AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **74(2)** (2019), Pages 240–252

Minimal unavoidable sets of cycles in plane graphs with restricted minimum degree and edge weight

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Abstract

A set S of cycles is minimal unavoidable in a graph family $\mathcal{G}$ if every graph $G \in \mathcal{G}$ contains a cycle from S and, for every nonempty proper subset $S' \subset S$, there exists an infinite subfamily $\mathcal{G}' \subseteq \mathcal{G}$ such that no graph from $\mathcal{G}'$ contains a cycle from S'. In this paper, we explore minimal unavoidable sets of cycles in planar graphs with prescribed minimum vertex degree or minimum edge-weight. In particular, we show that every planar graph with $\delta \geq 3$ and without adjacent 3-vertices always contains a 3- or 4-cycle (and a 3- or 6-cycle or else a 4- or 6-cycle); when the minimum edge-weight is at least 9, a 3-, 4- and a 5-cycle is always present. For planar graphs with $\delta \geq 4$, we show that they contain a 4- or 8-cycle, and a 4- or 9-cycle. Besides this, we describe constructions of infinite graph families whose members omit cycles of prescribed length lists.

1 Introduction

In this paper, we consider connected planar graphs without loops and multiple edges. A particular plane drawing D of a planar graph G is represented by triple (V, E, F) where V is the vertex set, E is the edge set and F is the set of faces. Each face $\alpha \in F$ is described by its *facial walk* which is a clockwise-oriented closed walk $v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k, e_k, v_1$ whose vertices and edges are incident with α and, for all $i \in \{1, \ldots, k\}$, e_i follows e_{i-1} (indices modulo k) in the counter-clockwise order of edges around v_i in D; in the sequel, we will consider facial walks simply as clockwise-ordered lists of their vertices. The number k is called the *size* of α, and is denoted by $\deg(\alpha)$. A face of size k (at least k) is further referred to as k-face (k^+-face); similarly, a vertex of degree k (at least k or at most k) is a k-vertex (k^+-vertex or k^--vertex, respectively).

ISSN: 2202-3518

By $C_k, k \geq 3$, we denote the cycle on k vertices. For positive integers $k_1, \ldots, k_\ell \geq 3$, we set $S_{k_1,\ldots,k_\ell} = \{C_{k_1}, \ldots, C_{k_\ell}\}$; in addition, $S_{k_1,\ldots,k_\ell,k^+} = \{C_{k_1}, \ldots, C_{k_\ell}\} \cup \{C_p : p \geq k\}$ (here we also allow $\ell = 0$).

In [9], the following definition was introduced: A set S of cycles is *minimal unavoidable* in a graph family $\mathcal{G}$ if

(i) every graph $G \in \mathcal{G}$ contains a cycle from S and

(ii) for every nonempty proper subset $S' \subset S$, there exists an infinite subfamily $\mathcal{G}' \subseteq \mathcal{G}$ such that no graph from $\mathcal{G}'$ contains a cycle from S'.

Continuing our research from [9], we explore minimal unavoidable sets of cycles in plane graphs under various constraints (but excluding 3-connectedness) on their minimum vertex degree or minimum edge weight. Compared to the family of plane graphs of minimum degree at least 3 where each minimal unavoidable set of cycles consists of at least three elements, the above mentioned additional requirements yield various smaller unavoidable cycles sets. For example, every plane graph G with $\delta(G) \geq 4$ contains C_3 (this follows easily from Euler's formula), C_5 [11], and C_6 [2]. Furthermore, in [3], it was shown that every plane graph G with $\delta(G) \geq 4$ contains C_4 or C_7. Together with the fact that there exist infinitely many plane graphs of minimum degree 4 which contain no 4-cycles (they can be obtained, for example, from cubic plane graphs of girth 5 when turning them into 4-regular plane graphs by replacing every trivalent vertex by a triangular face), and, also, infinitely many plane graphs of minimum degree 4 containing no 7-cycles (for example, the graphs obtained from t copies of the octahedron graph by selecting a single vertex in every such a copy and then identifying those vertices), we obtain that the set $S_{4,7}$ is minimal unavoidable in the family of planar graphs of minimum degree ≥ 4. We also note that various light graph theory results (for details, see [5, 1, 10, 8] or the survey [4]) imply the existence of cycles of lengths from 3 to 7 in plane graphs of minimum degree 5.

In addition to these results, we take a closer look on the families of plane graphs of minimum degree ≥ 4, or minimum degree ≥ 3 and the minimum edge-weight (that is, the sum of degrees of endvertices of edges) ≥ 7 (or ≥ 9). The negative results (that is, on finite sets of cycles which are not unavoidable) are contained in Section 2; we describe constructions of infinite sets of graphs whose members contain no cycles from prescribed cycle set. The positive results on minimal unavoidability of particular small cycle sets appear in Section 3. The paper concludes with several open problems.

2 Negative results

We recall the general outline (from [9]) of constructions which exclude selected particular cycles from infinitely many graphs of a family $\mathcal{G}$: consider a plane graph G without cycles of lengths $\ell_1, \ldots, \ell_k$ (note that it need not belong to $\mathcal{G}$), and with x

being arbitrary vertex on the outerface of G. Next, form the graph G_n by taking n copies of G and identifying all vertices which are counterparts of x. With some care when choosing G and x, the graph G_n is planar, it belongs to $\mathcal{G}$ and it does not contain cycles of lengths $\ell_1, \ldots, \ell_k$ and, also, the cycles of length $\ell \geq |V(G)| + 1$.

Sometimes we also use a similar construction, taking a plane graph H without cycles of the above specified lengths and choosing two distinct vertices x, y on its outerface. Next, we take a cycle C_n (with n greater than the length of the longest forbidden cycle) and replace every edge of this cycle with a copy of H (identifying the endvertices of this edge with the vertices x, y). Provided that H is chosen in such a way that the resulting graph belongs to $\mathcal{G}$, we obtain another infinite sequence of graphs of $\mathcal{G}$ which do not contain the considered cycles.

The source graphs G and H are often built from smaller plane graphs using several operations which replace parts of graphs (mostly vertices and edges) with other configurations. The general operations are the following ones:

- *truncation*: each edge of given plane graph is subdivided by two new vertices; then, each star with the center being an original vertex of degree k is replaced by a k-face. The resulting graph is cubic and plane.
- *rectification*: after the truncation is performed, each edge whose both endvertices were subdividing vertices is contracted. The resulting graph is 4-regular and plane.

Among small starter-graphs which are used in the subsequent constructions, there are well-known graphs of five Platonic polyhedra, and further particular graphs (see Figure 1):

- ◦ the rhombic dodecahedron graph.
- ◦ rhombic triacontahedron graph (the dual of rectified dodecahedron graph).

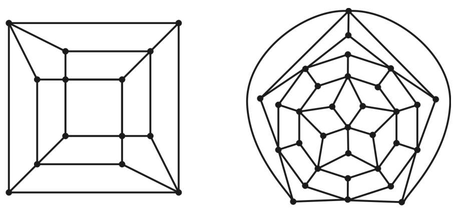

Figure 1: The rhombic dodecahedron and the rhombic triacontahedron graph

We use also other specialized graph operations which exclude some cycle lengths (see Figure 2):

- K_4-*substitution*: each 3-face is replaced by a copy of a plane drawing of the complete 4-vertex graph.

- $\triangle$-*substitution*: works similarly as K_4-substitution, just the replacement graph is the graph of 7-wheel.

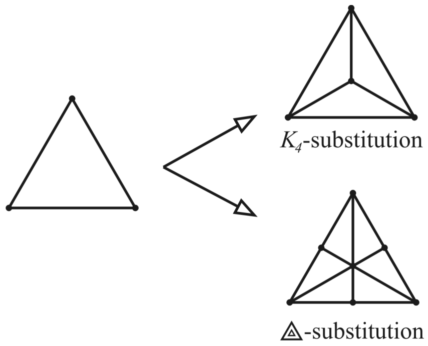

Figure 2: The K_4- and $\triangle$-substitutions

Now, the results on excluded cycles are the following:

Theorem 2.1 *Let $\mathcal{S} = \{S_{6^+}, S_{3,5,7,9,11,13^+}, S_{5,6,7,8,9,81^+}, S_{4,5,109^+}, S_{4,19^+}\}$. If S is a finite set of cycles such that $S \subset T$ for $T \in \mathcal{S}$, then S is not unavoidable in the family of plane graphs of minimum degree at least* 3 *and minimum edge-weight at least* 7*.*

Proof: For every $T \in \mathcal{S}$ and every finite $S \subset T$, we describe a graph yielding an infinite set of plane graphs of minimum degree at least 3 and minimum edge-weight at least 7 whose members contain no cycle from S:

- If $S \subset S_{6^+}$, then choose H to be the 5-wheel graph.
- If $S \subset S_{3,5,7,9,11,13^+}$, then G is the graph of rhombic dodecahedron.
- If $S \subset S_{5,6,7,8,9,81^+}$, then G is obtained from truncated dodecahedron graph by K_4-substitution.
- If $S \subset S_{4,5,109^+}$, then G is the graph from [7, Fig. 1]; it consists of 3- and 4-vertices such that no two 3-vertices are adjacent, and of 6-faces together with nonadjacent 3-faces.
- If $S \subset S_{4,19^+}$, then choose H to be the graph in Figure 3, with x, y being its vertices of degree 2; note that this graph is not hamiltonian.

$\square$

Theorem 2.2 *If S is a finite set of cycles such that $S \subset S_{7^+}$ or $S \subset S_{4,31^+}$, then S is not unavoidable in the family of plane graphs of minimum degree at least* 4*, and also in the family of plane graphs of minimum degree at least* 3 *and minimum edge-weight at least* 8*; for the latter family, $S \subset S_{3,5,7,\dots,25^+}$ is not unavoidable as well.*

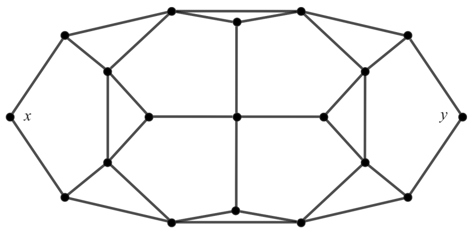

Figure 3: The graph H for excluding 4- and 19^{+}-cycles

Proof: The constructions of the corresponding infinite families of plane graphs are as follows:

- If $S \subset S_{7^+}$, then take G being the regular octahedron graph.
- If $S \subset S_{4,31^+}$ then choose G to be the rectified dodecahedron graph.
- If $S \subset S_{3,5,7,\dots,25^+}$ then choose G to be rhombic triacontahedron graph.

□

Theorem 2.3 *If S is a finite set of cycles such that $S \subset S_{10^+}$ or $S \subseteq S_{8,9}$, then S is not unavoidable in the family of plane graphs of minimum degree at least 3 and minimum edge-weight at least 9.*

Proof: To construct the corresponding infinite sets, we use the following graphs:

- If $S \subset S_{10^+}$, then choose H to be the graph obtained from 4-antiprism by inserting a new 4-vertex into one of its 4-faces.
- If $S \subset S_{8,9}$, then G is obtained from a rectified dodecahedron by ◬-substitution.

□

3 Minimal unavoidable sets

All unavoidability proofs of this section use a common technique called the Discharging Method which works in the following way: in order to prove a particular structural result, we assume the existence of a hypothetical counterexample graph $G = (V, E, F)$ of a specified family of plane graphs. The plane version of Euler's polyhedral formula implies that the following general equality holds for any positive a and any non-negative b:

$$\sum_{v \in V} (a \deg(v) - 2(a+b)) + \sum_{\alpha \in F} (b \deg(\alpha) - 2(a+b)) = -4(a+b).$$

According to this, we assign to vertices and faces of G initial charges $\mu : V \cup F \to \mathbb{Z}$ by setting $\mu(v) = a\deg(v) - 2(a+b)$ for every $v \in V$ and $\mu(\alpha) = b\deg(\alpha) - 2(a+b)$ for every $\alpha \in F$. Thus the sum of initial charges $\sum\limits_{x \in V \cup F} \mu(x)$ is negative.

Now, the initial charges of elements of G are locally redistributed in such a way that the total sum of charges remains, in the process of redistribution, the same. This is performed by a set of discharging rules specifying in which situations an element having a positive charge transfers certain amount of its charge to a close element which has negative charge. Finally, by case analysis, it is checked that, after the discharging, the new charges $\widetilde{\mu} : V \cup F \to \mathbb{Q}$ are non-negative; this means that $\sum\limits_{x \in V \cup F} \mu(x) = \sum\limits_{x \in V \cup F} \widetilde{\mu}(x) \geq 0$, a contradiction.

Theorem 3.1 *The sets $S_{3,4}$ and $S_{3,6}$ are minimal unavoidable in the family of planar graphs of minimum degree at least* 3 *and minimum edge weight at least* 7*.*

Proof: For the set $S_{3,4}$, we use the initial charge assignment set by $a = b = 1$, with the following discharging rule:

Rule 3.1.1: Every k-face, $k \geq 5$, sends $\frac{1}{2}$ to each incident 3-vertex.

It is enough to examine the final charge of 3-vertices and 5^+-faces. Every m-vertex, $m \geq 3$, of G is incident only to 5^+-faces, so, after applying the Rule 3.1.1 on a 3-vertex x, $\widetilde{\mu}(x) \geq 3 - 4 + 3 \cdot \frac{1}{2} > 0$. Further, the condition on minimum edge weight of G implies that every k-face, $k \geq 5$ is incident to at most $\left\lfloor \frac{k}{2} \right\rfloor$ 3-vertices; hence, for a k-face α of G, $k \geq 5$ we obtain, after applying the Rule 3.1.1, $\widetilde{\mu}(\alpha) \geq \mu(\alpha) - \left\lfloor \frac{k}{2} \right\rfloor \cdot \frac{1}{2} = k - 4 - \left\lfloor \frac{k}{2} \right\rfloor \cdot \frac{1}{2} \geq 0$.

The proof for the set $S_{3,6}$ uses the same initial charge assignment and the same redistribution rule, repeating the argument for nonnegativity of final charges of faces directly. Consider a 3-vertex x, and assume that it is incident with two adjacent 4-faces, say $x_1 x x_2 y$ and $x_2 x x_3 z$ (note that $y \neq z$). If $y = x_3$ or $z = x_1$, then a 3-cycle $x_1 x x_3$ is found; otherwise, these 4-faces together form a 6-cycle. Hence, x is incident with at least two 5^+-faces and, by Rule 3.1.1, $\widetilde{\mu}(x) \geq 3 - 4 + 2 \cdot \frac{1}{2} = 0$.

The fact that $S_{3,4}$ and $S_{3,6}$ are minimal unavoidable follows from Theorem 2.1 on avoidance of 3-, 4- and 6-cycle in plane graphs of minimum degree 3 and minimum edge weight 7. $\square$

Theorem 3.2 *The set $S_{4,6}$ is minimal unavoidable in the family of planar graphs of minimum degree at least* 3 *and minimum edge weight at least* 7*.*

Proof: Here, we proceed again by the Discharging Method, with the initial charge assignment given by $a = 1, b = \frac{4}{5}$; it follows that $\sum\limits_{x \in V \cup F} \mu(x) = -\frac{36}{5}$.

The following discharging rules are used:

Rule 3.2.1: Every 4^+-vertex sends $\frac{1}{5}$ to each incident 3-face.

Rule 3.2.2: Every 5-face sends $\frac{1}{5}$ to each incident 3-vertex.

Rule 3.2.3: Every 7^+-face sends $\frac{3}{10}$ to each incident 3-vertex.

Rule 3.2.4: Let α be an 7^+-face having a common edge xy with a 3-face β.

(a) If both x, y are 4^+-vertices then α sends $\frac{1}{5}$ to β.

(b) If one of x, y is a 3-vertex then α sends $\frac{3}{10}$ to β.

We analyze the final charges of all vertices and faces except 6-faces (note that their boundary is not a 6-cycle) whose charge is intact and positive.

Case 1: Let v be a 3-vertex. If v is incident only with 5^+-faces, then $\widetilde{\mu}(v) \geq 3 - \frac{18}{5} + 3 \cdot \frac{1}{5} = 0$. In the opposite case, v is incident with exactly one 3-face and the remaining two incident faces are 7^+-faces (notice that the presence of a 4-face, or a 3-face and a 5-face or a 6-face around v always yields a 4-cycle or a 6-cycle in G); thus, by Rule 3.2.3, $\widetilde{\mu}(v) \geq 3 - \frac{18}{5} + 2 \cdot \frac{3}{10} = 0$.

Case 2: Let v be a k-vertex, $k \geq 4$. Note that v is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-faces (otherwise two of them are adjacent, thereby forming a 4-cycle in G); therefore, $\widetilde{\mu}(v) \geq k - \frac{18}{5} - \lfloor \frac{k}{2} \rfloor \cdot \frac{1}{5} \geq \frac{9}{10}(k-4) \geq 0$.

Case 3: Let α be a 3-face. Similarly as in Case 1, observe that α is adjacent only with 7^+-faces. If α is incident only with 4^+-vertices, then, by Rules 3.2.1 and 3.2.4(a), $\widetilde{\mu}(\alpha) \geq \frac{4}{5} \cdot 3 - \frac{18}{5} + 3 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} = 0$; otherwise, α is incident with exactly one 3-vertex and two 4^+-vertices, and by Rules 3.2.1, 3.2.4(a) and 3.2.4(b), $\widetilde{\mu}(\alpha) \geq \frac{4}{5} \cdot 3 - \frac{18}{5} + 2 \cdot \frac{1}{5} + 2 \cdot \frac{3}{10} + \frac{1}{5} = 0$.

Case 4: Let α be a 5-face. As α is incident with at most two 3-vertices, we have, by Rule 3.2.2, $\widetilde{\mu}(\alpha) \geq \frac{4}{5} \cdot 5 - \frac{18}{5} - 2 \cdot \frac{1}{5} = 0$.

Case 5: Let α be a k-face, $k \geq 7$. Denote by a, b, c the numbers of transfers of charge from α by Rules 3.2.3, 3.2.4(a) and 3.2.4(b), respectively. Observe that $b + c \leq k - a$ (because no 3-vertex incident with α lies in two 3-faces). Therefore, $\widetilde{\mu}(\alpha) \geq \frac{4}{5}k - \frac{18}{5} - \frac{3}{10}a - \frac{3}{10}c - \frac{1}{5}b = \frac{4}{5}k - \frac{18}{5} - \frac{3}{10}a - \frac{3}{10}(b+c) + \frac{1}{10}b \geq \frac{4}{5}k - \frac{18}{5} - \frac{3}{10}a - \frac{3}{10}(k-a) + \frac{1}{10}b = \frac{k}{2} - \frac{18}{5} + \frac{b}{10}$. This is clearly satisfied for $k \geq 8$ or for $k = 7$ with $b \geq 1$. Thus, assume that α is a 7-face and Rule 3.2.4(a) is not applied. Then $c \leq a, a \leq 3$, so $\widetilde{\mu}(\alpha) \geq \frac{4}{5} \cdot 7 - \frac{18}{5} - \frac{3}{10}a - \frac{3}{10}c \geq 2 - 2 \cdot \frac{3}{10} \cdot 3 > 0$.

The fact that $S_{4,6}$ is minimal unavoidable follows from Theorem 2.1 on avoidance of 4- and 6-cycle in plane graphs of minimum degree 3 and minimum edge weight 7. □

Theorem 3.3 *The sets S_3, S_4 and S_5 are minimal unavoidable in the family of planar graphs of minimum degree at least* 3 *and minimum edge-weight at least* 9.

Proof: Concerning the set S_3, we use the initial charge assignment set by $a = b = 1$ and redistribute the charges of 6^+-vertices uniformly to all adjacent 3-vertices. Since a 3-vertex v is adjacent only with 6^+-vertices, we obtain $\widetilde{\mu}(v) \geq \mu(v) + 3 \cdot \frac{6-4}{6} = -1 + 3 \cdot \frac{1}{3} = 0$. $\square$

For S_4, we use the initial charge assignment set by $a = 1, b = 2$; the local redistribution of charges is performed according to

Rule 3.3.1: Each 5^+-face sends $\frac{3}{2}$ (or 1 or $\frac{1}{3}$) to each incident 3-vertex (or 4- or 5-vertex, respectively).

It is enough to analyze final charges of 5^+-faces and 5^--vertices.

Case 1: Let v be a 3-vertex. Since G contains no 4-cycles, v is incident with at most one 3-face and, consequently, with at least two 5^--faces, which gives $\widetilde{\mu}(v) \geq -3 + 2 \cdot \frac{3}{2} = 0$.

Case 2: Let v be a 4-vertex. Note that v is incident with at most two 3-faces; thus, it is incident with at least two 5^+-faces and $\widetilde{\mu}(v) \geq -2 + 2 \cdot 1 = 0$.

Case 3: Let v be a 5-vertex. Again, v is incident with at most two 3-faces, thus receiving $\frac{1}{3}$ from at least three incident 5^+-faces; then $\widetilde{\mu}(v) \geq -1 + 3 \cdot \frac{1}{3} = 0$.

Case 4: Let α be an r-face, $r \geq 5$. Denote by $t_i, i \in \{3, 4, 5\}$, the number of i-vertices incident with α. Due to the requirement on minimum edge weight of G, $t_3 + t_4 + t_5 \leq r, t_3 \leq \left\lfloor \frac{r}{2} \right\rfloor, t_4 \leq \left\lfloor \frac{r}{2} \right\rfloor$ and $2t_3 + t_4 + t_5 \leq r$. Then $\widetilde{\mu}(\alpha) \geq \mu(\alpha) - \frac{3}{2} \cdot t_3 - 1 \cdot t_4 - \frac{1}{3} \cdot t_5 \geq 2r - 6 - \frac{3}{2}t_3 - t_4 - \frac{1}{3}(r - t_4 - 2t_3) = \frac{5}{3}r - 6 - \frac{5}{6}t_3 - \frac{2}{3}t_4 \geq \frac{5}{3}r - 6 - \left\lfloor \frac{r}{2} \right\rfloor \left(\frac{5}{6} + \frac{2}{3}\right) \geq 0$ for $r \geq 7$. Now, if $r = 6$, then (t_3, t_4, t_5) is element-wise majorized by one of $(3, 0, 0), (2, 1, 0), (2, 0, 1), (1, 2, 1), (1, 1, 2), (1, 0, 3), (0, 3, 3)$, $(0, 2, 4)$, $(0, 1, 5)$ or $(0, 0, 6)$; among these triples, the worst charge consumption is associated with triples $(3, 0, 0), (1, 2, 1)$ and $(0, 3, 3)$, thus yielding $\widetilde{\mu}(\alpha) \geq 2 \cdot 6 - 6 - 3 \cdot \frac{3}{2} > 0$. If $r = 5$, then (t_3, t_4, t_5) is element-wise majorized by one of $(2, 0, 0), (1, 1, 1), (0, 2, 3), (0, 1, 4)$ or $(0, 0, 5)$ which gives $\widetilde{\mu}(\alpha) \geq 2 \cdot 4 - 6 - 2 \cdot \frac{3}{2} > 0$. $\square$

Finally, for S_5, use the initial charge assignment set again by $a = 1, b = 2$, and redistribute these charges according to five discharging rules:

Rule 3.3.2: Each 6^+-face sends 1 to each incident 5^--vertex.

Rule 3.3.3: Each 4-face sends 1 to each incident 3-vertex.

Rule 3.3.4: Each 4-face sends $\frac{1}{2}$ to each incident 4- or 5-vertex.

Rule 3.3.5: Let α be an 6^+-face which is adjacent with a 3-face $[vuw]$ with u being a 3-vertex and v being 6^+-vertex. Then α sends $\frac{1}{2}$ to u (through v).

Rule 3.3.6: Let α be an 6^+-face incident with an edge uv with u being a 3-vertex (and v being an 6^+-vertex). Then α sends additional $\frac{1}{2}$ to u (through v).

To check the nonnegativity of final charges of elements of counterexample, it is enough to examine 4^+-faces and 5^--vertices.

Case 1: Let v be a 3-vertex with neighbours v_1, v_2, v_3 in clockwise order. Now discuss the number of 3-faces around v:

Case 1.1: If v is incident with three 3-faces, then the other faces incident with edges v_1v_2, v_2v_3 and v_3v_1 are 6^+-faces (otherwise a 5-cycle is found); by Rule 3.3.5, $\widetilde{\mu}(v) \geq -3 + 6 \cdot \frac{1}{2} = 0$.

Case 1.2: If v is incident with exactly two 3-faces, say $[v_1vv_2]$ and $[v_1vv_3]$. Then at least one of the edges v_1v_2, v_1v_3 is incident with an 6^+-face, and the third nontriangular face incident with v is also 6^+-face. Hence, by Rules 3.3.2, 3.3.5 and 3.3.6, $\widetilde{\mu}(v) \geq -3 + 1 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 0$.

Case 1.3: Let v be incident to exactly one 3-face, say $[v_1vv_2]$. Then the other two faces around v are 6^+-faces, thus, by Rules 3.3.2 and 3.3.6, $\widetilde{\mu}(v) \geq -3+2\cdot 1+2\cdot \frac{1}{2} = 0$.

Case 1.4: Suppose that v is not incident with a 3-face. Then, by Rules 3.3.2 or 3.3.3, $\widetilde{\mu}(v) \geq -3 + 3 \cdot 1 = 0$.

Case 2: Let v be a 4-vertex. Then v is incident with at most two 3-faces. If v is incident with at least one 3-face, then at least two faces incident with v are 6^+-faces and $\widetilde{\mu}(v) \geq -2 + 2 \cdot 1 = 0$; otherwise, all faces around v contribute at least $\frac{1}{2}$ to v by Rule 3.3.2 or 3.3.4, and we can roughly estimate that $\widetilde{\mu}(v) \geq -2 + 4 \cdot \frac{1}{2} = 0$.

Case 3: Let v be a 5-vertex. Then v is incident with at most three 3-faces (otherwise a 5-cycle is found in the neighbourhood of v), thus it receives charge from at least two 4^+-faces yielding $\widetilde{\mu}(v) \geq -1 + 2 \cdot \frac{1}{2} = 0$.

Case 4: Let α be a 4-face. Then α is incident with at most two 3-vertices; by Rule 3.3.3 or 3.3.4, either $\widetilde{\mu}(\alpha) \geq 2 \cdot 4 - 6 - 2 \cdot 1 = 0$ (when a 3-vertex is incident with α) or $\widetilde{\mu}(\alpha) \geq 2 \cdot 4 - 6 - 4 \cdot \frac{1}{2} = 0$ (when no 3-vertex is incident with α – note that still there may be up to four 5^--vertices on α).

Case 5: Let α be an r-face $r, \geq 6$. For calculating the final charge of α, we use the following averaging argument: each time the Rule 3.3.5 or 3.3.6 is used, assign the charge $\frac{1}{2}$ (which is transferred to a 3-vertex) to the 6^+-vertex through which the transfer is performed. Now each 6^+-vertex on α is assigned with at most 1 while any transfer from α to an incident 5^--vertex by Rule 3.3.2 conducts the charge 1. Therefore, $\widetilde{\mu}(\alpha) \geq 2r - 6 - r \cdot 1 \geq 0$ since $r \geq 6$. □

Theorem 3.4 *The set $S_{4,8}$ is minimal unavoidable in the family of planar graphs of minimum degree at least* 4.

Proof: We use the initial charge assignment set by $a = b = 1$, and the following discharging rule:

Rule 3.4.1: Every k-face, $k \geq 5$, sends $\frac{1}{3}$ to each adjacent 3-face.

It is enough to discuss the final charge of faces. Let α be a 3-face of G. Then all three faces incident to α are 5^+-faces. After applying Rule 3.4.1, $\widetilde{\mu}(\alpha) \geq \mu(\alpha)+3\cdot \frac{1}{3} =$

$3-4+1=0$. Finally, let α be a k-face of G, $k \geq 5$. If $k=5$, then α is incident to at most two 3-faces (otherwise a 4-cycle or an 8-cycle is present) and, by Rule 3.4.1, $\widetilde{\mu}(\alpha) \geq 5-4-2\cdot\frac{1}{3} > 0$; otherwise $\widetilde{\mu}(\alpha) \geq \mu(\alpha) - k\cdot\frac{1}{3} = \frac{2}{3}k - 4 \geq 0$ for each $k \geq 6$.

The minimal unavoidability of $S_{4,8}$ follows from Theorem 2.2 on avoidance of 4- and 8-cycle in plane graphs of minimum degree at least 4. □

Theorem 3.5 *The set $S_{4,9}$ is minimal unavoidable in the family of planar graphs of minimum degree at least* 4.

Proof: Here, we use the initial charge assignment set by $a=1, b=2$, and the local redistribution of initial charges by the following discharging rule:

Rule 3.5.1: Every k-face, $k \geq 5$, sends $\frac{2k-6}{k}$ to each incident 4-vertex or 5-vertex.

To check the nonnegativity of final charges, it is enough to consider only 4- and 5-vertices.

Case 1: Let v be a 4-vertex with the neighbours v_1, v_2, v_3, v_4 in clockwise order. Note that v is incident to at most two 3-faces which are not adjacent. If the vertex v is incident with at least two 6^+-faces, then $\widetilde{\mu}(v) \geq \mu(v) + 2\cdot\frac{2\cdot 6-6}{6} = 0$, hence we can assume that there is at most one 6^+-face incident with v. Now consider the number of 3-faces around v:

Case 1.1: Let v be incident with at most one 3-face. As the remaining faces around v are 5^+-faces, we have $\widetilde{\mu}(v) \geq \mu(v) + 3\cdot\frac{2\cdot 5-6}{5} = -2 + \frac{12}{5} > 0$.

Case 1.2: Let v be incident with two 3-faces $[vv_1v_2]$ and $[vv_3v_4]$. Without loss of generality, let $\beta = [v_3vv_2xy]$ be a 5-face. Then x, y are distinct from v_1, v_4 (otherwise a 4-cycle is found in G). Consider the fourth face γ having the facial subwalk uv_4vv_1w; due to absence of 4-cycles in G, all these five vertices are distinct – thus, it is a facial path – and, moreover, u, w are distinct from vertices of β. Now, γ is neither a 5-face (otherwise a 9-cycle $uv_4vv_3yxv_2v_1wu$ is found) nor a 6-face $[tuv_4vv_1w]$ (t is distinct from vertices of β due to absence of 4-cycles, but then $tuv_4v_3yxv_2v_1wt$ is a 9-cycle) nor else a 7-face (such a face is necessarily bounded by a cycle, say $[stuv_4vv_1w]$. If none of its vertices coincides with a vertex of β, then $stuv_4v_3vv_2v_1ws$ is a 9-cycle; otherwise, taking into account the avoidance of a 4-cycle, we get $s=y$ or $t=x$, and another 9-cycle omitting x or y is found). Hence, γ is a 8^+-face and this gives that $\widetilde{\mu}(v) \geq \mu(v) + \frac{2\cdot 5-6}{5} + \frac{2\cdot 8-6}{8} = \frac{1}{20} > 0$.

Case 2: Let v be a 5-vertex. Then v is incident with at most two 3-faces, hence, it is incident with at least three 5^+-faces and $\widetilde{\mu}(v) \geq \mu(v) + 3\cdot\frac{2\cdot 5-6}{5} = -1 + \frac{12}{5} > 0$.

The minimal unavoidability of $S_{4,9}$ follows from Theorem 2.2 on avoidance of 4- and 9-cycle in plane graphs of minimum degree at least 4. □

4 Concluding remarks

The negative results of Section 2 exclude many sets of cycles from being unavoidable; nevertheless, still there are many undecided cases, even for sets consisting of two cycles. The following table summarizes the obtained results for two-element cycle sets $S_{k,l}$ (+ indicates minimal unavoidability, $-$ stands for non-unavoidability, ? for open problem; $*$ means that the cycle C_k or C_l is already unavoidable in the considered family).

$\delta \geq 3, w \geq 7$	3	4	5	6	7	8	9	10	11	12
3	-	+	-	+	-	?	-	?	-	?
4		-	-	+	?	?	?	?	?	?
5			-	-	-	-	-	?	-	?
6^+				-	-	-	-	-	-	-

$\delta \geq 4$	3	4	5	6	7	8	9	$10-30$	31^+	
3	+	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	
4		-	$*$	$*$	+	+	+	?	-	
5			+	$*$	$*$	$*$	$*$	$*$	$*$	
6				+	$*$	$*$	$*$	$*$	$*$	
7^+					-	-	-	-	-	

$\delta \geq 3, w \geq 9$	3	4	5	6	7	8	9	10^+		
3	+	$*$	$*$	$*$	$*$	$*$	$*$	$*$		
4		+	$*$	$*$	$*$	$*$	$*$	$*$		
5			+	$*$	$*$	$*$	$*$	$*$		
6				?	?	?	?	?		
7					?	?	?	?		
8						-	-	-		
9							-	-		
10^+								-		

Table 1: An overview of obtained results on unavoidability of $S_{k,l}$

For the family of plane graphs of minimum degree at least 4, the open cases are the sets $S_{4,k}$ for $10 \leq k \leq 30$, of which we conjecture that all of them are minimal unavoidable. For the family of plane graphs of minimum degree at least 3 and edge-weight at least 7, the first small open cases are the sets $S_{3,8}, S_{3,10}, S_{3,12}, S_{4,k}$ for $7 \leq k \leq 18$, and $S_{5,10}$; for the edge-weight at least 9, we conjecture that 6- and 7-cycles are unavoidable.

The family of plane graphs of minimum degree 5, or minimum degree 4 and minimum edge-weight at least 9 is the subject of our further research in [6].

It is a matter of discussion whether the definition of minimal unavoidable set S of cycles should involve infinitely many exceptional graphs for nonempty proper subsets of S. For graph families discussed in this paper, infinitely many exceptional graphs can be indeed formed from a single such graph, as described in Section 2.

However, the constructions used in Theorems 1–3 cannot be used for excluding particular cycles in the family of polyhedral graphs or its subfamilies. In this case, one shall develop other constructions where is less obvious that the existence of a single exceptional polyhedral graph rises to an infinite exceptional family. We also mention an open problem by Malkevitch (personal communication) that every hamiltonian polyhedral 5-regular plane graph is pancyclic, where it is not clear whether a possible counterexample still allows to hold that all such graphs, provided large enough, are pancyclic, or there exists an infinite family of such graphs omitting some cycle, or even a cycle with fixed length.

Acknowledgements

This work was supported by the Slovak Research and Development Agency under the Contract No. APVV-15-0116, by the Slovak VEGA Grant 1/0368/16, by P.J. Šafárik University grant VVGS-PF-2016-72604 and partially supported by DAAD, Germany (as part of BMBF) and by the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57320575.

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(Received 8 Jan 2018; revised 18 Dec 2018, 3 Apr 2019)