METHODUS NOVA

INTEGRALIUM VALORES

PER APPROXIMATIONEM INVENIENDI

AUCTORE

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1.

Inter methodos ad determinationem numericam approximatam integralium propositas insignem tenent locum regulae, quas praeeunte summo NEWTON evolutas dedit COTES. Scilicet si requiritur valor integralis $\int y \, \mathrm{d}x$ ab $x = g$ usque ad $x = h$ sumendus, valores ipsius y pro his valoribus extremis ipsius x et pro quotcunque aliis intermediis a primo ad ultimum incrementis aequalibus progredientibus, multiplicandi sunt per certos coëfficientes numericos, quo facto productorum aggregatum in $h - g$ ductum integrale quaesitum suppeditabit, eo maiore praecisione, quo plures termini in hac operatione adhibentur. Quum principia huius methodi, quae a geometris rarius quam par est in usum vocari videtur, nusquam quod sciam plenius explicata sint, pauca de his praemittere ab instituto nostro haud alienum erit.

2.

Sit $n+1$ multitudo terminorum, quos in usum vocare placuit, statuamusque $h - g = \Delta$, ita ut valores ipsius x sint g, $g + \frac{\Delta}{n}$, $g + \frac{2\Delta}{n}$, $g + \frac{3\Delta}{n}$ etc. usque ad $g + \Delta$, respondeantque iisdem resp. valores ipsius y hi A, A', A'', A''' etc. usque ad $A^{(n)}$: denique ponatur indefinite $x = g + \Delta t$, ita ut y etiam spectari possit tamquam functio ipsius t. Designemus per Y functionem sequentem

$$A.\frac{(nt-1)(nt-2)(nt-3)\dots(nt-n)}{(-1)\;(-2)\,.\,(-3)\dots(-n)}$$
$$+A'.\frac{nt.(nt-2).(nt-3)\dots\dots(nt-n)}{1\,.\,(-1)\;(-2)\dots\dots(1-n)}$$
$$+A''.\frac{nt.(nt-1).(nt-3)\dots\dots(nt-n)}{2.\;1.\;(-1)\dots\dots(2-n)}$$
$$+A'''.\frac{nt.(nt-1).(nt-2)\dots\dots(nt-n)}{3.\;2.\;1\dots\dots(3-n)}$$
$$+\text{ etc.}$$
$$+A^{(n)}.\frac{nt(nt-1).(nt-2)\dots\dots(nt-n+1)}{n.(n-1)\;(n-2)\dots\dots 1}$$

sive $\Sigma\frac{A^{(\mu)}T^{(\mu)}}{M^{(\mu)}}$, ubi repraesentante μ singulos integros $0, 1, 2, 3\dots n$,

$$T^{(\mu)}=\frac{nt(nt-1)(nt-2)(nt-3)\dots(nt-n)}{nt-\mu}$$
$M^{(\mu)}$ valor ipsius T pro $nt=\mu$.

Manifestum erit, Y exhibere functionem algebraicam integram ipsius t ordinis n, atque eius valores pro singulis $n+1$ valoribus ipsius t, puta $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}\dots 1$ aequales esse valoribus ipsius y. Porro patet, si Y' sit functio alia integra pro iisdem valoribus cum y conspirans, $Y'-Y$ pro iisdem evanescere, adeoque per factores $t, t-\frac{1}{n}, t-\frac{2}{n}, t-\frac{3}{n}\dots t-1$ et proin etiam per eorum productum (quod est ordinis $n+1$) divisibilem esse, unde patet, Y', nisi prorsus identica sit cum Y, certo ad altiorem ordinem ascendere debere, sive Y ex omnibus functionibus integris ordinem n haud egredientibus unicam esse, quae pro illis $n+1$ valoribus cum y conspiret. Quodsi itaque y, in seriem secundum potestates ipsius t progredientem evoluta, ante terminum qui implicat t^{n+1} omnino abrumpitur, cum Y identica erit: si vero saltem tam cito convergit, ut terminos sequentes spernere liceat, functio Y inter limites $t=0$, $t=1$ sive $x=g$, $x=h$ ipsius y vice fungi poterit.

3.

Iam integrale nostrum $\int y\,\mathrm{d}x$ transit in $\Delta\int y\,\mathrm{d}t$ a $t=0$ usque ad $t=1$ sumendum, cuius loco per ea, quae modo monuimus, adoptabimus $\Delta\int Y\mathrm{d}t$. Evolvendo itaque $T^{(\mu)}$ in

$$\alpha t^n+\beta t^{n-1}+\gamma t^{n-2}+\delta t^{n-3}+\text{ etc.}$$

erit $\int T^{(\mu)}\mathrm{d}t$, a $t=0$ usque ad $t=1$,

$$= \frac{\alpha}{n+1} + \frac{\beta}{n} + \frac{\gamma}{n-1} + \frac{\delta}{n-2} + \text{etc.}$$

qua quantitate posita $= M^{(\mu)} R^{(\mu)}$, erit integrale quaesitum

$$= \Delta(AR + A'R' + A''R'' + A'''R''' + \text{etc.} + A^{(n)}R^{(n)})$$

Exempli caussa apponemus computum coëfficientis R'' pro $n = 5$. Fit hic

$$T'' = 5^5 t^5 - 13.5^4 t^4 + 59.5^3 t^3 - 107.5^2 tt + 60.5.t$$
$$M'' = 2 \times 1 \times (-1) \times (-2) \times (-3) = -12$$

Hinc $-12R'' = \frac{3125}{6} - 1625 + \frac{7375}{4} - \frac{2675}{3} + 150 = -\frac{25}{12}$, adeoque $R'' = \frac{25}{144}$.

Computus aliquanto brevior evadit, statuendo $2t - 1 = u$. Tunc fit

$$T^{(\mu)} = \frac{(nu+n)(nu+n-2)(nu+n-4)\ldots(nu-n+4)(nu-n+2)(nu-n)}{2^n(nu+n-2\mu)}$$

Ponamus

$$\frac{(nnuu-nn).(nnuu-(n-2)^2).(nnuu-(n-4)^2).(nnuu-(n-6)^2)\ldots}{nnuu-(n-2\mu)^2} = U^{(\mu)}$$

ubi numerator desinere debet in $\ldots(nnuu-9)(nnuu-1)$, si n est impar, vel in $\ldots(nnuu-4)nu$, si n est par, eritque

$$T^{(\mu)} = \frac{(nu-n+2\mu)\,U^{(\mu)}}{2^n}$$

Iam integrale $\int T^{(\mu)} dt$ a $t = 0$ usque ad $t = 1$ acceptum aequale est integrali

$$\int \tfrac{1}{2} T^{(\mu)} du = \int \frac{nu\,U^{(\mu)} du}{2^{n+1}} + \int \frac{(2\mu-n)\,U^{(\mu)} du}{2^{n+1}}$$

ab $u = -1$ usque ad $u = +1$.

Statuendo itaque

$$U^{(\mu)} = \alpha u^{n-1} + \beta u^{n-3} + \gamma u^{n-5} + \delta u^{n-7} + \text{etc.}$$

(sponte enim patet, potestates u^{n-2}, u^{n-4}, u^{n-6} etc. abesse), integralis pars $\int \frac{nu\,U^{(n)} du}{2^{n+1}}$ evanescet pro valore impari ipsius n, pars altera $\int \frac{(2\mu-n)\,U^{(\mu)} du}{2^{n+1}}$ vero pro valore pari, unde integrale $\int T^{(\mu)} dt$ fiet pro n pari

$$= \frac{n}{2^n}\left(\frac{\alpha}{n+1} + \frac{\beta}{n-1} + \frac{\gamma}{n-3} + \frac{\delta}{n-5} + \text{etc.}\right)$$

pro n impari autem

$$= \frac{2\mu-n}{2^n}\left(\frac{\alpha}{n}+\frac{\beta}{n-2}+\frac{\gamma}{n-4}+\frac{\delta}{n-6}+\text{etc.}\right)$$

In exemplo nostro habetur

$$U'' = (25uu-25)(25uu-9) = 625u^4-850uu+225, \text{ adeoque}$$
$$-12R'' = -\tfrac{1}{32}(125-\tfrac{850}{3}+225) = -\tfrac{25}{12} \text{ ut supra.}$$

Observare convenit, fieri $U^{(n-\mu)} = U^{(\mu)}$, adeoque $\int T^{(n-\mu)}dt = \pm\int T^{(\mu)}dt$. signo superiore valente pro n pari, inferiore pro impari. Quare quum facile perspiciatur, perinde haberi $M^{(n-\mu)} = \pm M^{(\mu)}$, semper erit $R^{(n-\mu)} = R^{(\mu)}$, sive e coëfficientibus $R, R', R'' \ldots R^{(n)}$ ultimus primo aequalis, penultimus secundo et sic porro.

4.

Valores numericos horum coëfficientium a Cotesio usque ad $n = 10$ computatos ex *Harmonia Mensurarum* huc adscribimus.

Pro $n = 1$ sive terminis duobus.

$R = R' = \frac{1}{2}$

Pro $n = 2$ sive terminis tribus.

$R = R'' = \frac{1}{6}$, $R' = \frac{2}{3}$

Pro $n = 3$ sive terminis quatuor.

$R = R''' = \frac{1}{8}$, $R' = R'' = \frac{3}{8}$

Pro $n = 4$ sive terminis quinque.

$R = R'''' = \frac{7}{90}$, $R' = R''' = \frac{16}{45}$, $R'' = \frac{2}{15}$

Pro $n = 5$ sive terminis sex.

$R = R^{\mathrm{V}} = \frac{19}{288}$, $R' = R'''' = \frac{25}{96}$, $R'' = R''' = \frac{25}{144}$

Pro $n = 6$ sive terminis septem.

$R = R^{\mathrm{VI}} = \frac{41}{840}$, $R' = R^{\mathrm{V}} = \frac{9}{35}$, $R'' = R^{\mathrm{IV}} = \frac{9}{280}$, $R''' = \frac{34}{105}$

Pro $n = 7$ sive terminis octo.

$R = R^{\mathrm{VII}} = \frac{751}{17280}$, $R' = R^{\mathrm{VI}} = \frac{3577}{17280}$, $R'' = R^{\mathrm{V}} = \frac{49}{640}$, $R''' = R'''' = \frac{2989}{17280}$

Pro $n = 8$ sive terminis novem.

$R = R^{\mathrm{VIII}} = \frac{989}{28350}$, $R' = R^{\mathrm{VII}} = \frac{2944}{14175}$, $R'' = R^{\mathrm{VI}} = -\frac{464}{14175}$, $R''' = R^{\mathrm{V}} = \frac{5248}{14175}$,

$R^{\mathrm{IV}} = -\frac{454}{2835}$

Pro $n = 9$ sive terminis decem.

$$R = R^{\text{IX}} = \tfrac{2857}{89600},\; R' = R^{\text{VIII}} = \tfrac{15741}{89600},\; R'' = R^{\text{VII}} = \tfrac{27}{2240},\; R''' = R^{\text{VI}} = \tfrac{1209}{5600},$$
$$R^{\text{IV}} = R^{\text{V}} = \tfrac{2889}{44800}$$

Pro $n = 10$ sive terminis undecim.

$$R = R^{\text{X}} = \tfrac{16067}{598752},\; R' = R^{\text{IX}} = \tfrac{26575}{149688}.\; R'' = R^{\text{VIII}} = -\tfrac{16175}{199584},$$
$$R''' = R^{\text{VII}} = \tfrac{5675}{12474},\; R^{\text{IV}} = R^{\text{VI}} = -\tfrac{4825}{11088},\; R^{\text{V}} = \tfrac{17807}{24948}.$$

5.

Quum formula $\Delta(AR + A'R' + A'R' + A''R'' + \text{etc.} + A^{(n)}R^{(n)})$ integrale $\int y\,\mathrm{d}x$ ab $x = g$ usque ad $x = g + \Delta$, sive integrale $\Delta \int y\,\mathrm{d}t$ a $t = 0$ usque ad $t = 1$ exacte quidem exhibeat, quoties y in seriem evoluta potestatem t^n non transscendit, sed approximate tantum, quoties y ultra progreditur, superest, ut errorem, quem inducunt termini proxime sequentes, assignare doceamus. Designemus generaliter per $k^{(m)}$ differentiam inter valorem verum integralis $\int t^m \mathrm{d}t$ a $t = 0$ usque ad $t = 1$, atque valorem ex formula prodeuntem, ita ut sit

$$k = 1 - R - R' - R'' - R''' - \text{etc.} - R^{(n)}$$
$$k' = \tfrac{1}{2} - \tfrac{1}{n}(R' + 2R'' + 3R''' + \text{etc.} + nR^{(n)})$$
$$k'' = \tfrac{1}{3} - \tfrac{1}{nn}(R' + 4R'' + 9R''' + \text{etc.} + nnR^{(n)})$$
$$k''' = \tfrac{1}{4} - \tfrac{1}{n^3}(R' + 8R'' + 27R''' + \text{etc.}\; n^3 R^{(n)})$$

etc. Patet igitur, si y evolvatur in seriem

$$K + K't + K''tt + K'''t^3 + \text{etc.}$$

differentiam inter valorem verum integralis $\int y\,\mathrm{d}t$ atque valorem approximatum formulae exprimi per

$$Kk + K'k' + K''k'' + K'''k''' + \text{etc.}$$

Sed manifesto k, k', k'' etc. usque ad $k^{(n)}$ sponte fiunt $= 0$: correctio itaque formulae approximatae erit

$$K^{(n+1)}k^{(n+1)} + K^{(n+2)}k^{(n+2)} + K^{(n+3)}k^{(n+3)} + \text{etc.}$$

Indolem quantitatum $k^{(n+1)}$, $k^{(n+2)}$ etc. infra accuratius perscrutabimur; hic sufficiat, valores numericos primae aut secundae, pro singulis valoribus ipsius n, apposuisse, ut gradus praecisionis, quam formula approximata affert, inde aestimari possit.

Pro $n = 1$ habemus $k'' = -\frac{1}{6}$, $k''' = -\frac{1}{4}$, $k'''' = -\frac{3}{10}$
Pro $n = 2$ invenimus $k''' = 0$, $k'''' = -\frac{1}{120}$, $k^{\text{V}} = -\frac{1}{48}$
Pro $n = 3$ fit $k'''' = -\frac{1}{270}$, $k^{\text{V}} = -\frac{1}{108}$
Pro $n = 4 \ldots k^{\text{V}} = 0$, $k^{\text{VI}} = -\frac{1}{2688}$, $k^{\text{VII}} = -\frac{1}{768}$
Pro $n = 5 \ldots k^{\text{VI}} = -\frac{11}{52500}$, $k^{\text{VII}} = -\frac{11}{15000}$
Pro $n = 6 \ldots k^{\text{VII}} = 0$, $k^{\text{VIII}} = -\frac{1}{38880}$, $k^{\text{IX}} = -\frac{1}{8640}$
Pro $n = 7 \ldots k^{\text{VIII}} = -\frac{167}{105884160}$, $k^{\text{IX}} = -\frac{167}{2352980}$
Pro $n = 8 \ldots k^{\text{IX}} = 0$, $k^{\text{X}} = -\frac{37}{17301504}$, $k^{\text{XI}} = -\frac{37}{3145728}$
Pro $n = 9 \ldots k^{\text{X}} = -\frac{865}{631351908}$, $k^{\text{XI}} = -\frac{865}{114791256}$
Pro $n = 10 \ldots k^{\text{XI}} = 0$, $k^{\text{XII}} = -\frac{26927}{1365000000000}$, $k^{\text{XIII}} = -\frac{26927}{2100000000000}$

Pro valore pari ipsius n ubique hic fieri animadvertimus $k^{(n+1)} = 0$, ac praeterea $k^{(n+3)} = \frac{n+3}{2} k^{(n+2)}$; pro valore impari ipsius n autem ubique prodit $k^{(n+2)} = \frac{n+2}{2} k^{(n+1)}$. Ratio horum eventuum facile e considerationibus sequentibus depromitur.

Designemus generaliter per $l^{(m)}$ differentiam inter valorem verum huius integralis $\int (t - \frac{1}{2})^m \mathrm{d}t$ a $t = 0$ usque ad $t = 1$, atque valorem eum, quem formula approximata profert, ita ut habeatur

$$l^{(m)} = \int (t - \tfrac{1}{2})^m \mathrm{d}t - [(-\tfrac{1}{2})^m R + (\tfrac{1}{n} - \tfrac{1}{2})^m R' + (\tfrac{2}{n} - \tfrac{1}{2})^m R'' + (\tfrac{3}{n} - \tfrac{1}{2})^m R''' + \text{etc.} \\ + (\tfrac{1}{2} - \tfrac{1}{n})^m R^{(n-1)} + (\tfrac{1}{2})^m R^{(n)}]$$

integrali a $t = 0$ usque ad $t = 1$ accepto. Manifesto pro valore impari ipsius m evanescet tum valor verus integralis tum valor approximatus: erit itaque $l' = 0$, $l''' = 0$, $l^{\text{V}} = 0$, $l^{\text{VII}} = 0$ etc. sive generaliter $l^{(m)} = 0$ pro valore impari ipsius m. Pro valore pari autem ipsius m, formulae tribuimus formam hancce

$$l^{(m)} = \frac{1}{2^m(m+1)} - \frac{2}{n^m}\big((\tfrac{1}{2}n)^m R + (\tfrac{1}{2}n - 1)^m R' + (\tfrac{1}{2}n - 2)^m R'' + \text{etc.} \\ + 2^m R^{(\frac{1}{2}n-2)} + R^{(\frac{1}{2}n-1)}\big)$$

si simul fuerit n par; vel hanc

$$l^{(m)} = \frac{1}{2^m}\Big(\frac{1}{m+1} - \frac{2}{n^m}(n^m R + (n-2)^m R' + (n-4)^m R'' + \text{etc.} \\ + 3^m R^{(\frac{1}{2}n-\frac{3}{2})} + R^{(\frac{1}{2}n-\frac{1}{2})})\Big)$$

si simul fuerit n impar.

Si igitur per evolutionem ipsius y in seriem secundum potestates ipsius $t-\frac{1}{2}$ progredientem prodit

$$y = L + L'(t-\tfrac{1}{2}) + L''(t-\tfrac{1}{2})^2 + L'''(t-\tfrac{1}{2})^3 + \text{etc.}$$

correctio valori approximato integralis $\int y\,dt$ a $t=0$ usque ad $t=1$ applicanda erit

$$Ll + L''l'' + L''''l'''' + L^{\text{VI}}l^{\text{VI}} + \text{etc.}$$

aut potius, quum $l^{(m)}$ necessario evanescat pro valore quovis integro ipsius m haud maiori quam n, correctio erit

$$L^{(n+2)}\,l^{(n+2)} + L^{(n+4)}\,l^{(n+4)} + L^{(n+6)}\,l^{(n+6)} + \text{etc.}$$

pro n pari, vel

$$L^{(n+1)}\,l^{(n+1)} + L^{(n+3)}\,l^{(n+3)} + L^{(n+5)}\,l^{(n+5)} + \text{etc.}$$

pro n impari.

Facillime iam correctiones $l^{(m)}$ ad $k^{(m)}$ reducuntur et vice versa. Quum enim habeatur

$$(t-\tfrac{1}{2})^m = t^m - \tfrac{1}{2}m\,.\,t^{m-1} + \tfrac{1}{4}\,.\,\frac{m(m-1)}{1\,.\,2}\,t^{m-2} + \text{etc.}$$

erit

$$l^{(m)} = k^{(m)} - \tfrac{1}{2}m\,k^{(m-1)} + \tfrac{1}{4}\,.\,\frac{m(m-1)}{1\,.\,2}\,k^{(m-2)} + \text{etc.}$$

Et perinde fit

$$k^{(m)} = l^{(m)} + \tfrac{1}{2}m\,l^{(m-1)} + \tfrac{1}{4}\,.\,\frac{m(m-1)}{1\,.\,2}\,l^{(m-1)} + \text{etc.}$$

Ex posteriori formula eiicientur termini, ubi l afficitur indice impari: utraque autem continuanda est tantummodo usque ad indicem $n+1$ (inclus.). Manifesto itaque habebimus

pro n pari

$$k^{(n+1)} = 0$$

$$k^{(n+2)} = l^{(n+2)}$$

$$k^{(n+3)} = \frac{n+3}{2}\,.\,l^{(n+2)}$$

pro n impari

$$k^{(n+1)} = l^{(n+1)}$$

$$k^{(n+2)} = \frac{n+2}{2}\,.\,l^{(n+1)}$$

unde demanant observationes supra indicatae

6.

Generaliter itaque loquendo praestabit, in applicanda methodo Cotesiana ipsi n tribuere valorem parem, seu terminorum multitudinem imparem in usum vocare. Perparum scilicet praecisio augebitur, si loco valoris paris ipsius n ad imparem proxime maiorem ascendamus, quum error maneat eiusdem ordinis, licet coëfficiente aliquantulum minori affectus. Contra ascendendo a valore impari ipsius n ad parem proxime sequentem, error duobus ordinibus promovebitur, insuperque coëfficiens notabilius imminutus praecisionem augebit. Ita si quinque termini adhibentur, sive pro $n = 4$, error proxime exprimitur per $-\frac{1}{2688}K^6$ vel per $-\frac{1}{2688}L^6$; si statuitur $n = 5$, error erit proxime $-\frac{11}{52500}K^6$ vel $-\frac{11}{52500}L^6$, adeoque ne ad semissem quidem prioris depressus: contra faciendo $n = 6$, error fit proxime $= -\frac{1}{38880}K^8$ vel $= -\frac{1}{38880}L^8$, praecisioque tanto magis aucta, quo citius series, in quam functio evolvitur, iam per se convergit.

7.

Postquam haecce circa methodum Cotesii praemisimus, ad disquisitionem generalem progredimur, abiiciendo conditionem, ut valores ipsius x progressione arithmetica procedant. Problema itaque aggredimur, determinare valorem integralis $\int y\,dx$ inter limites datos ex aliquot valoribus datis ipsius y, vel exacte vel quam proxime. Supponamus, integrale sumendum esse ab $x = g$ usque ad $x = g + \Delta$, introducamusque loco ipsius x aliam variabilem $t = \frac{x-g}{\Delta}$, ita ut integrale $\Delta\int y\,dt$ a $t = 0$ usque ad $t = 1$ investigare oporteat. Respondeant $n+1$ valores dati ipsius y hi $A, A', A'', A''' \ldots . A^{(n)}$ valoribus ipsius t inaequalibus his $a, a', a'', a''' \ldots . a^{(n)}$, designemusque per Y functionem algebraicam integram ordinis n hancce:

$$\begin{aligned}
& A\,\frac{(t-a')(t-a'')(t-a''')\ldots.(t-a^{(n)})}{(a-a')(a-a'')(a-a''')\ldots.(a-a^{(n)})} \\
+ & A'\,\frac{(t-a)(t-a'')(t-a''')\ldots.(t-a^{(n)})}{(a'-a)(a'-a'')(a'-a''')\ldots.(a'-a^{(n)})} \\
+ & A''\,\frac{(t-a)(t-a')(t-a''')\ldots.(t-a^{(n)})}{(a''-a)(a''-a')(a''-a''')\ldots.(a''-a^{(n)})} \\
+ & \text{etc.} \\
+ & A^{(n)}\,\frac{(t-a)(t-a')(t-a'')\ldots.(t-a^{(n-1)})}{(a^{(n)}-a)(a^{(n)}-a')(a^{(n)}-a'')\ldots.(a^{(n)}-a^{(n-1)})}
\end{aligned}$$

Manifesto valores huius functionis, si t alicui quantitatum $a, a', a'', a''' \ldots . a^{(m)}$ aequalis ponitur, coincidunt cum valoribus respondentibus functionis y, unde perinde ut in art. 2. concludimus, Y cum y identicam esse, quoties y quoque sit functio algebraica integra ordinem n non transscendens, aut saltem ipsius y vice fungi posse, si y in seriem secundum potestates ipsius t progredientem conversa tantam convergentiam exhibeat, ut terminos altiorum ordinum negligere liceat.

8.

Iam ad eruendum integrale $\int Y\mathrm{d}t$ singulas partes ipsius Y consideremus. Designemus productum,

$$(t-a)(t-a')(t-a'')(t-a''') \ldots . (t-a^{(n)})$$

per T, fiatque per evolutionem huius producti

$$T = t^{n+1} + \alpha t^n + \alpha' t^{n-1} + \alpha'' t^{n-2} + \text{etc.} + \alpha^{(n)}$$

Numerator fractionis, per quam, in parte prima ipsius Y, multiplicata est A, fit $= \frac{T}{t-a}$; numeratores in partibus sequentibus perinde sunt $\frac{T}{t-a'}$, $\frac{T}{t-a''}$, $\frac{T}{t-a'''}$ etc. Denominatores vero nihil aliud sunt, nisi valores determinati horum numeratorum, si resp. statuitur $t = a$, $t = a'$, $t = a''$, $t = a'''$ etc.: denotemus hos denominatores resp. per M, M', M'', M''' etc., ita ut sit

$$Y = \frac{AT}{M(t-a)} + \frac{A'T}{M'(t-a')} + \frac{A''T}{M''(t-a'')} + \text{etc.} + \frac{A^{(n)}T}{M^{(n)}(t-a^{(n)})}$$

Quum fiat $T = 0$, pro $t = a$, habemus aequationem identicam

$$a^{n+1} + \alpha a^n + \alpha' a^{n-1} + \alpha'' a^{n-2} + \text{etc.} + \alpha^{(n)} = 0$$

adeoque

$$T = t^{n+1} - a^{n+1} + \alpha(t^n - a^n) + \alpha'(t^{n-1} - a^{n-1}) + \alpha''(t^{n-2} - a^{n-2}) + \text{etc.}$$
$$+ \alpha^{(n-1)}(t-a)$$

Hinc dividendo per $t-a$ fit

$$\begin{aligned}\frac{T}{t-a} = t^n + a t^{n-1} & + aa t^{n-2} + a^3 t^{n-3} + \text{etc.} + a^n \\ + \alpha t^{n-1} & + \alpha a t^{(n-2)} + \alpha aa t^{n-3} + \text{etc.} + \alpha a^{n-1} \\ & + \alpha' t^{n-2} + \alpha' a t^{n-3} + \text{etc.} + \alpha' a^{n-2} \\ & \qquad + \alpha'' t^{n-3} + \text{etc.} + \alpha'' a^{n-3} \\ & \qquad\qquad + \text{etc. etc.} \\ & \qquad\qquad\qquad + \alpha^{(n-1)}\end{aligned}$$

Valor huius functionis pro $t = a$ colligitur

$$= (n+1)a^n + n\alpha a^{n-1} + (n-1)\alpha' a^{n-2} + (n-2)\alpha'' a^{n-3} + \text{etc.} + \alpha^{(n-1)}$$

Hinc M aequalis valori ipsius $\frac{\mathrm{d}T}{\mathrm{d}t}$ pro $t = a$, uti etiam aliunde constat. Perinde M', M'', M''', etc. erunt valores ipsius $\frac{\mathrm{d}T}{\mathrm{d}t}$ pro $t = a'$, $t = a''$, $t = a'''$ etc.

Porro invenimus valorem integralis $\int\frac{T\mathrm{d}t}{t-a}$, a $t = 0$ usque ad $t = 1$,

$$\begin{aligned}
= \frac{1}{n+1} + \frac{a}{n} + \frac{aa}{n-1} + \frac{a^3}{n-2} &+ \text{etc.} + a^n \\
+ \frac{\alpha}{n} + \frac{\alpha a}{n-1} + \frac{\alpha a a}{n-2} &+ \text{etc.} + \alpha a^{n-1} \\
+ \frac{\alpha'}{n-1} + \frac{\alpha' a}{n-2} &+ \text{etc.} + \alpha' a^{n-2} \\
+ \frac{\alpha''}{n-2} &+ \text{etc.} + \alpha'' a^{n-3} \\
&+ \text{etc. etc.} \\
&\qquad + \alpha^{(n-1)}
\end{aligned}$$

quos terminos ordine sequenti disponemus:

$$\begin{aligned}
& a^n + \alpha a^{n-1} + \alpha' a^{n-2} + \alpha'' a^{n-3} + \text{etc.} + \alpha^{(n-1)} \\
& + \tfrac{1}{2}(a^{n-1} + \alpha a^{n-2} + \alpha' a^{n-3} + \text{etc.} + \alpha^{(n-2)}) \\
& + \tfrac{1}{3}(a^{n-2} + \alpha a^{n-3} + \alpha' a^{n-4} + \text{etc.} + \alpha^{(n-3)}) \\
& + \tfrac{1}{4}(a^{n-3} + \alpha a^{n-4} + \alpha' a^{n-5} + \text{etc.} + \alpha^{(n-4)}) \\
& + \text{etc.} \\
& + \frac{1}{n-1}(aa + \alpha a + \alpha') \\
& + \frac{1}{n}(a + \alpha) \\
& + \frac{1}{n+1}
\end{aligned}$$

Sed manifesto eadem quantitas prodit, si in producto e multiplicatione functionis T in seriem infinitam

$$t^{-1} + \tfrac{1}{2}t^{-2} + \tfrac{1}{3}t^{-3} + \tfrac{1}{4}t^{-4} \text{ etc.}$$

orto, reiectis omnibus terminis, qui implicant potestates ipsius t exponentibus negativis (sive brevius, in producti parte ea, quae est functio integra ipsius t) pro t scribitur a. Supponamus itaque fieri *)

$$T(t^{-1} + \tfrac{1}{2}t^{-2} + \tfrac{1}{3}t^{-3} + \tfrac{1}{4}t^{-4} + \text{etc.}) = T' + T''$$

---

*) Vix opus erit monere, characteres T, T', T'' alio sensu hic accipi, quam in art. 2.

ita ut T' sit functio integra ipsius t in hoc producto contenta, T'' vero pars altera, scilicet series descendens in infinitumque excurrens. Quo facto valor integralis $\int \frac{T\,\mathrm{d}t}{t-a}$ a $t=0$ usque ad $t=1$ aequalis erit valori functionis T' pro $t=a$. Quodsi itaque valores determinatos functionis

$$\frac{T'}{\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right)}$$

pro $t=a$, $t=a'$, $t=a''$, $t=a'''$ etc. usque ad $t=a^{(n)}$ resp. per R, R', R'', $R''' \ldots . R^{(n)}$ denotamus, integrale $\int Y \mathrm{d}t$ a $t=0$ usque ad $t=1$ fiet

$$= RA + R'A' + R''A'' + \text{etc.} + R^{(n)}A^{(n)}$$

quod per Δ multiplicatum exhibebit valorem vel verum vel approximatum integralis $\int y\,\mathrm{d}x$ ab $x=g$ usque ad $x=g+\Delta$.

9.

Hae operationes aliquanto facilius perficiuntur, si loco indeterminatae t introducitur alia $u=2t-1$. Scribimus quoque brevitatis caussa $b=2a-1$, $b'=2a'-1$, $b''=2a''-1$ etc. Transeat T, substituto pro t valore $\frac{1}{2}u+\frac{1}{2}$, in $\frac{U}{2^{n+1}}$, sive sit

$$U=(u-b)(u-b')(u-b'')\ldots.(u-b^n)$$

Erit itaque $\frac{\mathrm{d}T}{\mathrm{d}t}=\frac{1}{2^n}\cdot\frac{\mathrm{d}U}{\mathrm{d}u}$, adeoque M, M', M'' etc. valores determinati ipsius $\frac{1}{2^n}\cdot\frac{\mathrm{d}U}{\mathrm{d}u}$, si deinceps statuitur $u=b$, $u=b'$, $u=b''$ etc.

Quum series $t^{-1}+\frac{1}{2}t^{-2}+\frac{1}{3}t^{-3}+\frac{1}{4}t^{-4}+$ etc. nihil aliud sit quam $\log.\frac{1}{1-t^{-1}}=\log\frac{1+u^{-1}}{1-u^{-1}}$: per substitutionem $t=\frac{1}{2}u+\frac{1}{2}$ necessario transibit in $2u^{-1}+\frac{2}{3}u^{-3}+\frac{2}{5}u^{-5}+\frac{2}{7}u^{-7}+$ etc. Quodsi itaque statuimus

$$U(u^{-1}+\tfrac{1}{3}u^{-3}+\tfrac{1}{5}u^{-5}+\tfrac{1}{7}u^{-7}+\text{etc.})=U'+U''$$

ita ut U' sit functio integra ipsius u in hoc producto contenta, U'' vero pars altera, puta series descendens infinita, patet esse

$$T'+T''=\frac{1}{2^n}(U'+U'')$$

Sed manifesto T', tamquam functio integra ipsius t, per substitutionem $t=\frac{1}{2}u+\frac{1}{2}$ necessario functionem integram ipsius u producet: contra T'', quae non continet nisi potestates negativas ipsius t, per eandem substitutionem tantummodo potestates negativas ipsius u gignet. Quam ob rem U' nihil aliud erit quam $2^n T'$ per hanc substitutionem transformata, ac perinde U'' producta erit ex $2^n T''$. Nihil itaque intererit, sive in $\frac{T'}{(\frac{dT}{dt})}$ substituamus $t = a$, sive in $\frac{U'}{(\frac{dU}{du})}$ faciamus $u = b$, unde colligimus, R, R', R'', R''' etc. etiam esse valores determinatos functionis $\frac{U'}{(\frac{dU}{du})}$ pro $u = b$, $u = b'$, $u = b''$, $u = b'''$ etc.

10.

Antequam ulterius progrediamur, haecce praecepta per exemplum illustrabimus. Sit $n = 5$, statuamusque $a = 0$, $a' = \frac{1}{5}$, $a'' = \frac{2}{5}$, $a''' = \frac{3}{5}$, $a'''' = \frac{4}{5}$, $a''''' = 1$. Hinc fit

$$T = t^6 - 3t^5 + \tfrac{17}{5}t^4 - \tfrac{9}{5}t^3 + \tfrac{274}{625}tt - \tfrac{24}{625}t$$

Multiplicando per $t^{-1} + \frac{1}{2}t^{-2} + \frac{1}{3}t^{-3} + \frac{1}{4}t^{-4} +$ etc. obtinemus

$$T' = t^5 - \tfrac{5}{2}t^4 + \tfrac{67}{30}t^3 - \tfrac{17}{20}tt + \tfrac{913}{7500}t - \tfrac{19}{7500}$$

Valores itaque coëfficientium R, R', R'', R''', R'''', R''''' exprimuntur per functionem fractam

$$\frac{t^5 - \frac{5}{2}t^4 + \frac{67}{30}t^3 - \frac{17}{20}tt + \frac{913}{7500}t - \frac{19}{7500}}{6t^5 - 15t^4 + \frac{68}{5}t^3 - \frac{27}{5}tt + \frac{548}{625}t - \frac{24}{625}}$$

in qua pro t deinceps substituendi sunt valores $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$. Aliquanto brevior est methodus altera, quae suppeditat $b = -1$, $b' = -\frac{3}{5}$, $b'' = -\frac{1}{5}$, $b''' = \frac{1}{5}$, $b'''' = \frac{3}{5}$, $b''''' = 1$

$$U = u^6 - \tfrac{7}{5}u^4 + \tfrac{259}{625}uu - \tfrac{9}{625}$$
$$U' = u^5 - \tfrac{16}{15}u^3 + \tfrac{277}{1875}u$$

unde R. R', R'' etc. erunt valores functionis fractae

$$\frac{u^4 - \frac{16}{15}uu + \frac{277}{1875}}{6u^4 - \frac{28}{5}uu + \frac{518}{625}}$$

pro $u = -1$, $u = -\frac{3}{5}$, $u = -\frac{1}{5}$ etc. Utraque methodus eosdem numeros profert, quos in art. 4. ex Harmonia Mensurarum tradidimus. Ceterum in casu tali, qualem hocce exemplum sistit, ubi a, a', a'' etc. sunt quantitates rationales, valores denominatoris $\frac{dT}{dt}$ commodius in forma primitiva computantur, puta $(a-a')(a-a'').(a-a''')\ldots.(a-a^{(n)})$ pro $t = a$ ac perinde de reliquis. Idem valet de denominatore $\frac{dU}{du}$, qui pro $u = b$ fit $= (b-b')(b-b'')(b-b''')\ldots.(b-b^{(n)})$.

11.

Quoties a, a', a'' etc. vel ex parte vel omnes sunt irrationales, utilis erit transformatio functionis fractae, ex qua numeros R, R', R'' etc. derivamus, in functionem integram: principia talis transformationis, quum in libris algebraicis non inveniantur, hoc loco breviter explicabimus. Propositis scilicet tribus functionibus integris Z, ζ, ζ' indeterminatae z, quaeritur functio integra, quae fractae $\frac{Z}{\zeta}$ vice fungi possit, quatenus pro z accipitur radix quaecunque aequationis $\zeta'=0$. Supponemus autem, ζ pro nullo horum valorum ipsius z evanescere, sive quod eodem redit, ζ atque ζ' nullum divisorem communem indeterminatum implicare. Exponentes potestatum altissimarum ipsius z in ζ atque ζ' per k, k' denotabimus.

Dividatur sueto more ζ per ζ', donec residui ordo infra k' depressus sit; statuatur residuum $=\frac{\zeta''}{\lambda}$, eiusque ordo $=k''$, ita ut $\frac{1}{\lambda}z^{k''}$ sit residui terminus altissimus; divisionis quotientem ponemus $=\frac{p}{\lambda}$. Perinde ex divisione functionis ζ' per ζ'' prodeat residuum $\frac{\zeta'''}{\lambda'}$ ordinis k''', quotiens $\frac{p'}{\lambda'}$; dein rursus e divisione functionis ζ'' per ζ''' prodeat residuum $\frac{\zeta''''}{\lambda''}$ ordinis k'''' atque quotiens $\frac{p''}{\lambda''}$ et sic porro, donec in serie functionum ζ'', ζ''', ζ'''' etc., quae singulae terminum suum altissimum coëfficiente 1 affectum habebunt, perveniatur ad $\zeta^{(m)}=1$. Hoc tandem evenire debere facile perspicitur, quum quaelibet functionum ζ, ζ' ζ'', ζ''' etc. cum praecedenti divisorem communem indeterminatum habere nequeat, adeoque certo divisio absque residuo fieri nequeat, quamdiu divisor fuerit ordinis maioris quam 0. Habebimus igitur seriem aequationum

$$\zeta''=\lambda\zeta-p\zeta'$$
$$\zeta'''=\lambda'\zeta'-p'\zeta''$$
$$\zeta''''=\lambda''\zeta''-p''\zeta'''$$
$$\zeta'''''=\lambda'''\zeta'''-p'''\zeta''''$$
etc. usque ad
$$\zeta^{(m)}=\lambda^{(m-2)}\zeta^{(m-2)}-p^{(m-2)}\zeta^{(m-1)}$$

ubi ζ'', ζ''', $\zeta''''\ldots\zeta^{(m)}$ sunt functiones integrae ipsius z ordinis k'', k''', $k''''\ldots k^{(m)}$; numeri k', k'', $k'''\ldots k^{(m)}$ continuo decrescentes usque ad ultimum $k^{(m)}=0$; p, p', p'', p''' etc. quoque functiones integrae ipsius z ordinis $k-k'$, $k'-k''$, $k''-k'''$, $k'''-k''''$ etc. (excepto casu, ubi $k<k'$, in quo manifesto statui debet $p=0$).

His ita praeparatis formamus *secundam* seriem functionum integrarum ipsius z, puta η, η', η'', $\eta'''\ldots\eta^{(m)}$. Et quidem statuemus $\eta=1$, $\eta'=0$, reliquas vero singulas e binis praecedentibus per eandem legem derivamus, per quam functiones $\zeta, \zeta', \zeta'', \zeta'''$ etc. inter se nexae sunt, scilicet per aequationes

$$\begin{aligned}
\eta'' &= \lambda\eta \quad -p\eta' \\
\eta''' &= \lambda'\eta' \quad -p'\eta'' \\
\eta'''' &= \lambda''\eta'' \quad -p''\eta''' \\
\eta''''' &= \lambda'''\eta''' - p'''\eta'''' \text{ etc. usque ad} \\
\eta^{(m)} &= \lambda^{(m-2)}\eta^{(m-2)} - p^{(m-2)}\eta^{(m-1)}
\end{aligned}$$

Manifesto $\eta'' = \lambda$ hic est ordinis 0; $\eta''' = -\lambda p'$ ordinis $k' - k''$, et perinde sequentes η'''', η''''' etc. resp. ordinis $k' - k'''$, $k' - k''''$ etc., ita ut ultima $\eta^{(m)}$ ascendat ad ordinem $k' - k^{(m-1)}$.

Porro consideremus *tertiam* functionum seriem, $\zeta - \zeta\eta$, $\zeta' - \zeta\eta'$, $\zeta'' - \zeta\eta''$, $\zeta''' - \zeta\eta'''$ etc., inter cuius terminos quosvis ternos consequentes manifesto similis relatio intercedet, scilicet

$$\begin{aligned}
\zeta'' - \zeta\eta'' &= \lambda(\zeta - \zeta\eta) \quad - p(\zeta' - \zeta\eta') \\
\zeta''' - \zeta\eta''' &= \lambda'(\zeta' - \zeta\eta') \quad - p'(\zeta'' - \zeta\eta'') \\
\zeta'''' - \zeta\eta'''' &= \lambda''(\zeta'' - \zeta\eta'') - p''(\zeta''' - \zeta\eta''')
\end{aligned}$$

Iam prima harum functionum fit $= 0$, secunda $= \zeta'$: hinc facile colligitur, singulas per ζ' divisibiles fore.

Hinc autem nullo negotio sequitur, loco fractionis $\frac{Z}{\zeta}$ adoptari posse functionem integram $Z\eta^{(m)}$, quatenus quidem ipsi z non tribuantur alii valores nisi qui sint radices aequationis $\zeta' = 0$: manifesto enim differentia $\frac{Z(1-\zeta\eta^{(m)})}{\zeta}$ pro tali valore ipsius z necessario evanescit, quum $1 - \zeta\eta^{(m)} = \zeta^{(m)} - \zeta\eta^{(m)}$ per ζ' sit divisibilis.

Loco functionis $Z\eta^{(m)}$ etiam adoptari poterit eius residuum ex divisione per ζ' ortum, cuius ordo erit inferior ordine functionis ζ'.

Ceterum hocce residuum commodius per algorithmum sequentem immediate eruere licet. Formentur aequationes sequentes

$$\begin{aligned}
Z &= q'\zeta' + Z' \\
Z' &= q''\zeta'' + Z'' \\
Z'' &= q'''\zeta''' + Z''' \\
Z''' &= q''''\zeta'''' + Z'''' \text{ etc. usque ad} \\
Z^{(m-1)} &= q^{(m)}\zeta^{(m)} + Z^{(m)}
\end{aligned}$$

scilicet deinceps dividendo Z per ζ', dein residuum primae divisionis Z' per ζ'', tum residuum secundae divisionis per ζ''' ac sic porro. Quum residuum semper ad ordinem inferiorem pertineat quam divisor, ordo functionum Z', Z'', Z''', Z''''etc. erit resp. inferior quam k', k'', k''', k'''' etc.; ultima vero $Z^{(m)}$ necessario fit $= 0$, quum divisor $\zeta^{(m)}$ sit $= 1$. Habemus itaque

$$Z = q'\zeta' + q''\zeta'' + q'''\zeta''' + q''''\zeta'''' + \text{etc.} + q^{(m)}\zeta^{(m)}$$

Quatenus autem pro z solae radices aequationis $\zeta' = 0$ accipiuntur, fit $\zeta' = 0$, $\zeta'' = \zeta\eta''$, $\zeta''' = \zeta\eta'''$, $\zeta'''' = \zeta\eta''''$ etc., unde sub eadem restrictione erit

$$\frac{Z}{\zeta} = q''\eta'' + q'''\eta''' + q''''\eta'''' + \text{etc.} + q^{(m)}\eta^{(m)}$$

Ordo vero huius expressionis necessario erit infra k': quum enim ordo quotientium q'', q''', q''''etc. esse debeat infra $k' - k''$, $k'' - k'''$, $k''' - k''''$etc., ordo singularum partium $q''\eta'', q'''\eta''', q''''\eta''''$etc. erit infra $k' - k''$, $k' - k'''$, $k' - k''''$ etc.

Denique adhuc observamus, si forte inter valores indeterminatae z, quos in fractione $\frac{Z}{\zeta}$ substituere oporteat, rationales cum irrationalibus mixti reperiantur, magis e re fore, illos ab his separare atque hos solos in aequatione $\zeta' = 0$ comprehendere. Pro rationalibus enim valoribus calculi compendio opus non erit; pro irrationalibus autem calculus tanto simplicior erit, quo minor fuerit gradus functionis integrae, ad quam fractam reducere licet.

12.

Ecce nunc exemplum transformationis in art. praec explicatae. Proposita sit functio fracta

$$\frac{z^6 - \frac{50}{39}z^4 + \frac{283}{715}zz - \frac{256}{15015}}{7z^6 - \frac{105}{13}z^4 + \frac{315}{143}zz - \frac{35}{429}}$$

in qua z indefinite repraesentat radices aequationis

$$z^7 - \frac{21}{13}z^5 + \frac{105}{143}z^3 - \frac{35}{429}z = 0$$

Si hic omnes septem radices complecti vellemus, ad functionem integram sexti ordinis delaberemur. Manifesto autem pro valore rationali $z = 0$ computus fractionis obvius est, datque valorem $\frac{256}{1225}$: quapropter seposita hac radice in aequatione sexti gradus subsistemus:

$$z^6 - \tfrac{21}{13}z^4 + \tfrac{105}{143}zz - \tfrac{35}{429} = 0$$

quo pacto facile praevidemus orturam esse functionem integram quarti ordinis. Iam ex applicatione praeceptorum praecedentium prodeunt sequentia:

$$\begin{array}{ll}
\zeta\ = 7z^6 - \tfrac{105}{13}z^4 + \tfrac{315}{143}zz - \tfrac{35}{429} & \\
\zeta' = z^6 - \tfrac{21}{13}z^4 + \tfrac{105}{143}zz - \tfrac{35}{429} & \\
\zeta'' = z^4 - \tfrac{10}{11}zz + \tfrac{5}{33} & \\
\zeta''' = zz - \tfrac{3}{7} & \\
\zeta'''' = 1 & \\
\lambda\ = \tfrac{13}{42} & p\ = \tfrac{13}{6} \\
\lambda' = -\tfrac{4719}{280} & p' = -\tfrac{4719}{280}zz + \tfrac{3333}{280} \\
\lambda'' = -\tfrac{147}{8} & p'' = -\tfrac{147}{8}zz + \tfrac{777}{88} \\
\eta\ = 1 & \\
\eta' = 0 & \\
\eta'' = \tfrac{13}{42} & \\
\eta''' = \tfrac{20449}{3920}zz - \tfrac{14443}{3920} & \\
\eta'''' = \tfrac{61347}{640}z^4 - \tfrac{127413}{1120}zz + \tfrac{120263}{4480} & \\
Z = z^6 - \tfrac{50}{39}z^4 + \tfrac{283}{715}zz - \tfrac{256}{15015}; & q' = 1 \\
Z' = \tfrac{1}{3}z^4 - \tfrac{22}{65}zz + \tfrac{323}{5005} & q'' = \tfrac{1}{3} \\
Z'' = -\tfrac{76}{2145}zz + \tfrac{632}{45045} & q''' = -\tfrac{76}{2145} \\
Z''' = -\tfrac{4}{3465} & q'''' = -\tfrac{4}{3465}
\end{array}$$

Hinc tandem derivatur functio integra fractioni propositae aequivalens:

$$-\tfrac{1859}{16800}z^4 - \tfrac{1573}{29400}zz + \tfrac{7947}{39200}$$

13.

Ad determinandum gradum praecisionis, qua formula nostra integralis $RA + R'A' + R''A'' +$ etc. $+ R^{(n)}A^{(n)}$ gaudet, statuamus generaliter

$$Ra^m + R'a'^m + R''a''^m + \text{etc.} + R^{(n)}a^{(n)m} = \frac{1}{m+1} - k^{(m)}$$

ita ut $k^{(m)}$ sit differentia inter integralis $\int t^m \mathrm{d}t$ a $t=0$ usque ad $t=1$ sumti valorem verum atque approximatum. Habebimus itaque, singulis fractionibus in series evolutis,

$$\frac{R}{t-a}+\frac{R'}{t-a'}+\frac{R''}{t-a''}+\text{etc.}+\frac{R^{(n)}}{t-a^{(n)}}$$
$$=(1-k)t^{-1}+(\tfrac{1}{2}-k')t^{-2}+(\tfrac{1}{3}-k'')t^{-3}+(\tfrac{1}{4}-k''')t^{-4}+\text{etc.}$$
$$=t^{-1}+\tfrac{1}{2}t^{-2}+\tfrac{1}{3}t^{-3}+\tfrac{1}{4}t^{-4}+\text{etc.}-\Theta$$

si statuimus

$$\Theta=kt^{-1}+k't^{-2}+k''t^{-3}+k'''t^{-4}+\text{etc.}$$

sive potius (quum iam sciamus, k, k', k'', k''' etc. usque ad $k^{(n)}$ sponte evanescere debere)

$$\Theta=k^{(n+1)}t^{-(n+2)}+k^{(n+2)}t^{-(n+3)}+k^{(n+3)}t^{-(n+4)}+\text{etc.}$$

Multiplicando per T fit

$$T\left(\frac{R}{t-a}+\frac{R'}{t-a'}+\frac{R''}{t-a''}+\text{etc.}+\frac{R^{(n)}}{t-a^{(n)}}\right)=T'+T''-T\Theta$$

Pars prior huius aequationis est functio integra ipsius t ordinis n, eiusque valores determinati pro $t=a$, $t=a'$, $t=a''$ etc. resp. fiunt MR, $M'R'$, $M''R''$ etc.: quapropter, quum eadem valeant de functione T', uti ex ipso modo numeros R, R', R'' etc. determinandi perspicuum est, necessario illa pars prior aequationis identica esse debet cum T', adeoque $T''=T\Theta$. Oritur itaque Θ ex evolutione fractionis $\frac{T''}{T}$, quo pacto coëfficientes $k^{(n+1)}$, $k^{(n+2)}$ etc. quousque libet determinari poterunt. Quibus inventis correctio valoris nostri approximati integralis $\int y\,\mathrm{d}t$ erit

$$=k^{(n+1)}K^{(n+1)}+k^{(n+2)}K^{(n+2)}+\text{etc.}$$

si series, in quam evolvitur y, est

$$y=K+K't+K''tt+K'''t^3+\text{etc.}$$

14.

Si magis placet, correctionem exprimere per coëfficientes seriei secundum potestates ipsius $t-\frac{1}{2}$ progredientis

$$y=L+L'(t-\tfrac{1}{2})+L''(t-\tfrac{1}{2})^2+L'''(t-\tfrac{1}{2})^3+\text{etc.}$$

illa erit

$$= l^{(n+1)}L^{(n+1)} + l^{(n+2)}L^{(n+2)} + l^{(n+3)}L^{(n+3)} + \text{etc.}$$

si generaliter per $l^{(m)}$ exprimimus correctionem valoris approximati integralis $\int (t-\frac{1}{2})^m \, dt$. Hae correctiones $l^{(m)}$ cum correctionibus $k^{(m)}$ nexae erunt per aequationem

$$l^{(m)} = k^{(m)} - \tfrac{1}{2} m k^{(m-1)} + \tfrac{1}{4} \cdot \frac{m \,.\, m-1}{1 \,.\, 2} k^{(m-2)} - \tfrac{1}{8} \cdot \frac{m \,.\, m-1 \,.\, m-2}{1 \,.\, 2 \,.\, 3} k^{(m-3)} + \text{etc.}$$

Quo vero illas independenter eruere possimus, perpendamus, functionem Θ per substitutionem $t = \frac{1}{2}u + \frac{1}{2}$ transire in

$$\begin{aligned} & 2k(u^{-1} - u^{-2} + u^{-3} - u^{-4} + \text{etc.}) \\ + & \; 4k'(u^{-2} - 2u^{-3} + 3u^{-4} - 4u^{-5} + \text{etc.}) \\ + & \; 8k''(u^{-3} - 3u^{-4} + 6u^{-5} - 10u^{-6} + \text{etc.}) \\ + & \; 16k'''(u^{-4} - 4u^{-5} + 10u^{-6} - 20u^{-7} + \text{etc.}) \\ + & \; \text{etc.} \end{aligned}$$

sive in

$$\begin{aligned} 2ku^{-1} + 4(k' - \tfrac{1}{2})u^{-2} + 8(k'' - \tfrac{1}{2} \,.\, 2k' + \tfrac{1}{4}k)u^{-3} \\ + 16(k''' - \tfrac{1}{2} \,.\, 3k'' + \tfrac{1}{4} \,.\, 3k' - \tfrac{1}{8}k)u^{-4} + \text{etc.} \end{aligned}$$

sive in

$$2lu^{-1} + 4l'u^{-2} + 8l''u^{-3} + 16l'''u^{-4} + \text{etc.}$$

sive denique, quum a priori sciamus, l, l', l'', l''' etc. usque ad $l^{(n)}$ sponte evanescere, in

$$2^{n+2} l^{(n+1)} u^{-(n+2)} + 2^{n+3} l^{(n+2)} u^{-(n+3)} + 2^{n+4} l^{(n+4)} u^{-(n+4)} + \text{etc.}$$

At $\Theta = \frac{T'''}{T}$; quare quum T, T''' per substitutionem $t = \frac{1}{2}u + \frac{1}{2}$ transeant in $\frac{U}{2^{n+1}}$, $\frac{U''}{2^n}$, (art. 9), functio Θ per eandem substitutionem transibit in $\frac{2U''}{U}$. Quodsi itaque seriem ex evolutione fractionis $\frac{U''}{U}$ oriundam per Ω designamus, erit

$$\Omega = 2^{n+1} l^{(n+1)} u^{-(n+2)} + 2^{n+2} l^{(n+2)} u^{-(n+3)} + 2^{n+3} l^{(n+3)} u^{-(n+4)} + \text{etc.}$$

quo pacto coëfficientes $l^{(n+1)}$, $l^{(n+2)}$ etc. quousque lubet erui poterunt.

Ita in exemplo art. 10 invenimus

$$\begin{aligned} U'' &= -\tfrac{176}{13125}u^{-1} - \tfrac{304}{28125}u^{-3} - \tfrac{2576}{309375}u^{-5} - \text{etc.} \\ \Omega &= -\tfrac{176}{13125}u^{-7} - \tfrac{832}{28125}u^{-9} - \tfrac{189856}{4296875}u^{-11} - \text{etc.} \end{aligned}$$

adeoque correctio valoris approximati integralis

$$= -\tfrac{11}{52500}L^{\mathrm{VI}} - \tfrac{13}{112500}L^{\mathrm{VIII}} - \tfrac{5933}{1375000000}L^{\mathrm{X}} - \text{etc.}$$

15.

Coëfficiens $K^{(m)}$ functionis y in seriem evolutae fit, per theorema TAYLORI, aequalis valori ipsius

$$\frac{1}{1.2.3\ldots.m}\cdot\frac{d^m y}{dt^m} \quad \text{sive} \quad \frac{\Delta^m}{1.2.3\ldots.m}\cdot\frac{d^m y}{dx^m}$$

pro $t=0$ sive $x=g$; perinde coëfficiens $L^{(m)}$ est valor eiusdem expressionis pro $t=\frac{1}{2}$ sive $u=0$ sive $x=g+\frac{1}{2}\Delta$: utrique coëfficienti *ordinem* m tribuemus. Generaliter itaque loquendo integratio nostra usque ad ordinem n inclus. exacta erit, quicunque valores pro $a, a', a'' \ldots. a^{(n)}$ accipiantur. Attamen hinc nihil obstat, quominus pro valoribus harum quantitatum scite electis praecisio ad altiorem gradum evehatur. Ita iam supra vidimus, in methodo COTESII i. e. pro $a=0$, $a=\frac{1}{n}$, $a''=\frac{2}{n}$, $a'''=\frac{3}{n}$ etc. praecisionem sponte ad ordinem $n+1$ inclus. extendi, quoties n sit numerus par. Generaliter patet, si valores a, a', a'', a'''etc. ita fuerint electi, ut in functione T'' vel U'' ab initio excidat terminus unus pluresve, praecisionem totidem gradibus ultra ordinem n promotum iri, quot termini exciderint. Hinc facile colligitur, quum multitudo quantitatum quas eligere conceditur sit $n+1$, per idoneam earum determinationem praecisionem semper ad ordinem $2n+1$ inclus. evehi posse, quo pacto adiumento $n+1$ terminorum eundem praecisionis ordinem assequi licebit, ad quem attingendum $2n+1$ vel $2n+2$ terminos in usum vocare oporteret, si COTESII methodum sequeremur.

16.

Totum hoc negotium in eo vertitur, ut pro quovis valore dato ipsius n functionem T eruamus formae $t^{n+1}+\alpha t^n+\alpha' t^{n-1}+\alpha'' t^{n-2}$ etc. itaque comparatam, ut in producto

$$T(t^{-1}+\tfrac{1}{2}t^{-2}+\tfrac{1}{3}t^{-3}+\tfrac{1}{4}t^{-4}+\text{etc.})$$

evoluto potestates t^{-1}, t^{-2}, $t^{-3} \ldots. t^{-(n+1)}$ omnes nanciscantur coëfficientem 0; aut si magis placet, functionem U formae $u^{n+1}+\beta u^n+\beta' u^{n-1}+\beta'' u^{n-2}+$ etc., cuius productum per $u^{-1}+\frac{1}{3}u^{-3}+\frac{1}{5}u^{-5}+\frac{1}{7}u^{-7}+$ etc. liberum evadat a potestatibus u^{-1}, u^{-2}, u^{-3}, $u^{-4}\ldots u^{-(n+1)}$. Modus posterior aliquanto simplicior erit: quum enim facile perspiciatur, coëfficientes ipsius U, ut conditioni praescriptae satisfiat, alternatim evanescere debere, sive statui $\beta = 0$, $\beta'' = 0$, $\beta'''' = 0$ etc., laboris dimidia fere pars iam absoluta censenda erit. Evolvamus casus quosdam simpliciores.

I. Pro $n = 0$, coëfficiens unicus ipsius t^{-1} in producto

$$(t+\alpha)(t^{-1}+\tfrac{1}{2}t^{-2}+\tfrac{1}{3}t^{-3}+\text{ etc.})$$

evanescere debet. Qui quum fiat $=\frac{1}{2}+\alpha$, habemus $\alpha = -\frac{1}{2}$, sive $T = t-\frac{1}{2}$. Perinde $U = u$.

II. Pro $n = 1$, determinatio ipsius T pendet a duabus aequationibus

$$0 = \tfrac{1}{3}+\tfrac{1}{2}\alpha+\alpha'$$
$$0 = \tfrac{1}{4}+\tfrac{1}{3}\alpha+\tfrac{1}{2}\alpha'$$

unde deducimus $\alpha = -1$, $\alpha' = +\frac{1}{6}$, sive $T = tt-t+\frac{1}{6}$. Determinatio functionis U unicam aequationem affert

$$0 = \tfrac{1}{3}+\beta'$$

unde $\beta' = -\frac{1}{3}$, sive $U = uu-\frac{1}{3}$.

III. Pro $n = 2$, functio T determinatur adiumento trium aequationum

$$0 = \tfrac{1}{4}+\tfrac{1}{3}\alpha+\tfrac{1}{2}\alpha'+\alpha''$$
$$0 = \tfrac{1}{5}+\tfrac{1}{4}\alpha+\tfrac{1}{3}\alpha'+\tfrac{1}{2}\alpha''$$
$$0 = \tfrac{1}{6}+\tfrac{1}{5}\alpha+\tfrac{1}{4}\alpha'+\tfrac{1}{3}\alpha''$$

unde nanciscimur $\alpha = -\frac{3}{2}$, $\alpha' = \frac{3}{5}$, $\alpha'' = -\frac{1}{20}$, adeoque $T = t^3-\frac{3}{2}tt+\frac{3}{5}t-\frac{1}{20}$. Ad determinandam U unica aequatio sufficit

$$0 = \tfrac{1}{5}+\tfrac{1}{3}\beta'$$

unde $\beta' = -\frac{3}{5}$ sive $U = u^3-\frac{3}{5}u$.

Attamen hunc modum, qui calculos continuo molestiores adducit, hic ulterius non persequemur, sed ad fontem genuinum solutionis generalis progrediemur.

17.

Proposita fractione continua

$$\varphi = \cfrac{v}{w + \cfrac{v'}{w' + \cfrac{v''}{w'' + \cfrac{v'''}{w''' + \text{etc.}}}}}$$

constat, fractiones continuo magis appropinquantes inveniri per algorithmum sequentem. Formentur duae quantitatum series, V, V', V'', V''' etc., W, W', W'', W''' etc. per hasce formulas

$$\begin{array}{ll} V = 0 & W = 1 \\ V' = v & W' = wW \\ V'' = w'V' + v'V & W'' = w'W' + v'W \\ V''' = w''V'' + v''V' & W''' = w''W'' + v''W' \\ V'''' = w'''V''' + v'''V'' & W'''' = w'''W''' + v'''W'' \end{array}$$

etc. eritque

$$\frac{V}{W} = 0$$

$$\frac{V'}{W'} = \frac{v}{w}$$

$$\frac{V''}{W''} = \cfrac{v}{w + \cfrac{v'}{w'}}$$

$$\frac{V'''}{W'''} = \cfrac{v}{w + \cfrac{v'}{w' + \cfrac{v''}{w''}}}$$

et sic porro. Praeterea constat, vel facile ex ipsis aequationibus praecedentibus confirmatur, esse

$$\begin{aligned} VW' - V'W &= -v \\ V'W'' - V''W' &= +vv' \\ V''W''' - V'''W'' &= -vv'v'' \\ V'''W'''' - V''''W''' &= +vv'v''v''' \end{aligned}$$

etc. Hinc perspicuum est, seriei

$$\frac{v}{WW'} - \frac{vv'}{W'W''} + \frac{vv'v''}{W''W'''} - \frac{vv'v''v'''}{W'''W''''} + \text{etc.}$$

terminum primum esse $= \frac{V'}{W'}$

summam duorum terminorum primorum $= \frac{V''}{W''}$

summam trium terminorum primorum $= \frac{V'''}{W'''}$

summam quatuor terminorum primorum $= \frac{V''''}{W''''}$

et sic porro; quocirca series ipsa vel in infinitum vel usque dum abrumpatur continuata ipsam fractionem continuam φ exprimet. Simul hinc habetur differentia inter φ atque singulas fractiones appropinquantes $\frac{V'}{W'}, \frac{V''}{W''}, \frac{V'''}{W'''}$ etc.

E formula 33 art. 14 *Disquisitionum generalium circa seriem infinitam* mutando t in $\frac{1}{u}$, facile obtinemus transformationem seriei

$$\varphi = u^{-1} + \tfrac{1}{3}u^{-3} + \tfrac{1}{5}u^{-5} + \tfrac{1}{7}u^{-7} + \text{ etc.}$$

in fractionem continuam sequentem

$$\cfrac{1}{u - \cfrac{\frac{1}{3}}{u - \cfrac{\frac{2.2}{3.5}}{u - \cfrac{\frac{3.3}{5.7}}{u - \cfrac{\frac{4.4}{7.9}}{u - \text{ etc.}}}}}}$$

ita ut habeatur

$$v = 1,\ v' = -\tfrac{1}{3},\ v'' = -\tfrac{4}{15},\ v''' = -\tfrac{9}{35},\ v'''' = -\tfrac{16}{63} \text{ etc.}$$
$$w = w' = w'' = w''' = w'''' \text{ etc.} = u.$$

Hinc pro V, V', V'', V''' etc. W, W', W'', W''' etc. nanciscimur valores sequentes

$V = 0,$ $\qquad$ $W = 1$

$V' = 1,$ $\qquad$ $W' = u$

$V'' = u,$ $\qquad$ $W'' = uu - \tfrac{1}{3}$

$V''' = uu - \tfrac{4}{15},$ $\qquad$ $W''' = u^3 - \tfrac{3}{5}u$

$V'''' = u^3 - \tfrac{11}{21}u,$ $\qquad$ $W'''' = u^4 - \tfrac{6}{7}uu + \tfrac{3}{35}$

$V^{\text{V}} = u^4 - \tfrac{7}{9}uu + \tfrac{64}{945},$ $\qquad$ $W^{\text{V}} = u^5 - \tfrac{10}{9}u^3 + \tfrac{5}{21}u$

$V^{\text{VI}} = u^5 - \tfrac{34}{33}u^3 + \tfrac{1}{5}u,$ $\qquad$ $W^{\text{VI}} = u^6 - \tfrac{15}{11}u^4 + \tfrac{5}{11}uu - \tfrac{5}{231}$

$V^{\text{VII}} = u^6 - \tfrac{50}{39}u^4 + \tfrac{283}{715}uu - \tfrac{256}{15015},$ $\qquad$ $W^{\text{VII}} = u^7 - \tfrac{21}{13}u^5 + \tfrac{105}{143}u^3 - \tfrac{35}{429}u$ etc.

Leviattentione adhibita elucet, singulas V, V', V'', V''' etc. W, W', W'', W''' etc. fieri functiones integras indeterminatae u; terminum altissimum in $V^{(m)}$ fieri u^{m-1}, potestatesque u^{m-2}, u^{m-4}, u^{m-6} etc. abesse; terminum altissimum vero in $W^{(m)}$ fieri u^m, atque abesse potestates u^{m-1}, u^{m-3}, u^{m-5} etc. Per ea autem, quae supra demonstravimus, erit

$$\varphi = \frac{1}{WW'} + \frac{1}{3W'W''} + \frac{2.2}{3.3.5W''W'''} + \frac{2.2.3.3}{3.3.5.5.7W'''W''''} + \frac{2.2.3.3.4.4}{3.3.5.5.7.7.9W''''W'''''} + \text{etc.}$$

ac proin generaliter

$$\varphi - \frac{V^{(m)}}{W^{(m)}} = \frac{2.2.3.3.....m.m}{3.3.5.5....(2m-1)(2m+1)W^{(m)}W^{(m+1)}}$$
$$+\frac{2.2.3.3.....(m+1)(m+1)}{3.3.5.5.....(2m+1)(2m+3)W^{(m+1)}W^{(m+2)}}$$
$$+ \text{etc.}$$

Si igitur $\varphi - \frac{V^{(m)}}{W^{(m)}}$ in seriem descendentem convertitur, eius terminus primus erit

$$= \frac{2.2.3.3....m.mu^{-(2m+1)}}{3.3.5.5....(2m-1)(2m+1)}$$

Productum vero $\varphi W^{(m)}$ compositum erit e functione integra $V^{(m)}$ atque serie infinita, cuius terminus primus

$$= \frac{2.2.3.3.....mmu^{-(m+1)}}{3.3.5.5.....(2m-1)(2m+1)}$$

Hinc igitur sponte inventa est functio U ordinis $n+1$; quae conditioni in art. praec. stabilitae satisfacit, scilicet ut productum φU liberum evadat a potestatibus $u^{-1}, u^{-2}, u^{-3}.....u^{-(n+1)}$. Scilicet non est alia quam $W^{(n+1)}$, simulque patet, U' aequalem fieri ipsi $V^{(m+1)}$, nec non terminum primum ipsius U'' esse

$$= \frac{2.2.3.3.....(n+1)(n+1)}{3.3.5.5.....(2n+1)(2n+3)} \cdot u^{-(n+2)}$$

Quodsi igitur pro $b, b', b''.....b^{(n)}$ accipiuntur radices aequationis $W^{(n+1)} = 0$, valoresque coëfficientium $R, R', R''.....R^{(n)}$ per praecepta supra tradita eruuntur, formula nostra integralis praecisione gaudebit ad ordinem $2n+1$ ascendente, eiusque correctio exprimetur proxime per

$$\frac{1}{2^{2n+2}} \cdot \frac{2.2.3.3.....(n+1)(n+1)}{3.3.5.5.....(2n+1)(2n+3)} L^{(2n+2)} = \frac{1.1.2.2.3.3.....(n+1)(n+1)}{2.6.6.10.10.14.....(4n+2)(4n+6)} L^{(2n+2)}$$

18.

Disquisitiones art. praec. functiones idoneas U pro singulis valoribus numeri n invenire quidem docent, sed successive tantum, dum a valoribus minoribus ad maiores transeundum est. Facile autem animadvertimus, has functiones generaliter exprimi per

$$u^{n+1}-\frac{(n+1)n}{2.(2n+1)}u^{n-1}+\frac{(n+1)n(n-1)(n-2)}{2.4(2n+1)(2n-1)}u^{n-3}-\frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{2.4.6.(2n+1)(2n-1)(2n-3)}u^{n-5}$$
$$+\text{ etc.}$$

sive etiam, si characteristica F ad normam commentationis supra citatae utimur, per

$$u^{n+1}F(-\tfrac{1}{2}n,\ -\tfrac{1}{2}(n+1),\ -(n+\tfrac{1}{2}),\ u^{-2})$$

Haecce inductio facile in demonstrationem rigorosam convertitur per methodum vulgo notam, aut, si ita videtur, adiumento formulae 19 in comment. cit. Functio U, si magis placet, etiam ordine terminorum inverso, exprimi potest per

$$\pm\frac{3.5.7\ldots.(n+1)}{(n+3)(n+5)\ldots.(2n+1)}\cdot uF(-\tfrac{1}{2}n,\ \tfrac{1}{2}(n+3),\ \tfrac{3}{2},\ uu)$$

pro n pari, valente signo superiori vel inferiori, prout $\frac{1}{2}n$ par est vel impar aut per

$$\pm\frac{1.3.5\ldots.n}{(n+2)(n+4)\ldots.(2n+1)}F(-\tfrac{1}{2}(n+1),\ \tfrac{1}{2}n+1,\ \tfrac{1}{2},\ uu)$$

pro n impari, valente signo superiori vel inferiori, prout $\frac{1}{2}(n+1)$ par est vel impar.

Functio U' expressionem generalem aeque simplicem non admittit: facile tamen ex ipsa genesi quantitatum V, V', V'' etc. colligitur, terminum ultimum ipsius U' pro n pari fieri

$$=\pm\frac{2.2.4.4.6.6\ldots..n.n}{3.5.7.9.11.13\ldots..(2n-1)(2n+1)}$$

signo superiori vel inferiori valente, prout $\frac{1}{2}n$ par est vel impar.

Functio $U''=\varphi W^{(n+1)}-V^{(n+1)}$, cuius terminum primum iam in art. praec. assignare docuimus, etiam per algorithmum recurrentem evolvi potest, quum manifesto generaliter habeatur

$$\begin{aligned}\varphi W''-V''&=w'(\varphi W'-V')+v'(\varphi W-V)\\ \varphi W'''-V'''&=w''(\varphi W''-V'')+v''(\varphi W'-V')\\ \varphi W''''-V''''&=w'''(\varphi W'''-V''')+v'''(\varphi W''-V'')\end{aligned}$$

etc. adeoque eo quem tractamus casu

$$\varphi W^{(m+2)} - V^{(m+2)} = u(\varphi W^{(m+1)} - V^{(m+1)}) - \frac{(m+1)^2}{(2m-1)(2m+1)}(\varphi W^{(m)} - V^{(m)})$$

Ita invenimus

$$\begin{aligned}
\varphi W - V &= u^{-1} + \tfrac{1}{3}u^{-3} + \tfrac{1}{5}u^{-5} + \tfrac{1}{7}u^{-7} + \text{etc.}\\
\varphi W' - V' &= \tfrac{1}{3}u^{-2} + \tfrac{1}{5}u^{-4} + \tfrac{1}{7}u^{-6} + \tfrac{1}{9}u^{-8} + \text{etc.}\\
\varphi W'' - V'' &= \tfrac{4}{45}u^{-3} + \tfrac{8}{105}u^{-5} + \tfrac{4}{63}u^{-7} + \tfrac{112}{2079}u^{-9} + \text{etc.}\\
\varphi W''' - V''' &= \tfrac{4}{175}u^{-4} + \tfrac{8}{315}u^{-6} + \tfrac{4}{165}u^{-8} + \tfrac{16}{715}u^{-10} + \text{etc.}
\end{aligned}$$

etc. quas series ita quoque exhibere licet

$$\begin{aligned}
\varphi W - V &= u^{-1}\left(1 + \frac{1.2}{2.3}u^{-2} + \frac{1.2.3.4}{2.4.3.5}u^{-4} + \frac{1.2.3.4.5.6}{2.4.6.3.5.7}u^{-6} + \text{etc.}\right)\\
\varphi W' - V' &= \tfrac{1}{3}u^{-2}\left(1 + \frac{2.3}{2.5}u^{-4} + \frac{2.3.4.5}{2.4.5.7}u^{-4} + \frac{2.3.4.5.6.7}{2.4.6.5.7.9}u^{-6} + \text{etc.}\right)\\
\varphi W'' - V'' &= \tfrac{4}{45}u^{-3}\left(1 + \frac{3.4}{2.7}u^{-2} + \frac{3.4.5.6}{2.4.7.9}u^{-4} + \frac{3.4.5.6.7.8}{2.4.6.7.9.11}u^{-6} + \text{etc.}\right)\\
\varphi W''' - V''' &= \tfrac{4}{175}u^{-4}\left(1 + \frac{4.5}{2.9}u^{-2} + \frac{4.5.6.7}{2.4.9.11}u^{-4} + \frac{4.5.6.7.8.9}{2.4.6.9.11.13}u^{-6} + \text{etc.}\right)
\end{aligned}$$

etc. Hanc inductionem sequentes habebimus generaliter

$$U'' = \varphi W^{(n+1)} - V^{(n+1)} \text{ aequalem producto ex } \frac{2.2.3.3.4.4\ldots\ldots(n+1).(n+1)}{3.3.5.5.7.7.9\ldots\ldots(2n+1)(2n+3)}u^{-(n+2)}$$

in seriem infinitam

$$1 + \frac{(n+2)(n+3)}{2(2n+5)}u^{-2} + \frac{(n+2)(n+3)(n+4)(n+5)}{2.4.(2n+5)(2n+7)}u^{-4} + \text{etc.}$$

aut si magis placet in $F(\frac{1}{2}n+1,\ \frac{1}{2}n+\frac{3}{2},\ n+\frac{5}{2},\ u^{-2})$. Haec quoque inductio facillime ad plenam certitudinem evehitur vel per methodum vulgo notam vel adiumento formulae 19 in commentatione saepius citatae.

19.

Quum sufficiat, functionum T, U alterutram nosse, posterioris determinationem tamquam simpliciorem praetulimus. Quae quemadmodum evolutioni seriei $u^{-1} + \frac{1}{3}u^{-3} + \frac{1}{5}u^{-5} +$ etc. in fractionem continuam innixa est, per ratiocinia similia ex evolutione seriei $t^{-1} + \frac{1}{2}t^{-2} + \frac{1}{3}t^{-3} + \frac{1}{4}t^{-4} +$ etc. in fractionem continuam

$$\cfrac{1}{t-\cfrac{\frac{1}{2}}{1-\cfrac{\frac{1}{6}}{t-\cfrac{\frac{2}{6}}{1-\cfrac{\frac{2}{10}}{t-\cfrac{\frac{3}{10}}{1-\text{etc.}}}}}}}$$

derivare potuissemus algorithmum ad determinandam functionem T pro valoribus successivis numeri n. Ad eandem vero conclusionem pervenimus perpendendo, T nihil aliud esse quam $\frac{U}{2^{n+1}}$ seu $\frac{W^{(n+1)}}{2^{n+1}}$, si pro u scribitur $2t-1$, quo pacto functiones successive pro T adoptandae habebuntur per algorithmum sequentem:

$$\begin{aligned}
W &= 1\\
\tfrac{1}{2}W' &= t-\tfrac{1}{2}\\
\tfrac{1}{4}W'' &= (t-\tfrac{1}{2}).\tfrac{1}{2}W' - \frac{1.1}{2.6}W = tt-t+\tfrac{1}{6}\\
\tfrac{1}{8}W''' &= (t-\tfrac{1}{2}).\tfrac{1}{4}W'' - \frac{2.2}{6.10}.\tfrac{1}{2}W' = t^3-\tfrac{3}{2}tt+\tfrac{3}{5}t-\tfrac{1}{20}\\
\tfrac{1}{16}W'''' &= (t-\tfrac{1}{2}).\tfrac{1}{8}W''' - \frac{3.3}{10.14}.\tfrac{1}{4}W'' = t^4-2t^3+\tfrac{9}{7}tt-\tfrac{2}{7}t+\tfrac{1}{70}
\end{aligned}$$

etc. Per inductionem hinc resultat generaliter

$$T = t^{n+1} - \frac{(n+1)^2}{1.(2n+2)}t^n + \frac{(n+1)^2.nn}{1.2.(2n+2)(2n+1)}t^{n-1} - \frac{(n+1)^2.nn.(n-1)^2}{1.2.3.(2n+2)(2n+1).2n}t^{n-2} + \text{etc.}$$

sive $T = t^{n+1}F(-(n+1),\ -(n+1),\ -2(n+1),\ t^{-1})$, cui inductioni facile est demonstrationis vim conciliare. Si magis arridet, T ordine terminorum inverso etiam per

$$\pm\frac{1.2.3.4\ldots\ldots(n+1)}{2.6.10.14\ldots\ldots(4n+2)}F(n+2,\ -(n+1),\ 1,\ t)$$

exprimi potest, ubi signum superius valet pro n impari, inferius pro pari. Simili denique modo generaliter T'' aequalis invenitur producto ex

$$\frac{1.1.2.2.3.3\ldots\ldots(n+1).(n+1)}{2.6.6.10.10.14\ldots\ldots(4n+2).(4n+6)}t^{-(n+2)}$$

in seriem infinitam

$$1+\frac{(n+2)^2}{1.(2n+4)}t^{-1}+\frac{(n+2)^2(n+3)^2}{1.2.(2n+4)(2n+5)}t^{-2}+\frac{(n+2)^2.(n+3)^2(n+4)^2}{1.2.3.(2n+4)(2n+5)(2n+6)}t^{-3}+\text{etc.}$$

sive in $F(n+2,\ n+2,\ 2n+4,\ t^{-1})$

20.

Quum in functione U potestates u^n, u^{n-2}, u^{n-4} etc. absint, e radicibus aequationis $U = 0$ binae semper erunt magnitudine aequales signis oppositae, quibus pro valore pari ipsius n adhuc associare oportet radicem singularem 0. Inventis radicibus, valores coëfficientium R, R', R'' etc. secundum methodum art. 11 habebuntur per functionem integram ipsius u, quae pro valore impari ipsius n erit formae

$$\gamma u^{n-1} + \gamma' u^{n-3} + \gamma'' u^{n-5} + \text{etc.}$$

pro valore pari autem, si excluditur coëfficiens radici $u = 0$ respondens, formae

$$\gamma u^{n-2} + \gamma' u^{n-4} + \gamma'' u^{n-6} + \text{etc.}$$

Exemplum art. 12 ipsam hanc reductionem exhibet pro $n = 6$. Manifesto igitur valoribus oppositis ipsius u semper respondent coëfficientes aequales. Ceterum in casu eo, ubi n est par, coëfficiens radici $u = 0$ respondens facile generaliter a priori assignari potest. Habebitur hic coëfficiens, si in $\frac{U'}{\left(\frac{dU}{du}\right)}$ substituitur $u = 0$. Valorem numeratoris U' pro $u = 0$ iam in art. 18 tradidimus, valor denominatoris autem ibinde erit

$$= \pm \frac{3 \cdot 5 \cdot 7 \ldots\ldots (n+1)}{(n+3)(n+5) \ldots\ldots (2n+1)} = \pm \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \ldots\ldots (n+1)(n+1)}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \ldots\ldots (2n+1)}$$

adeoque coëfficiens quaesitus

$$= \left(\frac{2 \cdot 4 \cdot 6 \cdot 8 \ldots\ldots n}{3 \cdot 5 \cdot 7 \cdot 9 \ldots\ldots (n+1)}\right)^2$$

21.

Functio integra ipsius u coëfficientes R, R', R'' etc. repraesentans in eo quem hic tractamus casu etiam independenter a methodo generali art. 11 erui potest sequenti modo. Differentiando aequationem

$$\varphi - \frac{U'}{U} = \frac{U''}{U}$$

substituendo dein $\frac{d\varphi}{du} = \frac{1}{1-uu}$, ac multiplicando per $UU(uu-1)$, obtinemus

$$(uu-1)\, U' \frac{dU}{du} - U\left(\frac{dU'}{du} \cdot (uu-1) + U\right) = (uu-1)\, UU \frac{d\left(\frac{U''}{U}\right)}{du}$$

Termini huius aequationis ad laevam manifesto constituunt functionem integram ipsius u: itaque necessario in parte ad dextram coëfficientes potestatum ipsius u cum exponentibus negativis sese destruere debent.

Sed $\frac{\mathrm{d}\frac{U''}{U}}{\mathrm{d}u}$ producit seriem infinitam incipientem a termino

$$-\left(\frac{1.2.3.4\ldots\ldots(n+1)}{1.3.5.7\ldots\ldots(2n+1)}\right)^2 u^{-(2n+4)}$$

qua igitur per $(uu-1)UU$ multiplicata nihil aliud prodire poterit nisi quantitas constans

$$-\left(\frac{1.2.3.4\ldots\ldots(n+1)}{1.3.5.7\ldots\ldots(2n+1)}\right)^2$$

Hinc colligimus*)

$$(uu-1)U'\frac{\mathrm{d}U}{\mathrm{d}u}+\left(\frac{1.2.3.4\ldots\ldots(n+1)}{1.3.5.7\ldots\ldots(2n+1)}\right)^2$$

divisibilem esse per U, quamobrem functioni fractae $\frac{U'}{\left(\frac{\mathrm{d}U}{\mathrm{d}u}\right)}$, quae coëfficientes R, R', R'' etc. suggerit, aequivalebit functio integra

$$-\left(\frac{1.3.5.7\ldots\ldots(2n+1)}{1.2.3.4\ldots\ldots(n+1)}U'\right)^2.(uu-1)$$

Loco huius functionis, quae est ordinis $2n+2$, manifestoque solas potestates pares ipsius u implicat, adoptari poterit residuum ex eius divisione per U ortum, quod erit ordinis n, seu $n-1$, prout n par est seu impar. Si vero in casu priori coëfficientem eum, qui respondet radici $u=0$, excludere malumus, loco illius functionis eius residuum ex divisione per $\frac{U}{u}$ ortum adoptabimus, quod tantummodo ad ordinem $n-2$ ascendet.

22.

Ut praesto sint, quae ad applicationem methodi hucusque expositae requiruntur. adiungere visum est, pro valoribus successivis numeri n, valores numericos tum quantitatum a, a', a'' etc., tum coëfficientium R, R'. R'' etc. ad sedecim figuras computatos, una cum horum logarithmis ad decem figuras.

*) Simul hinc petitur demonstratio, quod U cum $\frac{\mathrm{d}U}{\mathrm{d}u}$ divisorem indeterminatum communem habere nequit, neque adeo aequatio $U=0$ radices *aequales*.

I. *Terminus unus*, $n = 0$.

$U = u$, $U' = 1$, $T = t - \frac{1}{2}$, $T' = 1$.

$a = 0,5$

$R = 1$

Correctio formulae integralis proxime $= \frac{1}{12} L''$.

II. *Termini duo*, $n = 1$.

$U = uu - \frac{1}{2}$, $U' = u$

$T = tt - t + \frac{1}{6}$, $T' = t - \frac{1}{2}$

$a = 0,2113248654\ 051871$

$a' = 0,7886751345\ 948129$

$R = R' = \frac{1}{2}$

Correctio proxime $= \frac{1}{180} L''''$

III. *Termini tres*, $n = 2$.

$U = u^3 - \frac{3}{5} u$, $U' = uu - \frac{4}{15}$

$T = t^3 - \frac{3}{2} tt + \frac{3}{5} t - \frac{1}{20}$, $T' = tt - t + \frac{11}{60}$

$a = 0,1127016653\ 792583$

$a' = 0,5$

$a'' = 0,8872983346\ 207417$

$R = R'' = \frac{5}{18}$

$R' = \frac{4}{9}$

Correctio proxime $= \frac{1}{2800} L^{\text{VI}}$.

IV. *Termini quatuor*, $n = 3$.

$U = u^4 - \frac{6}{7} uu + \frac{3}{35}$

$U' = u^3 - \frac{11}{21} u$

$T = t^4 - 2t^3 + \frac{9}{7} tt - \frac{2}{7} t + \frac{1}{70}$

$T' = t^3 - \frac{3}{2} tt + \frac{13}{21} t - \frac{5}{84}$

$a = 0,0694318442\ 029754$

$a' = 0,3300094782\ 075677$

$a'' = 0,6699905217\ 924323$

$a''' = 0,9305681557\ 970246$

$R = R''' = 0,1739274225\ 687284$ log. $= 9,2403680612$

$R' = R'' = 0,3260725774\ 312716$ log. $= 9,5133142764$

Horum coëfficientium expressio generalis $-\frac{35}{144} uu + \frac{17}{48}$

Correctio proxime $= \frac{1}{44100} L^{\text{VIII}}$

V. *Termini quinque*, $n = 4$.

$U = u^5 - \frac{10}{9} u^3 + \frac{5}{21} u$

$U' = u^4 - \frac{7}{9} uu + \frac{64}{945}$

$T = t^5 - \frac{5}{2} t^4 + \frac{20}{9} t^3 - \frac{5}{6} tt + \frac{5}{42} t - \frac{1}{252}$

$T' = t^4 - 2t^3 + \frac{47}{36} tt - \frac{11}{36} t + \frac{137}{75600}$

$a = 0,0469100770\ 306680$

$a' = 0,2307653449\ 471585$

$a'' = 0,5$

$a''' = 0,7692346550\ 528415$

$a'''' = 0,9530899229\ 693320$

$R = R'''' = 0,1184634425\ 280945$ log. $= 9,0735843490$

$R' = R''' = 0,2393143352\ 496832$ $\quad 9,3789687142$

$R'' = \frac{64}{225} = 0,2844444444\ 444444$ $\quad 9,4539974559$

Expressio generalis horum coëfficientium, excluso R'',

$$-\frac{91}{400} uu + \frac{1099}{3600}$$

Correctio proxime $= \frac{1}{698544} L^{\mathrm{X}}$

VI. *Termini sex*, $n = 5$.

$U = u^6 - \frac{15}{11} u^4 + \frac{5}{11} uu - \frac{5}{231}$

$U' = u^5 - \frac{34}{33} u^3 + \frac{1}{5} u$

$T = t^6 - 3t^5 + \frac{75}{22} t^4 - \frac{20}{11} t^3 + \frac{5}{11} tt - \frac{1}{22} t + \frac{1}{924}$

$T' = t^5 - \frac{5}{2} t^4 + \frac{74}{33} t^3 - \frac{19}{22} tt + \frac{29}{220} t - \frac{7}{1320}$

$a = 0,0337652428\ 984240$

$a' = 0,1693953067\ 668678$

$a'' = 0,3806904069\ 584015$

$a''' = 0,6193095930\ 415985$

$a'''' = 0,8306046932\ 331322$

$a''''' = 0,9662347571\ 015760$

$R = R''''' = 0,0856622461\ 895852$ log. $= 8,9327894580$

$R' = R'''' = 0,1803807865\ 240693$ $\quad 9,2561902763$

$R'' = R''' = 0,2339569672\ 863455$ $\quad 9,3691359831$

Coëfficientium expressio generalis

$$-\frac{77}{800} u^4 - \frac{7}{75} uu + \frac{23}{96}$$

Correctio proxime $= \frac{1}{11099088} L^{\mathrm{XII}}$

VII. *Termini septem*, $n = 6$.

$$U = u^7 - \tfrac{21}{13}u^5 + \tfrac{105}{143}u^3 - \tfrac{35}{429}u$$

$$U' = u^6 - \tfrac{50}{39}u^4 + \tfrac{283}{713}uu - \tfrac{256}{15015}$$

$$T = t^7 - \tfrac{7}{2}t^6 + \tfrac{63}{13}t^5 - \tfrac{175}{52}t^4 + \tfrac{175}{143}t^3 - \tfrac{63}{286}tt + \tfrac{7}{429}t - \tfrac{1}{3432}$$

$$T' = t^6 - 3t^5 + \tfrac{535}{156}t^4 - \tfrac{145}{78}t^3 + \tfrac{1377}{2860}tt - \tfrac{223}{4290}t + \tfrac{323}{240240}$$

$a = 0,0254460438\ 286202$

$a' = 0,1292344072\ 003028$

$a'' = 0,2970774243\ 113015$

$a''' = 0,5$

$a'''' = 0,7029225756\ 886985$

$a''''' = 0,8707655927\ 996972$

$a'''''' = 0,9745539561\ 713798$

$R = R'''''' = 0,0647424830\ 844348$ log. $= 8,8111893529$

$R' = R''''' = 0,1398526957\ 446384$ $\quad 9,1456708421$

$R'' = R'''' = 0,1909150252\ 525595$ $\quad 9,2808401093$

$R''' = \frac{256}{1225} = 0,2089795918\ 367347$ $\quad 9,3201038766$

Horum coëfficientium, R''' excluso, expressio generalis

$$-\tfrac{1859}{16800}u^4 - \tfrac{1573}{29400}uu + \tfrac{7947}{39200}$$

Correctio proxime $= \frac{1}{176679360}L^{\text{XIV}}$

23.

Coronidis loco methodi nostrae efficaciam ob oculos ponemus computando valorem integralis

$$\int \frac{dx}{\log x}$$

ab $x = 100000$ usque ad $x = 200000$.

I. Ex termino uno habemus $\Delta RA = 8390,394608$

II. Ex terminis duobus fit . . .
$\Delta RA = 4271,810097$
$\Delta R'A' = 4134,144502$
Summa $= 8405,954599$

III. Ex terminis tribus . . . . . .
$\Delta RA = 2390.572772$
$\Delta R'A' = 3729,064270$
$\Delta R''A'' = 2286,599733$
Summa $= 8406,236775$

IV. Ex terminis quatuor.....

$$\begin{cases} \Delta RA &= 1501{,}957053 \\ \Delta R'A' &= 2763{,}769240 \\ \Delta R''A'' &= 2711{,}454637 \\ \Delta R'''A''' &= 1429{,}062040 \\ \hline \text{Summa} &= 8406{,}242970 \end{cases}$$

V. Ex terminis quinque .....

$$\begin{cases} \Delta RA &= 1024{,}879445 \\ \Delta R'A' &= 2041{,}833335 \\ \Delta R''A'' &= 2386{,}601133 \\ \Delta R'''A''' &= 1980{,}509616 \\ \Delta R''''A'''' &= 972{,}419588 \\ \hline \text{Summa} &= 8406{,}243117 \end{cases}$$

VI. Ex terminis sex ....

$$\begin{cases} \Delta RA &= 741{,}912854 \\ \Delta R'A' &= 1545{,}757256 \\ \Delta R''A'' &= 1976{,}737668 \\ \Delta R'''A''' &= 1950{,}466223 \\ \Delta R''''A'''' &= 1488{,}588550 \\ \Delta R^{\text{v}}A^{\text{v}} &= 702{,}780570 \\ \hline \text{Summa} &= 8406{,}243121 \end{cases}$$

VII. Ex terminis septem.....

$$\begin{cases} \Delta RA &= 561{,}1213804 \\ \Delta R'A' &= 1202{,}0551998 \\ \Delta R''A'' &= 1621{,}6290819 \\ \Delta R'''A''' &= 1753{,}4212406 \\ \Delta R''''A'''' &= 1584{,}9790252 \\ \Delta R^{\text{v}}A^{\text{v}} &= 1152{,}0681116 \\ \Delta R^{\text{vi}}A^{\text{vi}} &= 530{,}9690816 \\ \hline \text{Summa} &= 8406{,}2431211 \end{cases}$$

E calculis clar. Bessel valor eiusdem integralis inventus est $= 8406{,}24312$.