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A New Approach to Constructing A Minimum Edge Cut Set Based on Maximum Flow

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Abstract. *The maximum flow algorithm has long been known for calculating the minimum edge cut of any two vertices of a connected graph. The original algorithm, however, does not tell us which edges should be taken exactly and therefore there could be more than one way to construct a minimum-edge-cut set. In this paper, we propose a new method to get a minimum-edge-cut set by selecting edges of a graph in the order of betweenness. By comparing with the augmenting path method, it tends to make the two divided parts of a graph more balanced.*
Keywords: Minimum cut set, Betweenness, Maximum flow.

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1. **Introduction.** Graph partitioning is one of the classical subjects in graph theory and has many practical applications in traditional aspects, such as chip and circuit design, robustness enhancement of communication networks, community detection [1] etc. The simplest graph partitioning problem is the division of a network into just two parts. Division into two parts is sometimes called graph bisection. Formally, the graph bisection problem is the problem of dividing the vertices of a network into two non-overlapping groups of given sizes such that the number of edges running between vertices in different groups is minimized. The number of edges between groups is called the cut size. Recently, the graph partition problem has gained more importance due to its application for clique detection [2] in social, and biological networks. Finding the minimum cut of any pair of nodes in undirected, edge-weighted graphs is a fundamental problem in graph partitioning. Precisely, the purpose is to find a way to set the pair of vertices separately into two parts such that the cut weight, i.e., the sum of the weights of the edges connecting the two parts, is minimum. The well-known approach to solve this problem is to use the Max-Flow-Min-Cut theorem by Ford and Fulkerson who showed the equivalence of the maximum flow and the minimum s-t-cut, where s and t are two vertices that are the source and the sink in the flow problem and have to be separated by the cut set. Based on this idea, the classical augmenting path method can provide us a construction of a minimum cut set. However, the augmenting path method only gives us a certain one of possible minimum cut sets and tells nothing else. Is there any approach better than the almost random method(i.e.

augmenting path method) when partitioned graphs are expected to have some structural characteristics? In next sections, we will discuss this problem in more detail and propose a new method to construct a minimum s-t-cut set.

2. **Problem Statement and Preliminaries.** In an undirected, unweighted network, a cut set, or more precisely an edge cut set, is a set of edges whose removal will disconnect a specified pair of vertices. Although the definitions here apply equally to directed and weighted ones, for simplicity, we just consider undirected, unweighted graphs in this paper. There is also anther definition of cut set called a vertex cut set which is a set of vertices whose removal will disconnect a specified pair of vertices. But in this paper we simply consider the former one, i.e., edge cut set. A minimum edge cut set is an edge cut set with the least edges. It is notable that a minimum cut set is not necessarily unique. For instance, there are different minimum edge cut sets of size two between the vertices A and B in a network as showed in Fig.1, where X,Y, W,Z and V,W are all minimum edge cut sets for this network. Of course, all the minimum cut sets must have the same size. Calculating the size of a minimum edge cut set directly is a rather difficult problem, thus it is usually translated to another simpler one, the so called maximum flow.

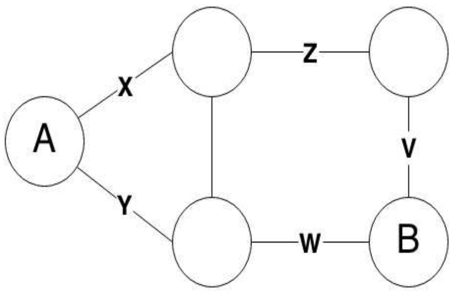

FIGURE 1. The vertices A and B in this network have more than one minimum edge cut sets, such as {X,Y}, {W,Z} and {V,W}.

Imagine a network consisting of water pipes, where the edges of the network correspond to the pipes and the vertices to junctions between pipes. Suppose that there is a maximum rate r, in terms of volume per unit time, at which water can flow through any pipe. What then is the maximum rate at which water can flow through the network from one vertex, A, to another, B? This is the maximum flow question.

The well-known max-flow-min-cut theorem indicates that the maximum flow between two vertices is always equal to the size of the minimum cut set times the capacity of a single pipe. Here the maximum rate r=1 corresponds to an unweighted network.

The equivalence of the maximum flow and the minimum cut set size has an important practical consequence. There are simple computer algorithms, such as the augmenting path algorithm [3] that can calculate maximum flows quite quickly ($O(|E|^2 log|V|)$) for any given network, and the equivalence means that we can use these same algorithms to quickly calculate a minimum cut set as well.

The augmenting path algorithm to construct a minimum edge cut set is defined as follow:

Step 1. Pick a pair of vertices, say S and T, in a graph.

Step 2. Calculate the maximum flow from S to T.

Step 3. In the final residual graph generated by step 2, let Vs be the subset of vertices reachable from S by some paths and let Vt be the set of all the other vertices in the graph that are not in Vs. By definition Vs and Vt do not share any vertices and all vertices in the graph belong to either Vs or Vt. Then the set of edges on the original graph that connect vertices in Vs to vertices in Vt constitutes a minimum cut set.

At last, we get one of possible minimum edge cut sets.

The drawback of this augmenting path method is that we cannot control the result of partition. In other words, the obtained minimum cut set depends on the constructing of the final residual graph which may not be unique for a graph. For instance, in Fig.1, if we get the minimum cut set {X,Y} or {V,W}, the size of the largest component of the divided graph is 4, 80% of the original graph size and if we get the minimum cut set {W,Z}, the size of the largest component of the divided graph is 3, 60% of the original graph size. One might imagine that one could simply look through all possible minimum cut sets and select what we want. For all but small networks, however, this exhaustive search turns out to be costly in terms of computer time. So we want to find another way to construct a minimum cut set if some characteristics of the graph to be divided are expected, such as the size of the largest component.

Betweenness centrality is a core concept for the analysis of social networks. It was introduced independently by Freeman [4] and Anthonisse whose work was never published [5] and measures a node's or an edge's centrality in a network. Over the past few years, betweenness centrality has become a popular strategy to deal with complex networks. Applications include social and computer networks [6, 7], transport [8], scientific cooperation [9] and so forth. In this paper we only concern ourselves with betweenness centrality in terms of edges. It is equal to the number of shortest paths from all vertices to all others that pass through that edge.

The betweenness of an edge e is given by the expression:

$$g(e) = \sum_{s \neq t} \frac{\sigma_{st}(e)}{\sigma_{st}} \tag{1}$$

Where σ_{st} is the total number of shortest paths from node s to node t and $\sigma_{st}(e)$ is the number of those paths that pass through edge e.

Edges with high betweenness centrality may have considerable influence within a communication network. They are the ones through which the largest number of messages pass. The edges with highest betweenness are also the ones whose removal from the network will most disrupt communications between other vertices because they lie on the largest number of paths taken by messages.

3. **Combination of Maximum Flow and Betweenness.** As can be seen above, the drawback of augmenting path method is that it does not fully utilize structural character of the graph and does not give us any insights into the impact of minimum cut sets on networks.

On that account, we propose our methodology which takes advantage of the structural feature, i.e., betweenness, of a network. The algorithm is described as follows:

Step 1: Pick a pair of vertices S and T in a graph.

Step 2: Calculate the maximum flow (denoted by MF) from S to T.

Step 3: Calculate the betweenness of every edge in the graph.

Step 4: Choose the edge with the highest betweenness and test whether it is in the minimum cut set. If MF is reduced after removing the edge, it is in a minimum cut set, otherwise it is not. If it is in the minimum cut set, the edge is therefore removed from the network and we recalculate the betweenness of all edges since the network has been changed. If it is not in the minimum cut set, we choose the edge with the second highest betweenness to test, and so forth.

Step 5: Repeat Step 4 until MF from S to T becomes zero.

The hightest betweenness can also be replaced by the lowest one, i.e., the edges with the lowest betweenness are tested first. In next section, we will show the differences between these methods.

4. **Experimental Results.** To demonstrate the superior of the hightest-betweenness-order based method to the augmenting path method and show the difference between the hightest-betweenness-order based method and the lowest-betweenness-order based method, we produce 100 connected community networks at random and apply the augmenting path method, the hightest-betweenness-order method and the lowest-betweenness-order method respectively. The parameters of these simple community networks are: The number of vertices $N = 100$; The number of groups $G = 8$; The probability of nodal connection in a group $P_i = 0.18$; The probability of nodal connection between groups $P_o = 0.018$. At first we select two vertices with the highest and second highest degree respectively. Then let n_1 and n_2 be the sizes of the two components produced by removal of the minimum edge cut set, where $N = n_1 + n_2$. Let $E = n_1 \times n_2$ which has its largest value when $n_1 = n_2$ and the smallest value when $n_1 = 1$ or $n_2 = 1$. We will measure E as an indication of the balance of the resulting networks.

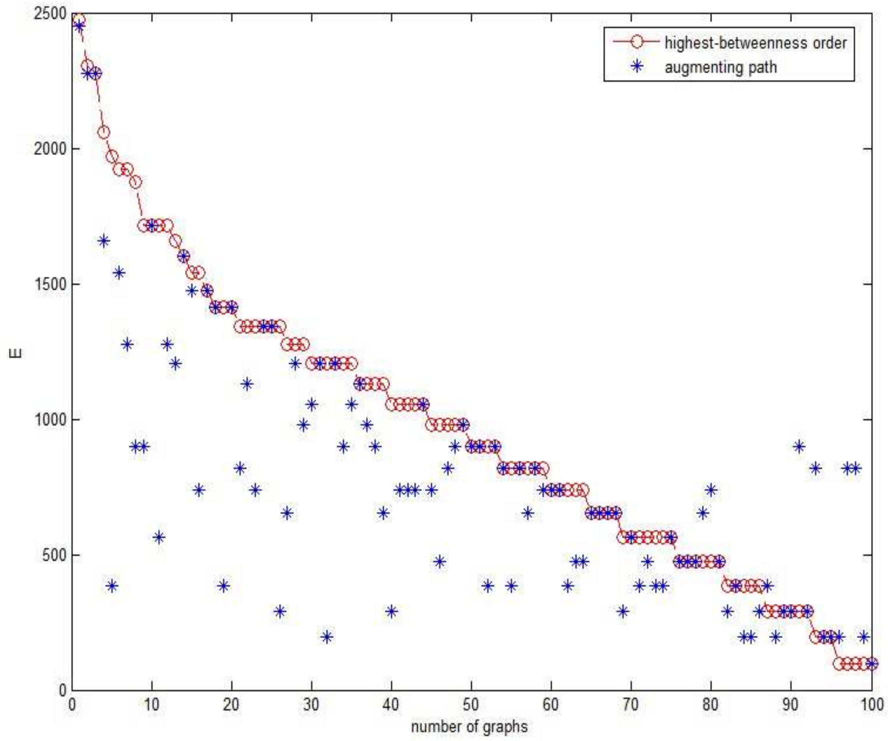

FIGURE 2. Comparison of highest-betweenness order and augmenting path method.

From Fig.1 and Fig.2, we can see that selecting edges in the highest-betweenness order tends to make the resulting two components more balanced, i.e., the difference of the number of vertices in the two components is relatively small. On the other hand, selecting edges in the lowest-betweenness order makes the two components drastically unbalanced compared to the highest-betweenness order method.

In Fig.1, compared with the E values produced by augmenting path method, those produced by the highest-betweenness method are bigger in 60% graphs, are equal in 30% graphs, and are only smaller in 10% graphs. This means that the difference between n_1

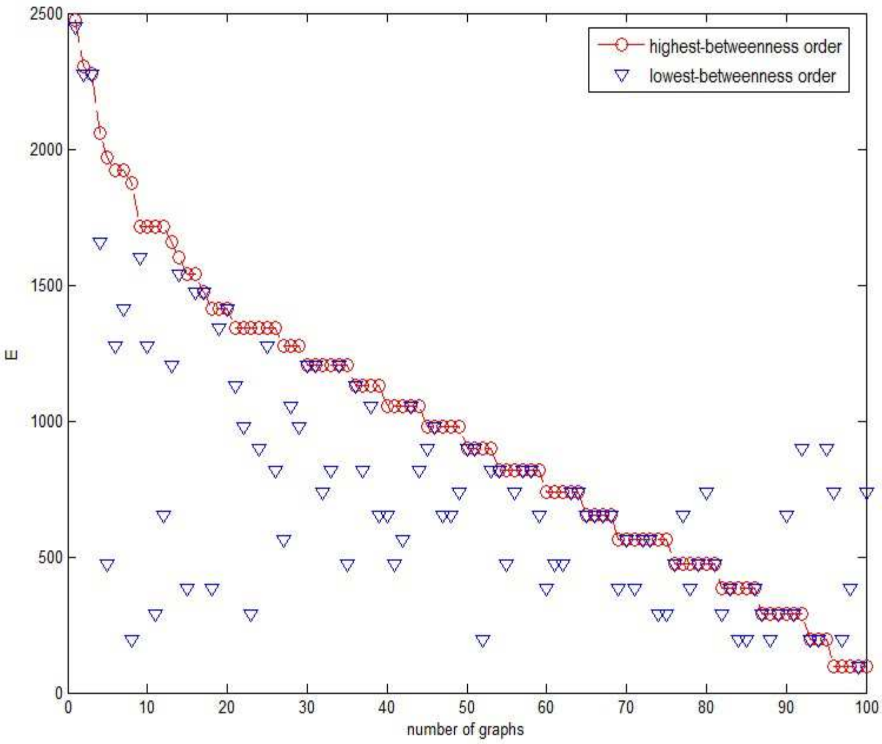

FIGURE 3. Comparison of highest-betweenness order and lowest-betweenness order.

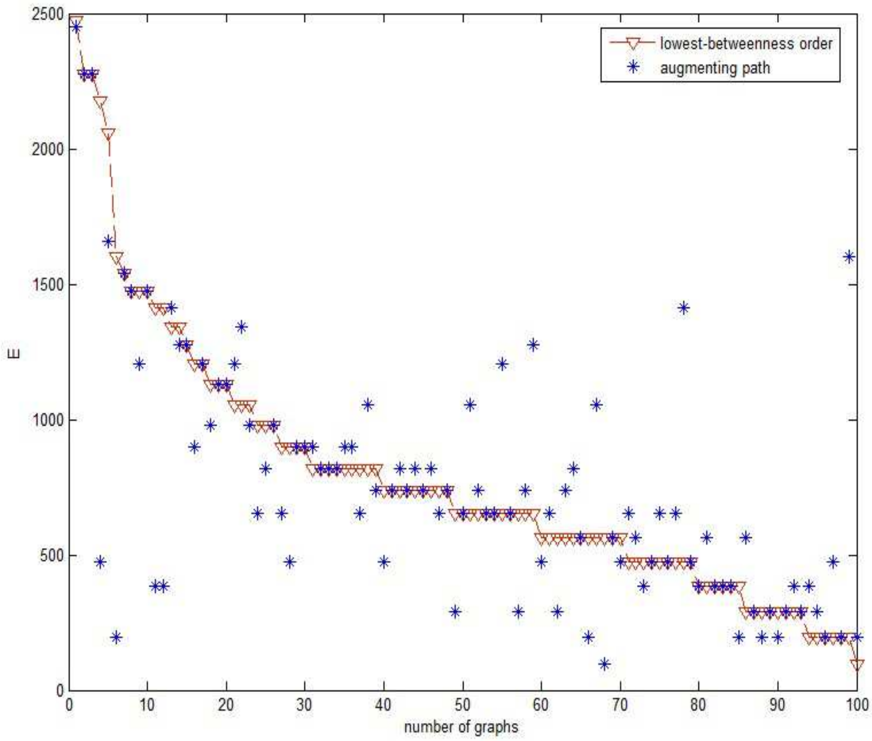

FIGURE 4. Comparison of lowest-betweenness order and augmenting path method.

and n_2 is smaller in 60% graphs when using the highest-betweenness-order method. In Fig.3, the difference of E values between the augmenting path method and the lowest-betweenness-order method is not significant since the E values of augmenting path method are smaller in 36% graphs, equal in 30% graphs and larger in 34% graphs. This result indicates that the highest-betweenness-order method is an advisable alternative to augmenting path method if we want to minimize the difference of the resulting two components.

5. **Conclusion.** This paper proposes a new method to construct a minimum edge cut set based on maximum flow. We make use of edge betweenness when finding a minimum edge cut set. Experimental results demonstrate that our highest-betweenness-order method is superior to the original augmenting path method when two divided parts of a graph are expected to be more balanced. Future work will focus on the combination of minimum vertex cut set and vertex betweenness and its impacts on networks.

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