

# Constrained $k$ -center and Movement to Independence

Adrian Dumitrescu\*

Minghui Jiang†

## Abstract

We obtain hardness results and approximation algorithms for two related geometric problems involving movement. The first is a constrained variant of the  $k$ -center problem, arising from a geometric client-server problem. The second is the problem of moving points towards an independent set.<sup>1</sup>

## 1 Introduction

Given a set  $S$  of  $n$  points in the plane, the  $k$ -center problem is to find  $k$  congruent disks of minimum radius  $r$  that cover  $S$ . We study the following constrained variant of the  $k$ -center problem:

CONSTRAINED  $k$ -CENTER

**Instance:** A set  $P = \{p_1, \dots, p_n\}$  of  $n$  black points and a set  $Q = \{q_1, \dots, q_k\}$  of  $k$  red points in the plane.  $P$  and  $Q$  are not necessarily disjoint.

**Problem:** Find a set  $\mathcal{D} = \{D_1, \dots, D_k\}$  of  $k$  disks *constrained* to the set  $Q = \{q_1, \dots, q_k\}$  of  $k$  red points (that is, for  $1 \leq j \leq k$ , the disk  $D_j$  contains the corresponding red point  $q_j$ ) such that all points in  $P$  are covered by the union of the disks in  $\mathcal{D}$ , and the maximum radius of the disks in  $\mathcal{D}$  is minimized.

The problem CONSTRAINED  $k$ -CENTER is the geometric version of a movement problem originally proposed by Demaine et al. [2] in the graph-theoretical setting: Given a connected graph  $G$  in which some vertices are occupied by clients and some vertices are occupied by servers, the problem FACILITY-LOCATION MOVEMENT is that of moving both the clients and the servers in the graph until each client occupies the same vertex as some server, such that the maximum movement of a client or a server is minimized; here the distance is the path length in the graph. The authors [2] observed that a 2-approximation can be achieved simply by keeping each server at its original location and moving each client to its nearest server. Friggstad and Salavatipour [5]

showed that this simple 2-approximation is in fact best possible: Unless  $P = NP$ , FACILITY-LOCATION MOVEMENT is NP-hard to approximate within  $2 - \varepsilon$  for any constant  $\varepsilon > 0$ .

Here we focus on the geometric version, where the clients and servers are points in the Euclidean plane (or more generally, in  $\mathbb{R}^d$ ), and the movement is measured as the Euclidean distance, rather than the number of edges of a path in the graph. The task is to determine a movement of the clients and servers, so that in the end, each client coincides with some server, and the maximum movement is minimized.

Let  $P$  be the set of clients, and  $Q$  be the set of servers, where  $|P| = n$  and  $|Q| = k$ . Usually  $k$  is much smaller than  $n$ . Let us first observe that our CONSTRAINED  $k$ -CENTER problem is essentially the same as the FACILITY-LOCATION MOVEMENT problem. Indeed, consider an optimal solution to the FACILITY-LOCATION MOVEMENT problem with maximum movement  $\lambda$ . Then the disks of radius  $\lambda$  centered at the server locations after the movement cover all clients and servers at their original locations. Conversely, consider a set of disks, say of radius  $\lambda$ , in an optimal solution to CONSTRAINED  $k$ -CENTER. Then moving the clients and the server contained in each disk (with ties broken arbitrarily) to its center, gives a solution to the FACILITY-LOCATION MOVEMENT problem with the maximum movement at most  $\lambda$ .

The afore-mentioned 2-approximation works in this setting as follows. Let  $d$  denote the maximum black-red (client-server) distance obtained by assigning each black point to its closest red point. Let OPT denote an optimal solution and ALG denote the solution returned by the algorithm. Then clearly

$$\text{OPT} \geq \frac{d}{2}, \text{ and } \text{ALG} = d, \quad (1)$$

and the ratio 2 immediately follows. It is worth observing that the algorithm which keeps fixed each red point achieves ratio 2 even on the line: place two red points at 0 and  $2 + \varepsilon$ , and two black points at  $1 + \varepsilon$  and  $3 + \varepsilon$ . Then  $\text{OPT} = (1 + \varepsilon)/2$ , while  $\text{ALG} = 1$  (this tight example can be easily extended for a larger number of points).

We first show that the approximation lower bound for the problem remains close to 2 already for the planar variant.

**Theorem 1** CONSTRAINED  $k$ -CENTER in the plane is NP-hard to approximate within 1.8279.

\*Department of Computer Science, University of Wisconsin–Milwaukee, USA. Email: [dumitres@uwm.edu](mailto:dumitres@uwm.edu). Supported in part by NSF CAREER grant CCF-0444188.

†Department of Computer Science, Utah State University, Logan, USA. Email: [mjiang@cc.usu.edu](mailto:mjiang@cc.usu.edu). Supported in part by NSF grant DBI-0743670.

<sup>1</sup>Due to space limitations, some proofs are omitted.

On the other hand, we have the following positive result showing that constant approximations for CONstrained  $k$ -CENTER can be obtained by a fixed parameter tractable algorithm [6] with  $k$  as the parameter.

**Theorem 2** *For any given  $\varepsilon > 0$ , there exists a  $(1+\varepsilon)$ -approximation algorithm for CONstrained  $k$ -CENTER in the plane that runs in  $O(\varepsilon^{-2k} \cdot n)$  time. Moreover, there exist: a 1.87-approximation algorithm that runs in  $O(3^k k \cdot n)$  time, a 1.71-approximation algorithm that runs in  $O(4^k k \cdot n)$  time, and a 1.61-approximation algorithm that runs in  $O(5^k k \cdot n)$  time.*

In the second part of the paper, we study another movement problem proposed by Demaine et al. [2]:

#### MOVEMENT TO INDEPENDENCE

**Instance:** A set  $P = \{p_1, \dots, p_n\}$  of  $n$  points in  $\mathbb{R}^d$ , and a threshold distance  $\Delta$ .

**Problem:** Find a set  $Q = \{q_1, \dots, q_n\}$  of  $n$  (target) points in  $\mathbb{R}^d$ , one point  $q_i \in Q$  for each point  $p_i \in P$ , such that the minimum pairwise distance  $\min_{i,j} |q_i q_j|$  among the points in  $Q$  is at least  $\Delta$ , and that the maximum movement  $\max_i |p_i q_i|$  from any point  $p_i \in P$  to the corresponding target point  $q_i \in Q$  is minimized.

There is a natural connection between MOVEMENT TO INDEPENDENCE and the dispersion problem in a set of congruent disks. The problem of *dispersion* in a given set of disks is that of selecting  $n$  points, one in each disk, such that the minimum inter-point distance is maximized.

#### DISPERSION IN CONGRUENT DISKS

**Instance:** A set  $\{D_1, \dots, D_n\}$  of  $n$  congruent disks.

**Problem:** Find a set  $Q = \{q_1, \dots, q_n\}$  of  $n$  points, one point  $q_i \in Q$  in each disk  $D_i$ , such that the minimum pairwise distance  $\min_{i,j} |q_i q_j|$  among the points in  $Q$  is maximized.

The dispersion problem was introduced by Fiala et al. [4] in a more general setting as “systems of distant representatives”, generalizing the classic problem “systems of distinct representatives”. See also [1]. Fiala et al. [4] showed that dispersion in unit disks is NP-hard. As a corollary we obtain

**Theorem 3** *MOVEMENT TO INDEPENDENCE in the plane (and in higher dimensions) is NP-hard.*

Let  $P = \{p_1, \dots, p_n\}$  be a set of  $n$  points in  $\mathbb{R}^2$ . Denote by  $\text{OPT}(x)$  the minimum maximum movement for the instance  $(P, x)$  of the problem MOVEMENT TO INDEPENDENCE in  $\mathbb{R}^2$  (that is, the value of the optimal

solution to this instance). Demaine et al. [2] presented a polynomial-time algorithm for MOVEMENT TO INDEPENDENCE on an instance  $(P, 1)$  with maximum movement at most  $\text{OPT}(1) + 1 + \frac{1}{\sqrt{3}}$ . Their algorithm moves the points to the grid points of an equilateral triangular lattice of unit side. By a scaling argument, this algorithm can be turned into an algorithm for  $(P, x)$  for any  $x > 0$ , with maximum movement at most  $\text{OPT}(x) + (1 + \frac{1}{\sqrt{3}})x$ . We have the following complementary result:

**Theorem 4** *There exists a polynomial-time approximation algorithm for MOVEMENT TO INDEPENDENCE in the plane that moves any given set  $P$  of  $n$  points in  $\mathbb{R}^2$  to another set  $Q$  of  $n$  points in  $\mathbb{R}^2$ , with a maximum movement no more than the minimum maximum movement necessary for a threshold distance of 1, and such that the minimum pairwise distance among the points in  $Q$  is at least  $c = \frac{1}{3+2/\sqrt{3}} = 0.24\dots$*

## 2 The Two Problems on the Line and on a Closed Curve

As a warm-up exercise, we first study the two problems CONstrained  $k$ -CENTER and MOVEMENT TO INDEPENDENCE on the line and on a closed curve. The *distance* between two points on a closed curve is the length of the shorter subcurve determined by the two points. In these two settings, both problems can be solved exactly in polynomial time.

**Proposition 1** *There exists an exact algorithm running in  $O((n+k)\log(n+k))$  time for CONstrained  $k$ -CENTER on the line.*

**Proposition 2** *There exists an exact algorithm running in  $O(\frac{1}{k}(n+k)^2 \log(n+k))$  time for CONstrained  $k$ -CENTER on a closed curve.*

**Proposition 3** *There exists a polynomial-time exact algorithm based on linear-programming for MOVEMENT TO INDEPENDENCE on the line.*

While MOVEMENT TO INDEPENDENCE on the line is always feasible, this is not the case for the new variant on a closed curve. Let  $\gamma$  be a closed curve of length  $L = |\gamma|$ . Obviously, MOVEMENT TO INDEPENDENCE on  $\gamma$  admits a solution if and only if  $L \geq n\Delta$ . We show next that an exact solution can still be found via linear-programming.

**Proposition 4** *There exists a polynomial-time exact algorithm based on linear-programming for MOVEMENT TO INDEPENDENCE on a closed curve.*

### 3 NP-hardness of Constrained $k$ -center

**Proof of Theorem 1.** We show that **CONSTRAINED  $k$ -CENTER** is NP-hard by a reduction from the NP-hard problem **PLANAR-3SAT** [7]. A reduction for  $k$ -center based on similar ideas, however from another problem **PLANAR-VERTEX-COVER**, appears in [3]. Let  $(V, C, G)$  be a **PLANAR-3SAT** instance, which consists of a set  $V$  of  $n$  boolean variables, a set  $C$  of  $m$  clauses that are disjunctions of three literals, and a planar embedding  $G$  of the bipartite graph with a vertex for each variable and each clause, and with an edge connecting a variable to a clause if and only if a literal of the variable occurs in the clause. We will construct a **CONSTRAINED  $k$ -CENTER** instance consisting of a gadget for each variable, clause, and literal.

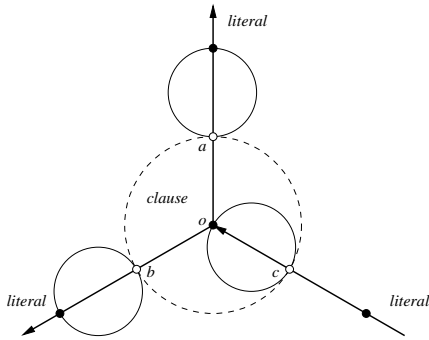


Figure 1: Connection between three literals in a clause. The three red points  $a, b, c$  (drawn as empty circles) come from three different literals, and are placed at the vertices of an equilateral triangle inscribed in a circle centered at the shared black point  $o$ ;  $|oa| = |ob| = |oc| = 2$ . The black point  $o$  in the clause gadget is covered only if at least one of the three literals is true. In this example, the literal of  $c$  is true, and the literals of  $a$  and  $b$  are false.

We now describe our construction. The gadget for each clause is a single black point. The gadget for each variable is a closed chain of alternating black and red points. The gadget for each literal is an open chain of alternating black and red points, with a black point at one end and a red point at the other end. The clause and variable gadgets model the vertices of the planar graph. The literal gadgets model the edges: each literal gadget is connected to the corresponding clause gadget at the end with a red point, and to the corresponding variable gadget at the end with a black point. We illustrate in Figure 1 the connection between the gadgets of a clause and its three literals, and in Figure 2 the connection between the gadgets of a literal and its variable. The distance between consecutive black and red points in each variable or clause gadget is exactly 2 (which is the diameter of unit-radius disk) except at the junctions where a literal gadget is connected to a variable gadget (between  $c$  and  $e$  in Figure 2).

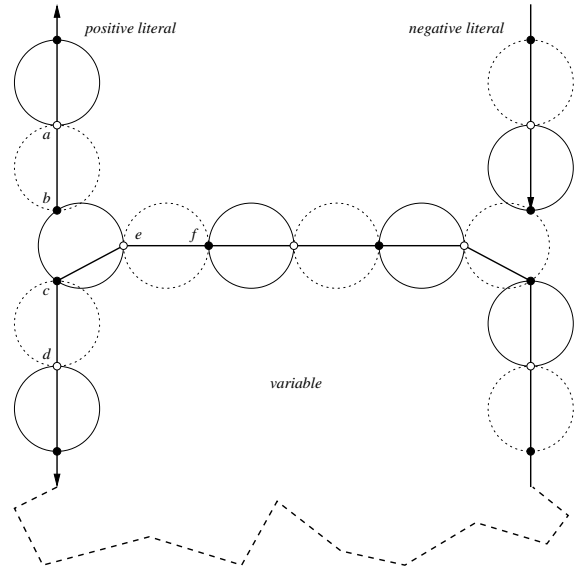


Figure 2: Connection between a variable and its literals. The four points  $d, c, e, f$  are part of a variable gadget; the two points  $a, b$  are part of a literal gadget;  $a, b, c, d$  are collinear;  $b, c, e$  are on a circle of unit radius;  $ef \perp ad$ ;  $|ab| = |cd| = |ef| = 2$ ,  $|ac| = |bd| = |bf| = |cf|$ . Red points are drawn as small empty circles. Large solid and dotted circles (of unit radius) correspond to true and false assignments, respectively.

Write  $x = |bc|/2$  for the configuration in Figure 2. Then  $|ac| = |bd| = 2 + 2x$ , and

$$|bf| = |cf| = \sqrt{(\sqrt{1-x^2} + 3)^2 + x^2}.$$

Let  $x$  be the solution to the equation

$$2 + 2x = \sqrt{(\sqrt{1-x^2} + 3)^2 + x^2},$$

and let  $y = 1 + x$ . Then  $|ac| = |bd| = |bf| = |cf| = 2y$ , and  $y$  is the solution to the following quartic equation

$$4y^4 - 11y^2 - 18y + 25 = 0.$$

A calculation shows that  $y = 1.8279\dots$ . Assume that  $1 \leq r < y$  in the following. It is easy to check that for any such  $r$ , a disk of radius  $r$  that contains a red point can contain at most one black point in our construction, except at the junction between each literal and its variable, where a disk may contain the red point  $e$  and the two black points  $b$  and  $c$  as in Figure 2. Now set the parameter  $k$  to the number of red points in the construction. Then the **PLANAR-3SAT** formula is satisfiable if and only if the **CONSTRAINED  $k$ -CENTER** instance has a feasible solution with  $k$  disks of radius  $r$ . The slackness in the disk radius  $r$  implies that **CONSTRAINED  $k$ -CENTER** is NP-hard to approximate within  $y - \varepsilon$  for any constant  $\varepsilon > 0$ .  $\square$

#### 4 Approximation for Constrained $k$ -center

**Proof of Theorem 2.** The idea of our approximation algorithm is very simple, namely to enumerate the approximate positions of an optimal constrained disk cover. Fix an optimal solution  $O = \{\Omega_1, \dots, \Omega_k\}$ . Suppose that the red point  $q_j$  is covered by a disk  $\Omega_j$  of radius  $r^*$  in the solution  $O$ . Then the center  $c_j$  of  $\Omega_j$  is contained in a disk  $D_j$  of radius  $r^*$  centered at  $q_j$ .

It is well-known that a disk of radius 1 can be covered by three smaller disks of radii  $\frac{\sqrt{3}}{2}$ , whose centers form an equilateral triangle, as shown in Figure 3. Now place three points around the red point  $q_j$  in an equilateral triangle formation (in some arbitrary orientation) such that the distance from  $q_j$  to each point is  $\frac{1}{2}r^*$ . Hence the disk  $D_j$  is covered by three smaller disks,  $E_{j1}, E_{j2}, E_{j3}$  of radius  $\frac{\sqrt{3}}{2}r^*$  centered at the three points. Recall that  $c_j$  is contained in  $D_j$ , so it is covered by one of the disks  $E_{j1}, E_{j2}, E_{j3}$ . Let  $F_{j1} \supset E_{j1}, F_{j2} \supset E_{j2}, F_{j3} \supset E_{j3}$ , be three larger concentric disks of radius  $(\frac{\sqrt{3}}{2} + 1)r^*$ . Since  $\Omega_j$  has radius  $r^*$ , it is covered by one of the larger disks  $F_{j1}, F_{j2}, F_{j3}$  of radius  $(\frac{\sqrt{3}}{2} + 1)r^*$ . So all black points covered by  $\Omega_j$  are also covered by one of these three larger disks.

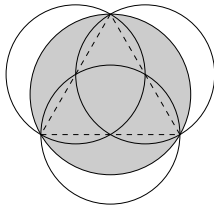


Figure 3: Covering a disk of radius 1 by three smaller disks of radius  $\sqrt{3}/2$ . The three sides of the equilateral triangle inscribed in the unit-radius disk are the diameters of the three smaller disks. The distance from the center of the unit-radius disk to the center of each smaller disk is  $1/2$ .

By the preceding observation, given any candidate radius  $r$ , we can either find a feasible solution of  $k$  disks of radii  $(\frac{\sqrt{3}}{2} + 1)r$  by enumerating one of three possible disks for each red point and testing the black points for containment, all in  $O(3^k k \cdot n)$  time, or decide (correctly) that there is no feasible solution with radius  $r$ . By (1), we can find a radius  $\bar{r}$  such that  $\frac{1}{2}r^* \leq \bar{r} \leq r^*$  in  $O(kn)$  time. Then, by a binary search in the range  $[\bar{r}, 2\bar{r}]$ , we can obtain a  $(\frac{\sqrt{3}}{2} + 1 + \varepsilon)$ -approximation in  $O(3^k k \cdot \log \frac{1}{\varepsilon} \cdot n)$  time, which is linear in  $n$  for any constants  $k$  and  $\varepsilon$ . In particular, since  $\frac{\sqrt{3}}{2} + 1 = 1.8660\dots$ , we have a 1.87-approximation algorithm that runs in  $O(3^k k \cdot n)$  time.

Finer approximations are obtained by covering the unit disk a larger number of smaller congruent disks. The details are omitted.  $\square$

#### 5 Approximation for Movement to Independence

**Proof of Theorem 4.** Let  $\delta$  be the minimum pairwise distance  $\min_{i,j} |p_i p_j|$  of the points in  $P$ . We claim that if  $\delta \leq x \leq 1$ , then

$$\text{OPT}(1) \geq \text{OPT}(x) + (1 - x)/2.$$

To see that the claim is true, imagine all points move from start configuration to target configuration with the same speed as in an optimal solution for  $\text{OPT}(1)$ . Pause the points as soon as their minimum pairwise distance is  $x$ . Then the movement is at least  $\text{OPT}(x)$  before the pause and at least  $(1 - x)/2$  after the pause.

Let  $c = \frac{1}{3+2/\sqrt{3}} = 0.24\dots$ . We now give an algorithm that moves the points to minimum pairwise distance at least  $c$  using maximum movement at most  $\text{OPT}(1)$ . Consider two cases:

- (1)  $\delta \geq c$  Stay put.
- (2)  $\delta < c$  Use the algorithm of Demaine et al. [2] with a smaller grid of size  $x = c$ . Then the maximum movement is at most

$$\begin{aligned} \text{OPT}(c) + \left(1 + \frac{1}{\sqrt{3}}\right)c &\leq \text{OPT}(1) - \frac{(1-c)}{2} \\ &+ \left(1 + \frac{1}{\sqrt{3}}\right)c = \text{OPT}(1). \quad \square \end{aligned}$$

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