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Asteroidal-triple-free interval k-graphs

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Abstract

A bipartite graph is a proper interval bigraph if and only if it is asteroidal-triple-free. An asteroidal triple (AT) is a triple of vertices such that any two are joined by a path that avoids the neighbors of the third. For $k \geq 3$, a k-partite graph that is AT-free is not necessarily an interval k-graph. In this paper, we prove that if a k-partite, AT-free graph has no induced 5-cycle and no vertex that is adjacent to two consecutive vertices of a 4-cycle, then it is an interval k-graph. We propose a conjecture for the characterization of k-partite, AT-free interval k-graphs.

1 Introduction

Throughout this paper, G will be a connected graph. A graph is called an **interval graph** if its vertices can be assigned intervals on the real line in such a way that two vertices are adjacent if and only if their corresponding intervals intersect. In [7], it was shown that *a graph is an interval graph if and only if it contains neither an asteroidal triple nor an induced cycle of length at least four*. An **asteroidal triple** (AT) is a triple of vertices such that any two are joined by a path that avoids the neighbors of the third. A bipartite graph is called an **interval bigraph** if its vertices can be assigned intervals on the real line in such a way that vertices in different parts of the

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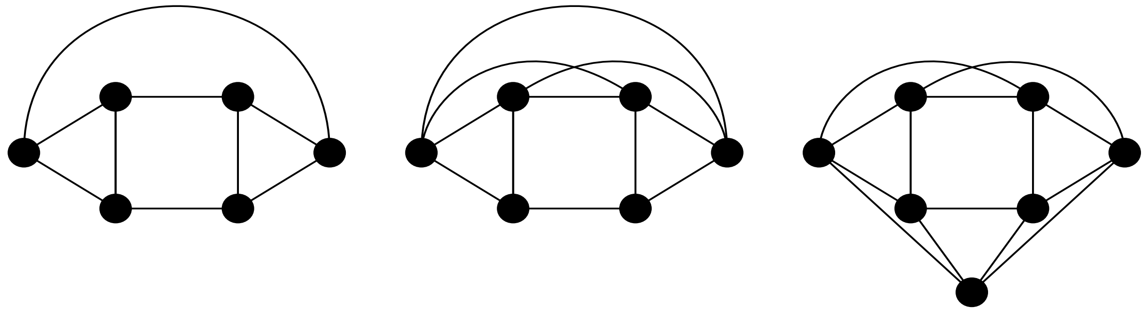

Figure 1: The forbidden subgraphs $\overline{C_6}$, $\overline{2P_3}$, and $\overline{C_7}$.

partition are adjacent if and only if their corresponding intervals intersect. There is no characterization of interval bigraphs in terms of forbidden subgraphs, but much else is known ([1] and [6]), including that *a bipartite, AT-free graph is an interval bigraph*[6]. The term **AT-free** refers to a graph that does not contain an asteroidal triple. Note that since induced cycles of length six or more contain asteroidal triples, a graph that is AT-free has no induced cycle of length six or more.

A generalization of the interval bigraph is the interval k-graph, which was introduced in [4] and has been studied further in [2], [3], and [5]. A k-partite graph is an **interval k-graph** if its vertices can be assigned intervals on the real line in such a way that vertices in different parts of the partition are adjacent if and only if their corresponding intervals intersect. From the above characterizations of interval graphs and interval bigraphs, we know that *a k-partite, AT-free graph with no induced cycle of length four or five is an interval k-graph*, and that *a k-partite, AT-free graph with no induced cycle of length three or five is an interval k-graph.* It is an easy exercise to show that an interval k-graph cannot have a 5-cycle as an induced subgraph. In this paper, we prove the following:

Theorem 1.1 *Let G be a k-partite, AT-free graph that does not contain a 5-cycle as an induced subgraph. Suppose that there is no vertex in G which is adjacent to two consecutive vertices of a 4-cycle. Then G is an interval k-graph.*

The graphs $\overline{C_6}$, $\overline{2P_3}$, and $\overline{C_7}$ are AT-free graphs without induced 5-cycles. These graphs are not, however, interval k-graphs (where $k = 3, 4, 4$, respectively), which is why we need the last assumption in Theorem 1.1. Indeed, in every proper coloring of $\overline{C_7}$ or $\overline{2P_3}$ there is an induced 4-cycle whose vertices are assigned four different colors, and hence they fail to be interval 4-graphs by the characterization of interval graphs. The graph $\overline{C_6}$ is the cartesian product of K_2 and K_3, which fails to be an interval 3-graph in the same way that C_4, the cartesian product of K_2 and K_2, fails to be an interval graph. Since $\overline{2P_3}$ is an induced subgraph of $\overline{C_j}$ for each $j \geq 8$, we propose the following conjecture:

Conjecture 1.2 *Let G be a k-partite, AT-free graph that does not contain a 5-cycle as an induced subgraph. Then G is an interval k-graph if and only if G does not contain $\overline{C_6}$, $\overline{2P_3}$, or $\overline{C_7}$ as an induced subgraph.*

1.1 Terminology and Notation

We use terminology and notation as in [8] except as indicated. When the context makes it clear, we use V for the vertex set $V(G)$ and E for the edge set $E(G)$ of the graph G. Let H be a subgraph of G. In the neighborhood of H we do not include any vertex from H. The subgraph of G induced by $X \subseteq V$ will be denoted $G[X]$. We use $G - X$ to denote the subgraph $G[V - X]$ of G, and $H + X$ to denote the supergraph $G[V(H) \cup X]$ of H. A j-cycle is a chordless cycle of length j and is denoted C_j. We call a graph acyclic if it does not contain C_j, $j \geq 4$, as an induced subgraph.

An **interval k-representation**, or **representation**, $\{I_x\}_{x \in G}$ (or just $\{I_x\}$, when the graph is understood) for the interval k-graph G is a collection of intervals that represent the vertices of G in a way that satisfies the definition of interval k-graph. Thus G is an interval k-graph if and only if it has an interval k-representation. The interval corresponding to the vertex x will be denoted I_x. The left and right endpoints of the interval I_x will be written, respectively, as x_L and x_R. Thus the intervals I_x and I_y intersect if and only if $x_L \leq y_R$ and $y_L \leq x_R$.

Say $G = (V_1, V_2, \ldots, V_k, E)$ is an interval k-graph, and let $v \in V_i$. Then I_v is an **end interval** of the representation $\{I_x\}$ if either

$$\begin{aligned} v_L \leq y_R \ \text{ for all } \ y \notin V_i \quad \text{or} \\ v_R \geq y_L \ \text{ for all } \ y \notin V_i. \end{aligned}$$

Thus I_v is an end interval of $\{I_x\}$ if and only if there exists a representation $\{I'_x\}$ in which the left(right) endpoint of I_v is further to the left(right) than all other intervals in $\{I'_x\}$.

When two vertices of a k-partite graph belong to different parts of the partition, we will say that they have different **colors**. Similarly, two vertices from the same part of the partition will have the same color. We use the same terminology when talking about intervals.

2 Lemmas

Lemma 2.1 *Let G be an interval k-graph. If there are vertices y and z in G such that $N(y) = \{x, z\}$ and $N(z) = \{x, y\}$ for some vertex x in G, and $G' = G - \{x, y, z\}$ is connected, then I_x is an end interval in every representation $\{I_v\}_{v \in G}$ of G.*

Proof: Suppose there is a representation $\{I_v\}$ in which I_x is not an end interval. Then there are some intervals I_a and I_b in $\{I_v\}$ such that $x_L > a_R$, $x_R < b_L$, and such that a and b have different colors than x. The vertices a and b can be neither y nor z because both y and z are adjacent to x in G, and thus a and b are in G'. Since G' is connected, there is a path $P = av_1 \ldots v_n b$ that avoids $\{x, y, z\}$. This path corresponds in $\{I_v\}$ to a chain of intervals $I_a, I_{v_1}, \ldots, I_{v_n}, I_b$ such that $a_R \geq (v_1)_L$,

$(v_n)_R \geq b_L$, and $(v_i)_R \geq (v_{i+1})_L$ for each $1 \leq i \leq n-1$. This implies that for each point $w \in [a_R, b_L]$ there is a j with $1 \leq j \leq n$ such that $w \in I_{v_j}$. Since $\{x, y, z\}$ induces a triangle, there is a point m that lies in the intersection of I_x, I_y, and I_z, so $m \in I_x \subset [a_R, b_L]$ is contained in some interval I_{v_j}, where j is between 1 and n. Thus the intersection of I_y, I_z, and I_{v_j} is nonempty, and because neither vertex y nor vertex z is adjacent to vertex v_j, we must use the same color for all of them. This is a contradiction, since $yz \in E$. ❏

Lemma 2.2 *Let G be an AT-free interval bigraph. If there is a vertex y in G with $N(y) = \{x\}$ for some $x \in V$ such that $G' = G - \{x, y\}$ is connected, then I_x is an end interval in every representation $\{I_v\}_{v \in G}$ of G.*

Proof: Suppose there is a representation $\{I_v\}$ in which I_x is not an end interval. Let the vertex set V be partitioned by A and B, and let $x \in A$. Then there are vertices $a, b \in B$ such that in $\{I_v\}$, $x_L > a_R$ and $x_R < b_L$. Since G' is connected, there is a path from a to b that avoids $N(y) = \{x\}$. If there is a path from a to x that avoids the neighborhood of b and a path from b to x that avoids the neighborhood of a then $\{a, b, y\}$ is an AT, which is a contradiction. Hence every path from, say, a to x, must include a vertex that is in the neighborhood of b.

Let $P = av_1v_2 \ldots v_nv_{n+1} \ldots x$ be an $a - x$ path, where v_n is the first vertex in P that is adjacent to b (note that $v_n \neq x$ because $I_x \cap I_b = \emptyset$). Since v_n is adjacent to $b \in B$, we must have $v_n \in A$, and therefore $v_{n-1} \in B$. Suppose that x is adjacent to some vertex v_j with $j \leq n-1$. Then $P' = av_1v_2 \ldots v_jx$ is an $a - x$ path and must therefore have a vertex $v_k \neq x$ that is adjacent to b. But $k \leq j \leq n-1$, which contradicts the fact that n is the smallest index such that v_n is adjacent to b. So x must not be adjacent to any vertex v_j with $j \leq n-1$. In particular, since $v_{n-1} \in B$, we must have $x_R < (v_{n-1})_L$ or $x_L > (v_{n-1})_R$.

Assume first that $x_R < (v_{n-1})_L$. Because $x \in A$ and x is not adjacent to v_j with $j \leq n-1$, I_x cannot intersect any interval I_{v_j} with $j \leq n-1$ and $v_j \in B$. Thus, since $x_R < (v_{n-1})_L$ and $x_L > a_R$, we must have either $a_R < x_L < x_R < (v_2)_L$ or $(v_{n-i-2})_R < x_L < x_R < (v_{n-i})_L$ for some $v_{n-i} \in B$ with $1 \leq i \leq n-4$. Then $I_x \subset I_{v_i}$ for some $1 \leq i \leq n-2$ with $v_i \in A$. Thus, since $xy \in E$ the intersection of I_y and I_{v_i} is nonempty, which is a contradiction as $y \in B$ and $v_i \notin N(y)$. Now assume $x_L > (v_{n-1})_R$. Since v_n and v_{n-1} are adjacent, we get $x_L > (v_{n-1})_R \geq (v_n)_L$, and since v_n is adjacent to b and $x_R < b_L$, we get $x_R < b_L \leq (v_n)_R$, so that I_x is contained in I_n. Thus we again get a contradiction, as $v_n \in A$ but $v_n \notin N(y)$. ❏

Let c be a cut vertex of the graph G, and let $G_1, G_2, \ldots, G_n$ be the components of $G - \{c\}$. The graphs $G_i + \{c\}$, $i = 1, \ldots, n$, are the **branches** of G at c. Let H and H' be subgraphs of G. We say that $c \in V(H)$ is a **DB cut-vertex** *in H of H'* if it is a cut vertex of H' with the additional property that there exist distinct branches $B_1 \subseteq H'$ and $B_2 \subseteq H'$ of H' at c such that $C_3 \subseteq B_1$ and $C_4 \subseteq B_2$ for some 3-cycle C_3 and 4-cycle C_4. If $H = G$ we will say just *DB cut-vertex of H'*, and if $H = H' = G$ we will say just *DB cut-vertex*. Note that graphs without 3-cycles and graphs without 4-cycles do not have DB cut-vertices. Since acyclic AT-free graphs are interval graphs by [7] and bipartite AT-free graphs are interval bigraphs by [6], we will use DB cut-vertices to separate G into components that are each either an interval graph or interval bigraph. If $X = v_1v_2 \dots v_jv_1$ is a (possibly chorded) cycle of length j in G, then we write $G[X]$ in place of $G[V(X)]$, and denote the edge set $\{v_iv_{i+1} \,|\, 1 \leq i \leq j-1\} \cup \{v_jv_1\}$ by $E(X)$.

Lemma 2.3 *Let G be a graph with no induced C_j, $j \geq 5$, and suppose that there is no vertex in G which is adjacent to two consecutive vertices of a C_4. Then for every 3-cycle C and 4-cycle L, there is a DB cut-vertex c such that C and L belong to different branches at c.*

Proof: Let C and L be a 3-cycle and 4-cycle, respectively. For contradiction, suppose that there is no cut vertex with C and L belonging to different branches. Then there are two disjoint paths from C to L. Since C and L are each connected, there is a cycle T such that $E(T) \cap E(C)$ is nonempty and $E(T) \cap E(L)$ is nonempty. Since G has no j-cycle for $j \geq 5$, $G[T]$ only has induced cycles of length three and four. Since $E(T) \cap E(C)$ is nonempty and $E(T) \cap E(L)$ is nonempty, this implies that some edge of $G[T]$ is shared by a 3-cycle and 4-cycle. But this contradicts the hypothesis that a vertex cannot be adjacent to two consecutive vertices of a C_4, so the proof is complete. ❑

3 Proof of Theorem 1.1

Proof: Let $G = (V, E)$ be a k-partite graph. We say that a subgraph H of G is **DB-nonseparable** if it has no DB cut-vertices (of H). By Lemma 2.3, a DB-nonseparable graph with no $C_\ell, \ell \geq 5$, and no vertex adjacent to two consecutive vertices of a 4-cycle is either bipartite or acyclic. A **DB cut** is a collection $\mathcal{X} \subset V$ of DB cut-vertices such that for each component G_i of $G - \mathcal{X}$, $G_i + \mathcal{X}$ is DB-nonseparable. If $|\mathcal{X}|$ is minimal in the set of all DB cuts, we call $\mathcal{X}$ a **minimal DB cut**, and call $|\mathcal{X}|$ the **DB-connectivity** of G. If G is an interval k-graph with subgraphs H and H', we write $\{I_x\}_{x \in H} < \{I_x\}_{x \in H'}$ if $x_R < y_L$ for each $x \in H$ and $y \in H'$.

Let G be AT-free with no C_5, and suppose no vertex in G is adjacent to two consecutive vertices of a 4-cycle. Let $\mathcal{X}$ be a minimal DB cut, and let $|\mathcal{X}| = j$. We prove Theorem 1.1 by induction on j. If $j = 0$ then by Lemma 2.3, G is either bipartite or acyclic and there is nothing to prove, so suppose $j \geq 1$.

Let $j = 1$, with $\mathcal{X} = \{x\}$. Consider the branches $B_i(x)$, $i = 1, 2, \dots, n$, and their corresponding components $G_i = B_i(x) - x$ of $G - x$. Since $j = 1$, each branch is either bipartite or acyclic and thus has a representation. Since G is AT-free, at most two branches may have a vertex that is not in the neighborhood of x. Without loss of generality let $B_1(x)$ be such a branch.

First assume that $B_1(x)$ is bipartite. There is a vertex $y \in G - B_1(x)$ with $y \notin N(G_1)$ such that $xy \in E$, since x is a DB cut-vertex. Because $B_1(x)$ is bipartite, clearly $B_1(x) + y$ is as well. Further, $B_1(x) + y$ is AT-free since it is an induced subraph of G. Hence $B_1(x) + y$ is an interval bigraph, and by Lemma 2.2 I_x is an end interval in each representation. Consequently, $B_1(x)$ is an interval bigraph and I_x is an end interval in $\{I_v\}_{v \in B_1(x)}$.

Now assume that $B_1(x)$ is acyclic. Create new vertices y and z and add them to $B_1(x)$ such that $N(y) = \{x, z\}$ and $N(z) = \{x, y\}$. Then $B_1(x) + \{y, z\}$ is acyclic, and is AT-free as well since $B_1(x) + y$ is AT-free and $N(z) = N(y)$. Therefore by Lemma 2.1, $B_1(x)$ is again an interval k-graph with I_x an end interval in each representation.

Now for each $B_i(x)$ such that $V(G_i) \subset N(x)$, we can construct a representation with $x_L < v_L$ and $x_R > v_R$ for each $v \in G_i$. For $B_1(x)$ we may construct a representation with $x_R > v_R$ for each $v \in G_1$, since I_x is an end interval. If without loss of generality $B_n(x)$ is another branch with a vertex that is not in the neighborhood of x, then we may construct a representation with $x_L < v_L$ for each $v \in G_n$. Thus, since $G_1, \ldots, G_n$ are distinct components, we may construct a representation for G by setting $\{I_v\}_{v \in G_1} < \{I_v\}_{v \in G_2} < \cdots < \{I_x\}_{x \in G_{n-1}} < \{I_v\}_{v \in G_n}$.

Now let $j \geq 2$, and suppose that the theorem holds for graphs with DB-connectivity less than j. Since G is connected and $j \geq 2$, there is a component H of $G - \mathcal{X}$ whose neighborhood in G includes at least two vertices from $\mathcal{X}$. Let $x, y \in N(H) \cap \mathcal{X}$, and let $H' = H + \{x, y\}$. Then H' is DB-nonseparable because $\mathcal{X}$ is a DB cut, so H' is acyclic or bipartite. Also, the branch $B(x)$ of G at x containing H' has DB-connectivity at most $j - 1$, since a subset of $\mathcal{X} - \{x\}$ is a minimal DB cut.

First suppose that H' is acyclic. Create new vertices a and b and add them to $B(x)$ such that $N(a) = \{b, x\}$ and $N(b) = \{a, x\}$. Then $B(x) + \{a, b\} = B'(x)$ has DB-connectivity at most $j - 1$. This is true because $H' + \{a, b\}$ is acyclic and $B(x)$ has DB-connectivity at most $j-1$. Hence $B'(x)$ has a representation, and by Lemma 2.1 we know that I_x is an end interval in $B(x)$.

Now suppose that H' is bipartite. Since x is a cut vertex of G, there is a vertex $z \in G - B(x)$ with $z \notin N(B(x) - \{x\})$ such that $zx \in E$. Hence because $B(x)$ contains H', there is a vertex $z \in G$ whose neighborhood in H' is just x, so as in the case $j = 1$ we know that I_x is an end interval in each representation of H'. Similarly, we know that I_y is an end interval in each representation of H'. Since I_x is an end interval, we may construct a representation $\{I_v\}_{v \in H'}$ of H' such that $x_L \leq v_R$ or $x_R \geq v_L$ for all $v \in H'$. Without loss of generality say $x_L \leq v_R$ for each $v \in V(H')$. Since I_y is an end interval in $\{I_v\}_{v \in H'}$, we may assume that either $y_L \leq v_R$ or $y_R \geq v_L$ for each vertex $v \in H'$. We will show that there is a representation of H' in which the left(or right) endpoint of I_x is further to the left(right) than all other intervals, and the right(left) endpoint of I_y is further to the right(left) than all other intervals. Thus we can assume that $y_R < u_L$ for some u in H' such that y and u have different colors, for otherwise we are done. Hence $y_L \leq v_R$ for each $v \in V(H')$, so we may similarly assume that $x_R < w_L$ for some vertex $w \in V(H')$ such that x and w have different colors.

Suppose that there is a vertex $v \in H$ whose neighborhood avoids a path from x to y. Since H is connected and x and y are cut vertices, we can use the same argument as in the preceding paragraph to find vertices x' and y' whose neighborhoods in H' are just x and y, respectively. But then $\{v, x', y'\}$ is an asteroidal triple, a contradiction. Thus each vertex in H must be adjacent to a vertex in each $x-y$ path. Assume that x and y are adjacent. Then every vertex in H is adjacent to either x or y, so since H' is bipartite every interval in $\{I_v\}_{v\in H'}$ either intersects I_y or has the same color as I_y. This is a contradiction, as y and u have different colors but their intervals do not intersect. Now assume that the shortest $x-y$ path is x, r, y. Then for each vertex v in H we have four possibilities for $N(v) \cap \{x, r, y\}$: $\{r\}$, $\{x\}$, $\{y\}$, and $\{x, y\}$. Suppose that $N(u_1) \cap \{x, r, y\} = \{x\}$ for some u_1 in H. Since u_1 and y have different colors and $y_L \leq v_R$ for each $v \in V(H')$, we must have $y_R < (u_1)_L$. Then, since $u_1x \in E$, we have $y_R < (u_1)_L \leq x_R$. If there is a vertex u_2 with $u_2x \notin E$ but $u_2y \in E$, then $x_R < (u_2)_L \leq y_R$, a contradiction. Hence there is no vertex adjacent to y but not adjacent to x. But then every vertex is either adjacent to x or the same color as x, which contradicts the vertex w. Thus the vertex u_1 does not exist, so each vertex in H is either adjacent to r or adjacent to y, which contradicts the vertex u. Therefore, the shortest $x-y$ path is P_i, $i \geq 4$. If i is even, then x and y are different colors and not adjacent, which contradicts the fact that $x_L \leq v_R$ and $y_L \leq v_R$ for each $v \in V(H')$. Thus i is odd with $i \geq 5$. Say the shortest $x-y$ path is $x = v_1v_2 \ldots v_{i-1}v_i = y$. Since $xv_{i-1} \notin E(H')$ and x has a different color than v_{i-1}, we have $x_R < (v_{i-1})_L$. Similarly, $y_R < (v_2)_L$. But then $(v_i)_R = y_R < (v_2)_L \leq (v_1)_R = x_R < (v_{i-1})_L$, a contradiction. Hence $x_L \leq v_R$ and $y_R \geq v_L$ for each v in H'.

Consider $B(x) - (H' - \{y\}) = B(y)$. $B(y)$ has DB-connectivity at most $j-1$, and has DB-connectivity at most $j-2$ if y is not a DB cut-vertex of $B(y)$. Thus we may add vertices a and b to $B(y)$ such that $N(a) = \{b, y\}$ and $N(b) = \{a, y\}$ to get a graph with DB-connectivity at most $j-1$, and by Lemma 2.1 conclude that I_y is an end interval in the representation of $B(y)$. Let $y_L \leq v_R$ for each $v \in V(B(y))$ (if this is not the case, reverse all of the inequalities in the representation). Then, since $y_R \geq v_L$ for each $v \in V(H')$ and since H and $B(y) - \{y\}$ are not connected, we can construct an interval k-representation $\{I_v\}$ for $B(x)$ by setting $\{I_v\}_{v\in H} < \{I_v\}_{v\in B(y)-\{y\}}$. Since $x_L \leq v_R$ for each $v \in V(H')$, we conclude that I_x is an end interval in $\{I_v\}$.

Let $B(x) = B_1(x)$, and label the other branches $B_2(x), \ldots, B_m(x)$. Each branch $B_i(x)$ for $i \geq 2$ has DB-connectivity at most $j-2$ and hence has an interval k-representation. Since G is AT-free there is at most one branch, say $B_m(x)$, that has a vertex that is not in the neighborhood of x. Since $B_m(x)$ has DB-connectivity at most $j-2$, the graph $B_m(x) + \{a, b\}$ for a, b with $N(a) = \{x, b\}$ and $N(b) = \{a, x\}$ has DB-connectivity at most $j-1$. Then by Lemma 2.1 I_x is an end interval in $\{I_v\}_{v\in B_m(x)}$, so as in the case $j = 1$ the proof is complete. ❑

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