_{103})(x_{81}, x_{94}, x_{107})(x_{82}, x_{95}, x_{105})(x_{83}, x_{93}, x_{106})$
$(x_{108}, x_{121}, x_{134})(x_{109}, x_{122}, x_{132})(x_{110}, x_{120}, x_{133})(x_{111}, x_{124}, x_{137})$
$(x_{112}, x_{125}, x_{135})(x_{113}, x_{123}, x_{136})(x_{114}, x_{127}, x_{140})(x_{115}, x_{128}, x_{138})$
$(x_{116}, x_{126}, x_{139})(x_{117}, x_{130}, x_{143})(x_{118}, x_{131}, x_{141})(x_{119}, x_{129}, x_{142})$, where $x \in \{p, B\}$.

**(6.7.2)** Let $f = (0)(1)(2)(3, 4, 5)(6, 7, 8)(9, 10, 11)$,
$g = (0, 1, 2)(3, 4, 5)(6, 7, 8)(9, 10, 11) \in \mathrm{Sym}\{0, 1, \ldots, 11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & {A_0}^{(f^2,1)} & {A_0}^{(f,1)} & A_1 & {A_1}^{(f^2,1)} & {A_1}^{(f,1)} & A_2 & {A_2}^{(f^2,1)} & {A_2}^{(f,1)} \\ S_2 & S_0 & S_1 & {A_0}^{(f,g^2)} & {A_0}^{(1,g^2)} & {A_0}^{(f^2,g^2)} & {A_1}^{(1,g^2)} & {A_1}^{(f^2,g^2)} & {A_1}^{(f,g^2)} & {A_2}^{(1,g^2)} & {A_2}^{(f^2,g^2)} & {A_2}^{(f,g^2)} \\ S_1 & S_2 & S_0 & {A_0}^{(f^2,g)} & {A_0}^{(f,g)} & {A_0}^{(1,g)} & {A_1}^{(1,g)} & {A_1}^{(f^2,g)} & {A_1}^{(f,g)} & {A_2}^{(1,g)} & {A_2}^{(f^2,g)} & {A_2}^{(f,g)} \end{array}\right),$$

where $S_0, S_1, S_2 \in \Lambda_2$ and $A_0,\ A_1,\ A_2$ are $12 \times 12$ permutation matrices.

## Type 8

**(6.8.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{16}, x_{29})$
$(x_4, x_{17}, x_{27})(x_5, x_{15}, x_{28})(x_6, x_{19}, x_{32})(x_7, x_{20}, x_{30})$
$(x_8, x_{18}, x_{31})(x_9, x_{22}, x_{35})(x_{10}, x_{23}, x_{33})(x_{11}, x_{21}, x_{34})$
$(x_{36}, x_{37}, x_{38})(x_{39}, x_{40}, x_{41})(x_{42}, x_{43}, x_{44})(x_{45}, x_{46}, x_{47})$
$(x_{48}, x_{49}, x_{50})(x_{51}, x_{52}, x_{53})(x_{54}, x_{55}, x_{56})(x_{57}, x_{58}, x_{59})$
$(x_{60}, x_{61}, x_{62})(x_{63}, x_{64}, x_{65})(x_{66}, x_{67}, x_{68})(x_{69}, x_{70}, x_{71})$
$(x_{72}, x_{97}, x_{86})(x_{73}, x_{98}, x_{84})(x_{74}, x_{96}, x_{85})(x_{75}, x_{100}, x_{89})$
$(x_{76}, x_{101}, x_{87})(x_{77}, x_{99}, x_{88})(x_{78}, x_{103}, x_{92})(x_{79}, x_{104}, x_{90})$
$(x_{80}, x_{102}, x_{91})(x_{81}, x_{106}, x_{95})(x_{82}, x_{107}, x_{93})(x_{83}, x_{105}, x_{94})$
$(x_{108}, x_{121}, x_{134})(x_{109}, x_{122}, x_{132})(x_{110}, x_{120}, x_{133})(x_{111}, x_{124}, x_{137})$
$(x_{112}, x_{125}, x_{135})(x_{113}, x_{123}, x_{136})(x_{114}, x_{127}, x_{140})(x_{115}, x_{128}, x_{138})$

$(x_{116}, x_{126}, x_{139})(x_{117}, x_{130}, x_{143})(x_{118}, x_{131}, x_{141})(x_{119}, x_{129}, x_{142})$, where $x \in \{p, B\}$.

**(6.8.2)** Let $f = (0)(1)(2)(3,4,5)(6,7,8)(9,10,11)$, $g = (0,1,2)(3,4,5)(6,7,8)(9,10,11) \in \mathrm{Sym}\{0,1,\ldots,11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & {A_0}^{(f^2,1)} & {A_0}^{(f,1)} & A_1 & {A_1}^{(f^2,1)} & {A_1}^{(f,1)} & A_2 & {A_2}^{(f^2,1)} & {A_2}^{(f,1)} \\ S_2 & S_0 & S_1 & {A_0}^{(f^2,g^2)} & {A_0}^{(f,g^2)} & {A_0}^{(1,g^2)} & {A_1}^{(f,g^2)} & {A_1}^{(1,g^2)} & {A_1}^{(f^2,g^2)} & {A_2}^{(1,g^2)} & {A_2}^{(f^2,g^2)} & {A_2}^{(f,g^2)} \\ S_1 & S_2 & S_0 & {A_0}^{(f,g)} & {A_0}^{(1,g)} & {A_0}^{(f^2,g)} & {A_1}^{(f^2,g)} & {A_1}^{(f,g)} & {A_1}^{(1,g)} & {A_2}^{(1,g)} & {A_2}^{(f^2,g)} & {A_2}^{(f,g)} \end{array}\right),$$

where $S_0, S_1, S_2 \in \Lambda_2$ and $A_0,\ A_1,\ A_2$ are $12 \times 12$ permutation matrices.

## Type 9

**(6.9.1)** $\varphi = (x_0)(x_1)(x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12})(x_{13})(x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24})(x_{25})(x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{16}, x_{29})$
$(x_4, x_{17}, x_{27})(x_5, x_{15}, x_{28})(x_6, x_{19}, x_{32})(x_7, x_{20}, x_{30})$
$(x_8, x_{18}, x_{31})(x_9, x_{22}, x_{35})(x_{10}, x_{23}, x_{33})(x_{11}, x_{21}, x_{34})$
$(x_{36}, x_{72}, x_{108})(x_{37}, x_{73}, x_{109})(x_{38}, x_{74}, x_{110})(x_{39}, x_{75}, x_{111})$
$(x_{40}, x_{76}, x_{112})(x_{41}, x_{77}, x_{113})(x_{42}, x_{78}, x_{114})(x_{43}, x_{79}, x_{115})$
$(x_{44}, x_{80}, x_{116})(x_{45}, x_{81}, x_{117})(x_{46}, x_{82}, x_{118})(x_{47}, x_{83}, x_{119})$
$(x_{48}, x_{84}, x_{120})(x_{49}, x_{85}, x_{121})(x_{50}, x_{86}, x_{122})(x_{51}, x_{87}, x_{123})$
$(x_{52}, x_{88}, x_{124})(x_{53}, x_{89}, x_{125})(x_{54}, x_{90}, x_{126})(x_{55}, x_{91}, x_{127})$
$(x_{56}, x_{92}, x_{128})(x_{57}, x_{93}, x_{129})(x_{58}, x_{94}, x_{130})(x_{59}, x_{95}, x_{131})$
$(x_{60}, x_{96}, x_{132})(x_{61}, x_{97}, x_{133})(x_{62}, x_{98}, x_{134})(x_{63}, x_{99}, x_{135})$
$(x_{64}, x_{100}, x_{136})(x_{65}, x_{101}, x_{137})(x_{66}, x_{102}, x_{138})(x_{67}, x_{103}, x_{139})$
$(x_{68}, x_{104}, x_{140})(x_{69}, x_{105}, x_{141})(x_{70}, x_{106}, x_{142})(x_{71}, x_{107}, x_{143})$, where $x \in \{p, B\}$.

**(6.9.2)** Let $f = (0)(1)(2)(3,4,5)(6,7,8)(9,10,11) \in \mathrm{Sym}\{0,1,\ldots,11\}$. Then

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc} S_0 & S_1 & S_2 & A_0 & {A_0}^{f^2} & {A_0}^{f} & {A_2}^{f^2} & {A_2}^{f} & A_2 & {A_1}^{f} & A_1 & {A_1}^{f^2} \\ S_2 & S_0 & S_1 & A_1 & {A_1}^{f^2} & {A_1}^{f} & {A_0}^{f^2} & {A_0}^{f} & A_0 & {A_2}^{f} & A_2 & {A_2}^{f^2} \\ S_1 & S_2 & S_0 & A_2 & {A_2}^{f^2} & {A_2}^{f} & {A_1}^{f^2} & {A_1}^{f} & A_1 & {A_0}^{f} & A_0 & {A_0}^{f^2} \end{array}\right),$$

where $S_0, S_1, S_2 \in \Lambda_2$ and $A_0,\ A_1,\ A_2$ are $12 \times 12$ permutation matrices.

**Lemma 6.2** *All matrices* $H_1$ *of* (6.1.2), (6.2.2), (6.3.2), (6.4.3), (6.5.2), (6.6.2), (6.7.2), (6.8.2) *and* (6.9.2) *do not exist. Therefore none of Types* 1 *to* 9 *can occur.*

Proof. Any matrix $H_1$ of (6.1.2), (6.2.2), (6.3.2), (6.4.3), (6.5.2), (6.6.2), (6.7.2), (6.8.2) and (6.9.2) must satisfy $H_1H_1^T = \begin{pmatrix} E_{12} & J_{12} & J_{12} \\ J_{12} & E_{12} & J_{12} \\ J_{12} & J_{12} & E_{12} \end{pmatrix}$, where $E_{12}$ is the identity matrix of degree 12 and $J_{12}$ is the all one $12 \times 12$ matrix by Lemma 2.8. But it follows that there do not exist matrices $H_1$ having these forms and satisfying this equation, using a computer. □

## 7 Types 10 to 15

In this section we consider Types 10 to 15 in Section 5 and we show that none of these types can occur.

**Definition 7.1** Let $m, n$ be positive integers. Let $R, S$ be $m \times n$ matrices with entries from $\mathbb{Z}$. Then we say that $R$ is *equivalent* to $S$ if there exist a permutation matrix $X$ of degree $m$ and a permutation matrix $Y$ of degree $n$ such that $S = XRY$.

The actions of $\varphi$ and $\tau$ on both $\mathcal{P}$ and $\mathcal{B}$ in Types 10 to 15 are determined explicitly from Section 5.

### Type 10

**(7.10.1)** $\varphi = (x_0, x_1, x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$, where $x \in \{p, B\}$,

$\tau = (p_0, p_{12}, p_{24})(p_1, p_{13}, p_{25})(p_2, p_{14}, p_{26})(p_3, p_{15}, p_{27})$
$(p_4, p_{16}, p_{28})(p_5, p_{17}, p_{29})(p_6, p_{18}, p_{30})(p_7, p_{19}, p_{31})$
$(p_8, p_{20}, p_{32})(p_9, p_{21}, p_{33})(p_{10}, p_{22}, p_{34})(p_{11}, p_{23}, p_{35})$
$(p_{36}, p_{72}, p_{108})(p_{37}, p_{73}, p_{109})(p_{38}, p_{74}, p_{110})(p_{39}, p_{75}, p_{111})$
$(p_{40}, p_{76}, p_{112})(p_{41}, p_{77}, p_{113})(p_{42}, p_{78}, p_{114})(p_{43}, p_{79}, p_{115})$
$(p_{44}, p_{80}, p_{116})(p_{45}, p_{81}, p_{117})(p_{46}, p_{82}, p_{118})(p_{47}, p_{83}, p_{119})$
$(p_{48}, p_{84}, p_{120})(p_{49}, p_{85}, p_{121})(p_{50}, p_{86}, p_{122})(p_{51}, p_{87}, p_{123})$
$(p_{52}, p_{88}, p_{124})(p_{53}, p_{89}, p_{125})(p_{54}, p_{90}, p_{126})(p_{55}, p_{91}, p_{127})$

$(p_{56}, p_{92}, p_{128})(p_{57}, p_{93}, p_{129})(p_{58}, p_{94}, p_{130})(p_{59}, p_{95}, p_{131})$
$(p_{60}, p_{96}, p_{132})(p_{61}, p_{97}, p_{133})(p_{62}, p_{98}, p_{134})(p_{63}, p_{99}, p_{135})$
$(p_{64}, p_{100}, p_{136})(p_{65}, p_{101}, p_{137})(p_{66}, p_{102}, p_{138})(p_{67}, p_{103}, p_{139})$
$(p_{68}, p_{104}, p_{140})(p_{69}, p_{105}, p_{141})(p_{70}, p_{106}, p_{142})(p_{71}, p_{107}, p_{143})$ and

$\tau = (B_0)(B_1)(B_2)(B_3, B_6, B_9)(B_4, B_7, B_{10})(B_5, B_8, B_{11})$
$(B_{12}, B_{14}, B_{13})(B_{15}, B_{18}, B_{21})(B_{16}, B_{19}, B_{22})(B_{17}, B_{20}, B_{23})$
$(B_{24}, B_{25}, B_{26})(B_{27}, B_{30}, B_{33})(B_{28}, B_{31}, B_{34})(B_{29}, B_{32}, B_{35})$
$(B_{36}, B_{72}, B_{108})(B_{37}, B_{73}, B_{109})(B_{38}, B_{74}, B_{110})(B_{39}, B_{75}, B_{111})$
$(B_{40}, B_{76}, B_{112})(B_{41}, B_{77}, B_{113})(B_{42}, B_{78}, B_{114})(B_{43}, B_{79}, B_{115})$
$(B_{44}, B_{80}, B_{116})(B_{45}, B_{81}, B_{117})(B_{46}, B_{82}, B_{118})(B_{47}, B_{83}, B_{119})$
$(B_{48}, B_{84}, B_{120})(B_{49}, B_{85}, B_{121})(B_{50}, B_{86}, B_{122})(B_{51}, B_{87}, B_{123})$
$(B_{52}, B_{88}, B_{124})(B_{53}, B_{89}, B_{125})(B_{54}, B_{90}, B_{126})(B_{55}, B_{91}, B_{127})$
$(B_{56}, B_{92}, B_{128})(B_{57}, B_{93}, B_{129})(B_{58}, B_{94}, B_{130})(B_{59}, B_{95}, B_{131})$
$(B_{60}, B_{96}, B_{132})(B_{61}, B_{97}, B_{133})(B_{62}, B_{98}, B_{134})(B_{63}, B_{99}, B_{135})$
$(B_{64}, B_{100}, B_{136})(B_{65}, B_{101}, B_{137})(B_{66}, B_{102}, B_{138})(B_{67}, B_{103}, B_{139})$
$(B_{68}, B_{104}, B_{140})(B_{69}, B_{105}, B_{141})(B_{70}, B_{106}, B_{142})(B_{71}, B_{107}, B_{143})$.

**(7.10.2)** There are the following 16 $G$-orbits on $\mathcal{P}$.

$\mathcal{Q}_0 = \{p_0, p_1, p_2, p_{12}, p_{13}, p_{14}, p_{24}, p_{25}, p_{26}\}$,
$\mathcal{Q}_1 = \{p_3, p_4, p_5, p_{15}, p_{16}, p_{17}, p_{27}, p_{28}, p_{29}\}$,
$\mathcal{Q}_2 = \{p_6, p_7, p_8, p_{18}, p_{19}, p_{20}, p_{30}, p_{31}, p_{32}\}$,
$\mathcal{Q}_3 = \{p_9, p_{10}, p_{11}, p_{21}, p_{22}, p_{23}, p_{33}, p_{34}, p_{35}\}$,
$\mathcal{Q}_4 = \{p_{36}, p_{48}, p_{60}, p_{72}, p_{84}, p_{96}, p_{108}, p_{120}, p_{132}\}$,
$\mathcal{Q}_5 = \{p_{37}, p_{49}, p_{61}, p_{73}, p_{85}, p_{97}, p_{109}, p_{121}, p_{133}\}$,
$\mathcal{Q}_6 = \{p_{38}, p_{50}, p_{62}, p_{74}, p_{86}, p_{98}, p_{110}, p_{122}, p_{134}\}$,
$\mathcal{Q}_7 = \{p_{39}, p_{51}, p_{63}, p_{75}, p_{87}, p_{99}, p_{111}, p_{123}, p_{135}\}$,
$\mathcal{Q}_8 = \{p_{40}, p_{52}, p_{64}, p_{76}, p_{88}, p_{100}, p_{122}, p_{124}, p_{136}\}$,
$\mathcal{Q}_9 = \{p_{41}, p_{53}, p_{65}, p_{77}, p_{89}, p_{101}, p_{113}, p_{125}, p_{137}\}$,
$\mathcal{Q}_{10} = \{p_{42}, p_{54}, p_{66}, p_{78}, p_{90}, p_{102}, p_{114}, p_{126}, p_{138}\}$,
$\mathcal{Q}_{11} = \{p_{43}, p_{55}, p_{67}, p_{79}, p_{91}, p_{103}, p_{115}, p_{127}, p_{139}\}$,
$\mathcal{Q}_{12} = \{p_{44}, p_{56}, p_{68}, p_{80}, p_{92}, p_{104}, p_{116}, p_{128}, p_{140}\}$,
$\mathcal{Q}_{13} = \{p_{45}, p_{57}, p_{69}, p_{81}, p_{93}, p_{105}, p_{117}, p_{129}, p_{141}\}$,
$\mathcal{Q}_{14} = \{p_{46}, p_{58}, p_{70}, p_{82}, p_{94}, p_{106}, p_{118}, p_{130}, p_{142}\}$,
$\mathcal{Q}_{15} = \{p_{47}, p_{59}, p_{71}, p_{83}, p_{95}, p_{107}, p_{119}, p_{131}, p_{143}\}$.

There are the following 18 $G$-orbits on $\mathcal{B}$.

$\mathcal{C}_0 = \{B_0, B_1, B_2\}$,
$\mathcal{C}_1 = \{B_{12}, B_{13}, B_{14}\}$,
$\mathcal{C}_2 = \{B_{24}, B_{25}, B_{26}\}$,
$\mathcal{C}_3 = \{B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, B_{11}\}$,
$\mathcal{C}_4 = \{B_{15}, B_{16}, B_{17}, B_{18}, B_{19}, B_{20}, B_{21}, B_{22}, B_{23}\}$,
$\mathcal{C}_5 = \{B_{27}, B_{28}, B_{29}, B_{30}, B_{31}, B_{32}, B_{33}, B_{34}, B_{35}\}$,
$\mathcal{C}_6 = \{B_{36}, B_{48}, B_{60}, B_{72}, B_{84}, B_{96}, B_{108}, B_{120}, B_{132}\}$,
$\mathcal{C}_7 = \{B_{37}, B_{49}, B_{61}, B_{73}, B_{85}, B_{97}, B_{109}, B_{121}, B_{133}\}$,
$\mathcal{C}_8 = \{B_{38}, B_{50}, B_{62}, B_{74}, B_{86}, B_{98}, B_{110}, B_{122}, B_{134}\}$,

$\mathcal{C}_9 = \{B_{39}, B_{51}, B_{63}, B_{75}, B_{87}, B_{99}, B_{111}, B_{123}, B_{135}\}$,
$\mathcal{C}_{10} = \{B_{40}, B_{52}, B_{64}, B_{76}, B_{88}, B_{100}, B_{122}, B_{124}, B_{136}\}$,
$\mathcal{C}_{11} = \{B_{41}, B_{53}, B_{65}, B_{77}, B_{89}, B_{101}, B_{113}, B_{125}, B_{137}\}$,
$\mathcal{C}_{12} = \{B_{42}, B_{54}, B_{66}, B_{78}, B_{90}, B_{102}, B_{114}, B_{126}, B_{138}\}$,
$\mathcal{C}_{13} = \{B_{43}, B_{55}, B_{67}, B_{79}, B_{91}, B_{103}, B_{115}, B_{127}, B_{139}\}$,
$\mathcal{C}_{14} = \{B_{44}, B_{56}, B_{68}, B_{80}, B_{92}, B_{104}, B_{116}, B_{128}, B_{140}\}$,
$\mathcal{C}_{15} = \{B_{45}, B_{57}, B_{69}, B_{81}, B_{93}, B_{105}, B_{117}, B_{129}, B_{141}\}$,
$\mathcal{C}_{16} = \{B_{46}, B_{58}, B_{70}, B_{82}, B_{94}, B_{106}, B_{118}, B_{130}, B_{142}\}$,
$\mathcal{C}_{17} = \{B_{47}, B_{59}, B_{71}, B_{83}, B_{95}, B_{107}, B_{119}, B_{131}, B_{143}\}$.

Set $q_0 = p_0,\ q_1 = p_3,\ q_2 = p_6,\ q_3 = p_9,\ q_4 = p_{36},\ q_5 = p_{37}, q_6 = p_{38},\ q_7 = p_{39},\ q_8 = p_{40},\ q_9 = p_{41},\ q_{10} = p_{42},\ q_{11} = p_{43}, q_{12} = p_{44},\ q_{13} = p_{45},\ q_{14} = p_{46},\ q_{15} = p_{47}$ and $C_0 = B_0,\ C_1 = B_{12},\ C_2 = B_{24},\ C_3 = B_3,\ C_4 = B_{15},\ C_5 = B_{27}, C_6 = B_{36},\ C_7 = B_{37},\ C_8 = B_{38},\ C_9 = B_{39},\ C_{10} = B_{40},\ C_{11} = B_{41}, C_{12} = B_{42},\ C_{13} = B_{43},\ C_{14} = B_{44},\ C_{15} = B_{45},\ C_{16} = B_{46},\ C_{17} = B_{47}$.

For $0 \le i \le 17$ and $0 \le j \le 15$ set $m_{i,j} = |\mathcal{C}_i \cap (q_j)|$ and $D_{i,j} = \{\alpha \in G |\ {C_i}^{\alpha} \in (q_j)\}$. Then $m_{i,j} = |D_{i,j}|\ \ (0 \le i \le 17,\ 0 \le j \le 15)$. Each $m_{i,j}$ depends only on $\mathcal{C}_i$ and $\mathcal{Q}_j$ not on $C_i$ and $q_j$. For a non-empty subset $X$ of $G$, set $\widehat{X} = \sum_{\alpha \in X} \alpha \in \mathbb{Z}[G]$.

Set $M = (m_{i,j})_{0 \le i \le 17,\ 0 \le j \le 15}$ and $A_{i,i'} = \sum_{j=0}^{15} \widehat{D_{i,j}} \widehat{D_{i',j}}^{(-1)}$ for $0 \le i, i' \le 17$.

**(7.10.3)** (i) For $0 \le i \ne i' \le 17$

$$A_{i,i'} = \begin{cases} 0 & \text{if } \{i,i'\} \in \{\{0,3\},\ \{1,4\},\ \{2,5\}\}, \\ \widehat{G \backslash \{1\}} & \text{if } 6 \le i \ne i' \le 17, \\ \widehat{G} & \text{otherwise.} \end{cases}.$$

(ii) For $0 \le i \le 17$

$$A_{i,i} = \begin{cases} 12\widehat{\langle \tau \rangle} & \text{if } i = 0, \\ 12\widehat{\langle \varphi\tau \rangle} & \text{if } i = 1, \\ 12\widehat{\langle \varphi^2\tau \rangle} & \text{if } i = 2, \\ 12 & \text{if } 3 \le i \le 5, \\ 12 + \widehat{G \backslash \{1\}} & \text{if } 6 \le i \le 17. \end{cases}.$$

PROOF. (i) Let $\alpha \in G$. Then there exist $0 \le j \le 15$ and $(\beta, \gamma) \in D_{i,j} \times D_{i',j}$ such that $\alpha = \beta\gamma^{-1}$, if and only if there exist $0 \le j \le 15$ and $\gamma \in G$ such that ${C_i}^{\alpha} \in ({q_j}^{\gamma^{-1}})$ and $C_{i'} \in ({q_j}^{\gamma^{-1}})$.

Suppose that $\{i, i'\} = \{0, 3\},\ \{1, 4\}$ or $\{2, 5\}$. Then there do not exist $0 \le j \le 15$ and $\gamma \in G$ such that ${C_i}^{\alpha} \in ({q_j}^{\gamma^{-1}})$ and $C_{i'} \in ({q_j}^{\gamma^{-1}})$. Therefore $A_{i,i'} = 0$.

Suppose that $6 \le i \ne i' \le 17$. If $\alpha = 1$, there do not exist $0 \le j \le 15$ and $\gamma \in G$ such that ${C_i}^{\alpha} \in ({q_j}^{\gamma^{-1}})$ and $C_{i'} \in ({q_j}^{\gamma^{-1}})$. If $\alpha \ne 1$, there exists only one $(j, \gamma) \in \{0, 1, \ldots, 15\} \times G$ such that ${C_i}^{\alpha} \in ({q_j}^{\gamma^{-1}})$ and $C_{i'} \in ({q_j}^{\gamma^{-1}})$. Therefore $A_{i,i'} = \widehat{G \backslash \{1\}}$.

Suppose that $0 \le i \ne i' \le 5$, $\{i, i'\} \notin \{\{0,3\},\ \{1,4\},\ \{2,5\}\}$ or $0 \le i \le 5,\ 6 \le i' \le 17$ or $0 \le i' \le 5,\ 6 \le i \le 17$. Then exists only one $(j, \gamma) \in \{0, 1, \cdots, 15\} \times G$ such that $C_i{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_{i'} \in (q_j{}^{\gamma^{-1}})$. Therefore $A_{i,i'} = \widehat{G}$

(ii) Let $\alpha \in G$. Then, there exist $0 \le j \le 15$ and $(\beta, \gamma) \in D_{i,j} \times D_{i',j}$ such that $\alpha = \beta\gamma^{-1}$, if and only if there exist $0 \le j \le 15$ and $\gamma \in G$ such that $C_i{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_i \in (q_j{}^{\gamma^{-1}})$.

If $\alpha \in \langle\tau\rangle$, there exist twelve $(j, \gamma) \in \{0, 1, \ldots, 15\} \times G$ such that $C_0 = C_0{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_0 \in (q_j{}^{\gamma^{-1}})$. If $\alpha \notin \langle\tau\rangle$, there do not exist $(j, \gamma) \in \{0, 1, \ldots, 15\} \times G$ such that $C_0 = C_i{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_0 \in (q_j{}^{\gamma^{-1}})$. Therefore $A_{0,0} = 12\widehat{\langle\tau\rangle}$.

By a similar argument, $A_{1,1} = 12\widehat{\langle\varphi\tau\rangle}$ and $A_{2,2} = 12\widehat{\langle\varphi\tau^2\rangle}$ hold.

Suppose that $3 \le i \le 5$. If $\alpha = 1$, there exist twelve $(j, \gamma) \in \{0, 1, \ldots, 15\} \times G$ such that $C_i = C_i{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_i \in (q_j{}^{\gamma^{-1}})$. If $\alpha \ne 1$, there do not exist $0 \le j \le 15$ and $\gamma \in G$ such that $C_i{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_i \in (q_j{}^{\gamma^{-1}})$. Therefore $A_{i,i} = 12$.

Suppose that $6 \le i \le 17$. If $\alpha = 1$, there exist twelve $(j, \gamma) \in \{0, 1, \ldots, 15\} \times G$ such that $C_i = C_i{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_i \in (q_j{}^{\gamma^{-1}})$. If $\alpha \in G\backslash\{1\}$, there exists only one $(j, \gamma) \in \{0, 1, \ldots, 15\} \times G$ such that $C_i{}^\alpha \in (q_j{}^{\gamma^{-1}})$ and $C_i \in (q_j{}^{\gamma^{-1}})$. Therefore $A_{i,i} = 12 + \widehat{G\backslash\{1\}}$ □

**(7.10.4)** (i) For $0 \le i \ne i' \le 17$

$$\sum_{j=0}^{15} m_{i,j} m_{i',j} = \begin{cases} 0 & \text{if } \{i, i'\} \in \{\{0,3\},\ \{1,4\},\ \{2,5\}\}, \\ 8 & \text{if } 6 \le i \ne i' \le 17, \\ 9 & \text{otherwise.} \end{cases}$$

(ii) For $0 \le i \le 17$

$$\sum_{j=0}^{15} m_{i,j}{}^2 = \begin{cases} 36 & \text{if } 0 \le i \le 2, \\ 12 & \text{if } 3 \le i \le 5, \\ 20 & \text{if } 6 \le i \le 17. \end{cases}$$

(iii) For $0 \le i \le 17$

$$\sum_{j=0}^{15} m_{i,j} = 12.$$

Proof. (i) and (ii) hold by acting the trivial character of $G$ on two equations in (7.10.3). Since there are twelve $(i, \alpha) \in \{0, 1, \ldots, 15\} \times G$ such that $C_i \in (q_j{}^{\alpha^{-1}})$, (iii) holds. □

**(7.10.5)** For $0 \le i \le 17$, the following hold, up to ordering of $m_{i,0}\ m_{i,1}\ \ldots\ m_{i,15}$.

(i) If $0 \le i \le 2$, then $(m_{i,0}\ m_{i,1}\ \ldots\ m_{i,15}) = (\underbrace{0\ 0 \ldots 0}_{12}\ 3\ 3\ 3\ 3)$, $(\underbrace{0\ 0 \ldots 0}_{11}\ 1\ 1\ 3\ 3\ 4)$, $(\underbrace{0\ 0 \ldots 0}_{10}\ 1\ 1\ 1\ 1\ 4\ 4)$ or $(\underbrace{0\ 0 \ldots 0}_{10}\ 1\ 1\ 1\ 1\ 2\ 2\ 5)$.

(ii) If $3 \le i \le 5$, then $(m_{i,0}\ m_{i,1}\ \ldots\ m_{i,15}) = (0\ 0\ 0\ 0\ \underbrace{1\ 1 \ldots 1}_{12})$.

(iii) If $6 \le i \le 17$, then $(m_{i,0}m_{i,1}\dots m_{i,15}) = (\underbrace{0\,0\dots0}_{8}\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2)$ or $(\underbrace{0\,0\dots0}_{7}\ \underbrace{1\,1\dots1}_{7}\ 2\ 3)$.

Proof. This assertion holds from (7.10.4) (ii), (iii). □

**(7.10.6)** $(m_{i,j})_{0\le i\le 5,\ 0\le j\le 15}$ coinsides with the folowing matrix, up to equivalence.

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&3&3&0&3&3\\
0&0&0&0&0&0&0&0&0&3&3&0&0&3&0&3\\
0&0&0&0&0&0&0&3&3&0&0&0&0&3&3&0\\
1&1&1&1&1&1&1&1&1&1&1&0&0&1&0&0\\
1&1&1&1&1&1&1&1&1&0&0&1&1&0&1&0\\
1&1&1&1&1&1&1&0&0&1&1&1&1&0&0&1
\end{pmatrix}.$$

Proof. This assertion holds from (7.10.4) and (7.10.5). □

**(7.10.7)** There exists the following unique $M$, up to equivalence.

$$M = \left(\begin{array}{cccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&3&3&0&3&3\\
0&0&0&0&0&0&0&0&0&3&3&0&0&3&0&3\\
0&0&0&0&0&0&0&3&3&0&0&0&0&3&3&0\\
1&1&1&1&1&1&1&1&1&1&1&0&0&1&0&0\\
1&1&1&1&1&1&1&1&1&0&0&1&1&0&1&0\\
1&1&1&1&1&1&1&0&0&1&1&1&1&0&0&1\\
\hline
0&0&0&0&1&1&2&0&3&1&1&1&1&0&0&1\\
1&1&1&1&0&0&3&1&0&0&0&0&0&1&1&2\\
0&0&0&2&0&2&1&1&0&0&2&0&2&1&1&0\\
0&0&2&0&0&2&1&1&0&2&0&2&0&1&1&0\\
0&0&2&2&2&0&0&0&1&0&0&1&1&2&0&1\\
0&2&0&0&2&0&1&1&0&0&2&2&0&1&1&0\\
0&2&0&2&1&1&0&0&1&2&0&0&0&0&2&1\\
0&2&2&0&0&0&0&2&1&1&1&0&2&0&0&1\\
2&0&0&0&2&0&1&1&0&2&0&0&2&1&1&0\\
2&0&0&2&0&0&0&2&1&1&1&2&0&0&0&1\\
2&0&2&0&1&1&0&0&1&0&2&0&0&0&2&1\\
2&2&0&0&0&2&0&0&1&0&0&1&1&2&0&1
\end{array}\right).$$

Proof. Using a computer, the assertion holds from (7.10.4), (7.10.5) and (7.10.6). □

**Lemma 7.2** *Type* 10 *does not occur.*

Proof. Using a computer, it follows that there does not exist $(D_{i,j})_{6\le i\le 11,\ 0\le j\le 15}$ corresponding to the submatrix $(m_{i,j})_{6\le i\le 11,\ 0\le j\le 15}$ of the matrix $M$ of (7.10.7). Therefore the lemma holds. □

The proofs of the following results in Types 11 to 15 are omitted, because they are similar to the results in Type 10.

## Type 11

**(7.11.1)** $\varphi = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{15}, x_{27})$
$(x_4, x_{16}, x_{28})(x_5, x_{17}, x_{29})(x_6, x_{18}, x_{30})(x_7, x_{19}, x_{31})$
$(x_8, x_{20}, x_{32})(x_9, x_{21}, x_{33})(x_{10}, x_{22}, x_{34})(x_{11}, x_{23}, x_{35})$

$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_1, x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{37}, x_{38})(x_{39}, x_{40}, x_{41})(x_{42}, x_{43}, x_{44})(x_{45}, x_{46}, x_{47})$
$(x_{48}, x_{49}, x_{50})(x_{51}, x_{52}, x_{53})(x_{54}, x_{55}, x_{56})(x_{57}, x_{58}, x_{59})$
$(x_{60}, x_{61}, x_{62})(x_{63}, x_{64}, x_{65})(x_{66}, x_{67}, x_{68})(x_{69}, x_{70}, x_{71})$
$(x_{72}, x_{85}, x_{98})(x_{73}, x_{86}, x_{96})(x_{74}, x_{84}, x_{97})(x_{75}, x_{88}, x_{101})$
$(x_{76}, x_{89}, x_{99})(x_{77}, x_{87}, x_{100})(x_{78}, x_{91}, x_{104})(x_{79}, x_{92}, x_{102})$
$(x_{80}, x_{90}, x_{103})(x_{81}, x_{94}, x_{107})(x_{82}, x_{95}, x_{105})(x_{83}, x_{93}, x_{106})$
$(x_{108}, x_{121}, x_{134})(x_{109}, x_{122}, x_{132})(x_{110}, x_{120}, x_{133})(x_{111}, x_{124}, x_{137})$
$(x_{112}, x_{125}, x_{135})(x_{113}, x_{123}, x_{136})(x_{114}, x_{127}, x_{140})(x_{115}, x_{128}, x_{138})$
$(x_{116}, x_{126}, x_{139})(x_{117}, x_{130}, x_{143})(x_{118}, x_{131}, x_{141})(x_{119}, x_{129}, x_{142})$, where $x \in \{p, B\}$.

**(7.11.2)** There are the following 16 $G$-orbits on $\mathcal{P}$ and on $\mathcal{B}$.

$\mathcal{Y}_0 = \{x_0, x_1, x_2, x_{12}, x_{13}, x_{14}, x_{24}, x_{25}, x_{26}\}$,
$\mathcal{Y}_1 = \{x_3, x_4, x_5, x_{15}, x_{16}, x_{17}, x_{27}, x_{28}, x_{29}\}$,
$\mathcal{Y}_2 = \{x_6, x_7, x_8, x_{18}, x_{19}, x_{20}, x_{30}, x_{31}, x_{32}\}$,
$\mathcal{Y}_3 = \{x_9, x_{10}, x_{11}, x_{21}, x_{22}, x_{23}, x_{33}, x_{34}, x_{35}\}$,
$\mathcal{Y}_4 = \{x_{36}, x_{37}, x_{38}, x_{48}, x_{49}, x_{50}, x_{60}, x_{61}, x_{62}\}$,
$\mathcal{Y}_5 = \{x_{39}, x_{40}, x_{41}, x_{51}, x_{52}, x_{53}, x_{63}, x_{64}, x_{65}\}$,
$\mathcal{Y}_6 = \{x_{42}, x_{43}, x_{44}, x_{54}, x_{55}, x_{56}, x_{66}, x_{67}, x_{68}\}$,
$\mathcal{Y}_7 = \{x_{45}, x_{46}, x_{47}, x_{57}, x_{58}, x_{59}, x_{69}, x_{70}, x_{71}\}$,
$\mathcal{Y}_8 = \{x_{72}, x_{73}, x_{74}, x_{84}, x_{85}, x_{86}, x_{96}, x_{97}, x_{98}\}$,
$\mathcal{Y}_9 = \{x_{75}, x_{76}, x_{77}, x_{87}, x_{88}, x_{89}, x_{99}, x_{100}, x_{101}\}$,
$\mathcal{Y}_{10} = \{x_{78}, x_{79}, x_{80}, x_{90}, x_{91}, x_{92}, x_{102}, x_{103}, x_{104}\}$,
$\mathcal{Y}_{11} = \{x_{81}, x_{82}, x_{83}, x_{93}, x_{94}, x_{95}, x_{105}, x_{106}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{109}, x_{110}, x_{120}, x_{121}, x_{122}, x_{132}, x_{133}, x_{134}\}$,
$\mathcal{Y}_{13} = \{x_{111}, x_{112}, x_{113}, x_{123}, x_{124}, x_{125}, x_{135}, x_{136}, x_{137}\}$,
$\mathcal{Y}_{14} = \{x_{114}, x_{115}, x_{116}, x_{126}, x_{127}, x_{128}, x_{138}, x_{139}, x_{140}\}$,
$\mathcal{Y}_{15} = \{x_{117}, x_{118}, x_{119}, x_{129}, x_{130}, x_{131}, x_{141}, x_{142}, x_{143}\}$, where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_3,\ q_2 = p_6,\ q_3 = p_9,\ q_4 = p_{36},\ q_5 = p_{39}, q_6 = p_{42},\ q_7 = p_{45},\ q_8 = p_{72},\ q_9 = p_{75},\ q_{10} = p_{78},\ q_{11} = p_{81}, q_{12} = p_{108},\ q_{13} = p_{111},\ q_{14} = p_{114},\ q_{15} = p_{117}$ and $C_0 = B_0,\ C_1 = B_3,\ C_2 = B_6,\ C_3 = B_9,\ C_4 = B_{36},\ C_5 = B_{39}, C_6 = B_{42},\ C_7 = B_{45},\ C_8 = B_{72},\ C_9 = B_{75},\ C_{10} = B_{78},\ C_{11} = B_{81}, C_{12} = B_{108},\ C_{13} = B_{111},\ C_{14} = B_{114},\ C_{15} = B_{117}$.

For $0 \leq i, j \leq 15$ set $m_{i,j} = |\mathcal{Q}_i \cap (C_j)|$ and $D_{i,j} = \{\alpha \in G |\ {q_i}^{\alpha} \in (C_j)\}$. Then $m_{i,j} = |D_{i,j}|$ $(0 \leq i, j \leq 15)$. Each $m_{i,j}$ depends only on $\mathcal{Q}_i$ and $\mathcal{C}_j$ not on $q_i$ and $C_j$. Set $M = (m_{i,j})_{0 \leq i,j \leq 15}$ and $A_{i,i'} = \sum_{j=0}^{15} \widehat{D_{i,j}} \widehat{D_{i',j}}^{(-1)}$ for $0 \leq i, i' \leq 15$.

**(7.11.3)**
Set $I_0 = \{0,1,2,3\}$, $I_1 = \{4,5,6,7\}$, $I_2 = \{8,9,10,11\}$ and $I_3 = \{12,13,14,15\}$.
(i) For $0 \leq i \neq i' \leq 15$

$$A_{i,i'} = \begin{cases} \widehat{G \backslash \langle \tau \rangle} & \text{if } i \neq i' \in I_k \quad \text{for some } k \in \{0,1\}, \\ \widehat{G \backslash \langle \varphi^2 \tau \rangle} & \text{if } i \neq i' \in I_k \quad \text{for some } k \in \{2,3\}, \\ \widehat{G} & \text{if } i \in I_k,\ i' \in I_l \quad \text{for some } k \neq l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \leq i \leq 15$

$$A_{i,i} = \begin{cases} 12 + \widehat{G \backslash \langle \tau \rangle} & \text{if } i \in I_k \quad \text{for some } k \in \{0,1\}, \\ 12 + \widehat{G \backslash \langle \varphi^2 \tau \rangle} & \text{if } i \in I_k \quad \text{for some } k \in \{2,3\}. \end{cases}$$

**(7.11.4)** Let $I_0, \ldots, I_3$ be the symbols used in (7.11.3).
(i) For $0 \leq i \neq i' \leq 15$

$$\sum_{j=0}^{15} m_{i,j} m_{i',j} = \begin{cases} 6 & \text{if } i \neq i' \in I_k \quad \text{for some } k \in \{0,1,2,3\}, \\ 9 & \text{if } i \in I_k, i' \in I_l \quad \text{for some } k \neq l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \leq i \leq 15$

$$\sum_{j=0}^{15} {m_{i,j}}^2 = 18.$$

(iii) For $0 \leq i \leq 15$

$$\sum_{j=0}^{15} m_{i,j} = 12.$$

**Lemma 7.3** *There does not exist an* $M = (m_{i,j})_{0 \leq i,j \leq 15}$*. Therefore Type* 11 *does not occur.*

## Type 12

**(7.12.1)** $\varphi = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{15}, x_{27})$
$(x_4, x_{16}, x_{28})(x_5, x_{17}, x_{29})(x_6, x_{18}, x_{30})(x_7, x_{19}, x_{31})$
$(x_8, x_{20}, x_{32})(x_9, x_{21}, x_{33})(x_{10}, x_{22}, x_{34})(x_{11}, x_{23}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$

$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and
$\tau = (x_0, x_1, x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{37}, x_{38})(x_{39}, x_{40}, x_{41})(x_{42}, x_{43}, x_{44})(x_{45}, x_{46}, x_{47})$
$(x_{48}, x_{49}, x_{50})(x_{51}, x_{52}, x_{53})(x_{54}, x_{55}, x_{56})(x_{57}, x_{58}, x_{59})$
$(x_{60}, x_{61}, x_{62})(x_{63}, x_{64}, x_{65})(x_{66}, x_{67}, x_{68})(x_{69}, x_{70}, x_{71})$
$(x_{72}, x_{85}, x_{98})(x_{73}, x_{86}, x_{96})(x_{74}, x_{84}, x_{97})(x_{75}, x_{88}, x_{101})$
$(x_{76}, x_{89}, x_{99})(x_{77}, x_{87}, x_{100})(x_{78}, x_{91}, x_{104})(x_{79}, x_{92}, x_{102})$
$(x_{80}, x_{90}, x_{103})(x_{81}, x_{94}, x_{107})(x_{82}, x_{95}, x_{105})(x_{83}, x_{93}, x_{106})$
$(x_{108}, x_{133}, x_{122})(x_{109}, x_{134}, x_{120})(x_{110}, x_{132}, x_{121})(x_{111}, x_{136}, x_{125})$
$(x_{112}, x_{137}, x_{123})(x_{113}, x_{135}, x_{124})(x_{114}, x_{139}, x_{128})(x_{115}, x_{140}, x_{126})$
$(x_{116}, x_{138}, x_{127})(x_{117}, x_{142}, x_{131})(x_{118}, x_{143}, x_{129})(x_{119}, x_{141}, x_{130})$, where $x \in \{p, B\}$.

**(7.12.2)** There are the following 16 $G$-orbits on $\mathcal{P}$ and on $\mathcal{B}$.

$\mathcal{Y}_0 = \{x_0, x_1, x_2, x_{12}, x_{13}, x_{14}, x_{24}, x_{25}, x_{26}\}$,
$\mathcal{Y}_1 = \{x_3, x_4, x_5, x_{15}, x_{16}, x_{17}, x_{27}, x_{28}, x_{29}\}$,
$\mathcal{Y}_2 = \{x_6, x_7, x_8, x_{18}, x_{19}, x_{20}, x_{30}, x_{31}, x_{32}\}$,
$\mathcal{Y}_3 = \{x_9, x_{10}, x_{11}, x_{21}, x_{22}, x_{23}, x_{33}, x_{34}, x_{35}\}$,
$\mathcal{Y}_4 = \{x_{36}, x_{37}, x_{38}, x_{48}, x_{49}, x_{50}, x_{60}, x_{61}, x_{62}\}$,
$\mathcal{Y}_5 = \{x_{39}, x_{40}, x_{41}, x_{51}, x_{52}, x_{53}, x_{63}, x_{64}, x_{65}\}$,
$\mathcal{Y}_6 = \{x_{42}, x_{43}, x_{44}, x_{54}, x_{55}, x_{56}, x_{66}, x_{67}, x_{68}\}$,
$\mathcal{Y}_7 = \{x_{45}, x_{46}, x_{47}, x_{57}, x_{58}, x_{59}, x_{69}, x_{70}, x_{71}\}$,
$\mathcal{Y}_8 = \{x_{72}, x_{73}, x_{74}, x_{84}, x_{85}, x_{86}, x_{96}, x_{97}, x_{98}\}$,
$\mathcal{Y}_9 = \{x_{75}, x_{76}, x_{77}, x_{87}, x_{88}, x_{89}, x_{99}, x_{100}, x_{101}\}$,
$\mathcal{Y}_{10} = \{x_{78}, x_{79}, x_{80}, x_{90}, x_{91}, x_{92}, x_{102}, x_{103}, x_{104}\}$,
$\mathcal{Y}_{11} = \{x_{81}, x_{82}, x_{83}, x_{93}, x_{94}, x_{95}, x_{105}, x_{106}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{109}, x_{110}, x_{120}, x_{121}, x_{122}, x_{132}, x_{133}, x_{134}\}$,
$\mathcal{Y}_{13} = \{x_{111}, x_{112}, x_{113}, x_{123}, x_{124}, x_{125}, x_{135}, x_{136}, x_{137}\}$,
$\mathcal{Y}_{14} = \{x_{114}, x_{115}, x_{116}, x_{126}, x_{127}, x_{128}, x_{138}, x_{139}, x_{140}\}$,
$\mathcal{Y}_{15} = \{x_{117}, x_{118}, x_{119}, x_{129}, x_{130}, x_{131}, x_{141}, x_{142}, x_{143}\}$,where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_3,\ q_2 = p_6,\ q_3 = p_9,\ q_4 = p_{36},\ q_5 = p_{39}, q_6 = p_{42},\ q_7 = p_{45},\ q_8 = p_{72},\ q_9 = p_{75},\ q_{10} = p_{78},\ q_{11} = p_{81}, q_{12} = p_{108},\ q_{13} = p_{111},\ q_{14} = p_{114},\ q_{15} = p_{117}$ and $C_0 = B_0,\ C_1 = B_3,\ C_2 = B_6,\ C_3 = B_9,\ C_4 = B_{36},\ C_5 = B_{39}, C_6 = B_{42},\ C_7 = B_{45},\ C_8 = B_{72},\ C_9 = B_{75},\ C_{10} = B_{78},\ C_{11} = B_{81}, C_{12} = B_{108},\ C_{13} = B_{111},\ C_{14} = B_{114},\ C_{15} = B_{117}$.

The symbols $m_{i,j},\ D_{i,j},\ M$ and $A_{i,i'}$ are the same as in Type 11.

**(7.12.3)**
Set $I_0 = \{0, 1, 2, 3\},\ I_1 = \{4, 5, 6, 7\},\ I_2 = \{8, 9, 10, 11\}$ and $I_3 = \{12, 13, 14, 15\}$.
(i) For $0 \le i \ne i' \le 15$

$$A_{i,i'} = \begin{cases} \widehat{G\backslash\langle\tau\rangle} & \text{if } \ i \neq i' \in I_k \quad \text{for some } k \in \{0,1\}, \\ \widehat{G\backslash\langle\varphi^2\tau\rangle} & \text{if } \ i \neq i' \in I_2, \\ \widehat{G\backslash\langle\varphi\tau\rangle} & \text{if } \ i \neq i' \in I_3, \\ \widehat{G} & \text{if } \ i \in I_k, i' \in I_l \quad \text{for some } k \neq l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \leq i \leq 15$

$$A_{i,i} = \begin{cases} 12 + \widehat{G\backslash\langle\tau\rangle} & \text{if } \ i \in I_k \quad \text{for some } k \in \{0,1\}, \\ 12 + \widehat{G\backslash\langle\varphi^2\tau\rangle} & \text{if } \ i \in I_2, \\ 12 + \widehat{G\backslash\langle\varphi\tau\rangle} & \text{if } \ i \in I_3. \end{cases}$$

**(7.12.4)** Let $I_0, \ldots, I_3$ be the symbols used in (7.12.3).
(i) For $0 \leq i \neq i' \leq 15$

$$\sum_{j=0}^{15} m_{i,j} m_{i',j} = \begin{cases} 6 & \text{if } \ i \neq i' \in I_k \quad \text{for some } k \in \{0,1,2,3\}, \\ 9 & \text{if } \ i \in I_k, i' \in I_l \quad \text{for some } k \neq l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \leq i \leq 15$

$$\sum_{j=0}^{15} {m_{i,j}}^2 = 18.$$

(iii) For $0 \leq i \leq 15$

$$\sum_{j=0}^{15} m_{i,j} = 12.$$

**Lemma 7.4** *There does not exist an* $M = (m_{i,j})_{0 \leq i,j \leq 15}$. *Therefore Type* 12 *does not occur.*

## Type 13

**(7.13.1)** $\varphi = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{15}, x_{27})$
$(x_4, x_{16}, x_{28})(x_5, x_{17}, x_{29})(x_6, x_{18}, x_{30})(x_7, x_{19}, x_{31})$
$(x_8, x_{20}, x_{32})(x_9, x_{21}, x_{33})(x_{10}, x_{22}, x_{34})(x_{11}, x_{23}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and
$\tau = (x_0, x_1, x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$

$(x_{24}, x_{25}, x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{37}, x_{38})(x_{39}, x_{40}, x_{41})(x_{42}, x_{43}, x_{44})(x_{45}, x_{46}, x_{47})$
$(x_{48}, x_{49}, x_{50})(x_{51}, x_{52}, x_{53})(x_{54}, x_{55}, x_{56})(x_{57}, x_{58}, x_{59})$
$(x_{60}, x_{61}, x_{62})(x_{63}, x_{64}, x_{65})(x_{66}, x_{67}, x_{68})(x_{69}, x_{70}, x_{71})$
$(x_{72}, x_{73}, x_{74})(x_{75}, x_{76}, x_{77})(x_{78}, x_{79}, x_{80})(x_{81}, x_{82}, x_{83})$
$(x_{84}, x_{85}, x_{86})(x_{87}, x_{88}, x_{89})(x_{90}, x_{91}, x_{92})(x_{93}, x_{94}, x_{95})$
$(x_{96}, x_{97}, x_{98})(x_{99}, x_{100}, x_{101})(x_{102}, x_{103}, x_{104})(x_{105}, x_{106}, x_{107})$
$(x_{108}, x_{121}, x_{134})(x_{109}, x_{122}, x_{132})(x_{110}, x_{120}, x_{133})(x_{111}, x_{124}, x_{137})$
$(x_{112}, x_{125}, x_{135})(x_{113}, x_{123}, x_{136})(x_{114}, x_{127}, x_{140})(x_{115}, x_{128}, x_{138})$
$(x_{116}, x_{126}, x_{139})(x_{117}, x_{130}, x_{143})(x_{118}, x_{131}, x_{141})(x_{119}, x_{129}, x_{142})$, where $x \in \{p, B\}$.

**(7.13.2)** There are the following 16 $G$-orbits on $\mathcal{P}$ and on $\mathcal{B}$.

$\mathcal{Y}_0 = \{x_0, x_1, x_2, x_{12}, x_{13}, x_{14}, x_{24}, x_{25}, x_{26}\}$,
$\mathcal{Y}_1 = \{x_3, x_4, x_5, x_{15}, x_{16}, x_{17}, x_{27}, x_{28}, x_{29}\}$,
$\mathcal{Y}_2 = \{x_6, x_7, x_8, x_{18}, x_{19}, x_{20}, x_{30}, x_{31}, x_{32}\}$,
$\mathcal{Y}_3 = \{x_9, x_{10}, x_{11}, x_{21}, x_{22}, x_{23}, x_{33}, x_{34}, x_{35}\}$,
$\mathcal{Y}_4 = \{x_{36}, x_{37}, x_{38}, x_{48}, x_{49}, x_{50}, x_{60}, x_{61}, x_{62}\}$,
$\mathcal{Y}_5 = \{x_{39}, x_{40}, x_{41}, x_{51}, x_{52}, x_{53}, x_{63}, x_{64}, x_{65}\}$,
$\mathcal{Y}_6 = \{x_{42}, x_{43}, x_{44}, x_{54}, x_{55}, x_{56}, x_{66}, x_{67}, x_{68}\}$,
$\mathcal{Y}_7 = \{x_{45}, x_{46}, x_{47}, x_{57}, x_{58}, x_{59}, x_{69}, x_{70}, x_{71}\}$,
$\mathcal{Y}_8 = \{x_{72}, x_{73}, x_{74}, x_{84}, x_{85}, x_{86}, x_{96}, x_{97}, x_{98}\}$,
$\mathcal{Y}_9 = \{x_{75}, x_{76}, x_{77}, x_{87}, x_{88}, x_{89}, x_{99}, x_{100}, x_{101}\}$,
$\mathcal{Y}_{10} = \{x_{78}, x_{79}, x_{80}, x_{90}, x_{91}, x_{92}, x_{102}, x_{103}, x_{104}\}$,
$\mathcal{Y}_{11} = \{x_{81}, x_{82}, x_{83}, x_{93}, x_{94}, x_{95}, x_{105}, x_{106}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{109}, x_{110}, x_{120}, x_{121}, x_{122}, x_{132}, x_{133}, x_{134}\}$,
$\mathcal{Y}_{13} = \{x_{111}, x_{112}, x_{113}, x_{123}, x_{124}, x_{125}, x_{135}, x_{136}, x_{137}\}$,
$\mathcal{Y}_{14} = \{x_{114}, x_{115}, x_{116}, x_{126}, x_{127}, x_{128}, x_{138}, x_{139}, x_{140}\}$,
$\mathcal{Y}_{15} = \{x_{117}, x_{118}, x_{119}, x_{129}, x_{130}, x_{131}, x_{141}, x_{142}, x_{143}\}$, where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_3,\ q_2 = p_6,\ q_3 = p_9,\ q_4 = p_{36},\ q_5 = p_{39}, q_6 = p_{42},\ q_7 = p_{45},\ q_8 = p_{72},\ q_9 = p_{75},\ q_{10} = p_{78},\ q_{11} = p_{81}, q_{12} = p_{108},\ q_{13} = p_{111},\ q_{14} = p_{114},\ q_{15} = p_{117}$ and $C_0 = B_0,\ C_1 = B_3,\ C_2 = B_6,\ C_3 = B_9,\ C_4 = B_{36},\ C_5 = B_{39}, C_6 = B_{42},\ C_7 = B_{45},\ C_8 = B_{72},\ C_9 = B_{75},\ C_{10} = B_{78},\ C_{11} = B_{81}, C_{12} = B_{108},\ C_{13} = B_{111},\ C_{14} = B_{114},\ C_{15} = B_{117}$.

The symbols $m_{i,j},\ D_{i,j},\ M$ and $A_{i,i'}$ are the same as in Type 11.

**(7.13.3)**
Set $I_0 = \{0, 1, 2, 3\},\ I_1 = \{4, 5, 6, 7\},\ I_2 = \{8, 9, 10, 11\}$ and $I_3 = \{12, 13, 14, 15\}$.
(i) For $0 \leq i \neq i' \leq 15$,

$$A_{i,i'} = \begin{cases} \widehat{G\backslash\langle\tau\rangle} & \text{if } \ i \neq i' \in I_k \quad \text{for some } k \in \{0,1,2\}, \\ \widehat{G\backslash\langle\varphi^2\tau\rangle} & \text{if } \ i \neq i' \in I_3, \\ \widehat{G} & \text{if } \ i \in I_k, i' \in I_l \quad \text{for some } k \neq l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \le i \le 15$

$$A_{i,i} = \begin{cases} 12 + \widehat{G\backslash\langle\tau\rangle} & \text{if } i \in I_k \text{ for some } k \in \{0,1,2\}, \\ 12 + \widehat{G\backslash\langle\varphi^2\tau\rangle} & \text{if } i \in I_3. \end{cases}$$

**(7.13.4)** Let $I_0, \dots, I_3$ be the symbols used in (7.13.3).
(i) For $0 \le i \ne i' \le 15$

$$\sum_{j=0}^{15} m_{i,j} m_{i',j} = \begin{cases} 6 & \text{if } i \ne i' \in I_k \text{ for some } k \in \{0,1,2,3\}, \\ 9 & \text{if } i \in I_k, i' \in I_l \text{ for some } k \ne l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \le i \le 15$,

$$\sum_{j=0}^{15} {m_{i,j}}^2 = 18.$$

(iii) For $0 \le i \le 15$

$$\sum_{j=0}^{15} m_{i,j} = 12.$$

**Lemma 7.5** *There does not exist an* $M = (m_{i,j})_{0 \le i,j \le 15}$. *Therefore Type* 13 *does not occur.*

## Type 14

**(7.14.1)** $\varphi = (x_0, x_1, x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{15}, x_{27})$
$(x_4, x_{16}, x_{28})(x_5, x_{17}, x_{29})(x_6, x_{18}, x_{30})(x_7, x_{19}, x_{31})$
$(x_8, x_{20}, x_{32})(x_9, x_{21}, x_{33})(x_{10}, x_{22}, x_{34})(x_{11}, x_{23}, x_{35})$
$(x_{36}, x_{37}, x_{38})(x_{39}, x_{40}, x_{41})(x_{42}, x_{43}, x_{44})(x_{45}, x_{46}, x_{47})$
$(x_{48}, x_{49}, x_{50})(x_{51}, x_{52}, x_{53})(x_{54}, x_{55}, x_{56})(x_{57}, x_{58}, x_{59})$
$(x_{60}, x_{61}, x_{62})(x_{63}, x_{64}, x_{65})(x_{66}, x_{67}, x_{68})(x_{69}, x_{70}, x_{71})$
$(x_{72}, x_{85}, x_{98})(x_{73}, x_{86}, x_{96})(x_{74}, x_{84}, x_{97})(x_{75}, x_{88}, x_{101})$
$(x_{76}, x_{89}, x_{99})(x_{77}, x_{87}, x_{100})(x_{78}, x_{91}, x_{104})(x_{79}, x_{92}, x_{102})$

$(x_{80}, x_{90}, x_{103})(x_{81}, x_{94}, x_{107})(x_{82}, x_{95}, x_{105})(x_{83}, x_{93}, x_{106})$
$(x_{108}, x_{133}, x_{122})(x_{109}, x_{134}, x_{120})(x_{110}, x_{132}, x_{121})(x_{111}, x_{136}, x_{125})$
$(x_{112}, x_{137}, x_{123})(x_{113}, x_{135}, x_{124})(x_{114}, x_{139}, x_{128})(x_{115}, x_{140}, x_{126})$
$(x_{116}, x_{138}, x_{127})(x_{117}, x_{142}, x_{131})(x_{118}, x_{143}, x_{129})(x_{119}, x_{141}, x_{130})$, where $x \in \{p, B\}$.

**(7.14.2)** There are the following 16 $G$-orbits on $\mathcal{P}$ and on $\mathcal{B}$.

$\mathcal{Y}_0 = \{x_0, x_1, x_2, x_{12}, x_{13}, x_{14}, x_{24}, x_{25}, x_{26}\}$,
$\mathcal{Y}_1 = \{x_3, x_4, x_5, x_{15}, x_{16}, x_{17}, x_{27}, x_{28}, x_{29}\}$,
$\mathcal{Y}_2 = \{x_6, x_7, x_8, x_{18}, x_{19}, x_{20}, x_{30}, x_{31}, x_{32}\}$,
$\mathcal{Y}_3 = \{x_9, x_{10}, x_{11}, x_{21}, x_{22}, x_{23}, x_{33}, x_{34}, x_{35}\}$,
$\mathcal{Y}_4 = \{x_{36}, x_{37}, x_{38}, x_{48}, x_{49}, x_{50}, x_{60}, x_{61}, x_{62}\}$,
$\mathcal{Y}_5 = \{x_{39}, x_{40}, x_{41}, x_{51}, x_{52}, x_{53}, x_{63}, x_{64}, x_{65}\}$,
$\mathcal{Y}_6 = \{x_{42}, x_{43}, x_{44}, x_{54}, x_{55}, x_{56}, x_{66}, x_{67}, x_{68}\}$,
$\mathcal{Y}_7 = \{x_{45}, x_{46}, x_{47}, x_{57}, x_{58}, x_{59}, x_{69}, x_{70}, x_{71}\}$,
$\mathcal{Y}_8 = \{x_{72}, x_{73}, x_{74}, x_{84}, x_{85}, x_{86}, x_{96}, x_{97}, x_{98}\}$,
$\mathcal{Y}_9 = \{x_{75}, x_{76}, x_{77}, x_{87}, x_{88}, x_{89}, x_{99}, x_{100}, x_{101}\}$,
$\mathcal{Y}_{10} = \{x_{78}, x_{79}, x_{80}, x_{90}, x_{91}, x_{92}, x_{102}, x_{103}, x_{104}\}$,
$\mathcal{Y}_{11} = \{x_{81}, x_{82}, x_{83}, x_{93}, x_{94}, x_{95}, x_{105}, x_{106}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{109}, x_{110}, x_{120}, x_{121}, x_{122}, x_{132}, x_{133}, x_{134}\}$,
$\mathcal{Y}_{13} = \{x_{111}, x_{112}, x_{113}, x_{123}, x_{124}, x_{125}, x_{135}, x_{136}, x_{137}\}$,
$\mathcal{Y}_{14} = \{x_{114}, x_{115}, x_{116}, x_{126}, x_{127}, x_{128}, x_{138}, x_{139}, x_{140}\}$,
$\mathcal{Y}_{15} = \{x_{117}, x_{118}, x_{119}, x_{129}, x_{130}, x_{131}, x_{141}, x_{142}, x_{143}\}$, where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_3,\ q_2 = p_6,\ q_3 = p_9,\ q_4 = p_{36},\ q_5 = p_{39}, q_6 = p_{42},\ q_7 = p_{45},\ q_8 = p_{72},\ q_9 = p_{75},\ q_{10} = p_{78},\ q_{11} = p_{81},\ q_{12} = p_{108},\ q_{13} = p_{111},\ q_{14} = p_{114},\ q_{15} = p_{117}$ and $C_0 = B_0\ C_1 = B_3,\ C_2 = B_6,\ C_3 = B_9,\ C_4 = B_{36},\ C_5 = B_{39}, C_6 = B_{42},\ C_7 = B_{45},\ C_8 = B_{72},\ C_9 = B_{75},\ C_{10} = B_{78},\ C_{11} = B_{81}, C_{12} = B_{108},\ C_{13} = B_{111},\ C_{14} = B_{114},\ C_{15} = B_{117}$.

The symbols $m_{i,j},\ D_{i,j},\ M$ and $A_{i,i'}$ are the same as in Type 11.

**(7.14.3)** Set $I_0 = \{0, 1, 2, 3\},\ I_1 = \{4, 5, 6, 7\},\ I_2 = \{8, 9, 10, 11\}$, and $I_3 = \{12, 13, 14, 15\}$.

(i) For $0 \leq i \neq i' \leq 15$,

$$A_{i,i'} = \begin{cases} \widehat{G\backslash\langle\varphi\rangle} & \text{if } i \neq i' \in I_0, \\ \widehat{G\backslash\langle\tau\rangle} & \text{if } i \neq i' \in I_1, \\ \widehat{G\backslash\langle\varphi^2\tau\rangle} & \text{if } i \neq i' \in I_2, \\ \widehat{G\backslash\langle\varphi\tau\rangle} & \text{if } i \neq i' \in I_3, \\ \widehat{G} & \text{if } i \in I_k, i' \in I_l \quad \text{for some } k \neq l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \leq i \leq 15$

$$A_{i,i} = \begin{cases} 12 + \widehat{G\backslash\langle\varphi\rangle} & \text{if } i \in I_0, \\ 12 + \widehat{G\backslash\langle\tau\rangle} & \text{if } i \in I_1, \\ 12 + \widehat{G\backslash\langle\varphi^2\tau\rangle} & \text{if } i \in I_2, \\ 12 + \widehat{G\backslash\langle\varphi\tau\rangle} & \text{if } i \in I_3. \end{cases}$$

**(7.14.4)** Let $I_0, \ldots, I_3$ be the symbols used in (7.14.3).
(i) For $0 \leq i \neq i' \leq 15$

$$\sum_{j=0}^{15} m_{i,j} m_{i',j} = \begin{cases} 6 & \text{if } i \neq i' \in I_k \quad \text{for some } k \in \{0,1,2,3\}, \\ 9 & \text{if } i \in I_k, i' \in I_l \quad \text{for some } k \neq l \in \{0,1,2,3\}. \end{cases}$$

(ii) For $0 \leq i \leq 15$

$$\sum_{j=0}^{15} {m_{i,j}}^2 = 18.$$

(iii) For $0 \leq i \leq 15$

$$\sum_{j=0}^{15} m_{i,j} = 12.$$

**Lemma 7.6** *There does not exist an* $M = (m_{i,j})_{0 \leq i,j \leq 15}$. *Therefore Type* 14 *does not occur.*

### Type 15

**(7.15.1)** $\varphi = (x_0, x_1, x_2)(x_3, x_4, x_5)(x_6, x_7, x_8)(x_9, x_{10}, x_{11})$
$(x_{12}, x_{13}, x_{14})(x_{15}, x_{16}, x_{17})(x_{18}, x_{19}, x_{20})(x_{21}, x_{22}, x_{23})$
$(x_{24}, x_{25}, x_{26})(x_{27}, x_{28}, x_{29})(x_{30}, x_{31}, x_{32})(x_{33}, x_{34}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})(x_{39}, x_{51}, x_{63})$
$(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})$
$(x_{44}, x_{56}, x_{68})(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})(x_{75}, x_{87}, x_{99})$
$(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})$
$(x_{80}, x_{92}, x_{104})(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132})(x_{109}, x_{121}, x_{133})(x_{110}, x_{122}, x_{134})(x_{111}, x_{123}, x_{135})$
$(x_{112}, x_{124}, x_{136})(x_{113}, x_{125}, x_{137})(x_{114}, x_{126}, x_{138})(x_{115}, x_{127}, x_{139})$
$(x_{116}, x_{128}, x_{140})(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143})$ and

$\tau = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})(x_3, x_{15}, x_{27})$
$(x_4, x_{16}, x_{28})(x_5, x_{17}, x_{29})(x_6, x_{18}, x_{30})(x_7, x_{19}, x_{31})$
$(x_8, x_{20}, x_{32})(x_9, x_{21}, x_{33})(x_{10}, x_{22}, x_{34})(x_{11}, x_{23}, x_{35})$
$(x_{36}, x_{72}, x_{108})(x_{37}, x_{73}, x_{109})(x_{38}, x_{74}, x_{110})(x_{39}, x_{75}, x_{111})$
$(x_{40}, x_{76}, x_{112})(x_{41}, x_{77}, x_{113})(x_{42}, x_{78}, x_{114})(x_{43}, x_{79}, x_{115})$
$(x_{44}, x_{80}, x_{116})(x_{45}, x_{81}, x_{117})(x_{46}, x_{82}, x_{118})(x_{47}, x_{83}, x_{119})$
$(x_{48}, x_{84}, x_{120})(x_{49}, x_{85}, x_{121})(x_{50}, x_{86}, x_{122})(x_{51}, x_{87}, x_{123})$
$(x_{52}, x_{88}, x_{124})(x_{53}, x_{89}, x_{125})(x_{54}, x_{90}, x_{126})(x_{55}, x_{91}, x_{127})$
$(x_{56}, x_{92}, x_{128})(x_{57}, x_{93}, x_{129})(x_{58}, x_{94}, x_{130})(x_{59}, x_{95}, x_{131})$
$(x_{60}, x_{96}, x_{132})(x_{61}, x_{97}, x_{133})(x_{62}, x_{98}, x_{134})(x_{63}, x_{99}, x_{135})$
$(x_{64}, x_{100}, x_{136})(x_{65}, x_{101}, x_{137})(x_{66}, x_{102}, x_{138})(x_{67}, x_{103}, x_{139})$
$(x_{68}, x_{104}, x_{140})(x_{69}, x_{105}, x_{141})(x_{70}, x_{106}, x_{142})(x_{71}, x_{107}, x_{143})$, where $x \in \{p, B\}$.

**(7.15.2)** There are the following 16 $G$-orbits on $\mathcal{P}$ and on $\mathcal{B}$.
$\mathcal{Y}_0 = \{x_0, x_1, x_2, x_{12}, x_{13}, x_{14}, x_{24}, x_{25}, x_{26}\}$,

$\mathcal{Y}_1 = \{x_3, x_4, x_5, x_{15}, x_{16}, x_{17}, x_{27}, x_{28}, x_{29}\}$,
$\mathcal{Y}_2 = \{x_6, x_7, x_8, x_{18}, x_{19}, x_{20}, x_{30}, x_{31}, x_{32}\}$,
$\mathcal{Y}_3 = \{x_9, x_{10}, x_{11}, x_{21}, x_{22}, x_{23}, x_{33}, x_{34}, x_{35}\}$,
$\mathcal{Y}_4 = \{x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96}, x_{108}, x_{120}, x_{132}\}$,
$\mathcal{Y}_5 = \{x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97}, x_{109}, x_{121}, x_{133}\}$,
$\mathcal{Y}_6 = \{x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98}, x_{110}, x_{122}, x_{134}\}$,
$\mathcal{Y}_7 = \{x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99}, x_{111}, x_{123}, x_{135}\}$,
$\mathcal{Y}_8 = \{x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100}, x_{112}, x_{124}, x_{136}\}$,
$\mathcal{Y}_9 = \{x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101}, x_{113}, x_{125}, x_{137}\}$,
$\mathcal{Y}_{10} = \{x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102}, x_{114}, x_{126}, x_{138}\}$,
$\mathcal{Y}_{11} = \{x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103}, x_{115}, x_{127}, x_{139}\}$,
$\mathcal{Y}_{12} = \{x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104}, x_{116}, x_{128}, x_{140}\}$,
$\mathcal{Y}_{13} = \{x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105}, x_{117}, x_{129}, x_{141}\}$,
$\mathcal{Y}_{14} = \{x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106}, x_{118}, x_{130}, x_{142}\}$,
$\mathcal{Y}_{15} = \{x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107}, x_{119}, x_{131}, x_{143}\}$, where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_3,\ q_2 = p_6,\ q_3 = p_9,\ q_4 = p_{36},\ q_5 = p_{37}, q_6 = p_{38},\ q_7 = p_{39},\ q_8 = p_{40},\ q_9 = p_{41},\ q_{10} = p_{42},\ q_{11} = p_{43}, q_{12} = p_{44},\ q_{13} = p_{45},\ q_{14} = p_{46},\ q_{15} = p_{47}$ and $C_0 = B_0,\ C_1 = B_3,\ C_2 = B_6,\ C_3 = B_9,\ C_4 = B_{36},\ C_5 = B_{37}, C_6 = B_{38},\ C_7 = B_{39},\ C_8 = B_{40},\ C_9 = B_{41},\ C_{10} = B_{42},\ C_{11} = B_{43}, C_{12} = B_{44},\ C_{13} = B_{45},\ C_{14} = B_{46},\ C_{15} = B_{47}$.

The symbols $m_{i,j},\ D_{i,j},\ M$ and $A_{i,i'}$ are the same as in Type 11.

**(7.15.3)** (i) For $0 \le i \ne i' \le 15$

$$A_{i,i'} = \begin{cases} \widehat{G \backslash \langle\varphi\rangle} & \text{if } 0 \le i \ne i' \le 3, \\ \widehat{G \backslash \{1\}} & \text{if } 4 \le i \ne i' \le 15, \\ \widehat{G} & \text{if } 0 \le i \le 3,\ 4 \le i' \le 15. \end{cases}$$

(ii) For $0 \le i \le 15$

$$A_{i,i} = \begin{cases} 12 + \widehat{G \backslash \langle\varphi\rangle} & \text{if } 0 \le i \le 3, \\ 12 + \widehat{G \backslash \{1\}} & \text{if } 4 \le i \le 15. \end{cases}$$

**(7.15.4)** (i) For $0 \le i \ne i' \le 15$

$$\sum_{j=0}^{15} m_{i,j} m_{i',j} = \begin{cases} 6 & \text{if } 0 \le i \ne i' \le 3, \\ 8 & \text{if } 4 \le i \ne i' \le 15, \\ 9 & \text{if } 0 \le i \le 3,\ 4 \le i' \le 15. \end{cases}$$

(ii) For $0 \le i \le 15$

$$\sum_{j=0}^{15} {m_{i,j}}^2 = \begin{cases} 18 & \text{if } 0 \le i \le 3, \\ 20 & \text{if } 4 \le i \le 15. \end{cases}$$

(iii) For $0 \le i \le 15$

$$\sum_{j=0}^{15} m_{i,j} = 12.$$

**(7.15.5)** For $0 \le i \le 15$, the following hold, up to ordering of $m_{i,0}, m_{i,1}, \ldots, m_{i,15}$.

(i) If $0 \le i \le 3$, $(m_{i,0}, m_{i,1}, \ldots, m_{i,15}) = (\underbrace{0\ 0 \ldots 0}_{7}\ \underbrace{1\ 1 \ldots 1}_{6}\ 2\ 2\ 2)$ or $(\underbrace{0\ 0 \ldots 0}_{6}\ \underbrace{1\ 1 \ldots 1}_{9}\ 3)$.

(ii) If $4 \le i \le 15$, $(m_{i,0}, m_{i,1}, \ldots, m_{i,15}) = (\underbrace{0\ 0 \ldots 0}_{8}\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2)$ or $(\underbrace{0\ 0 \ldots 0}_{7}\ \underbrace{1\ 1 \ldots 1}_{7}\ 2\ 3)$.

**(7.15.6)** There are exactly 119 $M$, up to equivalence. They are $M_1, M_2, \ldots, M_{119}$, where each matrix of $M_1, \ldots, M_{13}$ contains 3 as an entry but each matrix of $M_{14}, \ldots, M_{119}$ does not. $M_1, M_2, \ldots, M_{13}, M_{14}$ are given in the Appendix and the authors have the list of the remaining matrices $M_{15}, \ldots, M_{119}$.

**(7.15.7)** There does not exist $(D_{i,j})_{4 \le i \le 9, 0 \le j \le 15}$ corresponding to the submatrix $(m_{i,j})_{4 \le i \le 9, 0 \le j \le 15}$ of $M_k = (m_{i,j})_{0 \le i \le 9, 0 \le j \le 15}$ for $1 \le k \le 119$.

**Lemma 7.7** *Type* 15 *does not occur.*

**THEOREM** *There are no projective planes of order* 12 *admitting a collineation group of order* 9.

PROOF. The theorem holds from Lemmas 6.2, 7.2, 7.3, 7.4, 7.5, 7.6 and 7.7. □

The theorem and [3] yield the following corollary.

**Corollary** *If* $G$ *is a collineation group of a projective plane* $\pi$ *of order* 12, *then* $G$ *is cyclic and* $|G|$ *divides* 3 *or* 4.

# Appendix

$$M_1 = \left(\begin{array}{ccccc|ccccccccccc}
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 0 & 1 & 0 & 2 & 1 \\
1 & 0 & 2 & 1 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
1 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \\
0 & 1 & 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 3 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 2 & 0 \\
1 & 0 & 1 & 2 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\
1 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\
2 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$$M_2 = \left(\begin{array}{ccc|cccccccccccccc}
0&0&0&0&0&0&2&1&2&1&1&1&0&1&1&2\\
1&1&1&0&2&1&0&2&0&0&0&0&1&2&0&1\\
1&1&1&1&0&2&1&0&0&0&2&2&1&0&0&0\\
1&1&1&2&1&0&0&0&1&2&0&0&1&0&2&0\\
\hline
3&0&1&0&0&0&0&0&0&1&1&1&1&1&1&2\\
1&3&0&0&0&0&0&1&2&0&1&1&1&1&1&0\\
0&1&3&0&0&0&2&1&0&1&0&1&1&1&1&0\\
\hline
0&0&1&0&1&2&0&0&1&1&2&0&0&2&2&0\\
0&0&1&0&2&1&0&0&2&1&0&2&2&0&0&1\\
0&1&0&1&0&2&1&1&0&0&0&0&2&0&2&2\\
0&1&0&2&0&1&0&1&0&2&0&2&0&2&0&1\\
0&1&2&2&1&0&0&1&1&0&2&0&0&0&0&2\\
1&0&0&1&2&0&1&2&0&0&1&2&0&0&2&0\\
1&0&0&2&1&0&2&0&1&0&1&0&2&2&0&0\\
1&2&0&0&2&1&2&0&0&2&1&0&0&0&0&1\\
2&0&1&1&0&2&1&2&2&1&0&0&0&0&0&0
\end{array}\right)$$

$$M_3 = \left(\begin{array}{ccc|cccccccccccccc}
0&0&0&0&0&0&2&1&2&1&1&1&0&1&1&2\\
1&1&1&0&2&1&0&2&0&0&0&0&1&2&0&1\\
1&1&1&1&0&2&1&0&0&0&2&2&1&0&0&0\\
1&1&1&2&1&0&0&0&1&2&0&0&1&0&2&0\\
\hline
3&0&1&0&0&0&0&0&0&1&1&1&1&1&1&2\\
1&3&0&0&0&0&0&1&2&0&1&1&1&1&1&0\\
0&1&3&0&0&0&2&1&0&1&0&1&1&1&1&0\\
\hline
0&0&1&0&1&2&0&0&1&1&2&0&0&2&2&0\\
0&0&1&0&2&1&0&0&2&1&0&2&2&0&0&1\\
0&1&0&1&0&2&1&1&0&0&0&0&2&0&2&2\\
0&1&0&2&0&1&0&1&0&2&0&2&0&2&0&1\\
0&2&1&1&2&0&1&0&0&1&2&0&0&0&0&2\\
1&0&0&1&2&0&1&2&0&0&1&2&0&0&2&0\\
1&0&0&2&1&0&2&0&1&0&1&0&2&2&0&0\\
1&0&2&2&0&1&0&2&2&0&1&0&0&0&0&1\\
2&1&0&0&1&2&2&1&1&2&0&0&0&0&0&0
\end{array}\right)$$

$$M_4 = \left(\begin{array}{ccc|cccccccccccccc}
0&0&0&0&0&0&2&1&2&1&1&1&0&1&1&2\\
1&1&1&0&2&1&1&0&0&0&2&2&1&0&0&0\\
1&1&1&1&0&2&0&2&0&0&0&0&1&2&0&1\\
1&1&1&2&1&0&0&0&1&2&0&0&1&0&2&0\\
\hline
3&0&1&0&0&0&0&0&0&1&1&1&1&1&1&2\\
1&3&0&0&0&0&0&1&2&0&1&1&1&1&1&0\\
0&1&3&0&0&0&2&1&0&1&0&1&1&1&1&0\\
\hline
0&0&1&0&1&2&0&0&1&1&2&0&0&2&2&0\\
0&0&1&1&0&2&0&0&2&1&0&2&2&0&0&1\\
0&1&0&0&2&1&1&1&0&0&0&0&2&0&2&2\\
0&1&0&1&2&0&0&1&0&2&0&2&0&2&0&1\\
0&1&2&2&1&0&0&1&1&0&2&0&0&0&0&2\\
1&0&0&2&0&1&1&2&0&0&1&2&0&0&2&0\\
1&0&0&2&1&0&2&0&1&0&1&0&2&2&0&0\\
1&2&0&1&0&2&2&0&0&2&1&0&0&0&0&1\\
2&0&1&0&2&1&1&2&2&1&0&0&0&0&0&0
\end{array}\right)$$

$$M_5 = \left(\begin{array}{ccc|cccccccccccccc}
0&0&0&0&0&0&2&1&2&1&1&1&0&1&1&2\\
1&1&1&0&2&1&1&0&0&0&2&2&1&0&0&0\\
1&1&1&1&0&2&0&2&0&0&0&0&1&2&0&1\\
1&1&1&2&1&0&0&0&1&2&0&0&1&0&2&0\\
\hline
3&0&1&0&0&0&0&0&0&1&1&1&1&1&1&2\\
1&3&0&0&0&0&0&1&2&0&1&1&1&1&1&0\\
0&1&3&0&0&0&2&1&0&1&0&1&1&1&1&0\\
\hline
0&0&1&0&1&2&0&0&1&1&2&0&0&2&2&0\\
0&0&1&1&0&2&0&0&2&1&0&2&2&0&0&1\\
0&1&0&0&2&1&1&1&0&0&0&0&2&0&2&2\\
0&1&0&1&2&0&0&1&0&2&0&2&0&2&0&1\\
0&2&1&2&0&1&1&0&0&1&2&0&0&0&0&2\\
1&0&0&2&0&1&1&2&0&0&1&2&0&0&2&0\\
1&0&0&2&1&0&2&0&1&0&1&0&2&2&0&0\\
1&0&2&1&2&0&0&2&2&0&1&0&0&0&0&1\\
2&1&0&0&1&2&2&1&1&2&0&0&0&0&0&0
\end{array}\right)$$

$$M_6 = \left(\begin{array}{ccc|ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 2 \\
1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 3 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 \\
\hline
0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 \\
0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 1 \\
0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\
1 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 \\
1 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\
1 & 2 & 0 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\
2 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$$M_7 = \left(\begin{array}{ccc|ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 2 \\
1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 3 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 \\
\hline
0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 \\
0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 1 \\
0 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 \\
1 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 \\
1 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\
1 & 0 & 2 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$$M_8 = \left(\begin{array}{ccc|ccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
1 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 0 \\
0 & 1 & 3 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \\
0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\
0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 2 \\
1 & 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}\right)$$

$$M_9 = \left(\begin{array}{ccc|ccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
1 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 0 \\
0 & 1 & 3 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 \\
0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 1 \\
0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 2 \\
1 & 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{array}\right)$$

$$M_{10} = \left(\begin{array}{ccc|ccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
1 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 0 \\
0 & 1 & 3 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 0 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 1 \\
0 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2 & 1 \\
0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 2 \\
1 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{array}\right)$$

$$M_{11} = \left(\begin{array}{ccc|ccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 2 \\
1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 3 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}\right)$$

$$M_{12} = \left(\begin{array}{ccc|ccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 \\
1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 3 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 2 \\
0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\
2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right)$$

$$M_{13} = \left(\begin{array}{ccc|ccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 \\
1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 3 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right)$$

$$M_{14} = \left(\begin{array}{cccc|cccccccccccc}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 2 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 2 \\
\hline
0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 1 \\
0 & 1 & 2 & 0 & 2 & 2 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 1 \\
2 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 \\
1 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\
2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 2 & 2 & 0 \\
0 & 2 & 0 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\
2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 2 & 1 & 0 & 0 & 0
\end{array}\right)$$

## References


[1] A. Aguglia and A. Bonisoli, On the non-existence of a projective plane of order 15 with an $A_4$-invariant oval, *Discrete Math.* **288** (2004), 1–7.

[2] R. P. Anstee, M. Hall Jr. and J. G. Thompson, Planes of order 10 do not have a collineation of order 5, *J. Combin. Theory Ser. A* **29** (1980), 39–58.

[3] K. Akiyama and C. Suetake, The nonexistence of projective planes of order 12 with a collineation group of order 8, *J. Combin. Des.* **16** (2008), 411–430.

[4] K. Akiyama and C. Suetake, On projective planes of order 12 with a collineation group of order 9, *Australas. J. Combin.* **43** (2009), 133–162.

[5] T. Beth, D. Jungnickel and H. Lenz, *Design Theory,* Vols. I and II, Cambridge University Press, Cambridge, 1999.

[6] K. Horvatic-Baldasar, E. Kramer and I. Matulic-Bedenic, Projective planes of order 12 do not have an abelian group of order 6 as a collineation group, *Punime Mat.* **1** (1986), 75–81.

[7] K. Horvatic-Baldasar, E. Kramer and I. Matulic-Bedenic, On the full collineation group of projective planes of order 12, *Punime Mat.* **2** (1987), 9–11.

[8] R. H. Bruck and H. J. Ryser, The nonexistence of certain finite projective planes, *Canad. J. Math.* **1** (1949), 88–93.

[9] R. C. Bose and S. S. Shrikhande, On the falsity of Euler's conjecture about the non-existence of two orthogonal latin squares of order $4t + 2$, *Proc. Nat. Acad. Sci.* **45** (1959), 734–737.

[10] R. C. Bose, S. S. Shrikhande and E. T. Parker, Further results on the construction of mutually orthogonal latin squares and the falsity of Euler's conjecture, *Canad. J. Math.* **12** (1960), 189–203.

[11] D. Casiello, L. Indaco and G. P. Nagy, Sull'approccio computazionale al problema dell'esistenza di un piano proiettivo di ordine 10, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* **57** (2010), 69–88.

[12] GAP—Groups, Algorithms, Programming—System for Computational Discrete Algebra, GAP 4.10.0, https://www gap-system.org.

[13] C. Y. Ho, Projective planes of order 15 and other odd composite orders, *Geom. Dedicata* **27** (1988), 49–64.

[14] D. R. Hughes and F. C. Piper, *Projective Planes,* Springer-Verlag, Berlin, Heidelberg, New Vork, 1973.

[15] Z. Janko and T. van Trung, On projective planes of order 12 which have a subplane of order three I, *J. Combin. Theory Ser. A* **29** (1980), 254–256.

[16] Z. Janko and T. van Trung, Projective planes of order 12 do not have a non-abelian group of order 6 as a collineation group, *J. Reine Angew. Math.* **326** (1981), 152–157.

[17] Z. Janko and T. van Trung, Projective planes of order 10 do not have a collineation of order 3, *J. Reine Angew. Math.* **325** (1981), 189–209.

[18] Z. Janko and T. van Trung, Projective planes of order 12 do not possess an elation of order 3, *Stud. Sci. Math.* **16** (1981), 115–118.

[19] Z. Janko and T. van Trung, On projective planes of order 12 with an automorphism of order 13, Part I: Kirkman designs of order 27, *Geom. Dedicata* **11** (1981), 257–284.

[20] Z. Janko and T. van Trung, On projective planes of order 12 with an automorphism of order 13, Part II: Orbit matrices and conclusion, *Geom. Dedicata* **12** (1982), 87–99.

[21] Z. Janko and T. van Trung, The full collineation group of any projective plane of order 12 in a $\{2,3\}$ group, *Geom. Dedicata* **12** (1982), 101–110.

[22] Z. Janko and T. van Trung, A generalization of a result of L. Baumert and M. Hall about projective planes of order 12, *J. Combin. Theory Ser. A* **32** (1982), 378–385.

[23] Z. Janko and T. van Trung, Projective planes of order 12 do not have a four group as a collineation group, *J. Combin. Theory Ser. A* **32** (1982), 401–404.

[24] M. J. Kallaher, *Affine Planes with Transitive Collineation Groups,* North-Holland, New York, Amsterdam, Oxford, 1982.

[25] S. Kang and J.-H. Lee, An explicit formula and its fast algorithm for a class of symmetric balanced incomplete block designs, *J. Appl. Math. Comput.* **19** (2005), 105–125.

[26] C. W. H. Lam, The search for a finite projective plane of order 10, *Amer. Math. Monthly* **98** (1991), 305–318.

[27] E. T. Parker, Construction of some sets of mutually orthogonal Latin squares, *Proc. Amer. Math. Soc.* **10** (1959), 946–949.

[28] E. T. Parker, Orthogonal Latin squares, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 859–862.

[29] C. Suetake, On projective planes of order 15 admitting a collineation of order 7, *Geom. Dedicata* **81** (2000), 61–86.

[30] C. Suetake, The nonexistence of projective planes of order 12 with a collineation group of order 16, *J. Combin. Theory Ser. A* **107** (2004), 21–48.

[31] S. H. Whitesides, Collineations of projective planes of order 10, part I and II, *J. Combin. Theory Ser. A* **26** (1979), 249–277.

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