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Editorial

This volume contains the proceedings of the international conference "Advances in Group Theory and Applications 2011", held in Porto Cesareo in June 2011. This was the third in a series of such conferences. The detailed course notes and individual research papers, which the authors have so generously and carefully edited, provide a background and stimulus that will promote further efforts to resolve the many challenging questions raised.

In this regard, prompted by the success of the earlier volumes, the Scientific Committee decided to continue the conference format – "Lecture courses" complemented by individual "Research presentations". We express our sincerest thanks to all who attended the conference and trust that these proceedings will serve also as a special memento of an enjoyable meeting.

Without special financial support it would not be possible to undertake the many organizational tasks. We gratefully acknowledge the contribution made by "Università del Salento", "Università di Napoli Federico II", "Seconda Università di Napoli", "Università di Salerno", "Università dell'Aquila" and G.N.S.A.G.A. of INdAM.

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Discrete dynamical systems in group theory

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Abstract. In this expository paper we describe the unifying approach for many known entropies in Mathematics developed in [27].

First we give the notion of semigroup entropy $h_{\mathfrak{S}} : \mathfrak{S} \to \mathbb{R}_+$ in the category $\mathfrak{S}$ of normed semigroups and contractive homomorphisms, recalling also its properties from [27]. For a specific category $\mathfrak{X}$ and a functor $F : \mathfrak{X} \to \mathfrak{S}$ we have the entropy h_F, defined by the composition $h_F = h_{\mathfrak{S}} \circ F$, which automatically satisfies the same properties proved for $h_{\mathfrak{S}}$. This general scheme permits to obtain many of the known entropies as h_F, for appropriately chosen categories $\mathfrak{X}$ and functors $F : \mathfrak{X} \to \mathfrak{S}$.

In the last part we recall the definition and the fundamental properties of the algebraic entropy for group endomorphisms, noting how its deeper properties depend on the specific setting. Finally we discuss the notion of growth for flows of groups, comparing it with the classical notion of growth for finitely generated groups.

Keywords: entropy, normed semigroup, semigroup entropy, bridge theorem, algebraic entropy, growth, Milnor Problem.

MSC 2000 classification: 16B50,20M15,20K30,20F65,22D35, 37A35,37B40,54C70,55U30

1 Introduction

This paper covers the series of three talks given by the first named author at the conference "Advances in Group Theory and Applications 2011" held in June, 2011 in Porto Cesareo. It is a survey about entropy in Mathematics, the approach is the categorical one adopted in [27] (and announced in [16], see also [13]).

We start recalling that a *flow* in a category $\mathfrak{X}$ is a pair (X, ϕ), where X is an object of $\mathfrak{X}$ and $\phi : X \to X$ is a morphism in $\mathfrak{X}$. A morphism between two flows $\phi : X \to X$ and $\psi : Y \to Y$ is a morphism $\alpha : X \to Y$ in $\mathfrak{X}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ X \arrow[r, "\alpha"] & Y. \end{tikzcd}$$

commutes. This defines the category $\mathbf{Flow}_{\mathfrak{X}}$ of flows in $\mathfrak{X}$.

To classify flows in $\mathfrak{X}$ up to isomorphisms one uses invariants, and entropy is roughly a numerical invariant associated to flows. Indeed, letting $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$ and $\mathbb{R}_+ = \mathbb{R}_{\geq 0} \cup \{\infty\}$, by the term *entropy* we intend a function

$$h : \mathbf{Flow}_{\mathfrak{X}} \to \mathbb{R}_+, \tag{1}$$

obeying the invariance law $h(\phi) = h(\psi)$ whenever (X, ϕ) and (Y, ψ) are isomorphic flows. The value $h(\phi)$ is supposed to measure the degree to which X is "scrambled" by ϕ, so for example an entropy should assign 0 to all identity maps. For simplicity and with some abuse of notations, we adopt the following **Convention.** If $\mathfrak{X}$ is a category and h an entropy of $\mathfrak{X}$, writing $h : \mathfrak{X} \to \mathbb{R}_+$ we always mean $h : \mathbf{Flow}_{\mathfrak{X}} \to \mathbb{R}_+$ as in (1).

The first notion of entropy in Mathematics was the measure entropy h_{mes} introduced by Kolmogorov [53] and Sinai [74] in 1958 in Ergodic Theory. The topological entropy h_{top} for continuous self-maps of compact spaces was defined by Adler, Konheim and McAndrew [1] in 1965. Another notion of topological entropy h_B for uniformly continuous self-maps of metric spaces was given later by Bowen [11] (it coincides with h_{top} on compact metric spaces). Finally, entropy was taken also in Algebraic Dynamics by Adler, Konheim and McAndrew [1] in 1965 and Weiss [85] in 1974; they defined an entropy ent for endomorphisms of torsion abelian groups. Then Peters [64] in 1979 introduced its extension h_{alg} to automorphisms of abelian groups; finally h_{alg} was defined in [19] and [20] for any group endomorphism. Recently also a notion of algebraic entropy for module endomorphisms was introduced in [70], namely the algebraic i-entropy ent_i, where i is an invariant of a module category. Moreover, the adjoint algebraic entropy $\mathrm{ent}^\star$ for group endomorphisms was investigated in [26] (and its topological extension in [35]). Finally, one can find in [5] and [20] two "mutually dual" notions of entropy for self-maps of sets, namely the covariant set-theoretic entropy $\mathfrak{h}$ and the contravariant set-theoretic entropy $\mathfrak{h}^*$.

The above mentioned specific entropies determined the choice of the main cases considered in this paper. Namely, $\mathfrak{X}$ will be one of the following categories (other examples can be found in §§2.5 and 3.6):

(a) **Set** of sets and maps and its non-full subcategory $\mathbf{Set}_{\mathrm{fin}}$ of sets and finite-to-one maps (set-theoretic entropies $\mathfrak{h}$ and $\mathfrak{h}^*$ respectively);

(b) **CTop** of compact topological spaces and continuous maps (topological entropy h_{top});

(c) **Mes** of probability measure spaces and measure preserving maps (measure entropy h_{mes});

(d) **Grp** of groups and group homomorphisms and its subcategory **AbGrp** of abelian groups (algebraic entropy ent, algebraic entropy h_{alg} and adjoint algebraic entropy $\mathrm{ent}^\star$);

(e) $\mathbf{Mod}_R$ of right modules over a ring R and R-module homomorphisms (algebraic i-entropy ent_i).

Each of these entropies has its specific definition, usually given by limits computed on some "trajectories" and by taking the supremum of these quantities (we will see some of them explicitly). The proofs of the basic properties take into account the particular features of the specific categories in each case too. It appears that all these definitions and basic properties share a lot of common features. The aim of our approach is to unify them in some way, starting from a general notion of entropy of an appropriate category. This will be the semigroup entropy $h_{\mathfrak{S}}$ defined on the category $\mathfrak{S}$ of normed semigroups.

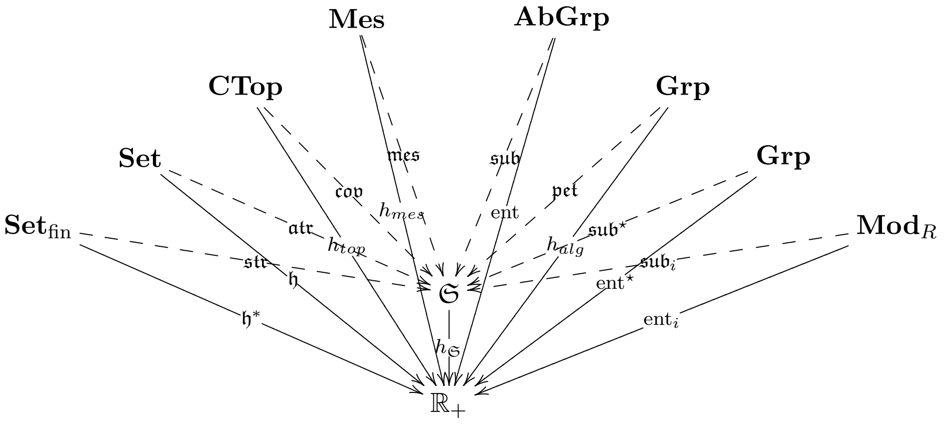
In Section 2 we first introduce the category $\mathfrak{S}$ of normed semigroups and related basic notions and examples mostly coming from [27]. Moreover, in §2.2 (which can be avoided at a first reading) we add a preorder to the semigroup and discuss the possible behavior of a semigroup norm with respect to this preorder. Here we include also the subcategory $\mathfrak{L}$ of $\mathfrak{S}$ of normed semilattices, as the functors given in Section 3 often have as a target actually a normed semilattice.

In §2.3 we define explicitly the semigroup entropy $h_{\mathfrak{S}} : \mathfrak{S} \to \mathbb{R}_+$ on the category $\mathfrak{S}$ of normed semigroups. Moreover we list all its basic properties, clearly inspired by those of the known entropies, such as Monotonicity for factors, Invariance under conjugation, Invariance under inversion, Logarithmic Law, Monotonicity for subsemigroups, Continuity for direct limits, Weak Addition Theorem and Bernoulli normalization.

Once defined the semigroup entropy $h_{\mathfrak{S}} : \mathfrak{S} \to \mathbb{R}_+$, our aim is to obtain all known entropies $h : \mathfrak{X} \to \mathbb{R}_+$ as a composition $h_F := h_{\mathfrak{S}} \circ F$ of a functor $F : \mathfrak{X} \to \mathfrak{S}$ and $h_{\mathfrak{S}}$:

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "h=h_F"] \arrow[d, "F"'] & & \mathbb{R}_+ \\ \mathfrak{S} \arrow[urr, "h_{\mathfrak{S}}"'] & & \end{tikzcd}$$

This is done explicitly in Section 3, where all specific entropies listed above are obtained in this scheme. We dedicate to each of them a subsection, each time giving explicitly the functor from the considered category to the category of normed semigroups. More details and complete proofs can be found in [27]. These functors and the entropies are summarized by the following diagram:



In this way we obtain a simultaneous and uniform definition of all entropies and uniform proofs (as well as a better understanding) of their general properties, namely the basic properties of the specific entropies can be derived directly from those proved for the semigroup entropy.

The last part of Section 3 is dedicated to what we call Bridge Theorem (a term coined by L. Salce), that is roughly speaking a connection between entropies $h_1 : \mathfrak{X}_1 \to \mathbb{R}_+$ and $h_2 : \mathfrak{X}_2 \to \mathbb{R}_+$ via functors $\varepsilon : \mathfrak{X}_1 \to \mathfrak{X}_2$. Here is a formal definition of this concept:

Definition 1. Let $\varepsilon : \mathfrak{X}_1 \to \mathfrak{X}_2$ be a functor and let $h_1 : \mathfrak{X}_1 \to \mathbb{R}_+$ and $h_2 : \mathfrak{X}_2 \to \mathbb{R}_+$ be entropies of the categories $\mathfrak{X}_1$ and $\mathfrak{X}_2$, respectively (as in the diagram below).

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[dr, "h_1"] \arrow[d, "\varepsilon"'] & \\ \mathfrak{X}_2 \arrow[r, "h_2"'] & \mathbb{R}_+ \end{tikzcd}$$

We say that the pair (h_1, h_2) satisfies the *weak Bridge Theorem* with respect to the functor ε if there exists a positive constant C_ε, such that for every endomorphism ϕ in $\mathfrak{X}_1$

$$h_2(\varepsilon(\phi)) \leq C_\varepsilon h_1(\phi). \tag{2}$$

If equality holds in (2) we say that (h_1, h_2) satisfies the *Bridge Theorem* with respect to ε, and we shortly denote this by (BT_ε).

In §3.10 we discuss the Bridge Theorem passing through the category $\mathfrak{S}$ of normed semigroups and so using the new semigroup entropy. This approach permits for example to find a new and transparent proof of Weiss Bridge Theorem (see Theorem 6) as well as for other Bridge Theorems.

A first limit of this very general setting is the loss of some of the deeper properties that a specific entropy may have. So in the last Section 4 for the algebraic entropy we recall the definition and the fundamental properties, which cannot be deduced from the general scheme.

We start Section 4 recalling the Algebraic Yuzvinski Formula (see Theorem 9) recently proved in [37], giving the values of the algebraic entropy of linear transformations of finite-dimensional rational vector spaces in terms of the Mahler measure. In particular, this theorem provides a connection of the algebraic entropy with the famous Lehmer Problem. Two important applications of the Algebraic Yuzvinski Formula are the Addition Theorem and the Uniqueness Theorem for the algebraic entropy in the context of abelian groups.

In §4.3 we describe the connection of the algebraic entropy with the classical topic of growth of finitely generated groups in Geometric Group Theory. Its definition was given independently by Schwarzc [72] and Milnor [56], and after the publication of [56] it was intensively investigated; several fundamental results were obtained by Wolf [89], Milnor [57], Bass [6], Tits [76] and Adyan [2]. In [58] Milnor proposed his famous problem (see Problem 1 below); the question about the existence of finitely generated groups with intermediate growth was answered positively by Grigorchuk in [42, 43, 44, 45], while the characterization of finitely generated groups with polynomial growth was given by Gromov in [47] (see Theorem 12).

Here we introduce the notion of finitely generated flows (G, ϕ) in the category of groups and define the growth of (G, ϕ). When $\phi = \mathrm{id}_G$ is the identical endomorphism, then G is a finitely generated group and we find exactly the classical notion of growth. In particular we recall a recent significant result from [22] extending Milnor's dichotomy (between polynomial and exponential growth) to finitely generated flows in the abelian case (see Theorem 13). We leave also several open problems and questions about the growth of finitely generated flows of groups.

The last part of the section, namely §4.4, is dedicated to the adjoint algebraic entropy. As for the algebraic entropy, we recall its original definition and its main properties, which cannot be derived from the general scheme. In particular, the adjoint algebraic entropy can take only the values 0 and ∞ (no finite positive value is attained) and we see that the Addition Theorem holds only restricting to bounded abelian groups.

A natural side-effect of the wealth of nice properties of the entropy $h_F =$

$h_{\mathfrak{S}} \circ F$, obtained from the semigroup entropy $h_{\mathfrak{S}}$ through functors $F : \mathfrak{X} \to \mathfrak{S}$, is the loss of some entropies that do not have all these properties. For example Bowen's entropy h_B cannot be obtained as h_F since $h_B(\phi^{-1}) = h_B(\phi)$ fails even for the automorphism $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = 2x$, see §3.6 for an extended comment on this issue; there we also discuss the possibility to obtain Bowen's topological entropy of measure preserving topological automorphisms of locally compact groups in the framework of our approach. For the same reason other entropies that cannot be covered by this approach are the intrinsic entropy for endomorphisms of abelian groups [25] and the topological entropy for automorphisms of locally compact totally disconnected groups [17]. This occurs also for the function $\phi \mapsto \log s(\phi)$, where $s(\phi)$ is the scale function defined by Willis [86, 87]. The question about the relation of the scale function to the algebraic or topological entropy was posed by T. Weigel at the conference; these non-trivial relations are discussed for the topological entropy in [8].

2 The semigroup entropy

2.1 The category $\mathfrak{S}$ of normed semigroups

We start this section introducing the category $\mathfrak{S}$ of normed semigroups, and other notions that are fundamental in this paper.

Definition 2. Let $(S, \cdot)$ be a semigroup.

(i) A *norm* on S is a map $v : S \to \mathbb{R}_{\geq 0}$ such that

$$v(x \cdot y) \leq v(x) + v(y) \text{ for every } x, y \in S.$$

A *normed semigroup* is a semigroup provided with a norm.

If S is a monoid, a *monoid norm* on S is a semigroup norm v such that $v(1) = 0$; in such a case S is called *normed monoid.*

(ii) A semigroup homomorphism $\phi : (S, v) \to (S', v')$ between normed semigroups is *contractive* if

$$v'(\phi(x)) \leq v(x) \text{ for every } x \in S.$$

Let $\mathfrak{S}$ be the category of normed semigroups, which has as morphisms all contractive semigroup homomorphisms. In this paper, when we say that S is a normed semigroup and $\phi : S \to S$ is an endomorphism, we will always mean that ϕ is a contractive semigroup endomorphism. Moreover, let $\mathfrak{M}$ be the non-full subcategory of $\mathfrak{S}$ with objects all normed monoids, where the morphisms are all (necessarily contractive) monoid homomorphisms.

We give now some other definitions.

Definition 3. A normed semigroup (S, v) is:

(i) *bounded* if there exists $C \in \mathbb{N}_+$ such that $v(x) \leq C$ for all $x \in S$;

(ii) *arithmetic* if for every $x \in S$ there exists a constant $C_x \in \mathbb{N}_+$ such that $v(x^n) \leq C_x \cdot \log(n+1)$ for every $n \in \mathbb{N}$.

Obviously, bounded semigroups are arithmetic.

Example 1. Consider the monoid $S = (\mathbb{N}, +)$.

(a) Norms v on S correspond to subadditive sequences $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ (i.e., $a_{n+m} \leq a_n + a_m$) via $v \mapsto (v(n))_{n\in\mathbb{N}}$. Then $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$ exists by Fekete Lemma [33].

(b) Define $v : S \to \mathbb{R}_+$ by $v(x) = \log(1+x)$ for $x \in S$. Then v is an arithmetic semigroup norm.

(c) Define $v_1 : S \to \mathbb{R}_+$ by $v_1(x) = \sqrt{x}$ for $x \in S$. Then v_1 is a semigroup norm, but $(S, +, v_1)$ is not arithmetic.

(d) For $a \in \mathbb{N}$, $a > 1$ let $v_a(n) = \sum_i b_i$, when $n = \sum_{i=0}^{k} b_i a^i$ and $0 \leq b_i < a$ for all i. Then v_a is an arithmetic norm on S making the map $x \mapsto ax$ an endomorphism in $\mathfrak{S}$.

2.2 Preordered semigroups and normed semilattices

A triple $(S, \cdot, \leq)$ is a *preordered semigroup* if the semigroup $(S, \cdot)$ admits a preorder $\leq$ such that

$$x \leq y \text{ implies } x \cdot z \leq y \cdot z \text{ and } z \cdot x \leq z \cdot y \text{ for all } x, y, z \in S.$$

Write $x \sim y$ when $x \leq y$ and $y \leq x$ hold simultaneously. Moreover, the *positive cone* of S is

$$P_+(S) = \{a \in S : x \leq x \cdot a \text{ and } x \leq a \cdot x \text{ for every } x \in S\}.$$

A norm v on the preordered semigroup $(S, \cdot, \leq)$ is *monotone* if $x \leq y$ implies $v(x) \leq v(y)$ for every $x, y \in S$. Clearly, $v(x) = v(y)$ whenever $x \sim y$ and the norm v of S is monotone.

Now we propose another notion of monotonicity for a semigroup norm which does not require the semigroup to be explicitly endowed with a preorder.

Definition 4. Let (S, v) be a normed semigroup. The norm v is *s-monotone* if

$$\max\{v(x), v(y)\} \leq v(x \cdot y) \text{ for every } x, y \in S.$$

This inequality may become a too stringent condition when S is close to be a group; indeed, if S is a group, then it implies that $v(S) = \{v(1)\}$, in particular v is constant.

If $(S, +, v)$ is a commutative normed monoid, it admits a preorder $\leq^a$ defined for every $x, y \in S$ by $x \leq^a y$ if and only if there exists $z \in S$ such that $x + z = y$. Then $(S, \cdot, \leq)$ is a preordered semigroup and the norm v is s-monotone if and only if v is monotone with respect to $\leq^a$.

The following connection between monotonicity and s-monotonicity is clear.

Lemma 1. *Let S be a preordered semigroup. If $S = P_+(S)$, then every monotone norm of S is also s-monotone.*

A *semilattice* is a commutative semigroup $(S, \vee)$ such that $x \vee x = x$ for every $x \in S$.

Example 2. (a) Each lattice $(L, \vee, \wedge)$ gives rise to two semilattices, namely $(L, \vee)$ and $(L, \wedge)$.

(b) A filter $\mathcal{F}$ on a given set X is a semilattice with respect to the intersection, with zero element the set X.

Let SL be the full subcategory of $\mathfrak{S}$ with objects all normed semilattices.

Every normed semilattice $(L, \vee)$ is trivially arithmetic, moreover the canonical partial order defined by

$$x \leq y \text{ if and only if } x \vee y = y,$$

for every $x, y \in L$, makes L also a partially ordered semigroup.

Neither preordered semigroups nor normed semilattices are formally needed for the definition of the semigroup entropy. Nevertheless, they provide significant and natural examples, as well as useful tools in the proofs, to justify our attention to this topic.

2.3 Entropy in $\mathfrak{S}$

For (S, v) a normed semigroup $\phi : S \to S$ an endomorphism, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th ϕ-trajectory of x*

$$T_n(\phi, x) = x \cdot \phi(x) \cdot \ldots \cdot \phi^{n-1}(x)$$

and let

$$c_n(\phi, x) = v(T_n(\phi, x)).$$

Note that $c_n(\phi, x) \leq n \cdot v(x)$. Hence the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Definition 5. Let S be a normed semigroup. An endomorphism $\phi : S \to S$ is said to have *logarithmic growth*, if for every $x \in S$ there exists $C_x \in \mathbb{N}_+$ with $c_n(\phi, x) \leq C_x \cdot \log(n+1)$ for all $n \in \mathbb{N}_+$.

Obviously, a normed semigroup S is arithmetic if and only if id_S has logarithmic growth.

The following theorem from [27] is fundamental in this context as it witnesses the existence of the semigroup entropy; so we give its proof also here for reader's convenience.

Theorem 1. *Let S be a normed semigroup and $\phi : S \to S$ an endomorphism. Then for every $x \in S$ the limit*

$$h_{\mathfrak{S}}(\phi, x) := \lim_{n\to\infty} \frac{c_n(\phi, x)}{n} \tag{3}$$

exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

Proof. The sequence $(c_n(\phi, x))_{n\in\mathbb{N}_+}$ is subadditive. Indeed,

$$\begin{aligned} c_{n+m}(\phi, x) &= v(x \cdot \phi(x) \cdot \ldots \cdot \phi^{n-1}(x) \cdot \phi^n(x) \cdot \ldots \cdot \phi^{n+m-1}(x)) \\ &= v((x \cdot \phi(x) \cdot \ldots \cdot \phi^{n-1}(x)) \cdot \phi^n(x \cdot \ldots \cdot \phi^{m-1}(x))) \\ &\leq c_n(\phi, x) + v(\phi^n(x \cdot \ldots \cdot \phi^{m-1}(x))) \\ &\leq c_n(\phi, x) + v(x \cdot \ldots \cdot \phi^{m-1}(x)) = c_n(\phi, x) + c_m(\phi, x). \end{aligned}$$

By Fekete Lemma (see Example 1 (a)), the limit $\lim_{n\to\infty} \frac{c_n(\phi,x)}{n}$ exists and coincides with $\inf_{n\in\mathbb{N}_+} \frac{c_n(\phi,x)}{n}$. Finally, $h_{\mathfrak{S}}(\phi, x) \leq v(x)$ follows from $c_n(\phi, x) \leq nv(x)$ for every $n \in \mathbb{N}_+$. $\boxed{QED}$

Remark 1. (a) The proof of the existence of the limit defining $h_{\mathfrak{S}}(\phi, x)$ exploits the property of the semigroup norm and also the condition on ϕ to be contractive. For an extended comment on what can be done in case the function $v : S \to \mathbb{R}_+$ fails to have that property see §2.5.

(b) With $S = (\mathbb{N}, +)$, $\phi = \mathrm{id}_{\mathbb{N}}$ and $x = 1$ in Theorem 1 we obtain exactly item (a) of Example 1.

Definition 6. Let S be a normed semigroup and $\phi : S \to S$ an endomorphism. The *semigroup entropy* of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x\in S} h_{\mathfrak{S}}(\phi, x).$$

If an endomorphism $\phi : S \to S$ has logarithmic growth, then $h_{\mathfrak{S}}(\phi) = 0$. In particular, $h_{\mathfrak{S}}(\mathrm{id}_S) = 0$ if S is arithmetic.

Recall that an endomorphism $\phi : S \to S$ of a normed semigroup S is *locally quasi periodic* if for every $x \in S$ there exist $n, k \in \mathbb{N}$, $k > 0$, such that $\phi^n(x) = \phi^{n+k}(x)$. If S is a monoid and $\phi(1) = 1$, then ϕ is *locally nilpotent* if for every $x \in S$ there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) = 1$.

Lemma 2. *Let S be a normed semigroup and $\phi : S \to S$ an endomorphism.*

(a) If S is arithmetic and ϕ is locally periodic, then $h_{\mathfrak{S}}(\phi) = 0$.

(b) If S is a monoid and $\phi(1) = 1$ and ϕ is locally nilpotent, then $h_{\mathfrak{S}}(\phi) = 0$.

Proof. (a) Let $x \in S$, and let $l, k \in \mathbb{N}_+$ be such that $\phi^l(x) = \phi^{l+k}(x)$. For every $m \in \mathbb{N}_+$ one has

$$T_{l+mk}(\phi, x) = T_l(\phi, x) \cdot T_m(\mathrm{id}_S, y) = T_l(\phi, x) \cdot y^m,$$

where $y = \phi^l(T_k(\phi, x))$. Since S is arithmetic, there exists $C_x \in \mathbb{N}_+$ such that

$$v(T_{l+mk}(\phi, x)) = v(T_l(\phi, x) \cdot y^m) \leq \\ v(T_l(\phi, x)) + v(y^m) \leq v(T_l(\phi, x)) + C_x \cdot \log(m + 1),$$

so $\lim_{m\to\infty} \frac{v(T_{l+mk}(\phi,x))}{l+mk} = 0$.
Therefore we have found a subsequence of $(c_n(\phi, x))_{n\in\mathbb{N}_+}$ converging to 0, so also $h_{\mathfrak{S}}(\phi, x) = 0$. Hence $h_{\mathfrak{S}}(\phi) = 0$.

(b) For $x \in S$, there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) = 1$. Therefore $T_{n+k}(\phi, x) = T_n(\phi, x)$ for every $k \in \mathbb{N}$, hence $h_{\mathfrak{S}}(\phi, x) = 0$. $\boxed{QED}$

We discuss now a possible different notion of semigroup entropy. Let (S, v) be a normed semigroup, $\phi : S \to S$ an endomorphism, $x \in S$ and $n \in \mathbb{N}_+$. One could define also the "left" n-th ϕ-trajectory of x as

$$T_n^{\#}(\phi, x) = \phi^{n-1}(x) \cdot \ldots \cdot \phi(x) \cdot x,$$

changing the order of the factors with respect to the above definition. With these trajectories it is possible to define another entropy letting

$$h_{\mathfrak{S}}^{\#}(\phi, x) = \lim_{n\to\infty} \frac{v(T_n^{\#}(\phi, x))}{n},$$

and

$$h_{\mathfrak{S}}^{\#}(\phi) = \sup\{h_{\mathfrak{S}}^{\#}(\phi, x) : x \in S\}.$$

In the same way as above, one can see that the limit defining $h_{\mathfrak{S}}^{\#}(\phi, x)$ exists.

Obviously $h^{\#}_{\mathfrak{S}}$ and $h_{\mathfrak{S}}$ coincide on the identity map and on commutative normed semigroups, but now we see that in general they do not take always the same values. Item (a) in the following example shows that it may occur the case that they do not coincide "locally", while they coincide "globally". Moreover, modifying appropriately the norm in item (a), J. Spevák found the example in item (b) for which $h^{\#}_{\mathfrak{S}}$ and $h_{\mathfrak{S}}$ do not coincide even "globally".

Example 3. Let $X = \{x_n\}_{n\in\mathbb{Z}}$ be a faithfully enumerated countable set and let S be the free semigroup generated by X. An element $w \in S$ is a word $w = x_{i_1}x_{i_2}\dots x_{i_m}$ with $m \in \mathbb{N}_+$ and $i_j \in \mathbb{Z}$ for $j = 1, 2, \dots, m$. In this case m is called the *length* $\ell_X(w)$ of w, and a subword of w is any $w' \in S$ of the form $w' = x_{i_k}x_{i_k+1}\dots x_{i_l}$ with $1 \leq k \leq l \leq n$.

Consider the automorphism $\phi : S \to S$ determined by $\phi(x_n) = x_{n+1}$ for every $n \in \mathbb{Z}$.

(a) Let $s(w)$ be the number of adjacent pairs (i_k, i_{k+1}) in w such that $i_k < i_{k+1}$. The map $v : S \to \mathbb{R}_+$ defined by $v(w) = s(w) + 1$ is a semigroup norm. Then $\phi : (S, v) \to (S, v)$ is an automorphism of normed semigroups.

It is straightforward to prove that, for $w = x_{i_1}x_{i_2}\dots x_{i_m} \in S$,

(i) $h^{\#}_{\mathfrak{S}}(\phi, w) = h_{\mathfrak{S}}(\phi, w)$ if and only if $i_1 > i_m + 1$;

(ii) $h^{\#}_{\mathfrak{S}}(\phi, w) = h_{\mathfrak{S}}(\phi, w) - 1$ if and only if $i_m = i_1$ or $i_m = i_1 - 1$.

Moreover,

(iii) $h^{\#}_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\phi) = \infty$.

In particular, $h_{\mathfrak{S}}(\phi, x_0) = 1$ while $h^{\#}_{\mathfrak{S}}(\phi, x_0) = 0$.

(b) Define a semigroup norm $\nu : S \to \mathbb{R}_+$ as follows. For $w = x_{i_1}x_{i_2}\dots x_{i_n} \in S$ consider its subword $w' = x_{i_k}x_{i_{k+1}}\dots x_{i_l}$ with maximal length satisfying $i_{j+1} = i_j + 1$ for every $j \in \mathbb{Z}$ with $k \leq j \leq l - 1$ and let $\nu(w) = \ell_X(w')$. Then $\phi : (S, \nu) \to (S, \nu)$ is an automorphism of normed semigroups.

It is possible to prove that, for $w \in S$,

(i) if $\ell_X(w) = 1$, then $\nu(T_n(\phi, w)) = n$ and $\nu(T^{\#}_n(\phi, w)) = 1$ for every $n \in \mathbb{N}_+$;

(ii) if $\ell_X(w) = k$ with $k > 1$, then $\nu(T_n(\phi, w)) < 2k$ and $\nu(T^{\#}_n(\phi, w)) < 2k$ for every $n \in \mathbb{N}_+$.

From (i) and (ii) and from the definitions we immediately obtain that

(iii) $h_{\mathfrak{S}}(\phi) = 1 \neq 0 = h^{\#}_{\mathfrak{S}}(\phi)$.

We list now the main basic properties of the semigroup entropy. For complete proofs and further details see [27].

Lemma 3 (Monotonicity for factors)**.** *Let S, T be normed semigroups and $\phi : S \to S$, $\psi : T \to T$ endomorphisms. If $\alpha : T \to S$ is a surjective homomorphism such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.*

Proof. Fix $x \in S$ and find $y \in T$ with $x = \alpha(y)$. Then $c_n(x, \phi) \leq c_n(\psi, y)$ for every $n \in \mathbb{N}_+$. Dividing by n and taking the limit gives $h_{\mathfrak{S}}(\phi, x) \leq h_{\mathfrak{S}}(\psi, y)$. So $h_{\mathfrak{S}}(\phi, x) \leq h_{\mathfrak{S}}(\psi)$. When x runs over S, we conclude that $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$. $\boxed{QED}$

Corollary 1 (Invariance under conjugation)**.** *Let S be a normed semigroup and $\phi : S \to S$ an endomorphism. If $\alpha : S \to T$ is an isomorphism, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.*

Lemma 4 (Invariance under inversion)**.** *Let S be a normed semigroup and $\phi : S \to S$ an automorphism. Then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.*

Theorem 2 (Logarithmic Law)**.** *Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then*

$$h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$$

for every $k \in \mathbb{N}_+$. Furthermore, equality holds if v is s-monotone. Moreover, if $\phi : S \to S$ is an automorphism, then

$$h_{\mathfrak{S}}(\phi^k) = |k| \cdot h_{\mathfrak{S}}(\phi)$$

for all $k \in \mathbb{Z} \setminus \{0\}$.

Proof. Fix $k \in \mathbb{N}_+$, $x \in S$ and let $y = x \cdot \phi(x) \cdot \ldots \cdot \phi^{k-1}(x)$. Then

$$h_{\mathfrak{S}}(\phi^k) \geq h_{\mathfrak{S}}(\phi^k, y) = \lim_{n\to\infty} \frac{c_n(\phi^k, y)}{n} = \lim_{n\to\infty} \frac{v(y \cdot \phi^k(y) \cdot \ldots \cdot \phi^{(n-1)k}(y))}{n} =$$
$$= k \cdot \lim_{n\to\infty} \frac{c_{nk}(\phi, x)}{nk} = k \cdot h_{\mathfrak{S}}(\phi, x).$$

This yields $h_{\mathfrak{S}}(\phi^k) \geq k \cdot h_{\mathfrak{S}}(\phi, x)$ for all $x \in S$, and consequently, $h_{\mathfrak{S}}(\phi^k) \geq k \cdot h_{\mathfrak{S}}(\phi)$.

Suppose v to be s-monotone, then

$$h_{\mathfrak{S}}(\phi, x) = \lim_{n\to\infty} \frac{v(x \cdot \phi(x) \cdot \ldots \cdot \phi^{nk-1}(x))}{n \cdot k} \geq$$
$$\lim_{n\to\infty} \frac{v(x \cdot \phi^k(x) \cdot \ldots \cdot (\phi^k)^{n-1}(x))}{n \cdot k} = \frac{h_{\mathfrak{S}}(\phi^k, x)}{k}$$

Hence, $k \cdot h_{\mathfrak{S}}(\phi) \geq h_{\mathfrak{S}}(\phi^k, x)$ for every $x \in S$. Therefore, $k \cdot h_{\mathfrak{S}}(\phi) \geq h_{\mathfrak{S}}(\phi^k)$.

If ϕ is an automorphism and $k \in \mathbb{Z} \setminus \{0\}$, apply the previous part of the theorem and Lemma 4. $\boxed{QED}$

The next lemma shows that monotonicity is available not only under taking factors:

Lemma 5 (Monotonicity for subsemigroups)**.** *Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. If T is a ϕ-invariant normed subsemigroup of (S, v), then $h_{\mathfrak{S}}(\phi) \geq h_{\mathfrak{S}}(\phi \restriction_T)$. Equality holds if S is ordered, v is monotone and T is cofinal in S.*

Note that T is equipped with the induced norm $v \restriction_T$. The same applies to the subsemigroups S_i in the next corollary:

Corollary 2 (Continuity for direct limits)**.** *Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. If $\{S_i : i \in I\}$ is a directed family of ϕ-invariant normed subsemigroup of (S, v) with $S = \varinjlim S_i$, then $h_{\mathfrak{S}}(\phi) = \sup h_{\mathfrak{S}}(\phi \restriction_{S_i})$.*

We consider now products in $\mathfrak{S}$. Let $\{(S_i, v_i) : i \in I\}$ be a family of normed semigroups and let $S = \prod_{i \in I} S_i$ be their direct product in the category of semigroups.

In case I is finite, then S becomes a normed semigroup with the max-norm $v_{\prod}$, so $(S, v_{\prod})$ is the product of the family $\{S_i : i \in I\}$ in the category $\mathfrak{S}$; in such a case one has the following

Theorem 3 (Weak Addition Theorem - products)**.** *Let (S_i, v_i) be a normed semigroup and $\phi_i : S_i \to S_i$ an endomorphism for $i = 1, 2$. Then the endomorphism $\phi_1 \times \phi_2$ of $S_1 \times S_2$ has $h_{\mathfrak{S}}(\phi_1 \times \phi_2) = \max\{h_{\mathfrak{S}}(\phi_1), h_{\mathfrak{S}}(\phi_2)\}$.*

If I is infinite, S need not carry a semigroup norm v such that every projection $p_i : (S, v) \to (S_i, v_i)$ is a morphism in $\mathfrak{S}$. This is why the product of the family $\{(S_i, v_i) : i \in I\}$ in $\mathfrak{S}$ is actually the subset

$$S_{\text{bnd}} = \{x = (x_i)_{i \in I} \in S : \sup_{i \in I} v_i(x_i) \in \mathbb{R}\}$$

of S with the norm $v_{\prod}$ defined by

$$v_{\prod}(x) = \sup_{i \in I} v_i(x_i) \text{ for any } x = (x_i)_{i \in I} \in S_{\text{bnd}}.$$

For further details in this direction see [27].

2.4 Entropy in $\mathfrak{M}$

We collect here some additional properties of the semigroup entropy in the category $\mathfrak{M}$ of normed monoids where also coproducts are available. If (S_i, v_i)

is a normed monoid for every $i \in I$, the direct sum

$$S = \bigoplus_{i \in I} S_i = \{(x_i) \in \prod_{i \in I} S_i : |\{i \in I : x_i \neq 1\}| < \infty\}$$

becomes a normed monoid with the norm

$$v_\oplus(x) = \sum_{i \in I} v_i(x_i) \text{ for any } x = (x_i)_{i \in I} \in S.$$

This definition makes sense since v_i are monoid norms, so $v_i(1) = 0$. Hence, $(S, v_\oplus)$ becomes a coproduct of the family $\{(S_i, v_i) : i \in I\}$ in $\mathfrak{M}$.

We consider now the case when I is finite, so assume without loss of generality that $I = \{1, 2\}$. In other words we have two normed monoids (S_1, v_1) and (S_2, v_2). The product and the coproduct have the same underlying monoid $S = S_1 \times S_2$, but the norms $v_\oplus$ and $v_\prod$ in S are different and give different values of the semigroup entropy $h_{\mathfrak{S}}$; indeed, compare Theorem 3 and the following one.

Theorem 4 (Weak Addition Theorem - coproducts). *Let (S_i, v_i) be a normed monoid and $\phi_i : S_i \to S_i$ an endomorphism for $i = 1, 2$. Then the endomorphism $\phi_1 \oplus \phi_2$ of $S_1 \oplus S_2$ has $h_{\mathfrak{S}}(\phi_1 \oplus \phi_2) = h_{\mathfrak{S}}(\phi_1) + h_{\mathfrak{S}}(\phi_2)$.*

For a normed monoid $(M, v) \in \mathfrak{M}$ let $B(M) = \bigoplus_{\mathbb{N}} M$, equipped with the above coproduct norm $v_\oplus(x) = \sum_{n \in \mathbb{N}} v(x_n)$ for any $x = (x_n)_{n \in \mathbb{N}} \in B(M)$. The *right Bernoulli shift* is defined by

$$\beta_M : B(M) \to B(M),\ \beta_M(x_0, \ldots, x_n, \ldots) = (1, x_0, \ldots, x_n, \ldots),$$

while the *left Bernoulli shift* is

$${}_M\beta : B(M) \to B(M),\ {}_M\beta(x_0, x_1, \ldots, x_n, \ldots) = (x_1, x_2, \ldots, x_n, \ldots).$$

Theorem 5 (Bernoulli normalization). *Let (M, v) be a normed monoid. Then:*

(a) $h_{\mathfrak{S}}(\beta_M) = \sup_{x \in M} v(x)$;

(b) $h_{\mathfrak{S}}({}_M\beta) = 0$.

Proof. (a) For $x \in M$ consider $\underline{x} = (x_n)_{n \in \mathbb{N}} \in B(M)$ such that $x_0 = x$ and $x_n = 1$ for every $n \in \mathbb{N}_+$. Then $v_\oplus(T_n(\beta_M, \underline{x})) = n \cdot v(x)$, so $h_{\mathfrak{S}}(\beta_M, \underline{x}) = v(x)$. Hence $h_{\mathfrak{S}}(\beta_M) \geq \sup_{x \in M} v(x)$. Let now $\underline{x} = (x_n)_{n \in \mathbb{N}} \in B(M)$ and let $k \in \mathbb{N}$ be the greatest index such that $x_k \neq 1$; then

$$v_\oplus(T_n(\beta_M, \underline{x})) = \sum_{i=0}^{k+n} v(T_n(\beta_M, \underline{x})_i) \leq$$

$$\sum_{i=0}^{k-1} v(x_0 \cdot \ldots \cdot x_i) + (n-k) \cdot v(x_1 \cdot \ldots \cdot x_k) + \sum_{i=1}^{k} v(x_i \cdot \ldots \cdot x_k).$$

Since the first and the last summand do not depend on n, after dividing by n and letting n converge to infinity we obtain

$$h_{\mathfrak{S}}(\beta_M, \underline{x}) = \lim_{n\to\infty} \frac{v_\oplus(T_n(\beta_M, \underline{x}))}{n} \leq v(x_1 \cdot \ldots \cdot x_k) \leq \sup_{x\in M} v(x).$$

(b) Note that ${}_M\beta$ is locally nilpotent and apply Lemma 2. $\boxed{QED}$

2.5 Semigroup entropy of an element and pseudonormed semigroups

One can notice a certain asymmetry in Definition 6. Indeed, for S a normed semigroup, the local semigroup entropy defined in (3) is a two variable function

$$h_{\mathfrak{S}} : \mathrm{End}(S) \times S \to \mathbb{R}_+.$$

Taking $h_{\mathfrak{S}}(\phi) = \sup_{x\in S} h_{\mathfrak{S}}(\phi, x)$ for an endomorphism $\phi \in \mathrm{End}(S)$, we obtained the notion of semigroup entropy of ϕ. But one can obviously exchange the roles of ϕ and x and obtain the possibility to discuss the entropy of an element $x \in S$. This can be done in two ways. Indeed, in Remark 2 we consider what seems the natural counterpart of $h_{\mathfrak{S}}(\phi)$, while here we discuss a particular case that could appear to be almost trivial, but actually this is not the case, as it permits to give a uniform approach to some entropies which are not defined by using trajectories. So, by taking $\phi = \mathrm{id}_S$ in (3), we obtain a map $h^0_{\mathfrak{S}} : S \to \mathbb{R}_+$:

Definition 7. Let S be a normed semigroup and $x \in S$. The *semigroup entropy* of x is

$$h^0_{\mathfrak{S}}(x) := h_{\mathfrak{S}}(\mathrm{id}_S, x) = \lim_{n\to\infty} \frac{v(x^n)}{n}.$$

We shall see now that the notion of semigroup entropy of an element is supported by many examples. On the other hand, since some of the examples given below cannot be covered by our scheme, we propose first a slight extension that covers those examples as well.

Let $\mathfrak{S}^*$ be the category having as objects of all pairs (S, v), where S is a semigroup and $v : S \to \mathbb{R}_+$ is an *arbitrary* map. A morphism in the category $\mathfrak{S}^*$ is a semigroup homomorphism $\phi : (S, v) \to (S', v')$ that is contracting with respect to the pair v, v', i.e., $v'(\phi(x)) \leq v(x)$ for every $x \in S$. Note that our starting category $\mathfrak{S}$ is simply a full subcategory of $\mathfrak{S}^*$, having as objects those pairs (S, v) such that v satisfies (i) from Definition 2. These pairs were called normed semigroups and v was called a semigroup norm. For the sake of convenience and in order to keep close to the current terminology, let us call the function v in the larger category $\mathfrak{S}^*$ a *semigroup pseudonorm* (although, we are imposing no condition on v whatsoever).

So, in this setting, one can define a local semigroup entropy $h_{\mathfrak{S}^*}: \mathrm{End}(S) \times S \to \mathbb{R}_+$ following the pattern of (3), replacing the limit by

$$h_{\mathfrak{S}^*}(\phi, x) = \limsup_{n\to\infty} \frac{v(T_n(\phi, x))}{n}.$$

In particular,

$$h^0_{\mathfrak{S}^*}(x) = \limsup_{n\to\infty} \frac{v(x^n)}{n}.$$

Let us note that in order to have the last lim sup a limit, one does not need (S, v) to be in $\mathfrak{S}$, but it suffices to have the semigroup norm condition (i) from Definition 2 fulfilled only for products of powers of the same element.

We consider here three different entropies, respectively from [55], [32] and [73], that can be described in terms of $h^0_{\mathfrak{S}}$ or its generalized version $h^0_{\mathfrak{S}^*}$. We do not go into the details, but we give the idea how to capture them using the notion of semigroup entropy of an element of the semigroup of all endomorphisms of a given object equipped with an appropriate semigroup (pseudo)norm.

(a) Following [55], let R be a Noetherian local ring and $\phi: R \to R$ an endomorphism of finite length; moreover, $\lambda(\phi)$ is the length of ϕ, which is a real number ≥ 1. In this setting the entropy of ϕ is defined by

$$h_\lambda(\phi) = \lim_{n\to\infty} \frac{\log \lambda(\phi^n)}{n}$$

and it is proved that this limit exists.

Then the set $S = \mathrm{End}_{\mathrm{fl}}(R)$ of all finite-length endomorphisms of R is a semigroup and $\log \lambda(-)$ is a semigroup norm on S. For every $\phi \in S$, we have

$$h_\lambda(\phi) = h_{\mathfrak{S}}(\mathrm{id}_S, \phi) = h^0_{\mathfrak{S}}(\phi).$$

In other words, $h_\lambda(\phi)$ is nothing else but the semigroup entropy of the element ϕ of the normed semigroup $S = \mathrm{End}_{\mathrm{fl}}(R)$.

(b) We recall now the entropy considered in [73], which was already introduced in [7]. Let $t \in \mathbb{N}_+$ and $\varphi : \mathbb{P}^t \to \mathbb{P}^t$ be a dominant rational map of degree d. Then the entropy of φ is defined as the logarithm of the dynamical degree, that is

$$h_\delta(\varphi) = \log \delta_\phi = \limsup_{n\to\infty} \frac{\log \deg(\varphi^n)}{n}.$$

Consider the semigroup S of all dominant rational maps of $\mathbb{P}^n$ and the function $\log\deg(-)$. In general this is only a semigroup pseudonorm on S and

$$h^0_{\mathfrak{S}^*}(\varphi) = h_\delta(\varphi).$$

Note that $\log\deg(-)$ is a semigroup norm when φ is an endomorphism of the variety $\mathbb{P}^t$.

(c) We consider now the growth rate for endomorphisms introduced in [10] and recently studied in [32]. Let G be a finitely generated group, X a finite symmetric set of generators of G, and $\varphi : G \to G$ an endomorphism. For $g \in G$, denote by $\ell_X(g)$ the length of g with respect to the alphabet X. The growth rate of φ with respect to $x \in X$ is

$$\log GR(\varphi, x) = \lim_{n\to\infty} \frac{\log \ell_X(\varphi^n(x))}{n}$$

(and the growth rate of φ is $\log GR(\varphi) = \sup_{x\in X} \log GR(\varphi, x)$).

Consider $S = \mathrm{End}(G)$ and, fixed $x \in X$, the map $\log GR(-, x)$. As in item (b) this is only a semigroup pseudonorm on S. Nevertheless, also in this case the semigroup entropy

$$\log GR(\varphi, x) = h^0_{\mathfrak{S}^*}(\varphi).$$

Remark 2. For a normed semigroup S, let $h_{\mathfrak{S}} : \mathrm{End}(S) \times S \to \mathbb{R}_+$ be the local semigroup entropy defined in (3). Exchanging the roles of $\phi \in \mathrm{End}(S)$ and $x \in S$, define the *global semigroup entropy* of an element $x \in S$ by

$$h_{\mathfrak{S}}(x) = \sup_{\phi\in\mathrm{End}(S)} h_{\mathfrak{S}}(\phi, x).$$

Obviously, $h^0_{\mathfrak{S}}(x) \leq h_{\mathfrak{S}}(x)$ for every $x \in S$.

3 Obtaining known entropies

3.1 The general scheme

Let $\mathfrak{X}$ be a category and let $F : \mathfrak{X} \to \mathfrak{S}$ be a functor. Define the entropy

$$h_F : \mathfrak{X} \to \mathbb{R}_+$$

on the category $\mathfrak{X}$ by

$$h_F(\phi) = h_{\mathfrak{S}}(F(\phi)),$$

for any endomorphism $\phi : X \to X$ in $\mathfrak{X}$. Recall that with some abuse of notation we write $h_F : \mathfrak{X} \to \mathbb{R}_+$ in place of $h_F : \mathrm{Flow}_{\mathfrak{X}} \to \mathbb{R}_+$ for simplicity.

Since the functor F preserves commutative squares and isomorphisms, the entropy h_F has the following properties, that automatically follow from the previously listed properties of the semigroup entropy $h_{\mathfrak{S}}$. For the details and for properties that need a further discussion see [27].

Let X, Y be objects of $\mathfrak{X}$ and $\phi : X \to X$, $\psi : Y \to Y$ endomorphisms in $\mathfrak{X}$.

(a) [Invariance under conjugation] If $\alpha : X \to Y$ is an isomorphism in $\mathfrak{X}$, then $h_F(\phi) = h_F(\alpha \circ \phi \circ \alpha^{-1})$.

(b) [Invariance under inversion] If $\phi : X \to X$ is an automorphism in $\mathfrak{X}$, then $h_F(\phi^{-1}) = h_F(\phi)$.

(c) [Logaritmic Law] If the norm of $F(X)$ is s-monotone, then $h_F(\phi^k) = k \cdot h_F(\phi)$ for all $k \in \mathbb{N}_+$.

Other properties of h_F depend on properties of the functor F.

(d) [Monotonicity for invariant subobjects] If F sends subobject embeddings in $\mathfrak{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then, if Y is a ϕ-invariant subobject of X, we have $h_F(\phi \restriction_Y) \leq h_F(\phi)$.

(e) [Monotonicity for factors] If F sends factors in $\mathfrak{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then, if $\alpha : T \to S$ is an epimorphism in $\mathfrak{X}$ such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_F(\phi) \leq h_F(\psi)$.

(f) [Continuity for direct limits] If F is covariant and sends direct limits to direct limits, then $h_F(\phi) = \sup_{i \in I} h_F(\phi \restriction_{X_i})$ whenever $X = \varinjlim X_i$ and X_i is a ϕ-invariant subobject of X for every $i \in I$.

(g) [Continuity for inverse limits] If F is contravariant and sends inverse limits to direct limits, then $h_F(\phi) = \sup_{i \in I} h_F(\overline{\phi}_i)$ whenever $X = \varprojlim X_i$ and (X_i, ϕ_i) is a factor of (X, ϕ) for every $i \in I$.

In the following subsections we describe how the known entropies can be obtained from this general scheme. For all the details we refer to [27]

3.2 Set-theoretic entropy

In this section we consider the category **Set** of sets and maps and its (non-full) subcategory $\mathbf{Set}_{\mathrm{fin}}$ having as morphisms all the finitely many-to-one maps. We construct a functor $\mathfrak{atr} : \mathbf{Set} \to \mathfrak{S}$ and a functor $\mathfrak{str} : \mathbf{Set}_{\mathrm{fin}} \to \mathfrak{S}$, which give the set-theoretic entropy $\mathfrak{h}$ and the covariant set-theoretic entropy $\mathfrak{h}^*$, introduced in [5] and [20] respectively. We also recall that they are related to invariants for self-maps of sets introduced in [34] and [3] respectively.

A natural semilattice with zero, arising from a set X, is the family $(\mathcal{S}(X), \cup)$ of all finite subsets of X with neutral element $\emptyset$. Moreover the map defined by $v(A) = |A|$ for every $A \in \mathcal{S}(X)$ is an s-monotone norm. So let $\mathfrak{atr}(X) = (\mathcal{S}(X), \cup, v)$. Consider now a map $\lambda : X \to Y$ between sets and define $\mathfrak{atr}(\lambda) : \mathcal{S}(X) \to \mathcal{S}(Y)$ by $A \mapsto \lambda(A)$ for every $A \in \mathcal{S}(X)$. This defines a covariant functor

$$\mathfrak{atr} : \mathbf{Set} \to \mathfrak{S}$$

such that

$$h_{\mathfrak{atr}} = \mathfrak{h}.$$

Consider now a finite-to-one map $\lambda : X \to Y$. As above let $\mathfrak{str}(X) = (\mathcal{S}(X), \cup, v)$, while $\mathfrak{str}(\lambda) : \mathfrak{str}(Y) \to \mathfrak{str}(X)$ is given by $A \mapsto \lambda^{-1}(A)$ for every $A \in \mathcal{S}(Y)$. This defines a contravariant functor

$$\mathfrak{str} : \mathbf{Set}_{\mathrm{fin}} \to \mathfrak{S}$$

such that

$$h_{\mathfrak{str}} = \mathfrak{h}^*.$$

3.3 Topological entropy for compact spaces

In this subsection we consider in the general scheme the topological entropy h_{top} introduced in [1] for continuous self-maps of compact spaces. So we specify the general scheme for the category $\mathfrak{X} = \mathbf{CTop}$ of compact spaces and continuous maps, constructing the functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$.

For a topological space X let $\mathfrak{cov}(X)$ be the family of all open covers $\mathcal{U}$ of X, where it is allowed $\emptyset \in \mathcal{U}$. For $\mathcal{U}, \mathcal{V} \in \mathfrak{cov}(X)$ let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\} \in \mathfrak{cov}(X)$. One can easily prove commutativity and associativity of $\vee$; moreover, let $\mathcal{E} = \{X\}$ denote the trivial cover. Then

$$(\mathfrak{cov}(X), \vee, \mathcal{E}) \text{ is a commutative monoid.}$$

For a topological space X, one has a natural preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathfrak{cov}(X)$; indeed, $\mathcal{V}$ *refines* $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. Note that this preorder has bottom element $\mathcal{E}$, and that it is not an order. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$, yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$, and more generally

$$\mathcal{U} \vee \mathcal{U} \vee \ldots \vee \mathcal{U} \sim \mathcal{U}. \tag{4}$$

For X, Y topological spaces, a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathfrak{cov}(Y)$, let $\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}$. Then, as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$, we have that $\mathfrak{cov}(\phi) : \mathfrak{cov}(Y) \to \mathfrak{cov}(X)$, defined by $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$, is a semigroup homomorphism. This defines a contravariant functor $\mathfrak{cov}$ from the category of all topological spaces to the category of commutative semigroups.

To get a semigroup norm on $\mathfrak{cov}(X)$ we restrict this functor to the subcategory **CTop** of compact spaces. For a compact space X and $\mathcal{U} \in \mathfrak{cov}(X)$, let

$$M(\mathcal{U}) = \min\{|\mathcal{V}| : \mathcal{V} \text{ a finite subcover of } \mathcal{U}\} \text{ and } v(\mathcal{U}) = \log M(\mathcal{U}).$$

Now (4) gives $v(\mathcal{U} \vee \mathcal{U} \vee \ldots \vee \mathcal{U}) = v(\mathcal{U})$, so

$$(\mathfrak{cov}(X), \vee, v) \text{ is an arithmetic normed semigroup.}$$

For every continuous map $\phi : X \to Y$ of compact spaces and $\mathcal{W} \in \mathfrak{cov}(Y)$, the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds. Consequently

$$\mathfrak{cov}(\phi) : \mathfrak{cov}(Y) \to \mathfrak{cov}(X), \text{ defined by } \mathcal{W} \mapsto \phi^{-1}(\mathcal{W}), \text{ is a morphism in } \mathfrak{S}.$$

Therefore the assignments $X \mapsto \mathfrak{cov}(X)$ and $\phi \mapsto \mathfrak{cov}(\phi)$ define a contravariant functor

$$\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}.$$

Moreover,

$$h_{\mathfrak{cov}} = h_{top}.$$

Since the functor $\mathfrak{cov}$ takes factors in **CTop** to embeddings in $\mathfrak{S}$, embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and inverse limits in **CTop** to direct limits in $\mathfrak{S}$, we have automatically that the topological entropy h_{top} is monotone for factors and restrictions to invariant subspaces, continuous for inverse limits, is invariant under conjugation and inversion, and satisfies the Logarithmic Law.

3.4 Measure entropy

In this subsection we consider the category **MesSp** of probability measure spaces $(X, \mathfrak{B}, \mu)$ and measure preserving maps, constructing a functor $\mathfrak{mes} : \mathbf{MesSp} \to \mathfrak{S}$ in order to obtain from the general scheme the measure entropy h_{mes} from [53] and [74].

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \ldots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. As $\xi \vee \xi = \xi$, with zero the cover $\xi_0 = \{X\}$,

$$(\mathfrak{P}(X), \vee) \text{ is a semilattice with } 0.$$

Moreover, for $\xi = \{A_1, A_2, \dots, A_k\} \in \mathfrak{P}(X)$ the *entropy* of ξ is given by Boltzmann's Formula

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k).$$

This is a monotone semigroup norm making $\mathfrak{P}(X)$ a normed semilattice and a normed monoid.

Consider now a measure preserving map $T : X \to Y$. For a cover $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all $i = 1, \dots, k$. Hence, $v(T^{-1}(\xi)) = v(\xi)$ and so

$$\mathfrak{mes}(T) : \mathfrak{P}(Y) \to \mathfrak{P}(X), \text{ defined by } \xi \mapsto T^{-1}(\xi), \text{ is a morphism in SL.}$$

Therefore the assignments $X \mapsto \mathfrak{P}(X)$ and $T \mapsto \mathfrak{mes}(T)$ define a contravariant functor

$$\mathfrak{mes} : \mathbf{MesSp} \to \mathrm{SL}\,.$$

Moreover,

$$h_{\mathfrak{mes}} = h_{mes}.$$

The functor $\mathfrak{mes} : \mathbf{MesSp} \to \mathrm{SL}$ is covariant and sends embeddings in **MesSp** to surjective morphisms in SL and sends surjective maps in **MesSp** to embeddings in SL. Hence, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone for factors and restrictions to invariant subspaces, continuous for inverse limits, is invariant under conjugation and inversion, satisfies the Logarithmic Law and the Weak Addition Theorem.

In the next remark we briefly discuss the connection between measure entropy and topological entropy.

Remark 3. (a) If X is a compact metric space and $\phi : X \to X$ is a continuous surjective self-map, by Krylov-Bogolioubov Theorem [9] there exist some ϕ-invariant Borel probability measures μ on X (i.e., making $\phi : (X, \mu) \to (X, \mu)$ measure preserving). Denote by h_μ the measure entropy with respect to the measure μ. The inequality $h_\mu(\phi) \leq h_{top}(\phi)$ for every $\mu \in M(X, \phi)$ is due to Goodwyn [41]. Moreover the *variational principle* (see [84, Theorem 8.6]) holds true:

$$h_{top}(\phi) = \sup\{h_\mu(\phi) : \mu \ \phi\text{-invariant measure on } X\}.$$

(b) In the computation of the topological entropy it is possible to reduce to surjective continuous self-maps of compact spaces. Indeed, for a compact space X and a continuous self-map $\phi : X \to X$, the set $E_\phi(X) = \bigcap_{n\in\mathbb{N}} \phi^n(X)$ is closed and ϕ-invariant, the map $\phi \restriction_{E_\phi(X)} : E_\phi(X) \to E_\phi(X)$ is surjective and $h_{top}(\phi) = h_{top}(\phi \restriction_{E_\phi(X)})$ (see [84]).

(c) In the case of a compact group K and a continuous surjective endomorphism $\phi : K \to K$, the group K has its unique Haar measure and so ϕ is measure preserving as noted by Halmos [49]. In particular both h_{top} and h_{mes} are available for surjective continuous endomorphisms of compact groups and they coincide as proved in the general case by Stoyanov [75].

In other terms, denote by **CGrp** the category of all compact groups and continuous homomorphisms, and by $\mathbf{CGrp}_e$ the non-full subcategory of **CGrp**, having as morphisms all epimorphisms in **CGrp**. So in the following diagram we consider the forgetful functor $V : \mathbf{CGrp}_e \to \mathbf{Mes}$, while i is the inclusion of $\mathbf{CGrp}_e$ in **CGrp** as a non-full subcategory and $U : \mathbf{CGrp} \to \mathbf{Top}$ is the forgetful functor:

$$\begin{tikzcd} \mathbf{CGrp}_e \arrow[r, "i"] \arrow[d, "V"] & \mathbf{CGrp} \arrow[r, "U"] & \mathbf{Top} \\ \mathbf{Mes} & & \end{tikzcd}$$

For a surjective endomorphism ϕ of the compact group K, we have then $h_{mes}(V(\phi)) = h_{top}(U(\phi))$.

3.5 Algebraic entropy

Here we consider the category **Grp** of all groups and their homomorphisms and its subcategory **AbGrp** of all abelian groups. We construct two functors $\mathfrak{sub} : \mathbf{AbGrp} \to \mathrm{SL}$ and $\mathfrak{pet} : \mathbf{Grp} \to \mathfrak{S}$ that permits to find from the general scheme the two algebraic entropies ent and h_{alg}. For more details on these entropies see the next section.

Let G be an abelian group and let $(\mathcal{F}(G), \cdot)$ be the semilattice consisting of all finite subgroups of G. Letting $v(F) = \log |F|$ for every $F \in \mathcal{F}(G)$, then

$$(\mathcal{F}(G), \cdot, v) \text{ is a normed semilattice}$$

and the norm v is monotone.

For every group homomorphism $\phi : G \to H$,

the map $\mathcal{F}(\phi) : \mathcal{F}(G) \to \mathcal{F}(H)$, defined by $F \mapsto \phi(F)$, is a morphism in SL.

Therefore the assignments $G \mapsto \mathcal{F}(G)$ and $\phi \mapsto \mathcal{F}(\phi)$ define a covariant functor

$$\mathfrak{sub} : \mathbf{AbGrp} \to \mathrm{SL}\,.$$

Moreover

$$h_{\mathfrak{sub}} = \mathrm{ent}\,.$$

Since the functor $\mathfrak{sub}$ takes factors in **AbGrp** to surjective morphisms in $\mathfrak{S}$, embeddings in **AbGrp** to embeddings in $\mathfrak{S}$, and direct limits in **AbGrp** to direct limits in $\mathfrak{S}$, we have automatically that the algebraic entropy ent is monotone for factors and restrictions to invariant subspaces, continuous for direct limits, invariant under conjugation and inversion, satisfies the Logarithmic Law.

For a group G let $\mathcal{H}(G)$ be the family of all finite non-empty subsets of G. Then $\mathcal{H}(G)$ with the operation induced by the multiplication of G is a monoid with neutral element $\{1\}$. Moreover, letting $v(F) = \log|F|$ for every $F \in \mathcal{H}(G)$ makes $\mathcal{H}(G)$ a normed semigroup. For an abelian group G the monoid $\mathcal{H}(G)$ is arithmetic since for any $F \in \mathcal{H}(G)$ the sum of n summands satisfies $|F+\ldots+F| \leq (n+1)^{|F|}$. Moreover, $(\mathcal{H}(G), \subseteq)$ is an ordered semigroup and the norm v is s-monotone.

For every group homomorphism $\phi : G \to H$,

the map $\mathcal{H}(\phi) : \mathcal{H}(G) \to \mathcal{H}(H)$, defined by $F \mapsto \phi(F)$, is a morphism in $\mathfrak{S}$.

Consequently the assignments $G \mapsto (\mathcal{H}(G), v)$ and $\phi \mapsto \mathcal{H}(\phi)$ give a covariant functor

$$\mathfrak{pet} : \mathbf{Grp} \to \mathfrak{S}.$$

Hence

$$h_{\mathfrak{pet}} = h_{alg}.$$

Note that the functor $\mathfrak{sub}$ is a subfunctor of $\mathfrak{pet} : \mathbf{AbGrp} \to \mathfrak{S}$ as $\mathcal{F}(G) \subseteq \mathcal{H}(G)$ for every abelian group G.

As for the algebraic entropy ent, since the functor $\mathfrak{pet}$ takes factors in **Grp** to surjective morphisms in $\mathfrak{S}$, embeddings in **Grp** to embeddings in $\mathfrak{S}$, and direct limits in **Grp** to direct limits in $\mathfrak{S}$, we have automatically that the algebraic entropy h_{alg} is monotone for factors and restrictions to invariant subspaces, continuous for direct limits, invariant under conjugation and inversion, satisfies the Logarithmic Law.

3.6 h_{top} and h_{alg} in locally compact groups

As mentioned above, Bowen introduced topological entropy for uniformly continuous self-maps of metric spaces in [11]. His approach turned out to be especially efficient in the case of locally compact spaces provided with some Borel measure with good invariance properties, in particular for continuous endomorphisms of locally compact groups provided with their Haar measure. Later Hood in [51] extended Bowen's definition to uniformly continuous self-maps of arbitrary uniform spaces and in particular to continuous endomorphisms of (not necessarily metrizable) locally compact groups.

On the other hand, Virili [80] extended the notion of algebraic entropy to continuous endomorphisms of locally compact abelian groups, inspired by Bowen's definition of topological entropy (based on the use of Haar measure). As mentioned in [20], his definition can be extended to continuous endomorphisms of arbitrary locally compact groups.

Our aim here is to show that both entropies can be obtained from our general scheme in the case of measure preserving topological automorphisms of locally compact groups. To this end we recall first the definitions of h_{top} and h_{alg} in locally compact groups. Let G be a locally compact group, let $\mathcal{C}(G)$ be the family of all compact neighborhoods of 1 and μ be a right Haar measure on G. For a continuous endomorphism $\phi : G \to G$, $U \in \mathcal{C}(G)$ and a positive integer n, the n-th cotrajectory $C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \ldots \cap \phi^{-n+1}(U)$ is still in $\mathcal{C}(G)$. The topological entropy h_{top} is intended to measure the rate of decay of the n-th cotrajectory $C_n(\phi, U)$. So let

$$H_{top}(\phi, U) = \limsup_{n\to\infty} -\frac{\log \mu(C_n(\phi, U))}{n}, \tag{5}$$

which does not depend on the choice of the Haar measure μ. The *topological entropy* of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{C}(G)\}.$$

If G is discrete, then $\mathcal{C}(G)$ is the family of all finite subsets of G containing 1, and $\mu(A) = |A|$ for subsets A of G. So $H_{top}(\phi, U) = 0$ for every $U \in \mathcal{C}(G)$, hence $h_{top}(\phi) = 0$.

To define the algebraic entropy of ϕ with respect to $U \in \mathcal{C}(G)$ one uses the n-th ϕ-trajectory $T_n(\phi, U) = U \cdot \phi(U) \cdot \ldots \cdot \phi^{n-1}(U)$ of U, that still belongs to $\mathcal{C}(G)$. It turns out that the value

$$H_{alg}(\phi, U) = \limsup_{n\to\infty} \frac{\log \mu(T_n(\phi, U))}{n} \tag{6}$$

does not depend on the choice of μ. The *algebraic entropy* of ϕ is

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, U) : U \in \mathcal{C}(G)\}.$$

The term "algebraic" is motivated by the fact that the definition of $T_n(\phi, U)$ (unlike $C_n(\phi, U)$) makes use of the group operation.

As we saw above (6) is a limit when G is discrete. Moreover, if G is compact, then $h_{alg}(\phi) = H_{alg}(\phi, G) = 0$.

In the sequel, G will be a locally compact group. We fix also a measure preserving topological automorphism $\phi : G \to G$.

To obtain the entropy $h_{top}(\phi)$ via semigroup entropy fix some $V \in \mathcal{C}(G)$ with $\mu(V) \leq 1$. Then consider the subset

$$\mathcal{C}_0(G) = \{U \in \mathcal{C}(G) : U \subseteq V\}.$$

Obviously, $\mathcal{C}_0(G)$ is a monoid with respect to intersection, having as neutral element V. To obtain a pseudonorm v on $\mathcal{C}_0(G)$ let $v(U) = -\log \mu(U)$ for any $U \in \mathcal{C}_0(G)$. Then ϕ defines a semigroup isomorphism $\phi^\# : \mathcal{C}_0(G) \to \mathcal{C}_0(G)$ by $\phi^\#(U) = \phi^{-1}(U)$ for any $U \in \mathcal{C}_0(G)$. It is easy to see that $\phi^\# : \mathcal{C}_0(G) \to \mathcal{C}_0(G)$ is a an automorphism in $\mathfrak{S}^*$ and the semigroup entropy $h_{\mathfrak{S}^*}(\phi^\#)$ coincides with $h_{top}(\phi)$ since $H_{top}(\phi, U) \leq H_{top}(\phi, U')$ whenever $U \supseteq U'$.

To obtain the entropy $h_{alg}(\phi)$ via semigroup entropy fix some $W \in \mathcal{C}(G)$ with $\mu(W) \geq 1$. Then consider the subset

$$\mathcal{C}_1(G) = \{U \in \mathcal{C}(G) : U \supseteq W\}$$

of the set $\mathcal{C}(G)$. Note that for $U_1, U_2 \in \mathcal{C}_1(G)$ also $U_1U_2 \in \mathcal{C}_1(G)$. Thus $\mathcal{C}_1(G)$ is a semigroup. To define a pseudonorm v on $\mathcal{C}_1(G)$ let $v(U) = \log \mu(U)$ for any $U \in \mathcal{C}_1(G)$. Then ϕ defines a semigroup isomorphism $\phi_\# : \mathcal{C}_1(G) \to \mathcal{C}_1(G)$ by $\phi_\#(U) = \phi(U)$ for any $U \in \mathcal{C}_1(G)$. It is easy to see that $\phi_\# : \mathcal{C}_1(G) \to \mathcal{C}_1(G)$ is a morphism in $\mathfrak{S}^*$ and the semigroup entropy $h_{\mathfrak{S}^*}(\phi_\#)$ coincides with $h_{alg}(\phi)$, since $\mathcal{C}_1(G)$ is cofinal in $\mathcal{C}(G)$ and $H_{alg}(\phi, U) \leq H_{alg}(\phi, U')$ whenever $U \subseteq U'$.

Remark 4. We asked above the automorphism ϕ to be "measure preserving". In this way one rules out many interesting cases of topological automorphisms that are not measure preserving (e.g., all automorphisms of $\mathbb{R}$ beyond $\pm\,\mathrm{id}_{\mathbb{R}}$). This condition is imposed in order to respect the definition of the morphisms in $\mathfrak{S}^*$. If one further relaxes this condition on the morphisms in $\mathfrak{S}^*$ (without asking them to be contracting maps with respect to the pseudonorm), then one can obtain a semigroup entropy that covers the topological and the algebraic entropy of arbitrary topological automorphisms of locally compact groups (see [26] for more details).

3.7 Algebraic i-entropy

For a ring R we denote by mod_R the category of right R-modules and R-module homomorphisms. We consider here the algebraic i-entropy introduced in [70], giving a functor $\mathfrak{sub}_i : \mathrm{mod}_R \to \mathrm{SL}$, to find ent_i from the general scheme. Here $i : \mathrm{mod}_R \to \mathbb{R}_+$ is an invariant of mod_R (i.e., $i(0) = 0$ and $i(M) = i(N)$ whenever $M \cong N$). Consider the following conditions:

(a) $i(N_1 + N_2) \leq i(N_1) + i(N_2)$ for all submodules N_1, N_2 of M;

(b) $i(M/N) \leq i(M)$ for every submodule N of M;

(b*) $i(N) \leq i(M)$ for every submodule N of M.

The invariant i is called *subadditive* if (a) and (b) hold, and it is called *preadditive* if (a) and (b*) hold.

For $M \in \operatorname{mod}_R$ denote by $\mathcal{L}(M)$ the lattice of all submodules of M. The operations are intersection and sum of two submodules, the bottom element is $\{0\}$ and the top element is M. Now fix a subadditive invariant i of mod_R and for a right R-module M let

$$\mathcal{F}_i(M) = \{\text{submodules } N \text{ of } M \text{ with } i(M) < \infty\},$$

which is a subsemilattice of $\mathcal{L}(M)$ ordered by inclusion. Define a norm on $\mathcal{F}_i(M)$ setting

$$v(H) = i(H)$$

for every $H \in \mathcal{F}_i(M)$. The norm v is not necessarily monotone (it is monotone if i is both subadditive and preadditive).

For every homomorphism $\phi : M \to N$ in mod_R,

$$\mathcal{F}_i(\phi) : \mathcal{F}_i(M) \to \mathcal{F}_i(N), \text{ defined by } \mathcal{F}_i(\phi)(H) = \phi(H), \text{ is a morphism in SL.}$$

Moreover the norm v makes the morphism $\mathcal{F}_i(\phi)$ contractive by the property (b) of the invariant. Therefore, the assignments $M \mapsto \mathcal{F}_i(M)$ and $\phi \mapsto \mathcal{F}_i(\phi)$ define a covariant functor

$$\mathfrak{sub}_i : \operatorname{mod}_R \to \mathrm{SL}\,.$$

We can conclude that, for a ring R and a subadditive invariant i of mod_R,

$$h_{\mathfrak{sub}_i} = \mathrm{ent}_i\,.$$

If i is preadditive, the functor $\mathfrak{sub}_i$ sends monomorphisms to embeddings and so ent_i is monotone under taking submodules. If i is both subadditive and preadditive then for every R-module M the norm of $\mathfrak{sub}_i(M)$ is s-monotone, so ent_i satisfies also the Logarithmic Law. In general this entropy is not monotone under taking quotients, but this can be obtained with stronger hypotheses on i and with some restriction on the domain of $\mathfrak{sub}_i$.

A clear example is given by vector spaces; the algebraic entropy $\mathrm{ent}_{\dim}$ for linear transformations of vector spaces was considered in full details in [36]:

Example 4. Let K be a field. Then for every K-vector space V let $\mathcal{F}_d(M)$ be the set of all finite-dimensional subspaces N of M.

Then $(\mathcal{F}_d(V), +)$ is a subsemilattice of $(\mathcal{L}(V), +)$ and $v(H) = \dim H$ defines a monotone norm on $\mathcal{F}_d(V)$. For every morphism $\phi : V \to W$ in mod_K

the map $\mathcal{F}_d(\phi):\mathcal{F}_d(V)\to\mathcal{F}_d(W)$, defined by $H\mapsto\phi(H)$, is a morphism in SL.

Therefore, the assignments $M\mapsto\mathcal{F}_d(M)$ and $\phi\mapsto\mathcal{F}_d(\phi)$ define a covariant functor

$$\mathfrak{sub}_d:\quad \mathrm{mod}_{\,K}\to\mathrm{SL}\,.$$

Then

$$h_{\mathfrak{sub}_d}=\mathrm{ent}_{\dim}\,.$$

Note that this entropy can be computed ad follows. Every flow $\phi:V\to V$ of $\mathrm{mod}_{\,K}$ can be considered as a $K[X]$-module V_ϕ letting X act on V as ϕ. Then $h_{\mathfrak{sub}_d}(\phi)$ coincides with the rank of the $K[X]$-module V_ϕ.

3.8 Adjoint algebraic entropy

We consider now again the category **Grp** of all groups and their homomorphisms, giving a functor $\mathfrak{sub}^\star:\mathbf{Grp}\to\mathrm{SL}$ such that the entropy defined using this functor coincides with the adjoint algebraic entropy $\mathrm{ent}^\star$ introduced in [24].

For a group G denote by $\mathcal{C}(G)$ the family of all subgroups of finite index in G. It is a subsemilattice of $(\mathcal{L}(G),\cap)$. For $N\in\mathcal{C}(G)$, let

$$v(N)=\log[G:N];$$

then

$$(\mathcal{C}(G),v)\text{ is a normed semilattice,}$$

with neutral element G; moreover the norm v is monotone.

For every group homomorphism $\phi:G\to H$

$$\text{the map }\mathcal{C}(\phi):\mathcal{C}(H)\to\mathcal{C}(G)\text{, defined by }N\mapsto\phi^{-1}(N)\text{, is a morphism in }\mathfrak{S}.$$

Then the assignments $G\mapsto\mathcal{C}(G)$ and $\phi\mapsto\mathcal{C}(\phi)$ define a contravariant functor

$$\mathfrak{sub}^\star:\mathbf{Grp}\to\mathrm{SL}\,.$$

Moreover

$$h_{\mathfrak{sub}^\star}=\mathrm{ent}^\star\,.$$

There exists also a version of the adjoint algebraic entropy for modules, namely the adjoint algebraic i-entropy $\mathrm{ent}_i^\star$ (see [79]), which can be treated analogously.

3.9 Topological entropy for totally disconnected compact groups

Let (G, τ) be a totally disconnected compact group and consider the filter base $\mathcal{V}_G(1)$ of open subgroups of G. Then

$$(\mathcal{V}_G(1), \cap) \text{ is a normed semilattice}$$

with neutral element $G \in \mathcal{V}_G(1)$ and norm defined by $v_o(V) = \log[G : V]$ for every $V \in \mathcal{V}_G(1)$.

For a continuous homomorphism $\phi : G \to H$ between compact groups,

$$\text{the map } \mathcal{V}_H(1) \to \mathcal{V}_G(1), \text{ defined by } V \mapsto \phi^{-1}(V), \text{ is a morphism in SL.}$$

This defines a contravariant functor

$$\mathfrak{sub}_o^\star : \mathbf{TdCGrp} \to \mathrm{SL},$$

which is a subfunctor of $\mathfrak{sub}^\star$.

Then the entropy $h_{\mathfrak{sub}_o^\star}$ coincides with the restriction to **TdCGrp** of the topological entropy h_{top}.

This functor is related also to the functor $\mathfrak{cov} : \mathbf{TdCGrp} \to \mathfrak{S}$. Indeed, let G be a totally disconnected compact group. Each $V \in \mathcal{V}_G(1)$ defines a cover $\mathcal{U}_V = \{x \cdot V\}_{x \in G}$ of G with $v_o(V) = v(\mathcal{U}_V)$. So the map $V \mapsto \mathcal{U}_V$ defines an isomorphism between the normed semilattice $\mathfrak{sub}_o^\star(G) = \mathcal{V}_G(1)$ and the subsemigroup $\mathfrak{cov}_s(G) = \{\mathcal{U}_V : V \in \mathcal{V}_G(1)\}$ of $\mathfrak{cov}(G)$.

3.10 Bridge Theorem

In Definition 1 we have formalized the concept of Bridge Theorem between entropies $h_1 : \mathfrak{X}_1 \to \mathbb{R}_+$ and $h_2 : \mathfrak{X}_2 \to \mathbb{R}_+$ via functors $\varepsilon : \mathfrak{X}_1 \to \mathfrak{X}_2$. Obviously, the Bridge Theorem with respect to the functor ε is available when each h_i has the form $h_i = h_{F_i}$ for appropriate functors $F_i : \mathfrak{X}_i \to \mathfrak{S}$ $(i = 1, 2)$ that commute with ε (i.e., $F_1 = F_2\varepsilon$), that is

$$h_2(\varepsilon(\phi)) = h_1(\phi) \ \text{ for all morphisms } \phi \text{ in } \ \mathfrak{X}_1.$$

Actually, it is sufficient that F_i commute with ε "modulo $h_{\mathfrak{S}}$" (i.e., $h_{\mathfrak{S}}F_1 = h_{\mathfrak{S}}F_2\varepsilon$) to obtain this conclusion:

$$\begin{tikzcd}
\mathfrak{X}_1 \arrow[dd, "\varepsilon"'] \arrow[dr, "F_1"] \arrow[drr, bend left, "h_1"] & & \\
& \mathfrak{S} \arrow[r, "h_{\mathfrak{S}}"] & \mathbb{R}^+ \\
\mathfrak{X}_2 \arrow[ur, "F_2"'] \arrow[urr, bend right, "h_2"'] & &
\end{tikzcd} \tag{7}$$

In particular the Pontryagin duality functor $\widehat{\ }$: **AbGrp** $\to$ **CAbGrp** connects the category of abelian groups and that of compact abelian groups so connects the respective entropies h_{alg} and h_{top} by a Bridge Theorem. Taking the restriction to torsion abelian groups and the totally disconnected compact groups one obtains:

Theorem 6 (Weiss Bridge Theorem)**.** [85] *Let K be a totally disconnected compact abelian group and $\phi : K \to K$ a continuous endomorphism. Then $h_{top}(\phi) = \mathrm{ent}(\widehat{\phi})$.*

Proof. Since totally disconnected compact groups are zero-dimensional, every open finite cover $\mathcal{U}$ of K admits a refinement consisting of clopen sets in K. Moreover, since K admits a local base at 0 formed by open subgroups, it is possible to find a refinement of $\mathcal{U}$ of the form $\mathcal{U}_V$ for some open subgroup $\mathcal{V}$. This proves that $\mathfrak{cov}_s(K)$ is cofinal in $\mathfrak{cov}(K)$. Hence, we have

$$h_{top}(\phi) = h_{\mathfrak{S}}(\mathfrak{cov}(\phi)) = h_{\mathfrak{S}}(\mathfrak{cov}_s(\phi)).$$

Moreover, we have seen above that $\mathfrak{cov}_s(K)$ is isomorphic to $\mathfrak{sub}_o^\star(K)$, so one can conclude that

$$h_{\mathfrak{S}}(\mathfrak{cov}_s(\phi)) = h_{\mathfrak{S}}(\mathfrak{sub}_o^\star(\phi)).$$

Now the semilattice isomorphism $L \to \mathcal{F}(\widehat{K})$ given by $N \mapsto N^\perp$ preserves the norms, so it is an isomorphism in $\mathfrak{S}$. Hence

$$h_{\mathfrak{S}}(\mathfrak{sub}_o^\star(\phi)) = h_{\mathfrak{S}}(\mathfrak{sub}(\widehat{\phi}))$$

and consequently

$$h_{top}(\phi) = h_{\mathfrak{S}}(\mathfrak{sub}(\widehat{\phi})) = \mathrm{ent}(\widehat{\phi}).$$

$\boxed{QED}$

The proof of Weiss Bridge Theorem can be reassumed by the following diagram.

$$
\begin{tikzcd}
(\widehat{K},\widehat{\phi}) \arrow[r, "\mathfrak{sub}"] \arrow[ddrr, bend left, "h_{\mathfrak{sub}}"] & ((\mathcal{F}(\widehat{K}),+);\mathfrak{sub}(\widehat{\phi})) \arrow[d, "\widehat{}"'] \arrow[ddr, "h_{\mathfrak{S}}"] & \\
 & ((\mathfrak{sub}_o^\star(K),\cap);\mathfrak{sub}_o^\star(\phi)) \arrow[d, "\gamma"'] & \\
 & ((\mathfrak{cov}_s(K),\vee);\phi) \arrow[d, hook, "\iota"'] & \mathbb{R}^+ \\
(K,\phi) \arrow[uuu, bend left, "\widehat{}"] \arrow[r, "\mathfrak{cov}"'] \arrow[urr, bend right, "h_{\mathfrak{cov}}"'] & ((\mathfrak{cov}(K),\vee);\mathfrak{cov}(\phi)) \arrow[ur, "h_{\mathfrak{S}}"] &
\end{tikzcd}
$$

Similar Bridge Theorems hold for other known entropies; they can be proved using analogous diagrams (see [27]). The first one that we recall concerns the algebraic entropy ent and the adjoint algebraic entropy $\mathrm{ent}^\star$:

Theorem 7. *Let $\phi : G \to G$ be an endomorphism of an abelian group. Then* $\mathrm{ent}^\star(\phi) = \mathrm{ent}(\widehat{\phi})$.

The other two Bridge Theorems that we recall here connect respectively the set-theoretic entropy $\mathfrak{h}$ with the topological entropy h_{top} and the contravariant set-theoretic entropy $\mathfrak{h}^*$ with the algebraic entropy h_{alg}.

We need to recall first the notion of generalized shift, which extend the Bernoulli shifts. For a map $\lambda : X \to Y$ between two non-empty sets and a fixed non-trivial group K, define $\sigma_\lambda : K^Y \to K^X$ by $\sigma_\lambda(f) = f \circ \lambda$ for $f \in K^Y$. For $Y = X$, λ is a self-map of X and σ_λ was called *generalized shift* of K^X (see [3, 5]). In this case $\bigoplus_X K$ is a σ_λ-invariant subgroup of K^X precisely when λ is finitely many-to-one. We denote $\sigma_\lambda \restriction_{\bigoplus_X K}$ by $\sigma_\lambda^\oplus$.

Item (a) in the next theorem was proved in [5] (see also [20, Theorem 7.3.4]) while item (b) is [20, Theorem 7.3.3] (in the abelian case it was obtained in [3]).

Theorem 8. [5] *Let K be a non-trivial finite group, let X be a set and $\lambda : X \to X$ a self-map.*

(a) Then $h_{top}(\sigma_\lambda) = \mathfrak{h}(\lambda)\log|K|$.

(b) If λ is finite-to-one, then $h_{alg}(\sigma_\lambda^\oplus) = \mathfrak{h}^*(\lambda)\log|K|$.

In terms of functors, fixed a non-trivial finite group K, let $\mathcal{F}_K$: **Set** $\to$ **TdCGrp** be the functor defined on flows, sending a non-empty set X to K^X,

$\emptyset$ to 0, a self-map $\lambda : X \to X$ to $\sigma_\lambda : K^Y \to K^X$ when $X \neq \emptyset$. Then the pair $(\mathfrak{h}, h_{top})$ satisfies $(BT_{\mathcal{F}_K})$ with constant $\log |K|$.

Analogously, let $\mathcal{G}_K : \mathbf{Set}_{\mathrm{fin}} \to \mathbf{Grp}$ be the functor defined on flows sending X to $\bigoplus_X K$ and a finite-to-one self-map $\lambda : X \to X$ to $\sigma_\lambda^\oplus : \bigoplus_X K \to \bigoplus_X K$. Then the pair $(\mathfrak{h}^*, h_{alg})$ satisfies $(BT_{\mathcal{G}_K})$ with constant $\log |K|$.

Remark 5. At the conference held in Porto Cesareo, R. Farnsteiner posed the following question related to the Bridge Theorem. Is h_{top} studied in non-Hausdorff compact spaces?

The question was motivated by the fact that the prime spectrum $\mathrm{Spec}(A)$ of a commutative ring A is usually a non-Hausdorff compact space. Related to this question and to the entropy h_λ defined for endomorphisms ϕ of local Noetherian rings A (see §2.5), one may ask if there is any relation (e.g., a weak Bridge Theorem) between these two entropies and the functor Spec; more precisely, one can ask whether there is any stable relation between $h_{top}(\mathrm{Spec}(\phi))$ and $h_\lambda(\phi)$.

4 Algebraic entropy and its specific properties

In this section we give an overview of the basic properties of the algebraic entropy and the adjoint algebraic entropy. Indeed, we have seen that they satisfy the general scheme presented in the previous section, but on the other hand they were defined for specific group endomorphisms and these definitions permit to prove specific features, as we are going to briefly describe. For further details and examples see [19], [24] and [20].

4.1 Definition and basic properties

Let G be a group and $\phi : G \to G$ an endomorphism. For a finite subset F of G, and for $n \in \mathbb{N}_+$, the *n-th ϕ-trajectory* of F is

$$T_n(\phi, F) = F \cdot \phi(F) \cdot \ldots \cdot \phi^{n-1}(F);$$

moreover let

$$\gamma_{\phi,F}(n) = |T_n(\phi, F)|. \tag{8}$$

The *algebraic entropy of ϕ with respect to F* is

$$H_{alg}(\phi, F) = \lim_{n\to\infty} \frac{\log \gamma_{\phi,F}(n)}{n};$$

This limit exists as $H_{alg}(\phi, F) = h_{\mathfrak{S}}(\mathcal{H}(\phi), F)$ and so Theorem 1 applies (see also [20] for a direct proof of the existence of this limit and [19] for the abelian case). The *algebraic entropy* of $\phi : G \to G$ is

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, F) : F \text{ finite subset of } G\} = h_{\mathfrak{S}}(\mathcal{H}(\phi)).$$

Moreover

$$\mathrm{ent}(\phi) = \sup\{H_{alg}(\phi, F) : F \text{ finite subgroup of } G\}.$$

If G is abelian, then $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_{t(G)}) = h_{alg}(\phi \restriction_{t(G)})$.

Moreover, $h_{alg}(\phi) = \mathrm{ent}(\phi)$ if G is locally finite, that is every finite subset of G generates a finite subgroup; note that every locally finite group is obviously torsion, while the converse holds true under the hypothesis that the group is abelian (but the solution of Burnside Problem shows that even groups of finite exponent fail to be locally finite).

For every abelian group G, the identity map has $h_{alg}(\mathrm{id}_G) = 0$ (as the normed semigroup $\mathcal{H}(G)$ is arithmetic, as seen above). Another basic example is given by the endomorphisms of $\mathbb{Z}$, indeed if $\phi : \mathbb{Z} \to \mathbb{Z}$ is given by $\phi(x) = mx$ for some positive integer m, then $h_{alg}(\phi) = \log m$. The fundamental example for the algebraic entropy is the right Bernoulli shift:

Example 5. (Bernoulli normalization) Let K be a group.

(a) The *right Bernoulli shift* $\beta_K : K^{(\mathbb{N})} \to K^{(\mathbb{N})}$ is defined by

$$(x_0, \ldots, x_n, \ldots) \mapsto (1, x_0, \ldots, x_n, \ldots).$$

Then $h_{alg}(\beta_K) = \log |K|$, with the usual convention that $\log |K| = \infty$ when K is infinite.

(b) The *left Bernoulli shift* ${}_K\beta : K^{(\mathbb{N})} \to K^{(\mathbb{N})}$ is defined by

$$(x_0, \ldots, x_n, \ldots) \mapsto (x_1, \ldots, x_{n+1}, \ldots).$$

Then $h_{alg}({}_K\beta) = 0$, as ${}_K\beta$ is locally nilpotent.

The following basic properties of the algebraic entropy are consequences of the general scheme and were proved directly in [20].

Fact 1. *Let G be a group and $\phi : G \to G$ an endomorphism.*

(a) [Invariance under conjugation] *If $\phi = \xi^{-1}\psi\xi$, where $\psi : H \to H$ is an endomorphism and $\xi : G \to H$ isomorphism, then $h_{alg}(\phi) = h_{alg}(\psi)$.*

(b) [Monotonicity] *If H is a ϕ-invariant normal subgroup of the group G, and $\overline{\phi} : G/H \to G/H$ is the endomorphism induced by ϕ, then $h_{alg}(\phi) \geq \max\{h_{alg}(\phi \restriction_H), h_{alg}(\overline{\phi})\}$.*

(c) [Logarithmic Law] *For every* $k \in \mathbb{N}$ *we have* $h_{alg}(\phi^k) = k \cdot h_{alg}(\phi)$*; if* ϕ *is an automorphism, then* $h_{alg}(\phi) = h_{alg}(\phi^{-1})$*, so* $h_{alg}(\phi^k) = |k| \cdot h_{alg}(\phi)$ *for every* $k \in \mathbb{Z}$*.*

(d) [Continuity] *If* G *is direct limit of* ϕ*-invariant subgroups* $\{G_i : i \in I\}$*, then* $h_{alg}(\phi) = \sup_{i \in I} h_{alg}(\phi \restriction_{G_i})$*.*

(e) [Weak Addition Theorem] *If* $G = G_1 \times G_2$ *and* $\phi_i : G_i \to G_i$ *is an endomorphism for* $i = 1, 2$*, then* $h_{alg}(\phi_1 \times \phi_2) = h_{alg}(\phi_1) + h_{alg}(\phi_2)$*.*

As described for the semigroup entropy in the previous section, and as noted in [20, Remark 5.1.2], for group endomorphisms $\phi : G \to G$ it is possible to define also a "left" algebraic entropy, letting for a finite subset F of G, and for $n \in \mathbb{N}_+$,

$$T_n^{\#}(\phi, F) = \phi^{n-1}(F) \cdot \ldots \cdot \phi(F) \cdot F,$$

$$H_{alg}^{\#}(\phi, F) = \lim_{n \to \infty} \frac{\log |T_n^{\#}(\phi, F)|}{n}$$

and

$$h_{alg}^{\#}(\phi) = \sup\{H_{alg}^{\#}(\phi, F) : F \text{ finite subset of } G\}.$$

Answering a question posed in [20, Remark 5.1.2], we see now that

$$h_{alg}(\phi) = h_{alg}^{\#}(\phi).$$

Indeed, every finite subset of G is contained in a finite subset F of G such that $1 \in F$ and $F = F^{-1}$; for such F we have

$$H_{alg}(\phi, F) = H_{alg}^{\#}(\phi, F),$$

since, for every $n \in \mathbb{N}_+$,

$$T_n(\phi, F)^{-1} = \phi^{n-1}(F)^{-1} \cdot \ldots \cdot \phi(F)^{-1} \cdot F^{-1} = \\ \phi^{n-1}(F^{-1}) \cdot \ldots \cdot \phi(F^{-1}) \cdot F^{-1} = T_n^{\#}(\phi, F)$$

and so $|T_n(\phi, F)| = |T_n(\phi, F)^{-1}| = |T_n^{\#}(\phi, F)|$.

4.2 Algebraic Yuzvinski Formula, Addition Theorem and Uniqueness

We recall now some of the main deep properties of the algebraic entropy in the abelian case. They are not consequences of the general scheme and are proved using the specific features of the algebraic entropy coming from the definition given above. We give here the references to the papers where these results were proved, for a general exposition on algebraic entropy see the survey paper [20].

The next proposition shows that the study of the algebraic entropy for torsion-free abelian groups can be reduced to the case of divisible ones. It was announced for the first time by Yuzvinski [91], for a proof see [19].

Proposition 1. *Let G be a torsion-free abelian group, $\phi : G \to G$ an endomorphism and denote by $\widetilde{\phi}$ the (unique) extension of ϕ to the divisible hull $D(G)$ of G. Then $h_{alg}(\phi) = h_{alg}(\widetilde{\phi})$.*

Let $f(t) = a_n t^n + a_1 t^{n-1} + \ldots + a_0 \in \mathbb{Z}[t]$ be a primitive polynomial and let $\{\lambda_i : i = 1, \ldots, n\} \subseteq \mathbb{C}$ be the set of all roots of $f(t)$. The *(logarithmic) Mahler measure* of $f(t)$ is

$$m(f(t)) = \log |a_n| + \sum_{|\lambda_i|>1} \log |\lambda_i|.$$

The Mahler measure plays an important role in number theory and arithmetic geometry and is involved in the famous Lehmer Problem, asking whether $\inf\{m(f(t)) : f(t) \in \mathbb{Z}[t] \text{ primitive}, m(f(t)) > 0\} > 0$ (for example see [31] and [50]).

If $g(t) \in \mathbb{Q}[t]$ is monic, then there exists a smallest positive integer s such that $sg(t) \in \mathbb{Z}[t]$; in particular, $sg(t)$ is primitive. The Mahler measure of $g(t)$ is defined as $m(g(t)) = m(sg(t))$. Moreover, if $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ is an endomorphism, its characteristic polynomial $p_\phi(t) \in \mathbb{Q}[t]$ is monic, and the Mahler measure of ϕ is $m(\phi) = m(p_\phi(t))$.

The formula (9) below was given a direct proof recently in [37]; it is the algebraic counterpart of the so-called Yuzvinski Formula for the topological entropy [91] (see also [54]). It gives the values of the algebraic entropy of linear transformations of finite dimensional rational vector spaces in terms of the Mahler measure, so it allows for a connection of the algebraic entropy with Lehmer Problem.

Theorem 9 (Algebraic Yuzvinski Formula)**.** [37] *Let $n \in \mathbb{N}_+$ and $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ an endomorphism. Then*

$$h_{alg}(\phi) = m(\phi). \tag{9}$$

The next property of additivity of the algebraic entropy was first proved for torsion abelian groups in [28], while the proof of the general case was given in [19] applying the Algebraic Yuzvinski Formula.

Theorem 10 (Addition Theorem)**.** [19] *Let G be an abelian group, $\phi : G \to G$ an endomorphism, H a ϕ-invariant subgroup of G and $\overline{\phi} : G/H \to G/H$ the endomorphism induced by ϕ. Then*

$$h_{alg}(\phi) = h_{alg}(\phi \restriction_H) + h_{alg}(\overline{\phi}).$$

Moreover, uniqueness is available for the algebraic entropy in the category of all abelian groups. As in the case of the Addition Theorem, also the Uniqueness Theorem was proved in general in [19], while it was previously proved in [28] for torsion abelian groups.

Theorem 11 (Uniqueness Theorem)**.** [19] *The algebraic entropy*

$$h_{alg} : \mathrm{Flow}_{\mathbf{AbGrp}} \to \mathbb{R}_+$$

is the unique function such that:

(a) h_{alg} is invariant under conjugation;

(b) h_{alg} is continuous on direct limits;

(c) h_{alg} satisfies the Addition Theorem;

(d) for K a finite abelian group, $h_{alg}(\beta_K) = \log |K|$;

(e) h_{alg} satisfies the Algebraic Yuzvinski Formula.

4.3 The growth of a finitely generated flow in Grp

In order to measure and classify the growth rate of maps $\mathbb{N} \to \mathbb{N}$, one need the relation $\preceq$ defined as follows. For $\gamma, \gamma' : \mathbb{N} \to \mathbb{N}$ let $\gamma \preceq \gamma'$ if there exist $n_0, C \in \mathbb{N}_+$ such that $\gamma(n) \leq \gamma'(Cn)$ for every $n \geq n_0$. Moreover $\gamma \sim \gamma$ if $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$ (then $\sim$ is an equivalence relation), and $\gamma \prec \gamma'$ if $\gamma \preceq \gamma'$ but $\gamma \not\sim \gamma'$.

For example, for every $\alpha, \beta \in \mathbb{R}_{\geq 0}$, $n^\alpha \sim n^\beta$ if and only if $\alpha = \beta$; if $p(t) \in \mathbb{Z}[t]$ and $p(t)$ has degree $d \in \mathbb{N}$, then $p(n) \sim n^d$. On the other hand, $a^n \sim b^n$ for every $a, b \in \mathbb{R}$ with $a, b > 1$, so in particular all exponentials are equivalent with respect to $\sim$.

So a map $\gamma : \mathbb{N} \to \mathbb{N}$ is called:

(a) *polynomial* if $\gamma(n) \preceq n^d$ for some $d \in \mathbb{N}_+$;

(b) *exponential* if $\gamma(n) \sim 2^n$;

(c) *intermediate* if $\gamma(n) \succ n^d$ for every $d \in \mathbb{N}_+$ and $\gamma(n) \prec 2^n$.

Let G be a group, $\phi : G \to G$ an endomorphism and F a non-empty finite subset of G. Consider the function, already mentioned in (8),

$$\gamma_{\phi,F} : \mathbb{N}_+ \to \mathbb{N}_+ \text{ defined by } \gamma_{\phi,F}(n) = |T_n(\phi, F)| \text{ for every } n \in \mathbb{N}_+.$$

Since

$$|F| \leq \gamma_{\phi,F}(n) \leq |F|^n \text{ for every } n \in \mathbb{N}_+,$$

the growth of $\gamma_{\phi,F}$ is always at most exponential; moreover, $H_{alg}(\phi, F) \leq \log |F|$. So, following [22] and [20], we say that ϕ has *polynomial* (respectively, *exponential*, *intermediate*) *growth at* F if $\gamma_{\phi,F}$ is polynomial (respectively, exponential, intermediate).

Before proceeding further, let us make an important point here. All properties considered above concern practically the ϕ-invariant subgroup $G_{\phi,F}$ of G generated by the trajectory $T(\phi, F) = \bigcup_{n\in\mathbb{N}_+} T_n(\phi, F)$ and the restriction $\phi \restriction_{G_{\phi,F}}$.

Definition 8. We say that the flow (G, ϕ) in **Grp** is *finitely generated* if $G = G_{\phi,F}$ for some finite subset F of G.

Hence, all properties listed above concern finitely generated flows in **Grp**. We conjecture the following, knowing that it holds true when G is abelian or when $\phi = if_G$: if the flow (G, ϕ) is finitely generated, and if $G = G_{\phi,F}$ and $G = G_{\phi,F'}$ for some finite subsets F and F' of G, then $\gamma_{\phi,F}$ and $\gamma_{\phi,F'}$ have the same type of growth.
In this case the growth of a finitely generated flow $G_{\phi,F}$ would not depend on the specific finite set of generators F (so F can always be taken symmetric). In particular, one could speak of growth of a finitely generated flow without any reference to a specific finite set of generators. Nevertheless, one can give in general the following.

Definition 9. Let (G, ϕ) be a finitely generated flow in **Grp**. We say that (G, ϕ) has

(a) *polynomial growth* if $\gamma_{\phi,F}$ is polynomial for every finite subset F of G;

(b) *exponential growth* if there exists a finite subset F of G such that $\gamma_{\phi,F}$ is exponential;

(c) *intermediate growth* otherwise.

We denote by Pol and Exp the classes of finitely generated flows in **Grp** of polynomial and exponential growth respectively. Moreover, $\mathcal{M} = \mathrm{Pol} \cup \mathrm{Exp}$ is the class of finitely generated flows of non-intermediate growth.

This notion of growth generalizes the classical one of growth of a finitely generated group given independently by Schwarzc [72] and Milnor [56]. Indeed, if G is a finitely generated group and X is a finite symmetric set of generators of G, then $\gamma_X = \gamma_{\mathrm{id}_G,X}$ is the classical *growth function* of G with respect to X. For a connection of the terminology coming from the theory of algebraic entropy and the classical one, note that for $n \in \mathbb{N}_+$ we have $T_n(\mathrm{id}_G, X) = \{g \in G : \ell_X(g) \leq n\}$, where $\ell_X(g)$ is the length of the shortest word w in the alphabet X such that $w = g$ (see §2.5 (c)). Since ℓ_X is a norm on G, $T_n(\mathrm{id}_G, X)$ is the ball of radius n centered at 1 and $\gamma_X(n)$ is the cardinality of this ball.

Milnor [58] proposed the following problem on the growth of finitely generated groups.

Problem 1 (Milnor Problem)**.** [58] Let G be a finitely generated group and X a finite set of generators of G.

(i) Is the growth function γ_X necessarily equivalent either to a power of n or to the exponential function 2^n?

(ii) In particular, is the growth exponent $\delta_G = \limsup_{n\to\infty} \frac{\log \gamma_X(n)}{\log n}$ either a well defined integer or infinity? For which groups is δ_G finite?

Part (i) of Problem 1 was solved negatively by Grigorchuk in [42, 43, 44, 45], where he constructed his famous examples of finitely generated groups $\mathbb{G}$ with intermediate growth. For part (ii) Milnor conjectured that δ_G is finite if and only if G is virtually nilpotent (i.e., G contains a nilpotent finite-index subgroup). The same conjecture was formulated by Wolf [89] (who proved that a nilpotent finitely generated group has polynomial growth) and Bass [6]. Gromov [47] confirmed Milnor's conjecture:

Theorem 12 (Gromov Theorem)**.** [47] *A finitely generated group G has polynomial growth if and only if G is virtually nilpotent.*

The following two problems on the growth of finitely generated flows of groups are inspired by Milnor Problem.

Problem 2. Describe the permanence properties of the class $\mathcal{M}$.

Some stability properties of the class $\mathcal{M}$ are easy to check. For example, stability under taking finite direct products is obviously available, while stability under taking subflows (i.e., invariant subgroups) and factors fails even in the classical case of identical flows. Indeed, Grigorchuk's group $\mathbb{G}$ is a quotient of a finitely generated free group F, that has exponential growth; so $(F, \mathrm{id}_F) \in \mathcal{M}$, while $(\mathbb{G}, \mathrm{id}_{\mathbb{G}}) \notin \mathcal{M}$. Furthermore, letting $G = \mathbb{G} \times F$, one has $(G, \mathrm{id}_G) \in \mathcal{M}$, while $(\mathbb{G}, \mathrm{id}_{\mathbb{G}}) \notin \mathcal{M}$, so $\mathcal{M}$ is not stable even under taking direct summands. On the other hand, stability under taking powers is available since $(G, \phi) \in \mathcal{M}$ if and only if $(G, \phi^n) \in \mathcal{M}$ for $n \in \mathbb{N}_+$.

Problem 3.

(i) Describe the finitely generated groups G such that $(G, \phi) \in \mathcal{M}$ for every endomorphism $\phi : G \to G$.

(ii) Does there exist a finitely generated group G such that $(G, \mathrm{id}_G) \in \mathcal{M}$ but $(G, \phi) \notin \mathcal{M}$ for some endomorphism $\phi : G \to G$?

In item (i) of the above problem we are asking to describe all finitely generated groups G of non-intermediate growth such that (G, ϕ) has still non-intermediate growth for every endomorphism $\phi : G \to G$. On the other hand, in item (ii) we ask to find a finitely generated group G of non-intermediate growth that admits an endomorphism $\phi : G \to G$ of intermediate growth.

The basic relation between the growth and the algebraic entropy is given by below Proposition 2. For a finitely generated group G, an endomorphism ϕ of G and a pair X and X' of finite generators of G, one has $\gamma_{\phi,X} \sim \gamma_{\phi,X'}$. Nevertheless, $H_{alg}(\phi, X) \neq H_{alg}(\phi, X')$ may occur; in this case (G, ϕ) has necessarily exponential growth. We give two examples to this effect:

Example 6. (a) [20] Let G be the free group with two generators a and b; then $X = \{a^{\pm 1}, b^{\pm 1}\}$ gives $H_{alg}(\mathrm{id}_G, X) = \log 3$ while for $X' = \{a^{\pm 1}, b^{\pm 1}, (ab)^{\pm 1}\}$ we have $H_{alg}(\mathrm{id}_G, X') = \log 4$.

(b) Let $G = \mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(x) = mx$ for every $x \in \mathbb{Z}$ and with $m > 3$. Let also $X = \{0, \pm 1\}$ and $X' = \{0, \pm 1, \ldots \pm m\}$. Then $H_{alg}(\phi, X) \leq \log |X| = \log 3$, while $H_{alg}(\phi, X') = h_{alg}(\phi) = \log m$.

Proposition 2. [20] *Let (G, ϕ) be a finitely generated flow in* **Grp**.

(a) Then $h_{alg}(\phi) > 0$ if and only if (G, ϕ) has exponential growth.

(b) If (G, ϕ) has polynomial growth, then $h_{alg}(\phi) = 0$.

In general the converse implication in item (b) is not true even for the identity. Indeed, if (G, ϕ) has intermediate growth, then $h_{alg}(\phi) = 0$ by item (a). So for Grigorchuk's group $\mathbb{G}$, the flow $(\mathbb{G}, \mathrm{id}_{\mathbb{G}})$ has intermediate growth yet $h_{alg}(\mathrm{id}_{\mathbb{G}}) = 0$. This motivates the following

Definition 10. Let $\mathcal{G}$ be a class of groups and Φ be a class of morphisms. We say that the pair $(\mathcal{G}, \Phi)$ satisfies Milnor Paradigm (briefly, MP) if no finitely generated flow (G, ϕ) with $G \in \mathcal{G}$ and $\phi \in \Phi$ can have intermediate growth.

In terms of the class $\mathcal{M}$,

$$(\mathcal{G}, \Phi) \text{ satisfies MP if and only if } (\mathcal{G}, \Phi) \in \mathcal{M}\ (\forall G \in \mathcal{G})(\forall \phi \in \Phi).$$

Equivalently, $(\mathcal{G}, \Phi)$ satisfies MP when $h_{alg}(\phi) = 0$ always implies that (G, ϕ) has polynomial growth for finitely generated flows (G, ϕ) with $G \in \mathcal{G}$ and $\phi \in \Phi$.

In these terms Milnor Problem 1 (i) is asking whether the pair $(\mathbf{Grp}, \mathcal{I}d)$ satisfies MP, where $\mathcal{I}d$ is the class of all identical endomorphisms. So we give the following general open problem.

Problem 4. (i) Find pairs $(\mathcal{G}, \Phi)$ satisfying MP.

(ii) For a given Φ determine the properties of the largest class $\mathcal{G}_\Phi$ such that $(\mathcal{G}_\Phi, \Phi)$ satisfies MP.

(iii) For a given $\mathcal{G}$ determine the properties of the largest class $\Phi_{\mathcal{G}}$ such that $(\mathcal{G}, \Phi_{\mathcal{G}})$ satisfies MP.

(iv) Study the Galois correspondence between classes of groups $\mathcal{G}$ and classes of endomorphisms Φ determined by MP.

According to the definitions, the class $\mathcal{G}_{\mathcal{I}d}$ coincides with the class of finitely generated groups of non-intermediate growth.

The following result solves Problem 4 (iii) for $\mathcal{G} = \mathbf{AbGrp}$, showing that $\Phi_{\mathbf{AbGrp}}$ coincides with the class $\mathcal{E}$ of all endomorphisms.

Theorem 13 (Dichotomy Theorem)**.** [22] *There exist no finitely generated flows of intermediate growth in* **AbGrp**.

Actually, one can extend the validity of this theorem to nilpotent groups. This leaves open the following particular case of Problem 4. We shall see in Theorem 14 that the answer to (i) is positive when $\phi = \mathrm{id}_G$.

Question 1. Let (G, ϕ) be a finitely generated flow in **Grp**.

(i) If G is solvable, does $(G, \phi) \in \mathcal{M}$?

(ii) If G is a free group, does $(G, \phi) \in \mathcal{M}$?

We state now explicitly a particular case of Problem 4, inspired by the fact that the right Bernoulli shifts have no non-trivial quasi-periodic points and they have uniform exponential growth (see Example 7). In [22] group endomorphisms $\phi : G \to G$ without non-trivial quasi-periodic points are called algebraically ergodic for their connection (in the abelian case and through Pontryagin duality) with ergodic transformations of compact groups.

Question 2. Let Φ_0 be the class of endomorphisms without non-trivial quasi-periodic points. Is it true that the pair $(\mathbf{Grp}, \Phi_0)$ satisfies MP?

For a finitely generated group G, the *uniform exponential growth rate* of G is defined as

$$\lambda(G) = \inf\{H_{alg}(\mathrm{id}_G, X) : X \text{ finite set of generators of } G\}$$

(see for instance [15]). Moreover, G has *uniform exponential growth* if $\lambda(G) > 0$. Gromov [48] asked whether every finitely generated group of exponential growth is also of uniform exponential growth. This problem was recently solved by Wilson [88] in the negative.

Since the algebraic entropy of a finitely generated flow (G, ϕ) in **Grp** can be computed as

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, F) : F \text{ finite subset of } G \text{ such that } G = G_{\phi,F}\},$$

one can give the following counterpart of the uniform exponential growth rate for flows:

Definition 11. For (G, ϕ) be a finitely generated flow in **Grp** let

$$\lambda(G, \phi) = \inf\{H_{alg}(\phi, F) : F \text{ finite subset of } G \text{ such that } G = G_{\phi,F}\}.$$

The flow (G, ϕ) is said to have *uniformly exponential growth* if $\lambda(G, \phi) > 0$.

Let $\mathrm{Exp_u}$ be the subclass of Exp of all finitely generated flows in **Grp** of uniform exponential growth.

Clearly $\lambda(G, \phi) \leq h_{alg}(\phi)$, so one has the obvious implication

$$h_{alg}(\phi) = 0 \ \Rightarrow \ \lambda(G, \phi) = 0. \tag{10}$$

To formulate the counterpart of Gromov's problem on uniformly exponential growth it is worth to isolate also the class $\mathcal{W}$ of the finitely generated flows in **Grp** of exponential but not uniformly exponential growth (i.e., $\mathcal{W} = \mathrm{Exp} \setminus \mathrm{Exp_u}$). Then $\mathcal{W}$ is the class of finitely generated flows (G, ϕ) in **Grp** for which (10) cannot be inverted, namely $h_{alg}(\phi) > 0 = \lambda(G, \phi)$.

We start stating the following problem.

Problem 5. Describe the permanence properties of the classes $\mathrm{Exp_u}$ and $\mathcal{W}$.

It is easy to check that $\mathrm{Exp_u}$ and $\mathcal{W}$ are stable under taking direct products. On the other hand, stability of $\mathrm{Exp_u}$ under taking subflows (i.e., invariant subgroups) and factors fails even in the classical case of identical flows. Indeed, Wilson's group $\mathbb{W}$ is a quotient of a finitely generated free group F, that has uniform exponential growth (see [15]); so $(F, \mathrm{id}_F) \in \mathrm{Exp_u}$, while $(\mathbb{W}, \mathrm{id}_{\mathbb{W}}) \in \mathcal{W}$. Furthermore, letting $G = \mathbb{W} \times F$, one has $(G, \mathrm{id}_G) \in \mathrm{Exp_u}$, while $(\mathbb{W}, \mathrm{id}_{\mathbb{W}}) \in \mathcal{W}$, so $\mathrm{Exp_u}$ is not stable even under taking direct summands.

In the line of MP, introduced in Definition 10, we can formulate also the following

Definition 12. Let $\mathcal{G}$ be a class of groups and Φ be a class of morphisms. We say that the pair $(\mathcal{G}, \Phi)$ satisfies Gromovr Paradigm (briefly, MP), if every finitely generated flow (G, ϕ) with $G \in \mathcal{G}$ and $\phi \in \Phi$ of exponential growth has has uniform exponential growth.

In terms of the class $\mathcal{W}$,

$$(\mathcal{G}, \Phi) \text{ satisfies GP if and only if } \ (\mathcal{G}, \Phi) \notin \mathcal{M} \ (\forall G \in \mathcal{G})(\forall \phi \in \Phi).$$

In these terms, Gromov's problem on uniformly exponential growth asks whether the pair $(\mathbf{Grp}, \mathcal{I}d)$ satisfies GP. In analogy to the general Problem 4, one can consider the following obvious counterpart for GP:

Problem 6. (i) Find pairs $(\mathcal{G}, \Phi)$ satisfying GP.

(ii) For a given Φ determine the properties of the largest class $\mathcal{G}_\Phi$ such that $(\mathcal{G}_\Phi, \Phi)$ satisfies GP.

(iii) For a given $\mathcal{G}$ determine the properties of the largest class $\Phi_\mathcal{G}$ such that $(\mathcal{G}, \Phi_\mathcal{G})$ satisfies GP.

(iv) Study the Galois correspondence between classes of groups $\mathcal{G}$ and classes of endomorphisms Φ determined by GP.

We see now in item (a) of the next example a particular class of finitely generated flows for which λ coincides with h_{alg} and they are both positive, so in particular these flows are all in $\mathrm{Exp_u}$. In item (b) we leave an open question related to Question 2.

Example 7. (a) For a finite group K, consider the flow $(\bigoplus_\mathbb{N} K, \beta_K)$. We have seen in Example 5 that $h_{alg}(\beta_K) = \log|K|$. In this case we have $\lambda(\bigoplus_\mathbb{N} K, \beta_K) = \log|K|$, since a subset F of $\bigoplus_\mathbb{N} K$ generating the flow $(\bigoplus_\mathbb{N} K, \beta_K)$ must contain the first copy K_0 of K in $\bigoplus_\mathbb{N} K$, and $H_{alg}(\beta_K, K_0) = \log|K|$.

(b) Is it true that $\lambda(G, \phi) = h_{alg}(\phi) > 0$ for every finitely generated flow (G, ϕ) in **Grp** such that $\phi \in \Phi_0$? In other terms, we are asking whether all finitely generated flows (G, ϕ) in **Grp** with $\phi \in \Phi_0$ have uniform exponential growth (i.e., are contained in $\mathrm{Exp_u}$).

One can also consider the pairs $(\mathcal{G}, \Phi)$ satisfying the conjunction MP & GP. For any finitely generated flow (G, ϕ) in **Grp** one has

$$(G, \phi) \text{ has polynomial growth } \overset{(1)}{\Longrightarrow} h_{alg}(\phi) = 0 \overset{(2)}{\Longrightarrow} \lambda(G, \phi) = 0. \qquad (11)$$

The converse implication of (1) (respectively, (2)) holds for all (G, ϕ) with $G \in \mathcal{G}$ and $\phi \in \Phi$ precisely when the pair $(\mathcal{G}, \Phi)$ satisfies MP (respectively, GP). Therefore, the pair $(\mathcal{G}, \Phi)$ satisfies the conjunction MP & GP precisely when the three conditions in (11) are all equivalent (i.e., $\lambda(G, \phi) = 0 \Rightarrow (G, \phi) \in \mathrm{Pol}$) for all finitely generated flows (G, ϕ) with $G \in \mathcal{G}$ and $\phi \in \Phi$.

A large class of groups $\mathcal{G}$ such that $(\mathcal{G}, \mathcal{I}d)$ satisfies MP & GP was found by Osin [62] who proved that a finitely generated solvable group G of zero uniform exponential growth is virtually nilpotent, and recently this result was generalized in [63] to elementary amenable groups. Together with Gromov Theorem and Proposition 2, this gives immediately the following

Theorem 14. *Let G be a finitely generated elementary amenable group. The following conditions are equivalent:*

(a) $h_{alg}(\mathrm{id}_G) = 0$*;*

(b) $\lambda(G) = 0$*;*

(c) G *is virtually nilpotent;*

(d) G *has polynomial growth.*

This theorem shows that the pair $\mathcal{G} = \{$elementary amenable groups$\}$ and $\Phi = \mathcal{I}d$ satisfies simultaneously MP and GP. In other words it proves that the three conditions in (11) are all equivalent when G is an elementary amenable finitely generated group and $\phi = \mathrm{id}_G$.

4.4 Adjoint algebraic entropy

We recall here the definition of the adjoint algebraic entropy $\mathrm{ent}^\star$ and we state some of its specific features not deducible from the general scheme, so beyond the "package" of general properties coming from the equality $\mathrm{ent}^\star = h_{\mathfrak{sub}^\star}$ such as Invariance under conjugation and inversion, Logarithmic Law, Monotonicity for factors (these properties were proved in [20] in the general case and previously in [24] in the abelian case applying the definition).

In analogy to the algebraic entropy ent, in [24] the adjoint algebraic entropy of endomorphisms of abelian groups G was introduced "replacing" the family $\mathcal{F}(G)$ of all finite subgroups of G with the family $\mathcal{C}(G)$ of all finite-index subgroups of G. The same definition was extended in [20] to the more general setting of endomorphisms of arbitrary groups as follows. Let G be a group and $N \in \mathcal{C}(G)$. For an endomorphism $\phi : G \to G$ and $n \in \mathbb{N}_+$, the *n-th ϕ-cotrajectory of N* is

$$C_n(\phi, N) = N \cap \phi^{-1}(N) \cap \ldots \cap \phi^{-n+1}(N).$$

The *adjoint algebraic entropy of ϕ with respect to N* is

$$H^\star(\phi, N) = \lim_{n\to\infty} \frac{\log[G : C_n(\phi, N)]}{n}.$$

This limit exists as $H^\star(\phi, N) = h_{\mathfrak{S}}(\mathcal{C}(\phi), N)$ and so Theorem 1 applies. The *adjoint algebraic entropy of ϕ* is

$$\mathrm{ent}^\star(\phi) = \sup\{H^\star(\phi, N) : N \in \mathcal{C}(G)\}.$$

The values of the adjoint algebraic entropy of the Bernoulli shifts were calculated in [24, Proposition 6.1] applying [34, Corollary 6.5] and the Pontryagin duality; a direct computation can be found in [35]. So, in contrast with what occurs for the algebraic entropy, we have:

Example 8 (Bernoulli shifts)**.** For K a non-trivial group,

$$\mathrm{ent}^\star(\beta_K) = \mathrm{ent}^\star({}_K\beta) = \infty.$$

As proved in [24], the adjoint algebraic entropy satisfies the Weak Addition Theorem, while the Monotonicity for invariant subgroups fails even for torsion abelian groups; in particular, the Addition Theorem fails in general. On the other hand, the Addition Theorem holds for bounded abelian groups:

Theorem 15 (Addition Theorem)**.** *Let G be a bounded abelian group, $\phi : G \to G$ an endomorphism, H a ϕ-invariant subgroup of G and $\overline{\phi} : G/H \to G/H$ the endomorphism induced by ϕ. Then*

$$\mathrm{ent}^\star(\phi) = \mathrm{ent}^\star(\phi\restriction_H) + \mathrm{ent}^\star(\overline{\phi}).$$

The following is one of the main results on the adjoint algebraic entropy proved in [24]. It shows that the adjoint algebraic entropy takes values only in $\{0, \infty\}$, while clearly the algebraic entropy may take also finite positive values.

Theorem 16 (Dichotomy Theorem)**.** [24] *Let G be an abelian group and $\phi : G \to G$ an endomorphism. Then*

$$\textit{either } \mathrm{ent}^\star(\phi) = 0 \textit{ or } \mathrm{ent}^\star(\phi) = \infty.$$

Applying the Dichotomy Theorem and the Bridge Theorem (stated in the previous section) to the compact dual group K of G one gets that for a continuous endomorphism ψ of a compact abelian group K either $\mathrm{ent}(\psi) = 0$ or $\mathrm{ent}(\psi) = \infty$. In other words:

Corollary 3. *If K is a compact abelian group, then every endomorphism $\psi : K \to K$ with $0 < \mathrm{ent}(\psi) < \infty$ is discontinuous.*

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Dynkin diagrams, support spaces and representation type

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Abstract. This survey article is an expanded version of a series of lectures given at the conference on Advances in Group Theory and its Applications which was held in Porto Cesareo in June of 2011. We are concerned with representations of finite group schemes, a class of objects that generalizes the more familiar finite groups. In the last 30 years, this discipline has enjoyed considerable attention. One reason is the application of geometric techniques that originate in Quillen's fundamental work concerning the spectrum of the cohomology ring [25, 26]. The subsequent developments pertaining to cohomological support varieties and representation-theoretic support spaces have resulted in many interesting applications. Here we will focus on those aspects of the theory that are motivated by the problem of classifying indecomposable modules. Since the determination of the simple modules is often already difficult enough, one can in general not hope to solve this problem in a naive sense. However, the classification problem has resulted in an important subdivision of the category of algebras, which will be our general theme.

The algebras we shall be interested in are the so-called cocommutative Hopf algebras, which are natural generalizations of group algebras of finite groups. The module categories of these algebras are richer than those of arbitrary algebras:

- They afford tensor products which occasionally allow the transfer of information between various blocks of the algebra.
- Their cohomology rings are finitely generated, making geometric methods amenable to application.

The purpose of these notes is to illustrate how a combination of these features with methods from the abstract representation theory of algebras and quivers provides insight into classical questions.

Keywords: Dynkin diagram, support variety, representation type, small quantum groups

MSC 2000 classification: Primary 16G70, Secondary 17B50

1 Motivation and basic examples

1.1 Motivation

We fix the following notation once and for all:

- k denotes an algebraically closed field.

- Unless mentioned otherwise all k-vector spaces are assumed to be finite-dimensional.

- Λ denotes an associative k-algebra.

Given $d \in \mathbb{N}$, we let mod_Λ^d be the affine variety of d-dimensional Λ-modules. More precisely, mod_Λ^d is the variety of Λ-module structures on a fixed d-dimensional k-vector space V. If $\{x_1, \ldots, x_n\} \subseteq \Lambda$ is a basis of Λ such that $x_1 = 1$ and $x_i x_j = \sum_{\ell=1}^n \alpha_{ij\ell} x_\ell$, then a representation of Λ on V is given by an n-tuple $(A_1, \ldots, A_n)$ of $(d \times d)$-matrices such that $A_1 = I_d$ and $A_i A_j = \sum_{\ell=1}^n \alpha_{ij\ell} A_\ell$. In this fashion, mod_Λ^d is a Zariski closed subspace of k^{nd^2}.

The algebraic group $\mathrm{GL}_d(k)$ acts on mod_Λ^d via conjugation. Thus, the orbits correspond to the isoclasses of Λ-modules. Note that the set ind_Λ^d of indecomposable modules of mod_Λ^d is $\mathrm{GL}_d(k)$-invariant. (The set ind_Λ^d is a constructible subset of mod_Λ^d.)

Definition 1. Given $d \in \mathbb{N}$, we let $C_d \subseteq \mathrm{mod}_\Lambda^d$ be a closed subset of minimal dimension subject to $\mathrm{ind}_\Lambda^d \subseteq \mathrm{GL}_d(k).C_d$. The algebra Λ is

(a) *representation-finite*, provided $\dim C_d = 0$ for every $d \in \mathbb{N}$,

(b) *tame*, provided Λ is not representation-finite and $\dim C_d \leq 1$ for all $d \in \mathbb{N}$,

(c) *wild*, otherwise.

Remark 1. (1) An algebra is representation-finite if and only if there are only finitely many isoclasses of indecomposable Λ-modules. This follows from the so-called second Brauer-Thrall conjecture for Artin algebras, which is known to hold in our context.

(2) If an algebra is wild, then its module category is at least as complicated as that of any other algebra. For such algebras the classification of its indecomposable modules is deemed hopeless [3].

Example 1. (1) Every semi-simple algebra is representation-finite.

(2) The algebra $k[X]/(X^n)$ is representation-finite.

(3) More generally, Nakayama algebras are representation-finite. By definition, the projective indecomposable and injective indecomposable modules of such algebras are uniserial.

(4) The Kronecker algebra $k[X,Y]/(X^2, Y^2)$ is tame.

One may ask what this subdivision looks like for certain classes of algebras. As the representation type of an algebra is an invariant of its Morita equivalence class, the criteria one is looking for are often given in terms of the associated basic algebras. Such algebras can be described by finite directed graphs.

Definition 2. Let Q be a quiver (a directed graph). The *path algebra kQ* of Q has an underlying vector space, whose basis consists of all paths. We multiply paths by concatenation if possible and postulate that their product be zero otherwise.

The above definition is meant to include paths e_i of length 0, labelled by the vertices of Q. They form a system of primitive orthogonal idempotents of kQ. If the quiver is finite, then $\sum_i e_i$ is the identity element of kQ. We shall only be concerned with finite quivers (i.e., Q has finitely many vertices and arrows). In that case, kQ is finite-dimensional if and only if Q does not afford any oriented cycles. The following basic result concerning Morita equivalence $\sim_M$ of k-algebras indicates an interesting connection between representations of quivers and Lie theory:

Theorem 1. *Let Λ be an associative k-algebra.*

(1) There exists a finite quiver Q_Λ and a certain ideal $I \trianglelefteq kQ_\Lambda$ such that $\Lambda \sim_M kQ_\Lambda/I$ [14].

(2) If Λ is hereditary (i.e., submodules of projectives are projective) and Q_Λ is connected, then $\Lambda \sim_M kQ_\Lambda$ and

  (a) Λ is representation-finite if and only if Q_Λ is a Dynkin diagram of type A, D, E [14].

  (b) Λ is tame if and only if Q_Λ is an extended Dynkin diagram of type $\tilde{A}, \tilde{D}, \tilde{E}$ [2, 24].

In either case, the indecomposable modules can be classified via the associated root system.

The quiver Q_Λ is the so-called Ext-*quiver* of Λ. Its vertices are formed by a complete set of representatives for the simple Λ-modules. There are $\dim_k \operatorname{Ext}^1_\Lambda(S,T)$ arrows from S to T. There is no general rule for the computation of the relations generating the non-unique ideal I.

While the above results are very satisfactory from the point of view of abstract representation theory, they do rely on the knowledge of the quiver and the relations of the given algebra. However, even if an algebra is basic to begin with (that is, if all simple modules are one-dimensional), the given presentation may not be suitable for our purposes. Let me illustrate this point by considering an easy example.

Example 2. Let $\operatorname{char}(k) = p > 0$, and consider the algebra given by

$$\Lambda = k\langle t, x\rangle/(tx - xt - x, t^p - t, x^p).$$

This is the natural presentation of the restricted enveloping algebra of the two-dimensional, non-abelian Lie algebra. The bound quiver presentation we are looking for is

$$\Lambda \cong k\tilde{A}_{p-1}/(k\tilde{A}_{p-1})_{\geq p},$$

where the quiver $\tilde{A}_{p-1}$ is the clockwise oriented circle with p vertices and $(k\tilde{A}_{p-1})_{\geq p}$ is the subspace with basis the set of all paths of length $\geq p$.

The more complicated quiver presentation contains more information. One readily sees that Λ is a Nakayama algebra, which is not apparent in the natural presentation.

In these notes we will show how a combination of geometric and representation theoretic methods affords the transition to such a more complicated presentation for certain Hopf algebras of positive characteristic. The classical examples of Hopf algebras are of course the group algebras of finite groups. Here we have the following situation:

Theorem 2. *Suppose that* $\mathrm{char}(k) = p > 0$. *Let kG be the group algebra of a finite group G, $P \subseteq G$ be a Sylow-p-subgroup.*

(1) kG is representation-finite $\Leftrightarrow$ P is cyclic [16].

(2) kG is tame $\Leftrightarrow$ $p = 2$, and P is dihedral, semidihedral, or generalized quaternion [1].

Like any algebra, the group algebra kG is the direct sum of indecomposable two-sided ideals of kG, the so-called *blocks* of kG. Each block is an algebra in its own right and the module category of kG is the direct sum of the module categories of the blocks. (The block decomposition corresponds to the connected components of the Ext-quiver.) The basic algebras of the representation-finite and tame blocks of kG are completely understood. The representation-finite blocks were determined in the late sixties. Almost 20 years later, Karin Erdmann classified blocks of tame representation type via the stable Auslander-Reiten quiver [5].

1.2 Finite algebraic groups and their Hopf algebras

We let M_k and Gr be the categories of not necessarily finite-dimensional commutative k-algebras and groups, respectively. A representable functor

$$\mathcal{G} : M_k \longrightarrow \mathrm{Gr} \quad ; \quad R \mapsto \mathcal{G}(R)$$

is called an *affine group scheme.* By definition, there exists a commutative k-algebra $k[\mathcal{G}]$ such that $\mathcal{G}(R)$ is the set of algebra homomorphisms $k[\mathcal{G}] \longrightarrow R$ for every $R \in M_k$. By Yoneda's Lemma, the group functor structure of

$\mathcal{G}$ corresponds to a Hopf algebra structure of the coordinate ring $k[\mathcal{G}]$, which renders $k[\mathcal{G}]$ a commutative Hopf algebra.

We say that $\mathcal{G}$ is an *algebraic group* if the representing object $k[\mathcal{G}]$ is finitely generated. If $k[\mathcal{G}]$ is finite-dimensional, then $\mathcal{G}$ is referred to as a *finite algebraic group*. In this case,

$$k\mathcal{G} := k[\mathcal{G}]^*$$

is a finite-dimensional, cocommutative Hopf algebra, the so-called *algebra of measures* on $\mathcal{G}$. In fact, the correspondence

$$\mathcal{G} \mapsto k\mathcal{G}$$

provides an equivalence between the categories of finite algebraic groups and finite-dimensional cocommutative Hopf algebras. In this equivalence, group algebras of finite groups correspond to reduced finite algebraic groups. An algebraic group $\mathcal{G}$ is called *reduced* or *smooth*, provided its coordinate ring $k[\mathcal{G}]$ does not possess any non-trivial nilpotent elements. If $\mathrm{char}(k) = 0$, then Cartier's Theorem asserts that any algebraic group is reduced, thus all cocommutative Hopf algebras are semisimple in this case. We shall therefore henceforth assume that $\mathrm{char}(k) = p > 0$.

Definition 3. A finite group scheme $\mathcal{G}$ is called *infinitesimal*, provided $\mathcal{G}(k) = \{ 1 \}$.

Let $\mathcal{G}$ be a finite algebraic group. General theory shows that

$$k\mathcal{G} = \Lambda * G$$

is a skew group algebra, where $G = \mathcal{G}(k)$ is the finite group of k-rational points of $\mathcal{G}$ and $\Lambda = k\mathcal{G}^0$ is the Hopf algebra of a certain infinitesimal normal subgroup $\mathcal{G}^0$ of $\mathcal{G}$.

Example 3. Let $r \in \mathbb{N}$.

(1) For $n \in \mathbb{N}$, let $\mathrm{GL}(n)_r : M_k \longrightarrow \mathrm{Gr}$ be given by

$$\mathrm{GL}(n)_r(R) := \{ (\zeta_{ij}) \in \mathrm{GL}(n)(R) \mid \zeta_{ij}^{p^r} = \delta_{ij} \}.$$

By general theory, every infinitesimal group $\mathcal{G}$ is a subgroup of a suitable $\mathrm{GL}(n)_r$.

(2) Consider $\mathbb{G}_{m(r)} := \mathrm{GL}(1)_r$, that is,

$$\mathbb{G}_{m(r)}(R) := \{ x \in R^\times \mid x^{p^r} = 1 \} \subseteq R^\times.$$

Then we have

$$k\mathbb{G}_{m(r)} \cong k^{p^r}.$$

(3) Let $\mathbb{G}_{a(r)} : M_k \longrightarrow \mathrm{Gr}$ be given by

$$\mathbb{G}_{a(r)}(R) := \{\, x \in R \mid x^{p^r} = 0 \,\} \subseteq (R, +).$$

Then we have

$$k\mathbb{G}_{a(r)} \cong k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p).$$

As an algebra, $k\mathbb{G}_{a(r)}$ is the group algebra of an elementary abelian p-group of rank r. In particular, we have

$$k\mathbb{G}_{a(r)} \text{ is representation-finite } \Leftrightarrow\ r = 1 \quad ;$$
$$k\mathbb{G}_{a(r)} \text{ is tame } \Leftrightarrow\ p = 2 \text{ and } r = 2.$$

(4) For $m = np^r$ with $(n, p) = 1$ we consider

$$\mathcal{Q}_{(m)}(R) := \{\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2)(R) \mid a^m = 1 = d^m \ ,\ b^p = 0 = c^p \,\}.$$

Then we have $\mathcal{Q}_{(m)}(k) = \{\, \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a^n = 1 \,\}$. Thus, $\mathcal{Q}_{(m)}$ is a finite algebraic group, which is infinitesimal if and only if $n = 1$. The infinitesimal group $\mathcal{Q}_{(p^r)} = \mathrm{SL}(2)_1 T_r$ is the product of the first Frobenius kernel of $\mathrm{SL}(2)$ with the r-th Frobenius kernel of its standard maximal torus T.

Let $\mathcal{G} \subseteq \mathrm{GL}(n)$ be an algebraic group, $r \in \mathbb{N}$. Then

$$\mathcal{G}_r := \mathcal{G} \cap \mathrm{GL}(n)_r$$

is the *r-th Frobenius kernel* of $\mathcal{G}$. Thus, $\mathcal{G}_r$ is an infinitesimal group. One can show that the definition does not depend on the choice of the inclusion $\mathcal{G} \subseteq \mathrm{GL}(n)$.

If $\mathcal{G}$ is infinitesimal, then there exists $r \in \mathbb{N}$ with $\mathcal{G} = \mathcal{G}_r$ and

$$\mathrm{ht}(\mathcal{G}) := \min\{\, r \mid \mathcal{G}_r = \mathcal{G} \,\}$$

is called the *height* of $\mathcal{G}$. The Hopf algebra $k\mathcal{G}$ possesses a co-unit $\varepsilon : k\mathcal{G} \longrightarrow k$. The unique block $\mathcal{B}_0(\mathcal{G}) \subseteq k\mathcal{G}$ with $\varepsilon(\mathcal{B}_0(\mathcal{G})) \neq (0)$ is called the *principal block* of $k\mathcal{G}$. **Problem**. Let $\mathcal{G}$ be a finite algebraic group. When is $\mathcal{B}_0(\mathcal{G})$ representation-finite or tame?

Roughly speaking, we shall pursue the following strategy. Using geometric tools we reduce the problem to the consideration of small examples that are amenable to the methods from abstract representation theory. The latter will enable us to see which of the examples have the desired representation type and what their quivers and relations are.

We conclude this section by stating the analogue of Maschke's Theorem in the context finite algebraic groups. Since the tensor product of a module with a projective module is projective, a Hopf algebra $k\mathcal{G}$ is semi-simple if and only if its principal block is simple.

Theorem 3 (Nagata). *Let $\mathcal{G}$ be a finite algebraic group. Then $k\mathcal{G}$ is semi-simple if and only if $p \nmid \mathrm{ord}(\mathcal{G}(k))$ and $\mathcal{G}^0 \cong \prod_{i=1}^n \mathbb{G}_{m(r_i)}$ for some $n \in \mathbb{N}_0$ and $r_i \in \mathbb{N}$.*

2 Support varieties and rank varieties of restricted Lie algebras

2.1 Cohomological support varieties

Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field k of characteristic $p > 0$. We shall study the category $\mathrm{mod}\,\mathcal{G}$ of finite-dimensional $k\mathcal{G}$-modules, whose objects will be referred to as $\mathcal{G}$-modules. Our tools will be geometric in nature; we begin by outlining the main features.

Let M be a $\mathcal{G}$-module. We denote by

$$\mathrm{Ext}^*_{\mathcal{G}}(M,M) := \bigoplus_{n\geq 0} \mathrm{Ext}^n_{\mathcal{G}}(M,M)$$

the *Yoneda algebra* of self-extensions of M. If $M = k$ is the trivial $\mathcal{G}$-module, then

$$\mathrm{H}^\bullet(\mathcal{G},k) := \bigoplus_{n\geq 0} \mathrm{Ext}^{2n}_{\mathcal{G}}(k,k)$$

is the *even cohomology ring* of $\mathcal{G}$. This is a commutative k-algebra.

A classical result due to Evens [6] and Venkov [28] asserts that $\mathrm{H}^\bullet(G,k)$ is finitely generated whenever G is a finite group. The most general result of this type is the following:

Theorem 4 ([13]). *Let M be a $\mathcal{G}$-module.*

(1) The commutative k-algebra $\mathrm{H}^\bullet(\mathcal{G},k)$ is finitely generated.

(2) The homomorphism

$$\Phi_M : \mathrm{H}^\bullet(\mathcal{G},k) \longrightarrow \mathrm{Ext}^*_{\mathcal{G}}(M,M) \quad ; \quad [f] \mapsto [f \otimes \mathrm{id}_M]$$

is finite.

This fundamental result enables us to introduce geometric techniques by associating varieties to modules. We denote by $\mathrm{Maxspec}(\mathrm{H}^\bullet(\mathcal{G},k)) := \{\,\mathfrak{M} \trianglelefteq \mathrm{H}^\bullet(\mathcal{G},k) \mid \mathfrak{M} \text{ maximal ideal}\,\}$ the *maximal ideal spectrum of* $\mathrm{H}^\bullet(\mathcal{G},k)$. For an arbitrary ideal $I \trianglelefteq \mathrm{H}^\bullet(\mathcal{G},k)$, we let $Z(I) := \{\,\mathfrak{M} \in \mathrm{Maxspec}(\mathrm{H}^\bullet(\mathcal{G},k)) \mid I \subseteq \mathfrak{M}\,\}$ be the *zero locus* of I. These sets form the closed sets of the Zariski topology of the affine variety $\mathrm{Maxspec}(\mathrm{H}^\bullet(\mathcal{G},k))$.

Definition 4. Let M be a $\mathcal{G}$-module. The affine variety

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker \Phi_M) \subseteq \operatorname{Maxspec}(\mathrm{H}^\bullet(\mathcal{G}, k))$$

is called the *cohomological support variety* of M.

Before looking at an example, let us see how varieties provide information about the representation type of the algebra $k\mathcal{G}$.

Theorem 5. *Let $\mathcal{B} \subseteq k\mathcal{G}$ be a block, $M \in \operatorname{mod} \mathcal{B}$.*

(1) If $\mathcal{B}$ is representation-finite, then $\dim \mathcal{V}_{\mathcal{G}}(M) \leq 1$ *[15].*

(2) If $\mathcal{B}$ is tame, then $\dim \mathcal{V}_{\mathcal{G}}(M) \leq 2$ *[8].*

Example 4. Let $k\mathcal{G} = k(\mathbb{Z}/(p))^r$ be the group algebra of a p-elementary abelian group of rank r. Then

$$\mathrm{H}^*(\mathcal{G}, k) := k[X_1, \ldots, X_r] \otimes_k \Lambda(Y_1, \ldots, Y_r) \qquad \deg(X_i) = 2,\ \deg(Y_i) = 1,$$

is the tensor product of a polynomial ring and an exterior algebra. We thus obtain:

- $\mathcal{V}_{\mathcal{G}}(k) = \operatorname{Maxspec}(\mathrm{H}^\bullet(\mathcal{G}, k)) \cong \mathbb{A}^r$.
- $k\mathcal{G}$ is representation-finite $\Rightarrow r = 1$.
- $k\mathcal{G}$ is tame $\Rightarrow r = 2$.

In view of Theorem 2 this tells us that homological methods alone can in general not be expected to give complete answers to the problem of finding blocks of a given representation type.

2.2 Lie algebras

We have seen that finite algebraic groups consist of two building blocks, reduced groups and infinitesimal groups. In this section we focus on infinitesimal groups of height 1. It turns out that this is equivalent to studying restricted Lie algebras. Given a finite group scheme $\mathcal{G}$, we let $\Delta : k\mathcal{G} \longrightarrow k\mathcal{G} \otimes_k k\mathcal{G}$ denote the comultiplication of $k\mathcal{G}$. Then

$$\operatorname{Lie}(\mathcal{G}) := \{\, x \in k\mathcal{G} \mid \Delta(x) = x \otimes 1 + 1 \otimes x \,\}$$

is called the *Lie algebra* of $\mathcal{G}$. Writing $[x, y] = xy - yx$, we have

(a) $[x, y] \in \operatorname{Lie}(\mathcal{G})$ for every $x, y \in \operatorname{Lie}(\mathcal{G})$, and

(b) $x^p \in \mathrm{Lie}(\mathcal{G})$ for every $x \in \mathrm{Lie}(\mathcal{G})$.

A subspace $\mathfrak{g} \subseteq \Lambda$ of an associative k-algebra Λ satisfying (a) and (b) is called a *restricted Lie algebra*. These algebras may also be defined axiomatically: A restricted Lie algebra is a pair $(\mathfrak{g}, [p])$ consisting of an abstract Lie algebra $\mathfrak{g}$ and an operator $\mathfrak{g} \longrightarrow \mathfrak{g}$; $x \mapsto x^{[p]}$ that satisfies the formal properties of an associative p-th power.

Given such a restricted Lie algebra $(\mathfrak{g}, [p])$ with universal enveloping algebra $U(\mathfrak{g})$, one defines the *restricted enveloping algebra* via

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{\, x^p - x^{[p]} \mid x \in \mathfrak{g} \,\}).$$

The algebra $U_0(\mathfrak{g})$ inherits the Hopf algebra structure from $U(\mathfrak{g})$ and we have

$$\mathfrak{g} = \{\, x \in U_0(\mathfrak{g}) \mid \Delta(x) = x \otimes 1 + 1 \otimes x \,\}.$$

The connection with infinitesimal groups of height 1 is given by:

Proposition 1. *Let $\mathcal{G}$ be an infinitesimal group of height* 1. *Then there exists an isomorphism*

$$k\mathcal{G} \cong U_0(\mathrm{Lie}(\mathcal{G}))$$

of Hopf algebras.

Many of our results to follow will depend on the following basic examples pertaining to solvable and simple restricted Lie algebras.

Example 5. (1) Let V be a k-vector space, $t : V \longrightarrow V$ be a non-zero linear transformation satisfying $t^p = t$. Then $\mathfrak{g}(t, V) := kt \oplus V$ obtains the structure of a restricted Lie algebra via

$$[(\alpha t, v), (\beta t, w)] := (0, \alpha t(w) - \beta t(v)) \;\; ; \;\; (\alpha t, v)^{[p]} = (\alpha^p t, \alpha^{p-1} t^{p-1}(v)).$$

For the corresponding restricted enveloping algebra one can compute the Ext-quiver and the relations. Abstract representation theory then shows:

- $U_0(\mathfrak{g}(t, V))$ is representation-finite $\Leftrightarrow \dim_k V \leq 1$.
- $U_0(\mathfrak{g}(t, V))$ is tame $\Leftrightarrow \dim_k V = 2$ and $p = 2$.

(2) Let $\mathfrak{g} := \mathfrak{sl}(2)$ be the restricted Lie algebra of trace zero (2×2)-matrices. The restricted enveloping algebra $U_0(\mathfrak{sl}(2))$ possesses exactly p simple modules $L(0), \ldots, L(p-1)$ with $\dim_k L(i) = i+1$. In the early 1980's Fischer [11], Drozd [4] and Rudakov [27] independently computed the quiver and the relations of $U_0(\mathfrak{sl}(2))$. For $p \geq 3$, the algebra $U_0(\mathfrak{sl}(2))$ has blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\frac{p-3}{2}}$, each $\mathcal{B}_i$ possessing two simple modules $L(i)$ and $L(p-2-i)$. There is one additional simple block $\mathcal{B}_{p-1}$ belonging to the Steinberg module $L(p-1)$. The non-simple blocks have bound quiver presentation given by the quiver Δ_1:

$$\begin{tikzcd} 0 \arrow[r, shift left=4.5, "\alpha_0"] \arrow[r, shift left=1.5, "\beta_0"] & 1, \arrow[l, shift left=1.5, "\alpha_1"'] \arrow[l, shift left=4.5, "\beta_1"'] \end{tikzcd}$$

and relations defining the ideal $J \trianglelefteq k\Delta_1$ generated by

$$\{\, \beta_{i+1}\alpha_i - \alpha_{i+1}\beta_i, \;\; \alpha_{i+1}\alpha_i, \;\; \beta_{i+1}\beta_i \;\mid\; i \in \mathbb{Z}/(2) \,\}.$$

These examples will turn out to be of major importance for our determination of the tame infinitesimal groups of odd characteristic. The first example is essentially the reason for the validity of the following result:

Proposition 2 ([9])**.** *Suppose that $p \geq 3$, and let $\mathcal{G}$ be a solvable infinitesimal group. Then $\mathcal{B}_0(\mathcal{G})$ is either representation-finite or wild.*

Turning to the second example, we observe that the algebra $k[\Delta_1]/J$ is tame. In fact, our algebra belongs to an important class of tame algebras, the so-called *special biserial algebras.* The uniformity of the presentation of these blocks is not accidental; it is a consequence of the so-called translation principle [18], which affords the passage between certain blocks. Roughly speaking, one proceeds as follows: Given two blocks $\mathcal{B}, \mathcal{C}$ of $U_0(\mathfrak{g})$ and a simple module S, one considers the functor

$$\mathrm{Tr}_S : \mathrm{mod}\,\mathcal{B} \longrightarrow \mathrm{mod}\,\mathcal{C} \;\;;\;\; M \mapsto e_{\mathcal{C}} \cdot (S \otimes_k M).$$

Here $e_{\mathcal{C}} \in U_0(\mathfrak{g})$ is the central idempotent defining the block $\mathcal{C}$. Under certain compatibility conditions on $\mathcal{B}, \mathcal{C}$ and S, this functor is in fact a Morita equivalence. The easiest instance of the translation principle is given by one-dimensional modules. In particular, all blocks of basic cocommutative Hopf algebras (i.e., those corresponding to group schemes of upper triangular matrices) are isomorphic.

2.3 Rank varieties

Although being of theoretical importance, support varieties are inherently intractable. Quillen's early work on the spectrum of the cohomology ring of a finite group and Chouinard's result on projective modules suggested that elementary abelian groups could play an important rôle. Dade noticed a further reduction to cyclic shifted subgroups. These observations led Jon Carlson to his representation-theoretic notion of a rank variety. A few years later a similar theory for restricted Lie algebras was developed by Friedlander-Parshall and Jantzen.

About 7 years ago, Eric Friedlander and Julia Pevtsova introduced a theory of representation-theoretic support spaces that applies to all finite group schemes. Since this approach is a bit technical, we confine our attention to restricted Lie algebras.

Definition 5. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. The conical variety

$$V(\mathfrak{g}) := \{\, x \in \mathfrak{g} \mid x^{[p]} = 0 \,\}$$

is called the *nullcone* of $\mathfrak{g}$. Let M be a $U_0(\mathfrak{g})$-module. Then

$$V(\mathfrak{g})_M := \{\, x \in V(\mathfrak{g}) \mid M|_{k[x]} \text{ is not free} \,\} \cup \{\, 0 \,\}$$

is referred to as the *rank variety* of M.

The name derives from the following alternative description of $V(\mathfrak{g})_M$: Given $x \in V(\mathfrak{g})$, we denote by $x_M : M \longrightarrow M \ ; \ m \mapsto x.m$ the left multiplication by x on M. Then we have $x \in V(\mathfrak{g})_M$ if and only if $\mathrm{rk}(x_M) < \frac{p-1}{p} \dim_k M$.

Example 6. Let $\mathfrak{g} = \mathfrak{sl}(2)$.

- Note that $V(\mathfrak{sl}(2))$ is the set of nilpotent (2×2)-matrices, so that

  $$V(\mathfrak{sl}(2)) = \{\, \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \mid a^2 + bc = 0 \,\}.$$

  Thus, $V(\mathfrak{sl}(2))$ is a two-dimensional, irreducible variety.

- Recall that there are exactly p simple $U_0(\mathfrak{sl}(2))$-modules $L(i)$ $0 \le i \le p-1$ with $\dim_k L(i) = i+1$. If $x \in V(\mathfrak{sl}(2)) \smallsetminus V(\mathfrak{sl}(2))_{L(i)}$, then $L(i)$ is a free module for the p-dimensional algebra $k[x]$. Thus, $p | \dim_k L(i)$ and $i = p-1$. Hence $L(i) = L(p-1)$ is the Steinberg module, which is projective. We therefore have (see also Corollary 2 below)

  $$V(\mathfrak{sl}(2))_{L(i)} = \begin{cases} V(\mathfrak{sl}(2)) & i \neq p-1 \\ \{0\} & i = p-1. \end{cases}$$

- The rank varieties of the baby Verma modules $Z(i) := U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_i$ are of dimension 0 or 1. Here $\mathfrak{b} \subseteq \mathfrak{sl}(2)$ is the Borel subalgebra of upper triangular matrices of trace zero, and k_i denotes the one-dimensional $U_0(\mathfrak{b})$-module with weight $i \in \{\, 0, \ldots, p-1 \,\}$.

Theorem 6 ([17, 12])**.** *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then there exists a homeomorphism*

$$\Psi : \mathcal{V}_{\mathfrak{g}}(k) \longrightarrow V(\mathfrak{g})$$

such that $\Psi(\mathcal{V}_{\mathfrak{g}}(M)) = V(\mathfrak{g})_M$ for every $M \in \mathrm{mod}\, U_0(\mathfrak{g})$.

This result tells us that for our intents and purposes rank varieties are as good a cohomological support varieties. Theorem 5 now implies:

Corollary 1. *Let $\mathcal{G}$ be a finite algebraic group with Lie algebra $\mathfrak{g}$.*

(1) If $\mathcal{B}_0(\mathcal{G})$ is representation-finite, then $\dim V(\mathfrak{g}) \leq 1$.

(2) If $\mathcal{B}_0(\mathcal{G})$ is tame, then $\dim V(\mathfrak{g}) \leq 2$.

So why did we introduce support varieties to begin with? Let us look at the following result:

Corollary 2. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, M be a $U_0(\mathfrak{g})$-module. Then the following statements are equivalent:*

(1) M is projective.

(2) $V(\mathfrak{g})_M = \{ 0 \}$.

Proof. (1) $\Rightarrow$ (2). Let $x \in V(\mathfrak{g})$. By the PBW-Theorem, $U_0(\mathfrak{g})$ is a free $k[x]$-module. Hence $M|_{k[x]}$ is projective, so that $x = 0$.

(2) $\Rightarrow$ (1). If $V(\mathfrak{g})_M = \{ 0 \}$, then $\mathcal{V}_{\mathfrak{g}}(M)$ is finite. Recall that

$$\Phi_M : \mathrm{H}^\bullet(\mathfrak{g}, k) \longrightarrow \mathrm{Ext}^*_{U_0(\mathfrak{g})}(M, M)$$

is a finite morphism. Since the Krull dimension $\dim \mathrm{H}^\bullet(\mathfrak{g}, k)/\ker \Phi_M = \dim \mathcal{V}_{\mathfrak{g}}(M)$ $= 0$, the algebra $\mathrm{Ext}^*(M, M)$ is finite-dimensional. It follows that there exists $n_0 \in \mathbb{N}$ such that $\mathrm{Ext}^n_{U_0(\mathfrak{g})}(M, -) = 0$ for all $n \geq n_0$. Hence M has finite projective dimension. But $U_0(\mathfrak{g})$ is a Hopf algebra and hence self-injective. This implies that M is projective. $\boxed{QED}$

The foregoing result suggests that $\dim V(\mathfrak{g})_M$ has a representation-theoretic interpretation. Indeed,

$$\dim V(\mathfrak{g})_M = \mathrm{cx}_{U_0(\mathfrak{g})}(M)$$

is the *complexity* of M, that is, the polynomial rate of growth of a minimal projective resolution of M.

3 Binary polyhedral groups, McKay quivers, and tame blocks

Throughout, $\mathcal{G}$ denotes a finite group scheme over an algebraically closed field k of characteristic $\mathrm{char}(k) = p > 0$. We want to know when the principal block $\mathcal{B}_0(\mathcal{G})$ has tame representation type. If G is a finite group, this happens precisely, when $p = 2$, and the Sylow-2-subgroups of G are dihedral, semidihedral, or generalized quaternion.

Recall that

$$k\mathcal{G} = \Lambda * G$$

is a skew group algebra, where $G = \mathcal{G}(k)$ is the finite group of k-rational points of $\mathcal{G}$, and $\Lambda = k\mathcal{G}^0$ is the Hopf algebra of an infinitesimal group scheme.

3.1 A basic reduction

In the sequel, we write $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ and assume that $p \geq 3$.

Definition 6. The group scheme $\mathcal{G}$ is *linearly reductive* if the associative algebra $k\mathcal{G}$ is semi-simple.

Recall that Nagata's Theorem 3 describes the structure of the linearly reductive groups. Using rank varieties, we obtain the following result:

Theorem 7 ([7, 9]). *If $\mathcal{B}_0(\mathcal{G})$ is tame, then*

(a) $p \nmid |\mathcal{G}(k)|$, and

(b) $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$, where $C(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.

In particular, $\mathfrak{g}$ is a central extension of $\mathfrak{sl}(2)$. Since the Chevalley-Eilenberg cohomology group $\mathrm{H}^2(\mathfrak{sl}(2), k)$ vanishes, such an extension splits, when considered as one of ordinary Lie algebras. General theory then shows that the structure of $\mathfrak{g} = \mathfrak{sl}(2) \oplus V$ is given as follows:

$$[(x,v),(y,w)] := ([x,y],0) \quad \text{and} \quad (x,v)^{[p]} = (x^{[p]}, \psi(x)+v^{[p]}),$$

where $\psi : \mathfrak{sl}(2) \longrightarrow V$ is p-semilinear. One can say when exactly $U_0(\mathfrak{g})$ is tame. Instead of going into the technical details, let us look at one particular example, that reveals fundamental differences between finite groups and restricted Lie algebras.

Example 7. Let $\{ e, h, f \}$ be the standard basis of $\mathfrak{sl}(2)$ and suppose that $V = kv$ is one-dimensional. We define the Lie algebra $\mathfrak{sl}(2)_s := \mathfrak{sl}(2) \oplus kv$ via

$$e^{[p]} = 0 = f^{[p]} \quad ; \quad h^{[p]} = h + v \quad ; \quad v^{[p]} = 0.$$

(This amounts to choosing the p-semilinear map $\psi_s : \mathfrak{sl}(2) \longrightarrow kv$; $\psi_s(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)) = a^p v$.) The algebra $U_0(\mathfrak{sl}(2)_s)$ turns out to be tame. However, the subalgebra $U_0(ke \oplus kv) \cong k(\mathbb{Z}/(p) \times \mathbb{Z}/(p))$ is wild. By contrast, Brauer's Third Main Theorem implies that subgroups of tame finite groups are always tame.

We are going to simplify matters a little and assume from now on that $C(\mathfrak{g}) = (0)$. In the context of finite groups this amounts to assuming that the Sylow-2-subgroup is a Klein four group. We are thus studying a Hopf algebra

$$\Lambda = U_0(\mathfrak{sl}(2)){*}G,$$

where G is a linearly reductive finite group that acts on $U_0(\mathfrak{sl}(2))$ via automorphisms of Hopf algebras. Hence G acts on $\mathfrak{sl}(2)$ via automorphisms. If N denotes the kernel of this action, then the principal block of Λ is isomorphic to that of $U_0(\mathfrak{sl}(2)){*}(G/N)$. Since $\mathrm{Aut}(\mathfrak{sl}(2)) \cong \mathrm{PSL}(2)(k)$, we may thus assume $G \subseteq \mathrm{PSL}(2)(k)$. Passage to the double cover does not change the principal block and thus yields $G \subseteq \mathrm{SL}(2)(k)$. In other words, G is a *binary polyhedral group*. These groups were classified by Klein around 1884.

3.2 Extended Dynkin diagrams and finite groups

Extended Dynkin diagrams are perhaps best known from Lie theory, where they appear in the structure theory of affine Kac-Moody algebras. We have seen in Section 1 another occurrence in the representation theory of hereditary algebras. In this case, these diagrams describe the Ext-quivers of hereditary algebras of tame representation type.

Extended Dynkin diagrams also appear in the representation theory of finite groups. In his seminal work, J. McKay [22, 23] associated to a finite group G and a complex G-module V a quiver $\Psi_V(G)$ that has since played a rôle in a number of contexts. Let's generalize this a little to cover our setting.

- Let $H = k\mathcal{G}$ be the Hopf algebra of a linearly reductive finite group scheme.
- $\{ S_1, \ldots, S_n \}$ denotes a complete set of representatives for the isoclasses of the simple H-modules.
- Fix an H-module V. Then V defines an $(n \times n)$-matrix $(a_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ such that

$$V \otimes_k S_j \cong \bigoplus_{i=1}^{n} a_{ij} S_i \qquad 1 \leq j \leq n.$$

In other words, the integral $(n \times n)$-matrix (a_{ij}) describes the left multiplication by V in the Grothendieck ring $K_0(H)$ of H relative to its standard basis of simple modules.

Definition 7. Let $\mathcal{G}$ be a linearly reductive finite group scheme, V be a $\mathcal{G}$-module. The *McKay quiver* $\Psi_V(\mathcal{G})$ *of* $\mathcal{G}$ *relative to* V is given by the following data:

- Vertices: $\{1, \dots, n\}$
- Arrows: $i \xrightarrow{a_{ij}} j$.

Example 8. (1) Let G be an abelian group with $p \nmid \mathrm{ord}(G)$. If V is a faithful G-module with simple constituents $k_{\lambda_1}, \dots, k_{\lambda_r}$, then the character group $X(G)$ is generated by $S := \{\lambda_1, \dots, \lambda_r\}$ and the McKay quiver of G relative to V is the Cayley graph of $X(G)$ relative to S.

(2) Let G be a finite group with $p \nmid \mathrm{ord}(G)$, V be a faithful G-module. By Burnside's classical theorem, every simple G-module is a direct summand of some tensor power $V^{\otimes n}$. This implies that the quiver $\Psi_V(G)$ is connected. There is a version of Burnside's result for finite group schemes.

Let us return to our simplified context. We thus have

$$\Lambda = U_0(\mathfrak{sl}(2)) {*} G,$$

with $G \subseteq \mathrm{SL}(2)(k)$ acting on $\mathfrak{sl}(2)$ via automorphisms, and p not dividing the order of G. This implies that the McKay quiver $\Psi_{L(1)}(G)$ of G relative to the two-dimensional standard representation $L(1) = k^2$ is connected.

It turns out that a binary polyhedral group is uniquely determined by its McKay graph $\Psi_{L(1)}(G)$. Here is the list of groups up to conjugation in $\mathrm{SL}(2)(k)$:

G	$\overline{\Psi_{L(1)}(G)}$
$\mathbb{Z}/(n)$	$\tilde{A}_{n-1}$
Q_n	$\tilde{D}_{n+2}$
T	$\tilde{E}_6$
O	$\tilde{E}_7$
I	$\tilde{E}_8$.

The left-hand column gives the isomorphism types of the finite groups. Here Q_n denotes the quaternion group of order $4n$, and T, O, and I refer to the binary tetrahedral group (of order 24), the binary octahedral group (of order 48) and the binary icosahedral group (of order 120), respectively. The quivers corresponding to the graphs in the right-hand column are obtained by replacing each bond by $\bullet \leftrightarrows \bullet$.

The above list will be sufficient for our simplified context. In general, one needs to deal with linearly reductive group schemes $\mathcal{G} \subseteq \mathrm{SL}(2)$.

Thus, modulo our simplifications, we know the groups that can occur, i.e., we understand the Hopf algebra structure. Moreover, the affine quivers describing the tame hereditary algebras also appear. How can we get the Ext-quiver of Λ? The first step consists of finding the simple modules.

Lemma 1. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup. Suppose that $L_1, \dots, L_n$ are simple $\mathcal{G}$-modules such that $\{ L_1|_{\mathcal{N}}, \dots, L_n|_{\mathcal{N}} \}$ is a complete set of representatives for the simple $\mathcal{N}$-modules.*

(1) Every simple $\mathcal{G}$-module S is of the form

$$S \cong L_i \otimes_k M$$

for a unique $i \in \{ 1, \dots, n \}$ and a unique simple $\mathcal{G}/\mathcal{N}$-module M.

(2) Suppose that $\mathrm{Ext}^1_{\mathcal{N}}(V, V) = (0)$ for every simple $\mathcal{N}$-module V. If M, N are simple $\mathcal{G}/\mathcal{N}$-modules, then

$$\mathrm{Ext}^1_{\mathcal{G}}(L_i \otimes_k M, L_j \otimes_k N) \cong \begin{cases} (0) & i = j \\ \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(M, \mathrm{Ext}^1_{\mathcal{N}}(L_i, L_j) \otimes_k N) & i \neq j. \end{cases}$$

If $\mathcal{G}/\mathcal{N}$ is linearly reductive, then the dimension of our Ext-group describes the multiplicity of M in the $\mathcal{G}/\mathcal{N}$-module $\mathrm{Ext}^1_{\mathcal{N}}(L_i, L_j) \otimes_k N$. We thus obtain a connection between the Ext-quiver of $k\mathcal{G}$ and the McKay quiver of $\mathcal{G}/\mathcal{N}$ relative to $\mathrm{Ext}^1_{\mathcal{N}}(L_i, L_j)$.

The technical conditions of the Lemma may seem somewhat contrived, but they do hold in classical contexts such as ours: There exist simple Λ-modules $L(0), \dots, L(p-1)$, whose restrictions to $U_0(\mathfrak{sl}(2))$ give all simple $U_0(\mathfrak{sl}(2))$-modules. Moreover, there are isomorphisms of G-modules

$$\mathrm{Ext}^1_{\mathfrak{sl}(2)}(L(i), L(j)) \cong \begin{cases} (0) & i+j \neq p-2 \\ L(1) & \text{otherwise.} \end{cases}$$

The Lemma now shows that the Ext-graph of Λ consists of the extended Dynkin diagrams that appear in the classification of the tame hereditary algebras.

Group algebras, or Hopf algebras in general, are self-injective and thus are hereditary only in case they are semi-simple (no arrows). The passage from hereditary algebras to self-injective algebras is given by the notion of *trivial extension.*

Given an algebra Λ, the *trivial extension* of Λ is the semidirect product $T(\Lambda) := \Lambda \ltimes \Lambda^*$ of Λ with its bimodule Λ^*:

$$(a, f) \cdot (b, g) := (ab, a.g + f.b) \quad \forall\, a, b \in \Lambda,\ f, g \in \Lambda^*.$$

The algebra $T(\Lambda)$ is symmetric, and one can often compute the quiver and the relations of $T(\Lambda)$. For instance, if $\Delta_n = \tilde{A}_{2n-1}$ is the quiver without paths of length 2, then $T(k\Delta_n) = kQ/I$, where Q is given by

$$
\begin{tikzcd}
 & 1 \arrow[r, shift left, "\alpha_1"] \arrow[dl, shift left, "\beta_1"] & 2 \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, dash] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-2 \arrow[l, shift left, dash] \arrow[r, shift left, "\alpha_{n-2}"] & n-1 \arrow[l, shift left, "\beta_{n-1}"] \arrow[dr, shift left, "\alpha_{n-1}"] & \\
0 \arrow[ur, shift left, "\alpha_0"] \arrow[dr, shift left, "\beta_0"] & & & & & & n \arrow[ul, shift left, "\beta_n"] \arrow[dl, shift left, "\alpha_n"] \\
 & 2n-1 \arrow[ul, shift left, "\alpha_{2n-1}"] \arrow[r, shift left, "\beta_{2n-1}"] & 2n-2 \arrow[l, shift left, "\alpha_{2n-2}"] \arrow[r, shift left, dash] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n+2 \arrow[l, shift left, dash] \arrow[r, shift left, "\beta_{n+2}"] & n+1 \arrow[l, shift left, "\alpha_{n+1}"] \arrow[ur, shift left, "\beta_{n+1}"] &
\end{tikzcd}
$$

and $I \subseteq kQ$ is the ideal generated by

$$\{\, \beta_{i+1}\alpha_i - \alpha_{i-1}\beta_i, \quad \alpha_{i+1}\alpha_i, \quad \beta_i\beta_{i+1} \ \mid \ i \in \mathbb{Z}/(2n)\,\}.$$

Thus, the effect of passing to the trivial extension is the familiar doubling process. For $n = 1$ we obtain the algebra of Example 5(2).

It turns out that the tame principal blocks of finite group schemes are algebras of this type:

Theorem 8 ([7]). *Let $\mathcal{G}$ be a finite group scheme of characteristic $p \geq 3$ such that $\mathcal{B}_0(\mathcal{G})$ tame.*

(1) There exists a linearly reductive group scheme $\tilde{\mathcal{G}} \subseteq \mathrm{SL}(2)$ such that the Ext*-quiver of $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the McKay quiver $\Psi_{L(1)}(\tilde{\mathcal{G}})$.*

(2) The block $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to a generalized trivial extension of a tame hereditary algebra.

Let us return to our example and consider $G = T_{(2n)}$, the cyclic group of order $2n$ contained in the standard maximal torus $T \subseteq \mathrm{SL}(2)$ of diagonal matrices. In that case, $\tilde{\mathcal{G}}$ is the reduced group with $\tilde{\mathcal{G}}(k) = T_{(2n)}$, and our Theorem says that

$$\mathcal{B}_0(\Lambda) \sim_M T(k\tilde{A}_{2n-1})$$

is Morita equivalent to the trivial extension, which we have considered above. The other binary polyhedral groups give rise to the trivial extensions of the corresponding affine quivers.

4 Small quantum groups

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra. Given a complex number $\zeta \in \mathbb{C}\smallsetminus\{0\}$, Drinfeld and Jimbo defined the quantum group $U_\zeta(\mathfrak{g})$ of $\mathfrak{g}$. Roughly speaking, this Hopf algebra is a deformation of the ordinary enveloping algebra $U(\mathfrak{g})$. Technically, it is defined via a Chevalley basis of $\mathfrak{g}$ and the so-called quantum Serre relations. If ζ is not a root of unity, all finite-dimensional $U_\zeta(\mathfrak{g})$-modules are completely reducible. Alternatively, its representation theory resembles that of Lie algebras in positive characteristic. Lusztig defined a finite-dimensional Hopf subalgebra $u_\zeta(\mathfrak{g})$ of $U_\zeta(\mathfrak{g})$, which can be thought of as an analogue of the restricted enveloping algebra of a restricted Lie algebra. If ζ is a primitive ℓ-th root of unity, then $\dim_k u_\zeta(\mathfrak{g}) = \ell^{\dim_k \mathfrak{g}}$. To cut down on subtle technicalities, we shall henceforth assume that $6 \nmid \ell$.

In order to develop a theory of supports for $u_\zeta(\mathfrak{g})$, one needs an analogue of the Friedlander-Suslin Theorem. Since there are other cases of Hopf algebras, where such a result is available, it is expedient to formulate the relevant properties in broader context. A rather detailed summary of the current state of the art can be found in [19].

We consider a (finite-dimensional) Hopf algebra Λ over a algebraically closed field k (of arbitrary characteristic). It is well-known that the cohomology ring $\mathrm{H}^*(\Lambda, k)$ is graded commutative, so that the even cohomology ring $\mathrm{H}^\bullet(\Lambda, k)$ is a commutative k-algebra.

Definition 8. Let Λ be a Hopf algebra. We say that Λ is an *fg-Hopf algebra*, provided

(a) the algebra $\mathrm{H}^\bullet(\Lambda, k)$ is finitely generated, and

(b) for every $M \in \operatorname{mod} \Lambda$, the algebra homomorphism $\Phi_M : \mathrm{H}^\bullet(\Lambda, k) \longrightarrow \operatorname{Ext}^*_\Lambda(M, M)$ is finite.

In this case, $\operatorname{Maxspec}(\mathrm{H}^\bullet(\Lambda, k))$ carries the structure of an affine variety and one defines the support variety

$$\mathcal{V}_\Lambda(M) := Z(\ker \Phi_M)$$

for every $M \in \operatorname{mod} \Lambda$. One can show that $M \mapsto \mathcal{V}_\Lambda(M)$ enjoys properties analogous to those known for finite group schemes. In particular, Feldvoss and Witherspoon [10] have generalized Theorem 5 to the present context. Using these techniques one obtains the following result:

Theorem 9 ([20]). *Let $\mathfrak{g}$ be simple and suppose that ℓ is good for the root system of $\mathfrak{g}$. If $\mathcal{B} \subseteq u_\zeta(\mathfrak{g})$ is a block, then the following statements hold:*

(1) $\mathcal{B}$ is representation-finite if and only if $\mathcal{B}$ is the simple block belonging to the Steinberg module.

(2) If $\mathcal{B}$ has tame representation type, then $\mathfrak{g} \cong \mathfrak{sl}(2)$ and $\mathcal{B}$ is Morita equivalent to $T(k(\bullet \rightrightarrows \bullet))$.

The representation theory of the trivial extension of the Kronecker quiver $\bullet \rightrightarrows \bullet$ is completely understood.

Support spaces and Dynkin diagrams also appear in the context of Auslander-Reiten theory. Given a self-injective Λ, one defines a quiver $\Gamma_s(\Lambda)$, which is an important invariant of its Morita equivalence class. The vertices of the so-called *stable Auslander-Reiten quiver* are the isoclasses of the non-projective indecomposable Λ-modules. Arrows are given by irreducible morphisms. Roughly, speaking such a non-isomorphism does not factor non-trivially through any indecomposable Λ-module. A third ingredient is the Auslander-Reiten translation $\tau : \Gamma_s(\Lambda) \longrightarrow \Gamma_s(\Lambda)$, which reflects homological properties. A fundamental result by Riedtmann states that the isomorphism class of a connected component $\Theta \subseteq \Gamma_s(\Lambda)$ is essentially determined by an undirected tree T_Θ, the *tree class* of Θ. For fg-Hopf algebras, the possible tree classes are finite Dynkin diagrams, Euclidean diagrams or infinite Dynkin diagrams of type A_∞, D_∞, A_∞^∞. In concrete cases, support varieties can be used to decide, which tree class a given component has.

Recall that $u_\zeta(\mathfrak{g}) \subseteq U_\zeta(\mathfrak{g})$ is a Hopf subalgebra. We let h denote the Coxeter number of $\mathfrak{g}$.

Theorem 10 ([21])**.** *Let $\ell \geq h$. Suppose that $\Theta \subseteq \Gamma_s(u_\zeta(\mathfrak{g}))$ is a component containing the restriction of a $U_\zeta(\mathfrak{g})$-module. If $\mathfrak{g} \neq \mathfrak{sl}(2)$ is simple, then $T_\Theta = A_\infty$.*

Proof. Given $M, N \in \Theta$ one can show that $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(N)$, so that we have the support variety $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\Theta)$. This variety corresponds to a Zariski closed subspace X_Θ of the nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$. Since Θ contains the restriction of a $U_\zeta(\mathfrak{g})$-module, X_Θ is invariant under the adjoint action of the algebraic group G of $\mathfrak{g}$. As $\mathfrak{g} \neq \mathfrak{sl}(2)$, a little more structure theory implies that $\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}(\Theta) \geq 3$. Such components are known to have tree class A_∞. $\boxed{QED}$

Using this result, one can for instance locate the simple $u_\zeta(\mathfrak{g})$-modules within the AR-quiver and show that they have precisely one predecessor. This in turn yields information concerning the structure of certain subfactors of principal indecomposable $u_\zeta(\mathfrak{g})$-modules.

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Some Trends in the Theory of Groups with Restricted Conjugacy Classes

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Abstract. This survey article is an expanded version of the lectures given at the conference "Advances in Group Theory and Applications 2011", concerning the effect of finiteness conditions on infinite (generalized) soluble groups.

Keywords: FC-group; subgroup lattice; strongly inertial group

MSC 2000 classification: 20E15, 20F24

To Cristiano and Reinhold

1 Introduction

This survey article is an expanded version of the lectures that I delivered on the occasion of the conference "Advances in Group Theory and Applications 2011", which took place in Porto Cesareo. The aim of my lectures was to describe the state of knowledge on the effect of the imposition of finiteness conditions on a (generalized) soluble group, with special attention to conjugacy classes, generalized normal subgroups and lattice properties. My interest in this topic started about thirtyfive years ago when - as a young student - I was attending the lectures of Mario Curzio, and my ideas about mathematics were strongly influenced by Federico Cafiero. It was an exciting time, especially because of the impressive development of the theory produced by the schools founded by Reinhold Baer, Philip Hall and Sergei N. Černikov. Since then I never stopped working on these topics, and even though the fashions have changed, I am absolutely convinced that the theory of groups is full of fascinating properties still waiting to be discovered and exciting results that are waiting to be proved (and of course I mean here the theory of infinite soluble groups). I hope that these short notes may be able to transmit - especially to young people - the passion for this discipline, which in turn was forwarded to me by mathematicians much better than me.

Most notation is standard and can be found in [31].

This paper is dedicated to my youngest sons Cristiano and Reinhold, who were born during the organization of this conference.

2 Generalized normal subgroups

In 1955, a relevant year for the development of the theory of infinite groups, Bernhard H. Neumann [29] proved the following important result. It shows that the imposition of suitable normality conditions to all subgroups can be used to characterize group classes generalizing the class of abelian groups.

Theorem 1. (B.H. Neumann [29]) *Let G be a group. Then each subgroup of G has only finitely many conjugates if and only if the centre $Z(G)$ has finite index in G.*

It follows in particular from Neumann's theorem that if a group G has finite conjugacy classes of subgroups, then there is an upper bound for the orders of such classes.

A group G is called an *FC-group* if every element of G has only finitely many conjugates, or equivalently if the index $|G : C_G(x)|$ is finite for each element x of G. Finite groups and abelian groups are obvious examples of groups with the property FC; moreover, it is also clear that all central-by-finite groups belong to the class of FC-groups. Since for each element x of a group G the centralizer of the normal subgroup $\langle x\rangle^G$ coincides with the core of the centralizer $C_G(x)$, it follows easily that a group is an FC-group if and only if it has finite conjugacy classes of cyclic subgroups.

If x is any element of a group G, we have $x^g = x[x, g]$ for each element g of G, and hence the conjugacy class of x is contained in the coset xG', where G' is the commutator subgroup of G. Thus groups with finite commutator subgroup are FC-groups.

The FC-property for groups have been introduced seventy years ago, and since their first appearance in the literature important contributions have been given by several authors. A special mention is due here to R. Baer, Y.M. Gorcakov, P. Hall, L.A. Kurdachenko, B.H. Neumann, M.J. Tomkinson for the particular relevance of their works.

The direct product of an arbitrary collection of finite groups has clearly the property FC, and so in particular the conjugacy classes of elements of an FC-group can have unbounded orders, in contrast to the behavior of groups with finite conjugacy classes of subgroups. In fact, groups with boundedly finite conjugacy classes form a very special class of FC-groups, as the following theorem shows.

Theorem 2. (B.H. Neumann [28]) *A group G has boundedly finite conju-*

gacy classes of elements if and only if its commutator subgroup G' is finite.

The proof of Theorem 2 essentially depends on (and contains as a special case) a celebrated result of I. Schur concerning the relation between the size of the centre and that of the commutator subgroup of a group. This is one of the most important theorems in the theory of infinite groups.

Theorem 3. (I. Schur [40]) *Let G be a group whose centre has finite index. Then the commutator subgroup G' of G is finite.*

An easy application of such result proves that if G is a group such that the factor group $G/Z(G)$ is locally finite, then the commutator subgroup G' of G is locally finite. Since a finitely generated group has the FC-property if and only if it is finite over its centre, it follows also that the commutator subgroup of any FC-group is locally finite; in particular, the elements of finite order of an arbitrary FC-group form a subgroup. Thus also Dietzmann's Lemma, stating that periodic FC-groups are covered by their finite normal subgroups, can be seen as a direct consequence of the theorem of Schur.

Schur's theorem has been extended by Baer to any term of the upper central series (with finite ordinal type) in the following way.

Theorem 4. (R. Baer [1]) *Let G be a group in which the term $Z_i(G)$ of the upper central series has finite index for some positive integer i. Then the $(i+1)$-th term $\gamma_{i+1}(G)$ of the lower central series of G is finite.*

Although finitely generated finite-by-abelian groups are central-by-finite, the consideration of any infinite extraspecial group shows that the converse of Schur's theorem is false in the general case. On the other hand, Philip Hall was able to obtain a relevant and useful partial converse of Theorem 4; of course, it provides as a special case a partial converse for the theorem of Schur.

Theorem 5. (P. Hall [23]) *Let G be a group such that the $(i+1)$-th term $\gamma_{i+1}(G)$ of the lower central series of G is finite. Then the factor group $G/Z_{2i}(G)$ is finite.*

Combining the theorems of Baer and Hall, we have the following statement.

Corollary 1. *A group G is finite-by-nilpotent if and only if it is finite over some term (with finite ordinal type) of its upper central series.*

If G is any group, the last term of its (transfinite) upper central series is called the *hypercentre* of G, and the group G is *hypercentral* if it coincides with its hypercentre.

The consideration of the locally dihedral 2-group $D(2^\infty)$ shows that Baer's theorem cannot be extended to terms with infinite ordinal type of the upper central series. In fact,

$$Z_{\omega+1}(D(2^\infty)) = D(2^\infty)$$

but

$$\gamma_2(D(2^\infty)) = \gamma_3(D(2^\infty)) = Z(2^\infty).$$

Similarly, free non-abelian groups show that Hall's result does not hold for terms with infinite ordinal type of the lower central series. In fact, if F is any free non-abelian group, then $\gamma_\omega(F) = Z(F) = \{1\}$. On the other hand, the statement of Corollary 1 can be generalized as follows.

Theorem 6. (M. De Falco - F. de Giovanni - C. Musella - Y.P. Sysak [13]) *Let G be a group. The hypercentre of G has finite index if and only if G contains a finite normal subgroup N such that the factor group G/N is hypercentral.*

For other types of generalizations of Schur's theorem see for instance [19].

It was mentioned above that any direct product of finite groups has the FC-property. Such direct products are obviously also periodic and residually finite, and it was conversely proved by P. Hall [24] that every periodic countable residually finite FC-group is isomorphic to a subgroup of a direct product of a collection of finite groups (recall here that a group is called *residually finite* if the intersection of all its subgroups of finite index is trivial). An easy example shows that this property does not hold in general for uncountable groups. The result of P. Hall was extended by L.A. Kurdachenko [27] to the case of metabelian groups with countable centre. Moreover, many authors have determined conditions under which an FC-group is isomorphic at least to a section of a direct product of finite groups. On this problem we mention here only the following interesting result.

Theorem 7. (M.J. Tomkinson [42]) *Let G be an FC-group. Then the commutator subgroup G' of G is isomorphic to a section of the direct product of a collection of finite groups.*

Clearly, a subgroup X of a group G has only finitely many conjugates if and only if it is normal in a subgroup of finite index of G. One may consider other generalized normality properties, in which the obstruction to normality is represented by a finite section of the group. In particular, another theorem of B.H. Neumann deals with the case of groups in which every subgroup has finite index in a normal subgroup.

Theorem 8. (B.H. Neumann [29]) *A group G has finite commutator subgroup if and only if the index $|X^G : X|$ is finite for each subgroup X of G.*

Note that it is also easy to prove that a group G has the property FC if and only if each cyclic subgroup of G has finite index in its normal closure.

For our purposes, it is convenient to introduce the following definitions.

Let G be a group, and let X be a subgroup of G. Then

- X is *almost normal* in G if it has finitely many conjugates in G, or equivalently if the index $|G : N_G(X)|$ is finite,

- X is *nearly normal* in G if it has finite index in its normal closure X^G,
- X is *normal-by-finite* in G if the core X_G has finite index in X.

Using this terminology, we can describe FC-groups in the following way: *For a group G the following properties are equivalent:*

- *G is an FC-group,*
- *all cyclic subgroups of G are almost normal,*
- *all cyclic subgroups of G are nearly normal.*

Clearly, any finite subgroup of an arbitrary group is normal-by-finite, and so the imposition of this latter condition to (cyclic) subgroups does not force the group to have the FC-property. However, it follows from Dietzmann's Lemma that if all subgroups of an FC-group G are normal-by-finite, then every subgroup of G is also almost normal, and hence $G/Z(G)$ is finite.

Although almost normality and near normality are equivalent for cyclic (and so even for finitely) subgroups, easy examples show that these two concepts are usually incomparable for arbitrary subgroups. On the other hand, it can be proved that each almost normal subgroup of an FC-group is nearly normal, and that nearly normal subgroups of finite rank are always almost normal (see [21]). Moreover, combining Neumann's results with the theorem of Schur, we obtain:

Corollary 2. *Let G be a group in which all subgroups are almost normal. Then every subgroup of G is nearly normal.*

The above corollary has been extended in [21], proving that if G is a group in which every abelian subgroup is either almost normal or nearly normal, then the commutator subgroup of G is finite, and so all subgroups of G are nearly normal. Observe also that the hypotheses in the statements of Theorem 1 and Theorem 8 can be weakened, requiring that only the abelian subgroups are almost normal or nearly normal, respectively.

Theorem 9. (I.I. Eremin [18]) *Let G be a group in which all abelian subgroups are almost normal. Then the factor group $G/Z(G)$ is finite.*

Theorem 10. (M.J. Tomkinson [43]) *Let G be a group in which all abelian subgroups are nearly normal. Then the commutator subgroup G' of G is finite.*

A group G is called a *BCF-group* if all its subgroups are normal-by-finite and have bounded order over the core. Thus a group is BCF if and only if there exists a positive integer k such that $|X : X_G| \leq k$ for each subgroup X of G.

Theorem 11. (J. Buckley - J.C. Lennox - B.H. Neumann - H. Smith - J. Wiegold [4]) *Let G be a locally finite BCF-group. Then G contains an abelian subgroup of finite index.*

The above theorem has been later extended to the case of locally graded BCF-groups by H. Smith and J. Wiegold [41].

The structure of groups in which all non-abelian subgroups are either almost normal or nearly normal has been investigated by M. De Falco, F. de Giovanni, C. Musella and Y.P. Sysak [12]. The basic situation in this case is that of groups whose non-abelian subgroups are normal. Such groups are called *metahamiltonian* and have been introduced by G.M. Romalis and N.F. Sesekin in 1966. Of course, Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) are metahamiltonian. On the other hand, within the universe of (generalized) soluble groups, any metahamiltonian group has (boundedly) finite conjugacy classes. In fact:

Theorem 12. (G.M. Romalis - N.F. Sesekin [34],[35],[36]) *Let G be a locally graded metahamiltonian group. Then the commutator subgroup G' of G is finite of prime-power order.*

Recall here that a group G is *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index. It is easy to see that any locally (soluble-by-finite) group is locally graded. Thus locally graded groups form a large class of generalized soluble groups, and the assumption for a group to be locally graded is enough to avoid Tarski groups and other similar pathologies.

Neumann's theorems suggest that the behavior of normalizers has a strong influence on the structure of a group. In fact, groups with few normalizers of subgroups with a given property are of a very special type.

Theorem 13. (Y.D.Polovickiĭ [30]) *Let G be a group with finitely many normalizers of abelian subgroups. Then the factor group $G/Z(G)$ is finite.*

This theorem is a special case of a later result by F. De Mari and F. de Giovanni [15] concerning groups with few normalizer subgroups.

The *norm* $N(G)$ of a group G is the intersection of all normalizers of subgroups of G. Thus the norm of a group consists of all elements which induces by conjugation a power automorphism (recall that an automorphism of a group G is called a *power automorphism* if it maps each subgroup of G onto itself). As power automorphisms are central (see [6]), it follows that the norm of an arbitrary group G is contained in the second centre $Z_2(G)$ of G. Moreover, as a consequence of Polovickii's theorem, we have that that if the index $|G : N(G)|$ is finite, then also the centre $Z(G)$ has finite index in G, confirming that the section $N(G)/Z(G)$ is usually small.

The last part of this section is devoted to the study of certain finiteness conditions that are strictly related to the property FC, at least within the universe of locally (soluble-by-finite) groups. On this type of problems see also [22] and [16].

A group G is said to have the property $\mathfrak{K}$ if for each element x of G the set

$$\{[x, H] \mid H \leq G\}$$

is finite. Similarly, a group G has the property $\mathfrak{K}_\infty$ if the set

$$\{[x, H] \mid H \leq G, \quad H \text{ infinite}\}$$

is finite for every element x of G. As the commutator subgroup of any FC-group is locally finite, it is easy to prove that all FC-groups have the property $\mathfrak{K}$. Although also Tarski groups have the property $\mathfrak{K}$, the situation is completely clear in the case of locally (soluble-by-finite) groups.

Theorem 14. (M. De Falco - F. de Giovanni - C. Musella [9]) *A group G is an FC-group if and only if it is locally (soluble-by-finite) and has the property $\mathfrak{K}$.*

Also groups in the class $\mathfrak{K}_\infty$ are not too far from having the FC-property.

Theorem 15. (M. De Falco - F. de Giovanni - C. Musella [9]) *A soluble-by-finite group G has the property $\mathfrak{K}_\infty$ if and only if it is either an FC-group or a finite extension of a group of type p^∞ for some prime number p.*

Finally, we say that a group G has the property $\mathfrak{N}$ if the set

$$\{[X, H] \mid H \leq G\}$$

is finite for each subgroup X of G. It turns out that for soluble groups the property $\mathfrak{N}$ is equivalent to the property BFC.

Theorem 16. (M. De Falco - F. de Giovanni - C. Musella [9]) *Let G be a soluble group with the property $\mathfrak{N}$. Then the commutator subgroup G' of G is finite.*

3 Lattice properties

If G is any group, the set $\mathfrak{L}(G)$ of all subgroups of G is a complete lattice with respect to the ordinary set-theoretic inclusion. In this lattice, the operations $\wedge$ and $\vee$ are given by the rules

$$X \wedge Y = X \cap Y$$

and

$$X \vee Y = \langle X, Y \rangle$$

for each pair (X, Y) of subgroups of G. There is a very large literature on the relations between the structure of a group and that of its subgroup lattice (see for instance the monograph [39], but also the papers [7],[8],[10],[11],[20] for recent developments). For instance, one of the first and significant results was proved by O. Ore, and shows that a group has distributive subgroup lattice if and only if it is locally cyclic.

Let G and G^* be groups. A *projectivity* from G onto G^* is an isomorphism from the lattice $\mathfrak{L}(G)$ of all subgroups of G onto the subgroup lattice $\mathfrak{L}(G^*)$ of G^*; if there exists such a map, G^* is said to be a *projective image* of G. A group class $\mathfrak{X}$ is *invariant under projectivities* if all projective images of groups in $\mathfrak{X}$ are likewise $\mathfrak{X}$-groups. Relevant examples of group classes invariant under projectivities are the following:

- the class of all finite groups
- the class of all periodic groups
- the class of all soluble groups (B. Yakovlev, 1970)
- the class of all groups with finite Prüfer rank
- the class of all soluble minimax groups (R. Baer, 1968)

Moreover, it is clear that any group class defined by a lattice-theoretic property is invariant under projectivities; in particular, it follows from Ore's theorem quoted above that the class of locally cyclic groups is invariant under projectivities. On the other hand, the class of all abelian groups does not have such property. In fact, the elementary abelian group of order 9 and the symmetric group of degree 3 have isomorphic subgroup lattices.

It is easy to understand that the main obstacle in the study of projective images of abelian groups is the fact that normality is not preserved under projectivities; actually, the behavior of images of normal subgroups under projectivities plays a central role in the investigations concerning projectivities of groups.

Let $\mathfrak{L}$ be any lattice. An element a of $\mathfrak{L}$ is said to be *modular* if

$$(a \vee x) \wedge y = a \vee (x \wedge y)$$

for all $x, y \in \mathfrak{L}$ such that $a \leq y$ and

$$(a \vee x) \wedge y = x \vee (a \wedge y)$$

for all $x, y \in \mathfrak{L}$ such that $x \leq y$. The lattice $\mathfrak{L}$ is *modular* if all its elements are modular, i.e. if the identity

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

holds in $\mathfrak{L}$, whenever x, y, z are elements such that $x \leq z$.

If N is any normal subgroup of a group G, and φ is a projectivity from G onto another group G^*, it follows from the Dedekind's modular law that the image N^φ of N is a modular element of the lattice $\mathfrak{L}(G^*)$. In particular, any projective image of an abelian group has modular subgroup lattice, and groups with modular subgroup lattice can be considered as suitable lattice approximations of abelian groups.

Locally finite groups with modular subgroup lattice have been completely classified by K. Iwasawa [25] almost seventy years ago, while a full description of periodic groups with modular subgroup lattice has been obtained by

R. Schmidt [38] in 1986. Moreover, it turns out that the commutator subgroup of any group with modular subgroup lattice is periodic, and hence a torsion-free group has modular subgroup lattice if and only if it is abelian.

In the study of finiteness conditions from a lattice point of view, the following relevant result by G. Zacher is crucial; it was independently proved also by I.A. Rips.

Theorem 17. (G. Zacher [45]) *Let φ be a projectivity from a group G onto a group G^*, and let H be a subgroup of finite index of G. Then the subgroup H^φ has finite index in G^*.*

As a direct consequence, we have:

Corollary 3. *The class of residually finite groups is invariant under projectivities.*

A few years later, R. Schmidt [37] obtained a lattice theoretic description of the finiteness of the index of a subgroup, which of course allows to recognize subgroups of finite index within the lattice of subgroups. On the other hand, the index of a subgroup is not preserved under projectivities, as for instance all groups of prime order obviously have isomorphic subgroup lattice. The following result clarifies this situation.

Theorem 18. (M. De Falco - F. de Giovanni - C. Musella - R. Schmidt [10]) *Let G be a group and let X be a subgroup of finite index of G. Then the number $\pi(|G : X|)$ of prime factors (with multiplicity) of the index $|G : X|$ can be described by means of lattice properties. In particular, $\pi(|G : X|)$ is invariant under projectivities.*

It is easy to show that there exists a group G with an infinite class of conjugate elements such that the subgroup lattice of G is isomorphic to that of an abelian group. Therefore the class of FC-groups is not invariant under projectivities. Obviously, the same example also proves that neither the class of central-by-finite groups is invariant under projectivities nor that of finite-by-abelian groups. On the other hand, it is possible to study lattice analogues of both central-by-finite groups and finite-by-abelian groups.

A subgroup M of a group G is said to be *modularly embedded* in G if the lattice $\mathfrak{L}(\langle x, M\rangle)$ is modular for each element x of G. This concept was introduced by P.G. Kontorovic and B.I. Plotkin [26] in order to characterize torsion-free nilpotent groups by their subgroup lattices. Of course, any subgroup of the centre of a group G is modularly embedded in G, and actually the modular embedding seems to be the best translation of centrality into the subgroup lattice. In fact, using modularly embedded subgroups, the following lattice interpretation of Schur's theorem can be obtained.

Theorem 19. (M. De Falco - F. de Giovanni - C. Musella [8]) *Let G be a* *group containing a modularly embedded subgroup of finite index. Then G has a finite normal subgroup N such that the subgroup lattice $\mathfrak{L}(G/N)$ is modular.*

Suitable iterations of the above concepts allow to introduce lattice analogues of nilpotent groups, and can probably be used in order to translate Theorem 4 and Theorem 5 in terms of lattice properties.

Also Neumann's theorems have been investigated from a lattice point of view. In order to describe the corresponding results we need the following definitions, which introduce certain relevant types of generalized normal subgroups.

Let G be a group, and let X be a subgroup of G.

- X is called *almost modular* if there exists a subgroup H of G containing X such that the index $|G : H|$ is finite and X is a modular subgroup of H
- X is called *nearly modular* if there exists a modular subgroup H of G containing X such hat the index $|H : X|$ is finite

It follows from Schmidt's lattice characterization of the finiteness of the index of a subgroup that the above definitions are purely lattice theoretic, and so they can be given in any (complete) lattice. A lattice $\mathfrak{L}$ is called *almost modular* (respectively, *nearly modular*) if all its elements are almost modular (respectively, nearly modular).

Groups with almost modular subgroup lattice can be seen as lattice analogues of central-by-finite groups, while groups with nearly modular subgroup lattice correspond in this context to finite-by-abelian groups.

Theorem 20. (F. de Giovanni - C. Musella - Y.P. Sysak [20]) *Let G be a periodic group. The subgroup lattice $\mathfrak{L}(G)$ is almost modular if and only if $G = M \times K$, where M is a group with modular subgroup lattice, K is an abelian-by-finite group containing a finite normal subgroup N such that the lattice $\mathfrak{L}(K/N)$ is modular and $\pi(M) \cap \pi(K) = \emptyset$.*

Since a group is central-by-finite if and only if it is both abelian-by-finite and finite-by-abelian, the above result provides a lattice corresponding of Theorem 1, at least in the case of periodic groups. The next statement provides a lattice translation of Theorem 8, again within the universe of periodic groups.

Theorem 21. (M. De Falco - F. de Giovanni - C. Musella - Y.P. Sysak [11]) *Let G be a periodic group. The subgroup lattice $\mathfrak{L}(G)$ is nearly modular if and only if G contains a finite normal subgroup N such that the lattice $\mathfrak{L}(G/N)$ is modular.*

The following natural problem is still unsolved.

Question A *Is it possible to obtain lattice translations of the theorems of Eremin and Tomkinson on groups whose abelian subgroups are almost normal or nearly normal (at least inside the class of locally finite groups)?*

We mention finally that also some results concerning the class of BCF-groups have been translated into the theory of subgroup lattices.

A group G is called a *BMF-group* if there exists a positive integer k such that $\pi(|X : core^*(X)|) \leq k$ for each subgroup X of G, where $core^*(X)$ denotes the largest modular subgroup of G which is contained in X.

Theorem 22. (M. De Falco - F. de Giovanni - C. Musella [7]) *Let G be a locally finite BMF-group. Then G contains a subgroup M of finite index such that the lattice $\mathfrak{L}(M)$ is modular.*

Question B *Is it possible to extend Theorem 22 to the case of locally graded BMF-groups?*

4 Inertial properties

A subgroup X of a group G is said to be *inert* if the index $|X : X \cap X^g|$ is finite for each element g of G. Clearly, every normal-by-finite subgroup is inert, so that in particular normal subgroups and finite subgroups of arbitrary groups are inert. A group is *inertial* if all its subgroups are inert. Thus CF-groups (i.e. groups in which all subgroups are normal-by-finite) are inertial. The first result of this section shows that within the universe of locally finite groups there are no infinite simple inertial groups.

Theorem 23. (V.V. Belyaev - M. Kuzucuoglu - E. Seckin [3]) *Let G be a simple locally finite group. If G is inertial, then it is finite.*

The latter theorem has recently been extended to the case of simple locally graded groups by M.R. Dixon, M. Evans and A. Tortora [17]. The structure of soluble groups in which all subgroups are inert has been investigated by D.J.S. Robinson [33]; in particular, he characterized finitely generated soluble inertial groups and soluble minimax inertial groups.

A subgroup X of a group G is said to be *strongly inert* if it has finite index in $\langle X, X^g \rangle$ for each element g of G, and the group G is called *strongly inertial* if all its subgroups are strongly inert. Thus nearly normal subgroups (and in particular all subgroups of finite index) are strongly inert; it is also clear that finite subgroups of locally finite groups are strongly inert. It is easy to prove that any strongly inert subgroup is also inert, and hence strongly inertial groups are inertial. Clearly, the subgroups of order 2 of the infinite dihedral group D_∞

are not strongly inert, although they are inert in D_∞. Thus strongly inertial groups have no infinite dihedral sections. It is easy to show that inert and strongly inert subgroups of arbitrary groups have some inheritance properties. In fact, we have:

Let G be a group, and let X and Y be subgroups of G such that $Y \leq X$ and the index $|X : Y|$ is finite.

- *X is inert in G if and only if Y is inert in G;*
- *if X is strongly inert in G, then Y is strongly inert in G.*

Moreover, abelian strongly inert subgroups have the following useful behavior.

Lemma 1. *Let X be an abelian strongly inert subgroup of a group G. Then the subgroup $[X, X^g]$ is finite for each element g of G.*

Proof. As the indices $|\langle X, X^g\rangle : X|$ and $|\langle X, X^g\rangle : X^g|$ are finite, the subgroup $X \cap X^g$ has finite index in $\langle X, X^g\rangle$. Moreover, $X \cap X^g$ is contained in $Z(\langle X, X^g\rangle)$ since X is abelian, and so it follows from Schur's theorem that the commutator subgroup $\langle X, X^g\rangle' = [X, X^g]$ is finite. $\boxed{QED}$

Next result shows that all groups with finite conjugacy classes are strongly inertial, and so also inertial.

Lemma 2. *Every FC-group G is strongly inertial.*

Proof. Let X be any subgroup of G, and $H = \langle X, g\rangle$ for some g in G. The centralizer $C = C_H(\langle g\rangle^H)$ has finite index in H, so that the index $|X : X \cap C|$ is likewise finite. Moreover, $(X \cap C)^g = X \cap C$ and $X \cap C$ is normal in H. Application of Dietzmann's Lemma yields that the normal closure of $X/X \cap C$ in $H/X \cap C$ is finite. Therefore X has finite index in X^H and so X is strongly inert in G. $\boxed{QED}$

It is also possible to prove that in an arbitrary group G the elements of finite order form a locally finite subgroup if and only if all finite subgroups of G are strongly inert. In particular, the elements of finite order of any strongly inertial group form a locally finite subgroup.

Finitely generated strongly inertial groups are characterized by the following result. In particular, it turns out that in this case strongly inertial groups coincide with groups having the FC-property.

Theorem 24. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *Let G be a finitely generated strongly inertial group. Then the factor group $G/Z(G)$ is finite.*

Corollary 4. *The commutator subgroup of any strongly inertial group is locally finite.*

In particular, any perfect strongly inertial group is locally finite, and so Theorem 23 has the following consequence.

Corollary 5. *Simple strongly inertial groups are finite.*

Recall that a group G is *minimax* if it has a series of finite length whose factors satisfy either the minimal or the maximal condition on subgroups. If G is any soluble-by-finite minimax group, then its finite residual J is the largest divisble subgroup, and J is the direct product of finitely many Prüfer subgroups. Moreover, the Fitting subgroup F/J of G/J is nilpotent and G/F is finitely generated and abelian-by-finite. If G is any soluble-by-finite minimax group, the set of all prime numbers p such that G has a section of type p^∞ is an invariant, called the *spectrum* of G. A soluble-by-finite minimax group is called *p-primary* if its finite residual is a p-group. For detailed informations on the structure of soluble-by-finite minimax groups see [31], Chapter 10.
It is easy to prove the following result.

Lemma 3. *Let G be a minimax residually finite group. If G is strongly inertial, then the factor group $G/Z(G)$ is finite.*

In the above lemma, the hypothesis that the minimax group G is residually finite cannot be dropped out, even in the non-periodic case. To see this, consider the semidirect product $G = \langle x\rangle \ltimes P$, where the normal subgroup P is of type p^∞ for some prime number p and $\langle x\rangle$ an infinite cyclic subgroup such that $a^x = a^{1+p}$ for each a in P. Then G is a minimax strongly inertial group and its centre has order p.

Let Q be a torsion-free abelian minimax group, and let D be a divisible abelian p-group of finite rank (where p is a prime number). Consider D as a trivial Q-module. Since $Ext(Q, D) = \{0\}$, it follows from the Universal Coefficients Theorem that the cohomology group $H^2(Q, D)$ is isomorphic to the homomorphism group $Hom(M(Q), D)$, where $M(Q)$ is the Schur multiplier of Q (in this case $M(Q)$ is the exterior square $Q \bigwedge Q$ of Q). Let δ be an element of infinite order of $Hom(M(Q), D)$, and for all subgroups $E < D$ and $R < Q$ denote by δ_E and δ_R the homomorphisms naturally induced by δ from $M(Q)$ to D/E and from $M(R)$ to D, respectively. Assume also that the image $(R \wedge y)^{\delta_E}$ is finite for all elements y of Q, whenever $\delta_{E,R} = 0$. In this situation a central extension of D by Q with cohomology class δ is called a *central extension of type IIa* (see [33]).

Theorem 25. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *A soluble-by-finite p-primary minimax group is strongly inertial if and only if it satisfies one of the following conditions:*

(a) *The factor group $G/Z(G)$ is finite.*

(b) *G is a Černikov group and all subgroups of its finite residual are normal in G.*

(c) *G contains a finite normal subgroup N such that the group $H = G/N$ is the semidirect product of a divisible abelian normal p-subgroup D by a subgroup Q with finite commutator subgroup such that Q/Q' is torsion-free, all subgroups of D are normal in H and p does not belong to the spectrum of the centralizer of D in Q.*

(d) *G contains a finite normal subgroup N such that G/N is a central extension of type IIa.*

Corollary 6. *A soluble-by-finite minimax group is strongly inertial if and only if it is inertial and has no infinite dihedral sections.*

Next two results describe the behavior of groups in which all cyclic subgroups are strongly inert and those in which all infinite subgroups are strongly inert.

Theorem 26. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *Let G be a locally (soluble-by-finite) group. Then all cyclic subgroups of G are strongly inert if and only if every finitely generated subgroup of G is central-by-finite.*

Theorem 27. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *Let G be a locally (soluble-by-finite) group. Then all infinite subgroups of G are strongly inert if and only if G is either strongly inertial or a finite extension of an infinite cyclic subgroup.*

Groups in which all subnormal subgrouops are nearly normal have been investigated by C. Casolo [5]. We leave here as an open question the following corresponding problem.

Question C *Describe the structure of finitely generated soluble groups in which all subnormal subgroups are strongly inert.*

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Graphs and Classes of Finite Groups

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Abstract. There are different ways to associate to a finite group a certain graph. An interesting question is to analyse the relations between the structure of the group, given in group-theoretical terms, and the structure of the graph, given in the language of graph theory. This survey paper presents some contributions to this research line.

Keywords: finite groups, classes of groups, graphs

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All groups in this paper are finite. We will consider only graphs which are undirected, simple (that is, with no parallel edges), and without loops. These graphs will be characterised by the set of vertices and the adjacency relation between the vertices. Only basic concepts about graphs will be needed for this paper. They can be found in any book about graph theory or discrete mathematics, for example [12].

Given a group G, there are many ways to associate a graph with G by taking families of elements or subgroups as vertices and letting two vertices be joined by an edge if and only if they satisfy a certain relation. We may ask about characterising group structural properties by means of the properties of the associated graph. In recent years, there has been considerable interest in this line of research (see [1, 2, 3, 5, 7, 10]).

If $\mathfrak{X}$ is a class of groups, Delizia, Moravec, and Nicotera [10] associate a graph $\Gamma_{\mathfrak{X}}(G)$ with a group G by taking the nontrivial elements of G as vertices and letting $a, b \in G\setminus\{1\}$ be joined by an edge if $\langle a, b\rangle \in \mathfrak{X}$. If we choose $\mathfrak{X} = \mathfrak{A}$, where $\mathfrak{A}$ is the class of all abelian groups, then the graph is just the *commuting graph* of G and so we may think of this graph as a generalisation of the commuting graph. This graph has been used to study simple groups since the paper of Stellmacher [18].

For an arbitrary class, the graph considered by Delizia, Moravec, and Nicotera is too general to give much information and so we restrict the classes we consider. We suppose that our class $\mathfrak{X}$ *is subgroup closed and contains* $\mathfrak{A}$.

Delizia, Moravec, and Nicotera restrict their attention to groups which are $\mathfrak{X}$-transitive. A group G is said to be $\mathfrak{X}$-transitive, or an $\mathfrak{X}T$-group, if whenever $\langle a, b\rangle \in \mathfrak{X}$ and $\langle b, c\rangle \in \mathfrak{X}$, then $\langle a, c\rangle \in \mathfrak{X}$ $(a, b, c \in G)$. Note that the $\mathfrak{A}$-transitive groups are just the CT-groups, or groups in which the centraliser of any non-identity element is an abelian subgroup. CT-groups are of historical importance as an early example of the type of classification that would be used in the FeitThompson theorem and the classification of simple groups and were classified by several mathematicians (Weisner [20], Suzuki [19], Wu [23]). Characterisations of $\mathfrak{X}T$-groups are known for several classes of groups (see [11] for a survey).

Let G be an $\mathfrak{X}T$-group. If $\mathcal{C}$ is a connected component of $\Gamma_{\mathfrak{X}}(G)$, then $\mathcal{C}$ is a complete graph, $\mathcal{C} \cup \{1\}$ is a subgroup of G and $\{\mathcal{C} \cup \{1\} : \mathcal{C}$ a connected component of $\Gamma_{\mathfrak{X}}(G)\}$ is a partition of G ([17] p.145). As a consequence of the classification of groups with a partition (see for example [17, Theorems 3.5.10 and 3.5.1]) we have the following result.

Theorem 1. Let G be an $\mathfrak{X}T$-group.

(1) The connected components of $\Gamma_{\mathfrak{X}}(G)$ form a normal partition of $G \setminus \{1\}$, that is, conjugates of connected components are again connected components.

(2) Either $\Gamma_{\mathfrak{X}}(G)$ is connected or G is one of the following groups: a Frobenius group, $\mathrm{PSL}(2, 2^h)$, $\mathrm{Sz}(2^h)$ (a Suzuki group, $h = 2k + 1 > 1$).

Note that a group in the partition has the property that every 2-generator subgroup is an $\mathfrak{X}$-group. In general this will not ensure that the group is an $\mathfrak{X}$-group. For instance, $\mathfrak{N}_2$, the class of (finite) nilpotent groups of class at most 2, is an example, since the free group of exponent 3 on 3 generators has class 3, but every 2-generator subgroup has class at most 2 (see [16]). A class $\mathfrak{X}$ of groups has been called 2-recognisable (2-erkennbar) by Brandl [9] if a group G is in $\mathfrak{X}$ if and only if every 2-generator subgroup of G is in $\mathfrak{X}$.

Observe that G is an $\mathfrak{X}T$-group and $\Gamma_{\mathfrak{X}}(G)$ is connected, then every 2-generator subgroup of G is an $\mathfrak{X}$-group and so if $\mathfrak{X}$ is 2-recognisable then $G \in \mathfrak{X}$.

Many classes of groups are known to be 2-recognisable; abelian, nilpotent, supersoluble and soluble among them. Some of the classes of supersoluble groups that have been extensively investigated in recent years are 2-recognisable.

We consider here the following classes: the class $\mathfrak{T}$ of soluble T-groups (soluble groups in which normality is transitive), the class $\mathfrak{PT}$ of soluble PT-groups (soluble groups in which permutability is transitive) and the class $\mathfrak{PST}$ of soluble PST-groups (soluble groups in which Sylow permutability is transitive). That each of these classes is subgroup closed, it is contained in the class of all supersoluble groups and contains all abelian groups is well known (a description of these classes and their properties can be found in [6]).

Lemma 2. The classes $\mathfrak{T}$, $\mathfrak{PT}$, and $\mathfrak{PST}$ are all 2-recognisable.

We now define $\mathfrak{D}$ to be the class of groups G with all Sylow subgroups cyclic (these are just the groups with a cyclic normal subgroup whose quotient is cyclic and whose order and index are coprime, see [24, Theorem V.3.11]). We then define $\mathfrak{F}_{\mathfrak{X}}$ to be the class of Frobenius groups with the property that the kernel K is an $\mathfrak{X}$-group, $G/K \in \mathfrak{D}$ and for each prime $p \mid |G/K|$ and each prime $q \mid |K|$, p does not divide $q-1$.

Theorem 3. Let $\mathfrak{X}$ be a 2-recognisable class of soluble groups containing $\mathfrak{A} \cup \mathfrak{D}$. Then the class of all $\mathfrak{X}T$-groups is contained in $\mathfrak{X} \cup \mathfrak{F}_{\mathfrak{X}}$.

Theorem 4. Let $\mathfrak{X}$ be a 2-recognisable class of soluble groups containing $\mathfrak{A} \cup \mathfrak{D}$ and G be an $\mathfrak{X}T$-group.

(1) $G \in \mathfrak{X}$ if and only if $\Gamma_{\mathfrak{X}}(G)$ is connected.

(2) $G \notin \mathfrak{X}$ if and only if G is a Frobenius group and $K \backslash \{1\}$ is a connected component of $\Gamma_{\mathfrak{X}}(G)$ (where K is the kernel of G).

Bearing in mind the above results, it is natural to ask for the smallest 2-recognisable class of groups containing $\mathfrak{A} \cup \mathfrak{D}$.

Proposition 5. Let $\mathfrak{T}_0$ be the class of $\mathfrak{T}$-groups G with G/G' cyclic or G abelian. Then $\mathfrak{T}_0$ is the smallest 2-recognisable class of groups containing $\mathfrak{A} \cup \mathfrak{D}$.

The following variation of the commuting graph gives a characterisation for the groups in which all subgroups are permutable. We will call it the *graph of permutability of cyclic subgroups* (see [4]). Given a group G, consider the graph in which the vertices are the cyclic subgroups of G and in which every two vertices are adjacent when they permute. A group has all subgroups permutable if and only if the graph of permutability of cyclic subgroups is complete. A related graph whose vertices are the non-normal subgroups was studied by Bianchi, Gillio, and Verardi (see [7, 8, 13]).

The *prime graph* of a group has also attracted the attention of many researchers. The vertices of this graph are the prime numbers dividing the order of the group G and two different vertices p and q are connected if and only if G has an element of order pq. For instance, in the cyclic group of order 6, this graph is complete, but in the symmetric group of degree 3, this graph consists of two isolated vertices. The first references known to the authors of this graph correspond to Gruenberg and Kegel, in an unpublished manuscript, and to Williams, who studied the number of connected components of the prime graph of a finite group (see [14, 21, 22]). Abe and Iiyori studied in [2] a generalisation of the prime graph in the following way: given a group G, they construct the graph Γ_G whose vertices are the prime numbers dividing the order of G and in which given two different vertices p and q, they are adjacent if and only if G possesses a soluble group of order divisible by pq. Abe and Iiyori [2] proved:

Theorem 6. If G is a non-abelian simple group, then Γ_G is connected, but not complete.

Herzog, Longobardi, and Maj [15] have considered the graph whose vertices are the non-trivial conjugacy classes of a group G and in which two non-trivial conjugacy classes C and D of G are adjacent if and only if there exists $c \in C$ and $d \in D$ such that $cd = dc$. They show that if G is a soluble group, then this graph has at most two connected components, each of diameter at most 15. They also study the structure of the groups for which there exist no edges between non-central conjugacy classes and the relation between this graph and the prime graph.

In [5] (see also [4]), the authors define for a group G a graph $\Gamma(G)$ whose vertices are the conjugacy classes of cyclic subgroups of G and in which two vertices $\mathrm{Cl}_G(\langle x\rangle)$ and $\mathrm{Cl}_G(\langle y\rangle)$ are adjacent if and only if we can find an element $g \in G$ such that $\langle x\rangle$ permutes with $\langle y^g\rangle$. The main result of [5] is:

Theorem 7. A group G is a soluble PT-group if and only if the graph $\Gamma(G)$ is complete.

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Regular groups, radical rings, and Abelian Hopf Galois structures on prime-power Galois field extensions

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This is a report on joint work of the author with S. C. Featherstonhaugh and L. N. Childs, and expands slightly on the presentation given by the author at Porto Cesareo, on June 9, 2011, at the Conference on Advances in Group Theory and Applications. The full results appear in [5].

1 Hopf Galois structures

The concept of *Hopf Galois structures* arose in the study of purely inseparable field extensions, and was introduced by Chase and Sweedler, in their 1969 work [4]. It was later recognized that the Hopf algebras in question were too small to be able to describe the full automorphism structure of a purely inseparable extension of arbitrary height. However, Greither and Pareigis gave new life to the subject in 1987 [6], showing that the concept of Hopf Galois extension could be profitably applied to separable and Galois field extensions.

We will not give a definition of Hopf Galois structures (we refer to [3] for that), because we take advantage of the following result, which provides a translation in terms of regular subgroups of symmetric groups.

Theorem 1 (Greither-Pareigis). *Let L/K be a separable extension with normal closure E.*

[i]The author is very grateful to the organizers of the Conference *Advances in Group Theory and Applications 2011* for inviting him to give the talk of which this paper is a reworking. The author has been supported by MIUR–Italy via PRIN 2008 *Lie algebras, groups, computational methods, combinatorial identities, derivations.*

Let $G = \mathrm{Gal}(E/K)$, $G' = \mathrm{Gal}(E/L)$, and $X = G/G' = \{\, aG' : a \in G \,\}$ be the space of left coset.

Then there is a bijection between

(1) Hopf Galois structures on L/K, and

(2) regular subgroups N of $\mathrm{Sym}(X)$ normalized by $\lambda(G)$.

Here $\lambda : G \to \mathrm{Sym}(X)$ is the usual action of G on the left cosets:

$$g \mapsto (aG' \mapsto gaG').$$

It might be remarked that in this context the only Hopf algebras that occur are the group algebras EN.

Byott [1] was able to rephrase and refine Theorem 1.

Theorem 2 (Byott)**.** *Let $G' \leq G$ be finite groups, $X = G/G'$, and N a group of order $|X|$. There is a correspondence between*

(1) injective morphisms $\alpha : N \to \mathrm{Sym}(X)$ such that $\alpha(N)$ is regular, and

(2) injective morphisms $\beta : G \to \mathrm{Sym}(N)$ such that $\beta(G')$ is the stabilizer of the identity of N.

Here $\alpha_1(N) = \alpha_2(N)$ if and only if $\beta_1(G)$ and $\beta_2(G)$ are conjugate under $\mathrm{Aut}(N)$*. Moreover $\alpha(N)$ is normalized by $\lambda(G)$ if and only if $\beta(G) \leq \mathrm{Hol}(N)$.*

These results can be summed up as follows.

Theorem 3. *Let L/K be a separable field extension with normal closure E. Let $G = \mathrm{Gal}(E/K)$, $G' = \mathrm{Gal}(E/L)$. Let $\mathcal{S}$ be the set of isomorphism classes of groups N of order $|G/G'|$.*

Then the number of Hopf Galois structures on L/K is

$$\sum_{N \in \mathcal{S}} e(G, N),$$

where $e(G, N)$ is the number of equivalence classes, modulo conjugation under $\mathrm{Aut}(N)$, *of regular embeddings $\beta : G \to \mathrm{Hol}(N)$ such that $\beta(G')$ is the stabilizer of the identity of N.*

The main goal of [5] is to prove the following vanishing result for the summand $e(G, N)$ in Theorem 3.

Theorem 4. *Suppose G and N are non-isomorphic* abelian *p-groups, where N has rank m, and $p > m + 1$.*

Then

$$e(G, N) = 0,$$

that is, all abelian regular subgroups of $\mathrm{Hol}(N)$ are isomorphic to N.

It follows that if L/K is a Galois extension of fields with abelian Galois group G, and if L/K is H-Hopf Galois, where the K-Hopf algebra H has associated group N, then N is isomorphic to G.

2 Regular abelian subgroups

The key to our proof is the following result of [2].

Theorem 5. *Let F be an arbitrary field, and $(V, +)$ a vector space of arbitrary dimension over F.*

There is a one-to-one correspondence between

(1) abelian regular subgroups T of $\mathrm{AGL}(V)$, and

(2) commutative, associative F-algebra structures $(V, +, \cdot)$ that one can impose on the vector space structure $(V, +)$, such that the resulting ring is radical.

In this correspondence, isomorphism classes of F-algebras correspond to conjugacy classes under the action of $\mathrm{GL}(V)$ of abelian regular subgroups of $\mathrm{AGL}(V)$.

Now $\mathrm{AGL}(V)$ is the split extension of V by $\mathrm{GL}(V)$. This acts naturally on V. The above result holds verbatim if one replaces V by any abelian group N, and $\mathrm{AGL}(V)$ by the holomorph $\mathrm{Hol}(N)$ of N, that is the split extension of N by $\mathrm{Aut}(N)$. This also acts naturally on N. Thus we have

Theorem 6. *Let $(N, +)$ be an abelian group.*

There is a one-to-one correspondence between

(1) abelian regular subgroups T of $\mathrm{Hol}(N)$, and

(2) commutative, associative ring structures $(N, +, \cdot)$ that one can impose on the abelian group structure $(N, +)$, such that the resulting ring is radical.

In this correspondence, isomorphism classes of rings correspond to conjugacy classes under the action of $\mathrm{Aut}(N)$ of abelian regular subgroups of $\mathrm{Hol}(N)$.

Note how the equivalence classes fit perfectly with those of Theorem 2, involved in counting Hopf Galois structures.

3 An elementary result

Let p be a prime. Let $(N, +)$ be an elementary abelian group of order p^m. Let $(N, +, \cdot)$ be a commutative, associative, nilpotent ring based on the group $(N, +)$. Then $(N, \circ)$ is also a group, where

$$u \circ v = u + v + u \cdot v.$$

Because of the result above, each regular subgroup G of $\mathrm{Hol}(N)$ is isomorphic to such a $(N, \circ)$.

We begin with

Lemma 1. *If $(N, +)$ is elementary abelian of order p^m, with $p > m$, then $(N, \circ)$ is also elementary abelian.*

Note that this is simply stating the obvious fact that a p-element of $\mathrm{GL}(m, p)$, with $m < p$, has order p. However, we are using this simple instance as a first illustration of the way we are using Theorem 6 in the proof of Theorem 4.

We will be using repeatedly the simple relation

$$\begin{aligned} p_\circ a &= \sum_{i=1}^{p} \binom{p}{i} a^i \\ &= pa + \sum_{i=2}^{p-1} \binom{p}{i} a^i + a^p, \end{aligned}$$

where we use the notation $k_\circ a = \underbrace{a \circ \cdots \circ a}_{k \text{ times}}$.

Proof of Lemma 1. $(N, +, \cdot)$ is a nilpotent ring of order p^m. $p \geq m + 1$. Thus $N^p \subseteq N^{m+1} = \{ 0 \}$. It follows that $a^p = 0$ for $a \in N$. Now

$$p_\circ a = \sum_{i=1}^{p-1} \binom{p}{i} a^i + a^p$$

implies that $(N, \circ)$ is also elementary abelian. $\boxed{QED}$

4 Two examples

In constructing examples, the idea is to start with a suitable ring. Let F be the field with p elements, p a prime. Consider the ring of order p^p

$$(N, +, \cdot) = xF[x]/x^{p+1}F[x],$$

where $F[x]$ is the ring of polynomials in the indeterminate x. Now $(N, \circ)$ is (isomorphic to) a regular abelian subgroup of $\mathrm{Hol}(N, +)$, where $u \circ v = u + v + u \cdot v$. Let a be the image of x in the ring N. Then

$$p_\circ a = \sum_{i=1}^{p-1} \binom{p}{i} a^i + a^p = a^p \neq 0,$$

so that $(N, \circ)$ has exponent (at least) p^2. (It would be easy to see that $(N, \circ)$ has type $(p^2, p, \dots, p)$.)

This shows that the result of Lemma 1 is sharp.

For a "converse", start with the ring $x\mathbb{Z}[x]/x^{p+1}\mathbb{Z}[x]$, where p is a prime. Consider the quotient ring $(N, +, \cdot)$ of it modulo the ideal spanned by the image of $px + x^p$. Write a for the image of x in N. Then N has order p^p,

$$pa + a^p = 0, \qquad pa^i = 0, \text{ for } i > 1,$$

and $(N, +)$ has $m = p - 1$ generators $a, a^2, \dots, a^{p-1}$, and type $(p^2, p, \dots, p)$. Then there is an abelian regular subgroup of $\mathrm{Hol}(N, +)$ which is isomorphic to $(N, \circ)$. In $(N, \circ)$ we have $p_\circ a^i = 0$ for $i > 1$, and

$$p_\circ a = pa + \sum_{i=2}^{p-1} \binom{p}{i} a^i + a^p = pa + a^p = 0,$$

so that $(N, \circ)$ is elementary abelian.

In this example, $m = p + 1$, and the conclusion of Theorem 4 fails. This shows that the result of Theorem 4 is sharp.

5 Proof of Theorem 4

Because of the correspondence established in Theorem 6, we have to prove that, under the assumptions of Theorem 4, if $(N, +, \cdot)$ is any associative, nilpotent ring, then $(N, +)$ and $(N, \circ)$ are isomorphic.

We will show that the two finite abelian groups $(N, +)$ and $(N, \circ)$ have the same number of elements of each order, from which isomorphism follows.

Consider the subgroups of $(N, +)$

$$\Omega_i(N, +) = \left\{ x \in N : p^i x = 0 \right\}.$$

These are ideals of $(N, +, \cdot)$, so that they are also subgroups of $(N, \circ)$, as $x \circ y = x + y + x \cdot y$. We want to show that for each i the following *equalities* hold

$$\Omega_{i+1}(N, +) \setminus \Omega_i(N, +) = \Omega_{i+1}(N, \circ) \setminus \Omega_i(N, \circ) \tag{5.1}$$

between the set of elements of order p^{i+1} in $(N, +)$, respectively $(N, \circ)$.

However, we only need to prove the *inequalities*

$$\Omega_{i+1}(N, +) \setminus \Omega_i(N, +) \subseteq \Omega_{i+1}(N, \circ) \setminus \Omega_i(N, \circ). \tag{5.2}$$

In fact, suppose all of the (5.2) hold. If this is the case, note that N is the disjoint union of the left-hand terms of (5.2) (plus $\{ 0 \}$). Since N is finite, it follows that all inequalities in (5.2) are equalities, that is, all of the (5.1) also hold.

Consider the sections of the group $(N, +)$

$$S_i = \Omega_{i+1}(N, +)/\Omega_{i-1}(N, +),$$

for $1 \leq i < e$, where p^e is the exponent of $(N, +)$. These sections have exponent p^2 as groups with respect to $+$. Note that these are also sections of the ring $(N, +, \cdot)$ and of the group $(N, \circ)$.

We will now prove the following

Lemma 2. *The orders of the elements of each S_i are the same with respect to $+$ and $\circ$.*

From this the inequalities (5.2) will follow, and thus the main result. In fact, the cases $i = 0, 1$ of (5.2) are taken care directly by the Lemma for $i = 1$, as in this case $S = \Omega_2(N, +)$. Proceeding by induction, if $a \in \Omega_{i+1}(N, +) \setminus \Omega_i(N, +)$, the Lemma states that $p_\circ a \in \Omega_i(N, +) \setminus \Omega_{i-1}(N, +)$. By the inductive hypothesis, this is contained in $\Omega_i(N, \circ) \setminus \Omega_{i-1}(N, \circ)$, so that $a \in \Omega_{i+1}(N, \circ) \setminus \Omega_i(N, \circ)$.

Proof of Lemma 2. Clearly $T = \Omega_1(S, +) = \Omega_i(N, +)/\Omega_{i-1}(N, +)$, and $pS \subseteq T$.

Consider first an element $0 \neq a \in T$, so that a has order p with respect to $+$. We want to show that a has order p also with respect to $\circ$. Since T is an elementary abelian section of $(N, +)$, it has order at most p^m. Since $p > m + 1 > m$, Lemma 1 implies that $(T, \circ)$ is also elementary abelian.

Suppose now $a \in S \setminus T$, so that a has order p^2 with respect to $+$. We want to show that a has order p^2 also with respect to $\circ$.

Note that $(S/T, +)$ is an elementary abelian section of $(N, +)$, and thus S/T has order at most p^m. Now $(S/T, +, \cdot)$ is a nilpotent ring of order at most $p^m < p^p$, so that $S^p \subseteq S^{m+1} \subseteq T$. Using this, and the fact that $pS \subseteq T$, in the formula

$$p_\circ a = \sum_{i=1}^{p-1} \binom{p}{i} a^i + a^p,$$

we obtain that $p_\circ a \in T$, and so a has order *at most* p^2 with respect to $\circ$.

We will now show that $p_\circ a \neq 0$, so that a will have order *exactly* p^2 also with respect to $\circ$. Since we are only working in the subring of S spanned by a, we redefine S to be just that. If $pa \notin S^2$, then it is clear from

$$p_\circ a = pa + \sum_{i=2}^{p} \binom{p}{i} a^i$$

that $p_\circ a \equiv pa \not\equiv 0$ modulo S^2, so that $p_\circ a \neq 0$, and we are done.

So assume $pa \in S^2$, and let $k \geq 2$ be such that $pa \in S^k \setminus S^{k+1}$. Since S is generated by a, we will have $pS \subseteq S^k$. This means that $(S/S^k, +)$ is elementary abelian. Now $S^k \neq \{0\}$, as it contains $pa \neq 0$. Thus in the nilpotent ring S we have the proper inclusions $S \supset S^2 \supset \cdots \supset S^k \supset \{0\}$. It follows that the elementary abelian section $(S/S^k, +)$ of $(N, +)$ has a basis given by $a, a^2, \ldots, a^{k-1}$, so that it has order p^{k-1}, and thus $k - 1 \leq m$.

Consider once more

$$p_\circ a = pa + \sum_{i=2}^{p-1} \binom{p}{i} a^i + a^p.$$

Since $pa \in S^k$, for $2 \leq i \leq p-1$ we have

$$\binom{p}{i} a^i \in S^k S = S^{k+1}.$$

Since $p \geq m + 2 \geq k + 1$, we have also $a^p \in S^p \subseteq S^{k+1}$. Now the formula above yields $p_\circ a \equiv pa \not\equiv 0$ modulo S^{k+1}, so that $p_\circ a \neq 0$, and we are done. $\boxed{QED}$

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Generalisations of T-groups

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Abstract. This paper discusses work with Adolfo Ballester-Bolinches, James Beidleman, M.C. Pedraza-Aguilera, and M. F. Ragland. Let f be a subgroup embedding functor such that for every finite group G, $f(G)$ contains the set of normal subgroups of G and is contained in the set of Sylow-permutable subgroups of G. We say $H\ f\ G$ if H is an element of $f(G)$. Given such an f, let fT denote the class of finite groups in which $H\ f\ G$ if and only if H is subnormal in G; because Sylow-permutable subgroups are subnormal, this is the class in which f is a transitive relation. Thus if $f(G)$ is, respectively, the set of normal subgroups, permutable subgroups, or Sylow-permutable subgroups of G, then fT is, respectively, the class of T-groups, PT-groups, or PST-groups. Let $\mathcal{F}$ be a formation of finite groups. A subgroup M of a finite group G is said to be $\mathcal{F}$-normal in G if $G/Core_G(M)$ belongs to $\mathcal{F}$. A subgroup U of a finite group G is called a K-$\mathcal{F}$-subnormal subgroup of G if either $U = G$ or there exist subgroups $U = U_0 \leq U_1 \leq \cdots \leq U_n = G$ such that U_{i-1} is either normal or $\mathcal{F}$-normal in U_i, for $i = 1, 2, \ldots, n$. We call a finite group G an $fT_{\mathcal{F}}$-group if every K-$\mathcal{F}$-subnormal subgroup of G is in $f(G)$. When $\mathcal{F}$ is the class of all finite nilpotent groups, the $fT_{\mathcal{F}}$-groups are precisely the fT-groups. We analyse the structure of $fT_{\mathcal{F}}$-groups for certain classes of formations, particularly where the fT-groups are the T-, PT-, and PST-groups.

Keywords: T-groups, formations

MSC 2000 classification: 20D99

This paper includes work done with A. Ballester-Bolinches, M.C. Pedraza-Aguilera, M. Ragland, and J. Beidleman. See [1] for results on the situation in which $f(G)$ is the set of normal subgroups of G and [2] for results about T-, PT-, and PST-groups.

All groups treated are finite.

Definitions

A subgroup H is *subnormal* in G if $H = G$ or there exists a chain of subgroups $H = H_0 < H_1 < H_2 < \ldots < H_k = G$ such that H_{i-1} is normal in H_i for $1 \leq i \leq k$. Clearly subnormality is transitive: If H is subnormal in J and J is subnormal in G, then H is subnormal in G.

A *subgroup embedding functor* is a function f that associates a set of subgroups $f(G)$ to each group G such that if ι is an isomorphism from G onto G', then $H \in f(G)$ if and only if $\iota(H) \in f(G')$.

If f is a subgroup embedding functor and H is a subgroup of G, we say $H\ f\ G$ if $H \in f(G)$.

We assume f contains n, where $n(G)$ is the set of normal subgroups of G, and is contained in pS, where $pS(G)$ is the set of Sylow permutable subgroups of G – these are the subgroups H of G such that $HP = PH$ for every Sylow subgroup P of G.

Let $p(G)$ be the set of permutable subgroups of G, i.e. those subgroups H such that $HK = KH$ for all subgroups K of G.

We define fW to be the class of groups such that $H \leq G$ implies $H\ f\ G$, and fT to be the class of groups such that $H\ sn\ G$ implies $H\ f\ G$. Thus fT contains fW.

If $f = n$, then nW is the class of Dedekind groups, i.e. the groups such that all subgroups are normal, while nT is the class of T-groups, the groups in which every subnormal subgroup is normal. Hence the $(n)T$-groups are those in which normality is transitive.

If $f = p$, then pW is the class of Iwasawa groups, i.e. the groups such that all subgroups are permutable, while pT is the class of PT-groups, the groups in which every subnormal subgroup is permutable. Because normal implies permutable implies subnormal, the PT-groups are those in which permutability is transitive.

If $f = pS$, then pSW is the class of nilpotent groups, while pST is the class of PST-groups, the groups in which every subnormal subgroup is Sylow permutable. Because normal implies Sylow permutable implies subnormal, the PST-groups are those in which Sylow permutability is transitive.

The *nilpotent residual* of a group G is the unique smallest normal subgroup X of G such that the quotient group G/X is nilpotent. This nilpotent residual is denoted $G^{\mathfrak{N}}$; here $\mathfrak{N}$ denotes the class of finite nilpotent groups. (This residual exists because if X and Y are normal subgroups of G such that G/X and G/Y are nilpotent, then $G/X \cap Y$ is nilpotent, also.)

Theorem 1. *(Gaschütz, Zacher, Agrawal) [2] If $f = n$, p, or pS, then G is a finite soluble fT-group if and only if $G^{\mathfrak{N}}$ is abelian of odd order; $G^{\mathfrak{N}}$ and $G/G^{\mathfrak{N}}$ are of relatively prime order; $G/G^{\mathfrak{N}} \in fW$; and every subgroup of $G^{\mathfrak{N}}$ is normal in G.*

H is *pronormal* in G if for each $g \in G$, H and its conjugate H^g are conjugate in the join $< H, H^g >$, i.e $H^g = H^x$, where $x \in < H, H^g >$.

It is also possible to show that H is pronormal in G if and only if for each $g \in G$, H and H^g are conjugate via an element of $< H, H^g >^{\mathfrak{N}}$.

Examples:

Sylow p-subgroups are pronormal; so are maximal subgroups.

A subgroup that is both subnormal and pronormal is normal.

A *formation* $\mathfrak{F}$ is a class of groups such that:

(1) If $G \in \mathfrak{F}$ and X is a normal subgroup of G, then $G/X \in \mathfrak{F}$.

(2) If G/X, $G/Y \in \mathfrak{F}$ for X and Y normal subgroups in G, then $G/X \cap Y \in \mathfrak{F}$.

Here (2) is the property of $\mathfrak{N}$ guaranteeing the existence of the $\mathfrak{N}$-residual $G^{\mathfrak{N}}$. We can define $G^{\mathfrak{F}}$ similarly.

Let $\mathfrak{F}$ be a formation of finite groups containing all nilpotent groups such that any normal subgroup of any fT-group in $\mathfrak{F}$ and any subgroup of any soluble fT-group in $\mathfrak{F}$ belongs to $\mathfrak{F}$. We say such an $\mathfrak{F}$ has *Property* $f*$.

A subgroup M of a finite group G is said to be $\mathfrak{F}$-*normal* in G if $G/Core_G(M)$ belongs to $\mathfrak{F}$. A subgroup U of a finite group G is called a K-$\mathfrak{F}$-*subnormal* subgroup of G if either $U = G$ or there exist subgroups $U = U_0 \leq U_1 \leq \cdots \leq U_n = G$ such that U_{i-1} is either normal or $\mathfrak{F}$-normal in U_i, for $i = 1, 2, \ldots, n$.

We call a finite group G an $fT_{\mathfrak{F}}$-group if every K-$\mathfrak{F}$-subnormal subgroup of G is in $f(G)$. When $\mathfrak{F} = \mathfrak{N}$, the $fT_{\mathfrak{N}}$-groups are precisely the fT-groups. (This is because an $\mathfrak{N}$-normal subgroup is subnormal, so K-$\mathfrak{N}$-subnormal is the same as subnormal.)

H is $\mathfrak{F}$-*pronormal* in G if for each $g \in G$, H and H^g are conjugate via an element of $< H, H^g >^{\mathfrak{F}}$.

Just as K-$\mathfrak{N}$-subnormality is the same as subnormality, $\mathfrak{N}$-pronormality is the same as pronormality.

Results

Theorem 2. *[3] If $\mathfrak{F}$ is a subgroup-closed saturated formation containing $\mathfrak{N}$, a soluble group is in $\mathfrak{F}$ if and only if each of its subgroups is $\mathfrak{F}$-subnormal. (This generalises the well known fact for $\mathfrak{N}$.)*

If $\mathfrak{F}_1 \supseteq \mathfrak{F}_2$, every K-$\mathfrak{F}_2$-subnormal subgroup is K-$\mathfrak{F}_1$-subnormal, and every $\mathfrak{F}_1$-pronormal subgroup is $\mathfrak{F}_2$-pronormal.

Thus all our $fT_{\mathfrak{F}}$-groups are fT-groups, because K-$\mathfrak{N}$-subnormal subgroups are K-$\mathfrak{F}$-subnormal.

Theorem 3. *[3] If $\mathfrak{F}$ is a subgroup-closed saturated formation containing $\mathfrak{N}$, then a soluble group is a $T_{\mathfrak{F}}$-group if and only if each of its subgroups is $\mathfrak{F}$-pronormal.*

If $\mathfrak{F}$ contains $\mathfrak{N}$, then $G \in \mathfrak{F}$ is a $T_{\mathfrak{F}}$-group if and only if G is Dedekind.

Theorem 4. *If $\mathfrak{F}$ contains $\mathfrak{U}$, the formation of supersoluble groups, then the soluble $T_{\mathfrak{F}}$-groups are just the Dedekind groups.*

Proof. Each soluble $T_{\mathfrak{F}}$-group, being a soluble T-group, is in $\mathfrak{U}$, which is contained in $\mathfrak{F}$. Thus by Theorem 3, such a group is Dedekind.

Let $\mathfrak{O}$ be the set of ordered pairs (p, q) where p and q are prime numbers such that q divides $p - 1$, and for (p, q) in $\mathfrak{O}$, denote by $X_{(p,q)}$ a non-abelian group of order pq.

Let $\mathfrak{X}$ be the class consisting of every group that is isomorphic to $X_{(p,q)}$ for some $(p,q) \in \mathfrak{O}$ and denote by $\mathfrak{X}_{\mathfrak{F}}$ the class $\mathfrak{X} \cap \mathfrak{F}$.

Let $\mathfrak{Y}$ be the class of non-abelian simple groups, and let $\mathfrak{Y}_{\mathfrak{F}}$ be the class $\mathfrak{Y} \cap \mathfrak{F}$, and denote by $\mathfrak{S}$ the class of finite soluble groups.

Definition.

A group G is said to be an $fR_{\mathfrak{F}}$-group if G is an fT-group and

[i] No section of $G/G^{\mathfrak{S}}$ is isomorphic to an element of $\mathfrak{X}_{\mathfrak{F}}$.

[ii] No chief factor of $G^{\mathfrak{S}}$ is isomorphic to an element of $\mathfrak{Y}_{\mathfrak{F}}$.

Theorem 5. *If G is a group and $\mathfrak{F}$ has Property $f*$, then $G \in fT_{\mathfrak{F}}$ if and only if $G \in fR_{\mathfrak{F}}$.*

Theorem 6. *Let G be a group and $\mathfrak{F}$ be a formation containing $\mathfrak{N}$. If G is a soluble $fT_{\mathfrak{F}}$-group then Conditions (i), (ii), and (iii) below hold, and if (i), (ii) and (iii) hold and $\mathfrak{S} \cap \mathfrak{F}$ has Property $f*$ where $f = n$, p, or pS, then G is a soluble $fT_{\mathfrak{F}}$-group.*

[i] $G^{\mathfrak{F}}$ is a normal abelian Hall subgroup of G with odd order;

[ii] $X/X^{\mathfrak{F}}$ is an fW-group for every X sn G;

[iii] Every subgroup of $G^{\mathfrak{F}}$ is normal in G.

Definition. $he(G)$ is the set of hypercentrally embeddded subgroups of G, i.e. the set of subgroups H such that $H/H_G \leq Z_\infty(G/H_G)$, the hypercentre of G/H_G.

Lemma 1. *For all G, $p(G)$ is contained in $he(G)$, which is contained in $pS(G)$. However, these subgroup embedding functors are all distinct.*

Theorem 7. *If $\mathfrak{F}$ is a formation, then $\mathfrak{S} \cap \mathfrak{F}$ satisfies $pS*$ if and only if it satisfies $he*$. If G is a soluble group and $\mathfrak{S} \cap \mathfrak{F}$ possesses this property, then $G \in pST_{\mathfrak{F}}$ if and only if $G \in heT_{\mathfrak{F}}$.*

Thus it is possible for distinct functors f and g to yield the same generalisations $fT_{\mathfrak{F}}$ and $gT_{\mathfrak{F}}$, leading to the following:

Question. What other possibilities for f lead to new fT and fW and therefore potentially new $fT_{\mathfrak{F}}$?

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Representation growth and representation zeta functions of groups

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Abstract. We give a short introduction to the subject of representation growth and representation zeta functions of groups, omitting all proofs. Our focus is on results which are relevant to the study of arithmetic groups in semisimple algebraic groups, such as the group $\mathrm{SL}_n(\mathbb{Z})$ consisting of $n \times n$ integer matrices of determinant 1. In the last two sections we state several results which were recently obtained in joint work with N. Avni, U. Onn and C. Voll.

Keywords: Representations, characters, arithmetic groups, p-adic Lie groups, zeta functions.

MSC 2000 classification: primary 22E55, secondary 22E50, 20F69

1 Introduction

Let G be a group. For $n \in \mathbb{N}$, let $r_n(G)$ denote the number of isomorphism classes of n-dimensional irreducible complex representations of G. We suppose that G is *representation rigid*, i.e., that $r_n(G) < \infty$ for all positive integers n.

If the group G is finite then G is automatically representation rigid and the sequence $r_n(G)$ has only finitely many non-zero terms, capturing the distribution of irreducible character degrees of G. The study of finite groups by means of their irreducible character degrees and conjugacy classes is a well established research area; e.g., see [10] and references therein. Interesting asymptotic phenomena are known to occur when one considers the irreducible character degrees of suitable infinite families of finite groups, for instance, families of finite groups H of Lie type as $|H|$ tends to infinity; see [14].

In the present survey we are primarily interested in the situation where G is infinite, albeit G may sometimes arise as an inverse limit of finite groups. Two fundamental questions in this case are: what are the arithmetic properties of the sequence $r_n(G)$, $n \in \mathbb{N}$, and what is the asymptotic behaviour of $R_N(G) = \sum_{n=1}^{N} r_n(G)$ as N tends to infinity? To a certain degree this line of investigation is inspired by the subject of subgroup growth and subgroup zeta functions which,

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in a similar way, is concerned with the distribution of finite index subgroups; e.g., see [16, 8].

In order to streamline the investigation it is convenient to encode the arithmetic sequence $r_n(G)$, $n \in \mathbb{N}$, in a suitable generating function. The *representation zeta function* of G is the Dirichlet generating function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G)\, n^{-s} \qquad (s \in \mathbb{C}).$$

If the group G is such that there is a one-to-one correspondence between isomorphism classes of irreducible representations and irreducible characters then, writing $\mathrm{Irr}(G)$ for the space of irreducible characters of G, we can express the zeta function also in the suggestive and slightly more algebraic form

$$\zeta_G(s) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s} \qquad (s \in \mathbb{C}).$$

The function $\zeta_G(s)$ is a suitable vehicle for studying the distribution of character degrees of the group G whenever the representation growth of G is 'not too fast', a condition which is made precise in Section 3. Groups which meet this requirement include, for instance, arithmetic groups in semisimple algebraic groups with the Congruence Subgroup Property and open compact subgroups of semisimple p-adic Lie groups. In recent years, several substantial results have been obtained concerning the representation growth and representation zeta functions of these types of groups; see [12, 13, 2, 1, 3, 4, 5, 6]. In the present survey we discuss some of these results and we indicate what kinds of methods are involved in proving them.

2 Finite groups of Lie type

Our primary focus is on infinite groups, but it is instructive to briefly touch upon representation zeta functions of finite groups of Lie type. For example, the representation theory of the general linear group $\mathrm{GL}_2(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ is well understood and one deduces readily that

$$\zeta_{\mathrm{GL}_2(\mathbb{F}_q)}(s) = (q-1)\left(1 + q^{-s} + \tfrac{q-2}{2}(q+1)^{-s} + \tfrac{q}{2}(q-1)^{-s}\right). \qquad (2.1)$$

It is remarkable that the formula (2.1) is uniform in q in the sense that both the irreducible character degrees and their multiplicities can be expressed in terms of polynomials in q over the rational field $\mathbb{Q}$. In general, Deligne-Lusztig theory provides powerful and sophisticated tools to study the irreducible characters of finite groups of Lie type. In [14], Liebeck and Shalev obtained, for instance, the following general asymptotic result.

Theorem 1 (Liebeck and Shalev)**.** *Let L be a fixed Lie type and let h be the Coxeter number of the corresponding simple algebraic group $\mathbf{G}$, i.e., $h+1 = \dim \mathbf{G}/\mathrm{rk}\mathbf{G}$. Then for the finite quasi-simple groups $L(q)$ of type L over $\mathbb{F}_q$,*

$$\zeta_{L(q)}(s) \to \begin{cases} 1 & \text{for } s \in \mathbb{R}_{>2/h} \\ \infty & \text{for } s \in \mathbb{R}_{<2/h} \end{cases} \qquad \text{as } q \to \infty.$$

The Coxeter number h is computed easily. For example, for $\mathbf{G} = \mathrm{SL}_n$ and $L(q) = \mathrm{SL}_n(\mathbb{F}_q)$ one has $h = n$. In the smallest interesting case $n = 2$ and for odd q, the zeta function of $\mathrm{SL}_2(\mathbb{F}_q)$ is

$$\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s) = 1+q^{-s}+\tfrac{q-3}{2}(q+1)^{-s}+\tfrac{q-1}{2}(q-1)^{-s}+2(\tfrac{q+1}{2})^{-s}+2(\tfrac{q-1}{2})^{-s}, \quad (2.2)$$

which is approximately the expression in (2.1) divided by $(q-1)$. From the explicit formula one can verify directly the assertion of Theorem 1 in this special case.

3 Abscissa of convergence and polynomial representation growth

In Section 1 we introduced the zeta function $\zeta_G(s)$ of a representation rigid group G as a formal Dirichlet series. Clearly, if G is finite – or more generally if G has only finitely many irreducible complex representations – then the Dirichlet polynomial $\zeta_G(s)$ defines an analytic function on the entire complex plane.

Now suppose that G is infinite and that $r_n(G)$ is non-zero for infinitely many $n \in \mathbb{N}$. Naturally, we are interested in the convergence properties of $\zeta_G(s)$ for $s \in \mathbb{C}$. The general theory of Dirichlet generating functions shows that the region of convergence is always a right half plane of $\mathbb{C}$, possibly empty, and that the resulting function is analytic. If the region of convergence is non-empty, one is also interested in meromorphic continuation of the function to a larger part of the complex plane.

The *abscissa of convergence* $\alpha(G)$ of $\zeta_G(s)$ is the infimum of all $\alpha \in \mathbb{R}$ such that the series $\zeta_G(s)$ converges (to an analytic function) on the right half plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$. The abscissa $\alpha(G)$ is finite if and only if G has *polynomial representation growth*, i.e., if $R_N(G) = \sum_{n=1}^{N} r_n(G)$ grows at most polynomially in N. In fact, if the growth sequence $R_N(G)$, $N \in \mathbb{N}$, is unbounded then

$$\alpha(G) = \limsup_{N\to\infty} \frac{\log R_N(G)}{\log N}$$

gives the *polynomial degree of growth*: $R_N(G) = O\left(N^{\alpha(G)+\varepsilon}\right)$ for every $\varepsilon > 0$.

Two fundamental problems in the subject are: to characterise groups of polynomial representation growth – motivated by Gromow's celebrated theorem on groups of polynomial word growth – and to link the actual value of the abscissa of convergence $\alpha(G)$ of a group G to structural properties of G. In general these questions are still very much open. However, in the context of semisimple algebraic groups and their arithmetic subgroups a range of results have been obtained. A selection of these are discussed in the following sections.

4 Witten zeta functions

In [20], Witten initiated in the context of quantum gauge theories the study of certain representation zeta functions. Let $\mathbf{G}$ be a connected, simply connected, complex almost simple algebraic group and let $G = \mathbf{G}(\mathbb{C})$. It is natural to focus on rational representations of the algebraic group G and one can show that G is representation rigid in this restricted sense. The *Witten zeta function* $\zeta_G(s)$ counts irreducible rational representations of the complex algebraic group G. These zeta functions also appear naturally as archimedean factors of representation zeta functions of arithmetic groups, as explained in Section 8.

For example, the group $\mathrm{SL}_2(\mathbb{C})$ has a unique irreducible representation of each possible degree. Hence

$$\zeta_{\mathrm{SL}_2(\mathbb{C})}(s) = \sum_{n=1}^{\infty} n^{-s},$$

the famous Riemann zeta function. In particular, the abscissa of convergence is 1 and there is a meromorphic continuation to the entire complex plane.

In general, the irreducible representations V_λ of G are parametrised by their highest weights $\lambda = \sum_{i=1}^{r} a_i \overline{\omega}_i$, where $\overline{\omega}_1, \ldots, \overline{\omega}_r$ denote the fundamental weights and the coefficients $a_1, \ldots, a_r$ range over all non-negative integers. Moreover, $\dim V_\lambda$ is given by the Weyl dimension formula. By a careful analysis, Larsen and Lubotzky prove in [13] the following result.

Theorem 2 (Larsen and Lubotzky). *Let $\mathbf{G}$ be a connected, simply connected, complex almost simple algebraic group and let $G = \mathbf{G}(\mathbb{C})$. Then $\alpha(G) = 2/h$, where h is the Coxeter number of $\mathbf{G}$.*

It is known that Witten zeta functions can be continued meromorphically to the entire complex plane. Further analytic properties of these functions, such as the location of singularities and functional relations, have been investigated in some detail using multiple zeta functions; e.g., see [18, 11]. It is remarkable that the same invariant $2/h$ features in Theorems 1 and 2. Currently there appears to be no conceptual explanation for this.

5 The group $\mathbf{SL}_2(R)$ for discrete valuation rings R

If G is a topological group it is natural to focus attention on continuous representations. A finitely generated profinite group G is representation rigid in this restricted sense if and only if it is *FAb*, i.e., if every open subgroup H of G has finite abelianisation $H/[H,H]$. This is a consequence of Jordan's theorem on abelian normal subgroups of bounded index in finite linear groups. We tacitly agree that the representation zeta function $\zeta_G(s)$ of a finitely generated FAb profinite group G counts irreducible continuous complex representations of G.

Let R be a complete discrete valuation ring, with residue field $\mathbb{F}_q$ of odd characteristic. This means that R is either a finite integral extension of the ring of p-adic integers $\mathbb{Z}_p$ for some prime p or a formal power series ring $\mathbb{F}_q[\![t]\!]$ over a finite field of cardinality q.

In [12], Jaikin-Zapirain showed by a hands-on computation of character degrees that the representation zeta function $\zeta_{\mathrm{SL}_2(R)}(s)$ equals

$$\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s) + \left(4q\left(\tfrac{q^2-1}{2}\right)^{-s} + \tfrac{q^2-1}{2}(q^2-q)^{-s} + \tfrac{(q-1)^2}{2}(q^2+q)^{-s}\right)/(1-q^{1-s}),$$

where the Dirichlet polynomial $\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s)$ is described in (2.2). It is remarkable that the above formula is uniform in q, irrespective of the characteristic, absolute ramification index or isomorphism type of the ring R. In the case where R has characteristic 0, Lie-theoretic techniques combined with Clifford theory can be used to gain an insight into the features of this specific example which hold more generally; see Sections 6 and 9.

Clearly, the explicit formula for the function $\zeta_{\mathrm{SL}_2(R)}(s)$ provides a meromorphic extension to the entire complex plane. The abscissa of convergence is 1 and, in view of Theorems 1 and 2, this value could be interpreted as $2/h$, the Coxeter number of SL_2 being $h = 2$. But such an interpretation is misleading, as can be seen from the following general result obtained in [13].

Theorem 3 (Larsen and Lubotzky)**.** *Let $\mathbf{G}$ be a simple algebraic group over a non-archimedean local field F. Suppose that $\mathbf{G}$ is F-isotropic, i.e., $\mathrm{rk}_F \mathbf{G} \geq 1$. Let H be a compact open subgroup of $\mathbf{G}(F)$. Then $\alpha(H) \geq 1/15$.*

Taking $\mathbf{G} = \mathrm{SL}_n$ and $F = \mathbb{Q}_p$, we may consider the compact p-adic Lie groups $\mathrm{SL}_n(\mathbb{Z}_p)$. For these groups $2/h = 2/n \to 0$ as $n \to \infty$, whereas $\alpha(\mathrm{SL}_n(\mathbb{Z}_p))$ is uniformly bounded away from 0. Currently, the only explicit values known for $\alpha(\mathrm{SL}_n(\mathbb{Z}_p))$ are: 1 for $n = 2$ (as seen above), and 2/3 for $n = 3$ (see [4]). Unfortunately, these do not yet indicate the general behaviour.

6 FAb compact p-adic Lie groups

Let G be a compact p-adic Lie group. Then one associates to G a $\mathbb{Q}_p$-Lie algebra as follows. The group G contains a uniformly powerful open pro-p subgroup U. By the theory of powerful pro-p groups, U gives rise to a $\mathbb{Z}_p$-Lie lattice $L = \log(U)$ and the induced $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ does not depend on the specific choice of U. It is a fact that G is FAb if and only if $\mathcal{L}(G)$ is perfect, i.e., if $[\mathcal{L}(G), \mathcal{L}(G)] = \mathcal{L}(G)$. Conversely, for any $\mathbb{Q}_p$-Lie algebra $\mathcal{L}$ one can easily produce compact p-adic Lie groups G such that $\mathcal{L}(G) = \mathcal{L}$, using the exponential map. This supplies a large class of compact p-adic Lie groups which are FAb and hence have polynomial representation growth.

Using the Kirillov orbit method and techniques from model theory, Jaikin-Zapirain established in [12] that the representation zeta function of a FAb compact p-adic analytic pro-p group can always be expressed as a rational function in p^{-s} over $\mathbb{Q}$. More generally, he proved the following result, which is illustrated by the explicit example $G = \mathrm{SL}_2(R)$ given in Section 5.

Theorem 4 (Jaikin-Zapirain)**.** *Let G be an FAb compact p-adic Lie group, and suppose that $p > 2$. Then there are finitely many positive integers $n_1, \ldots, n_k$ and rational functions $f_1, \ldots, f_k \in \mathbb{Q}(X)$ such that*

$$\zeta_G(s) = \sum_{i=1}^{k} f_i(p^{-s})\, n_i^{-s}.$$

In particular, the theorem shows that the zeta function of a FAb compact p-adic Lie group G extends meromorphically to the entire complex plane. The invariant $\alpha(G)$ is the largest real part of a pole of $\zeta_G(s)$. It is natural to investigate the whole spectrum of poles and zeros of $\zeta_G(s)$.

Currently, very little is known about the location of the zeros of representation zeta functions. In 2010 Kurokawa and Kurokawa observed from the explicit formula given in Section 5 that $\zeta_{\mathrm{SL}_2(\mathbb{Z}_p)}(s) = 0$ for $s \in \{-1, -2\}$. We note that if G is a finite group then $\zeta_G(-2) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$. Based on this fact and the results in [12] one can prove the following general result.

Theorem 5 (Jaikin-Zapirain and Klopsch)**.** *Let G be an infinite FAb compact p-adic Lie group and suppose that $p > 2$. Then $\zeta_G(-2) = 0$.*

7 Rational representations of the infinite cyclic group

Before considering arithmetic subgroups of semisimple algebraic groups, let us look at representations of the simplest infinite group, i.e., the infinite cyclic group C_∞. The group C_∞ has already infinitely many 1-dimensional representations. Hence in order to say anything meaningful we need to slightly adapt our basic definitions.

We make two modifications: firstly let us only consider representations with finite image and secondly let us consider irreducible representations over $\mathbb{Q}$ rather than $\mathbb{C}$. More precisely, for any finitely generated nilpotent group Γ let $\hat{r}_n^{\mathbb{Q}}(\Gamma)$ denote the number of n-dimensional irreducible representations of Γ over $\mathbb{Q}$ with finite image. Then it turns out that $\hat{r}_n^{\mathbb{Q}}(\Gamma)$ is finite for every $n \in \mathbb{N}$ and we can define the $\mathbb{Q}$-*rational representation zeta function*

$$\zeta_\Gamma^{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \hat{r}_n^{\mathbb{Q}}(\Gamma)\, n^{-s}.$$

Using that every finite nilpotent group is the direct product of its Sylow p-subgroups and basic facts from character theory, one can show that $\zeta_\Gamma^{\mathbb{Q}}(s)$ admits an *Euler product decomposition*

$$\zeta_\Gamma^{\mathbb{Q}}(s) = \prod_{p \text{ prime}} \zeta_{\Gamma,p}^{\mathbb{Q}}(s), \tag{7.1}$$

where for each prime p the local factor $\zeta_{\Gamma,p}^{\mathbb{Q}}(s) = \sum_{k=0}^{\infty} \hat{r}_{p^k}^{\mathbb{Q}}(\Gamma)\, p^{-ks}$, enumerating irreducible representations of p-power dimension, can be re-interpreted as the $\mathbb{Q}$-rational representation zeta function of the pro-p completion $\widehat{\Gamma}_p$ of Γ. For more details and deeper results in this direction we refer to the forthcoming article [9].

Let us now return to the simplest case: $\Gamma = C_\infty$, the infinite cyclic group. Since the group C_∞ is abelian, its irreducible representations over $\mathbb{Q}$ with finite image can be effectively described by means of Galois orbits of irreducible complex characters. In the general setting, one would also need to keep track of Schur indices featuring in the computation of $\zeta_{\Gamma,2}^{\mathbb{Q}}(s)$. A short analysis yields

$$\zeta_{C_\infty}^{\mathbb{Q}}(s) = \sum_{m=1}^{\infty} \varphi(m)^{-s},$$

where φ denotes Euler's function familiar from elementary number theory.

The Dirichlet series $\psi(s) = \sum_{m=1}^{\infty} \varphi(m)^{-s}$ is of independent interest in analytic number theory and has been studied by many authors; e.g., see [7]. The Euler product decomposition (7.1) can be established directly

$$\psi(s) = \prod_{p \text{ prime}} \left(1 + (p-1)^{-s}/(1-p^{-s})\right).$$

The abscissa of convergence of $\psi(s)$, which can be interpreted as the degree $\alpha^{\mathbb{Q}}(C_\infty)$ of $\mathbb{Q}$-rational representation growth, is equal to 1. In fact, writing

$$\psi(s) = \underbrace{\prod_{p \text{ prime}} \big(1 + (p-1)^{-s} - p^{-s}\big) \cdot}_{\text{converges for } \mathrm{Re}(s) > 0} \underbrace{\prod_{p \text{ prime}} (1 - p^{-s})^{-1}}_{\text{Riemann zeta function } \zeta(s)} \quad ,$$

one sees that $\psi(s)$ admits a meromorphic continuation to $\mathrm{Re}(s) > 0$ (but not to the entire complex plane) and has a simple pole at $s = 1$ with residue $c = \zeta(2)\zeta(3)/\zeta(6) = 1.9435964\ldots$ This yields very precise asymptotics for the $\mathbb{Q}$-rational representation growth of C_∞; in particular,

$$\sum_{n=1}^{N} \hat{r}_n^{\mathbb{Q}}(C_\infty) = \#\{m \mid \varphi(m) \leq N\} \sim cN \qquad \text{as } N \to \infty.$$

One may regard this simple case and its beautiful connections to classical analytic number theory as a further motivation for studying representation zeta functions of arithmetic groups.

8 Arithmetic lattices in semisimple groups

In this section we turn our attention to lattices in semisimple locally compact groups. These lattices are discrete subgroups of finite co-volume and often, but not always, have arithmetic origin. For instance, $\mathrm{SL}_n(\mathbb{Z})$ is an arithmetic lattice in the real Lie group $\mathrm{SL}_n(\mathbb{R})$. More generally, let Γ be an arithmetic irreducible lattice in a semisimple locally compact group G of characteristic 0. Then Γ is commensurable to $\mathbf{G}(\mathcal{O}_S)$, where $\mathbf{G}$ is a connected, simply connected absolutely almost simple algebraic group defined over a number field k and $\mathcal{O}_S$ is the ring of S-integers for a finite set S of places of k. By a theorem going back to Borel and Harish-Chandra, any such $\mathbf{G}(\mathcal{O}_S)$ forms an irreducible lattice in the semisimple locally compact group $G = \prod_{\wp \in S} \mathbf{G}(k_\wp)$ under the diagonal embedding, as long as S is non-empty and contains all archimedean places $\wp$ such that $\mathbf{G}(k_\wp)$ is non-compact. Examples of this construction are $\mathrm{SL}_n(\mathbb{Z}[\sqrt{2}]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$. Margulis has shown that in the higher rank situation all irreducible lattices are arithmetic and arise in this way. For precise notions and a more complete description see [17].

Throughout the following we assume, for simplicity of notation, that $\Gamma = \mathbf{G}(\mathcal{O}_S)$ as above. In [15], Lubotzky and Martin gave a characterisation of arithmetic groups of polynomial representation growth, linking them to the classical Congruence Subgroup Problem.

Theorem 6 (Lubotzky and Martin)**.** *Let* Γ *be an arithmetic group as above. Then* $\alpha(\Gamma)$ *is finite if and only if* Γ *has the Congruence Subgroup Property.*

The group Γ has the Congruence Subgroup Property (CSP) if, essentially, all its finite index subgroups arise from the arithmetic structure of the group. Technically, this means that the congruence kernel $\ker(\widehat{\mathbf{G}(\mathcal{O}_S)} \to \overline{\mathbf{G}(\mathcal{O}_S)})$ is finite; here $\widehat{\mathbf{G}(\mathcal{O}_S)}$ is the profinite completion and $\overline{\mathbf{G}(\mathcal{O}_S)} \cong \prod_{\mathfrak{p} \notin S} \mathbf{G}(\mathcal{O}_{\mathfrak{p}})$, with $\mathfrak{p}$ running over non-archimedean places, denotes the congruence completion of $\mathbf{G}(\mathcal{O}_S)$. For instance, it was shown by Bass-Lazard-Serre and Mennicke that the group $\mathrm{SL}_n(\mathbb{Z})$ has the CSP if and only if $n \geq 3$. That $\mathrm{SL}_2(\mathbb{Z})$ does not have the CSP was discovered by Fricke and Klein. Retrospectively this is not surprising, because $\mathrm{SL}_2(\mathbb{Z})$ contains a free subgroup of finite index. We refer to [19] for a comprehensive survey of the Congruence Subgroup Problem, i.e., the problem to decide precisely which arithmetic groups have the CSP.

Suppose that Γ has the CSP. Using Margulis' super-rigidity theorem, Larsen and Lubotzky derived in [13] an Euler product decomposition for $\zeta_\Gamma(s)$, which takes a particularly simple form whenever the congruence kernel is trivial.

Theorem 7 (Larsen and Lubotzky)**.** *Let* Γ *be an arithmetic group as above and suppose that* Γ *has the CSP. Then* $\zeta_\Gamma(s)$ *admits an Euler product decomposition. In particular, if the congruence kernel for* $\Gamma = \mathbf{G}(\mathcal{O}_S)$ *is trivial then*

$$\zeta_\Gamma(s) = \zeta_{\mathbf{G}(\mathbb{C})}(s)^{[k:\mathbb{Q}]} \prod_{\mathfrak{p} \notin S} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s). \tag{8.1}$$

For instance, for the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, the Euler product takes the form

$$\zeta_{\mathrm{SL}_n(\mathbb{Z})}(s) = \zeta_{\mathrm{SL}_n(\mathbb{C})}(s) \prod_{p \text{ prime}} \zeta_{\mathrm{SL}_n(\mathbb{Z}_p)}(s).$$

In Sections 4 and 5 we already encountered individually the factors of these Euler products: $\zeta_{\mathbf{G}(\mathbb{C})}(s)$ is the Witten zeta function capturing rational representations of the algebraic group $\mathbf{G}(\mathbb{C})$ and, for each $\mathfrak{p}$, the function $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ enumerates continuous representations of the compact p-adic Lie group $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$. Larsen and Lubotzky's results for the abscissae of convergence of these local zeta functions include Theorems 2 and 3 stated above.

Regarding the abscissa of convergence of the global representation zeta function, Avni employed in [1] model-theoretic techniques to prove that the abscissa of convergence of $\zeta_\Gamma(s)$ is always a rational number. In [13], Larsen and Lubotzky made the following conjecture, which can be regarded as a refinement of Serre's conjecture on the Congruence Subgroup Problem.

Conjecture 1 (Larsen and Lubotzky)**.** *Let* G *be a higher-rank semisimple locally compact group. Then, for any two irreducible lattices* Γ_1 *and* Γ_2 *in* G, $\alpha(\Gamma_1) = \alpha(\Gamma_2)$.

Roughly speaking, the conjecture states that the ambient semisimple locally compact group does not only control whether lattices contained in it have the CSP (as in Serre's conjecture), but also what their polynomial degree of representation growth is. A concrete example of a lattice in $\mathrm{SL}_n(\mathbb{R})$ which is rather different from the most familiar one $\mathrm{SL}_n(\mathbb{Z})$ is the special unitary group $\mathrm{SU}_n(\mathbb{Z}[\sqrt{2}], \mathbb{Z})$, consisting of all matrices $A = (a_{ij})$ over the ring $\mathbb{Z}[\sqrt{2}]$ with $\det A = 1$ and $A^{-1} = (a_{ji}^{\sigma})$, where σ is the Galois automorphism of $\mathbb{Q}(\sqrt{2})$ swapping $\sqrt{2}$ and $-\sqrt{2}$.

9 New results for arithmetic groups and compact p-adic Lie groups

The short announcement [2] summarises a number of results obtained recently by the author in joint work with Avni, Onn and Voll. Details are appearing in [3, 4, 6]. The toolbox which we use to prove our results comprises a variety of techniques which can only be hinted at: they include, for instance, the Kirillov orbit method for p-adic analytic pro-p groups, methods from $\mathfrak{p}$-adic integration and the study of generalised Igusa zeta functions, the theory of sheets of simple Lie algebras, resolution of singularities in characteristic 0, aspects of the Weil conjectures regarding zeta functions of smooth projective varieties over finite fields, approximative and exact Clifford theory.

In summary our main results are

- a global *Denef formula* for the zeta functions of principal congruence subgroups of compact p-adic Lie groups, such as $\mathrm{SL}_n^m(\mathbb{Z}_p) \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$;
- local *functional equations* for the zeta functions of principal congruence subgroups of compact p-adic Lie groups, such as $\mathrm{SL}_n^m(\mathbb{Z}_p) \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$;
- *candidate pole sets* for the non-archimedean factors occurring in the Euler product (8.1), e.g., the zeta functions $\zeta_{\mathrm{SL}_n(\mathbb{Z}_p)}(s)$;
- *explicit formulae* for the zeta functions of compact p-adic Lie groups of type A_2, such as $\mathrm{SL}_3(\mathbb{Z}_p)$ and $\mathrm{SU}_3(\mathfrak{O}, \mathbb{Z}_p)$ for unramified $\mathfrak{O}$;
- *meromorphic continuation* of zeta functions and a precise *asymptotic description* of the representation growth for arithmetic groups of type A_2, such as $\mathrm{SL}_3(\mathbb{Z})$.

These results are clearly relevant in the context of the Euler product (8.1). Moreover, a large part of our work applies in a more general context than discussed so far. We recall from Section 6 that a compact p-adic Lie group G is representation rigid if and only if its $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G)$ is perfect. Let k be a number field, and let $\mathcal{O}$ be its ring of integers. Let Λ be an $\mathcal{O}$-Lie lattice such that $k \otimes_{\mathcal{O}} \Lambda$ is perfect of dimension d. Let $\mathfrak{o}$ be the completion $\mathcal{O}_{\mathfrak{p}}$ of $\mathcal{O}$ at a non-archimedean place $\mathfrak{p}$. Let $\mathfrak{O}$ be a finite integral extension of $\mathfrak{o}$, corresponding to a place $\mathfrak{P}$ lying above $\mathfrak{p}$. For $m \in \mathbb{N}$, let $\mathfrak{g}^m(\mathfrak{O})$ denote the mth principal congruence Lie sublattice of the $\mathfrak{O}$-Lie lattice $\mathfrak{O} \otimes_{\mathcal{O}} \Lambda$. For sufficiently large m, let $\mathsf{G}^m(\mathfrak{O})$ be the p-adic analytic pro-p group $\exp(\mathfrak{g}^m(\mathfrak{O}))$.

Using the Kirillov orbit method for permissible $\mathsf{G}^m(\mathfrak{O})$, e.g., $\mathrm{SL}_n^1(\mathbb{Z}_p)$, we can 'linearise' the problem of enumerating irreducible characters of the group $\mathsf{G}^m(\mathfrak{O})$ by their degrees. We then set up a generalised Igusa zeta function, i.e., a p-adic integral of the form

$$\mathcal{Z}_{\mathfrak{O}}(r,t) = \int_{(x,\mathbf{y}) \in V(\mathfrak{O})} |x|_{\mathfrak{P}}^t \prod_{j=1}^{\lfloor d/2 \rfloor} \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y})x^2\|_{\mathfrak{P}}^r}{\|F_{j-1}(\mathbf{y})\|_{\mathfrak{P}}^r} \, d\mu(x,\mathbf{y}),$$

where $V(\mathfrak{O}) \subset \mathfrak{O}^{d+1}$ is a union of cosets modulo $\mathfrak{P}$, $F_j(\mathbf{Y}) \subset \mathcal{O}[\mathbf{Y}]$ are polynomial sets defined in terms of the structure constants of the underlying $\mathcal{O}$-Lie lattice Λ, $\|\cdot\|_{\mathfrak{P}}$ is the $\mathfrak{P}$-adic maximum norm and μ is the additive Haar measure on $\mathfrak{O}^{d+1}$ with $\mu(\mathfrak{O}^{d+1}) = 1$. The integral $\mathcal{Z}_{\mathfrak{O}}(r,t)$ allows us to treat 'uniformly' the representation zeta functions of the different groups $\exp(\mathsf{G}^m(\mathfrak{O}))$ arising from the global $\mathcal{O}$-Lie lattice Λ under variation of the place $\mathfrak{p}$ of $\mathcal{O}$, the local ring extension $\mathfrak{O}$ of $\mathcal{O}_{\mathfrak{p}}$ and the congruence level m. In particular, we derive from our analysis a Denef formula and local functional equations.

Theorem 8 (Avni, Klopsch, Onn and Voll [4]). *In the setup described, there exist $r \in \mathbb{N}$ and a rational function $R(X_1, \ldots, X_r, Y) \in \mathbb{Q}(X_1, \ldots, X_r, Y)$ such that for almost every non-archimedean place $\mathfrak{p}$ of k the following holds.*

There are algebraic integers $\lambda_1, \ldots, \lambda_r$ such that for all finite extensions $\mathfrak{O}$ of $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ and all permissible m one has

$$\zeta_{\mathsf{G}^m(\mathfrak{O})}(s) = q_{\mathfrak{p}}^{fdm} R(\lambda_1^f, \ldots, \lambda_r^f, q_{\mathfrak{p}}^{-fs}),$$

where $q_{\mathfrak{p}}$ is the residue field cardinality of $\mathfrak{o}$, f denotes the inertia degree of $\mathfrak{O}$ over $\mathfrak{o}$ and $d = \dim_k(k \otimes_{\mathcal{O}} \Lambda)$. Moreover, there is the functional equation

$$\zeta_{\mathsf{G}^m(\mathfrak{O})}(s)\big|_{\substack{q_{\mathfrak{p}} \to q_{\mathfrak{p}}^{-1} \\ \lambda_i \to \lambda_i^{-1}}} = q_{\mathfrak{p}}^{fd(1-2m)} \zeta_{\mathsf{G}^m(\mathfrak{O})}(s).$$

Furthermore, we obtain candidate pole sets and we show that, locally, abscissae of convergence are monotone under ring extensions.

Theorem 9 (Avni, Klopsch, Onn and Voll [4]). *In the setup described, there exists a finite set $P \subset \mathbb{Q}_{>0}$ such that the following is true.*

For all non-archimedean places $\mathfrak{p}$ of k, all finite extensions $\mathfrak{O}$ of $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ and all permissible m one has

$$\big\{\mathrm{Re}(z) \mid z \in \mathbb{C} \text{ a pole of } \zeta_{\mathsf{G}^m(\mathfrak{O})}(s)\big\} \subseteq P.$$

In particular, one has $\alpha(\mathsf{G}^m(\mathfrak{O})) \leq \max P$, and equality holds for a set of positive Dirichlet density.

Furthermore, if $\mathfrak{p}$ is any non-archimedean place of k and if $\mathcal{O}_\mathfrak{p} = \mathfrak{o} \subseteq \mathfrak{O}_1 \subseteq \mathfrak{O}_2$ is a tower of finite ring extensions, then for every permissible m one has

$$\alpha(\mathsf{G}^m(\mathfrak{O}_1)) \leq \alpha(\mathsf{G}^m(\mathfrak{O}_2)).$$

By a more detailed study of groups of type A_2, we obtain the following theorems addressing, in particular, the conjecture of Larsen and Lubotzky stated in Section 8. Analysing the unique subregular sheet of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ and using approximative Clifford theory, we prove the next result.

Theorem 10 (Avni, Klopsch, Onn and Voll [4])**.** *Let Γ be an arithmetic subgroup of a connected, simply connected simple algebraic group of type A_2 defined over a number field. If Γ has the CSP, then $\alpha(\Gamma) = 1$.*

Employing exact Clifford theory, we obtain the following more detailed result for the special linear group $\mathrm{SL}_3(\mathcal{O})$ over the ring of integers of a number field.

Theorem 11 (Avni, Klopsch, Onn and Voll [6])**.** *Let $\mathcal{O}$ be the ring of integers of a number field k. Then there exists $\varepsilon > 0$ such that the representation zeta function of $\mathrm{SL}_3(\mathcal{O})$ admits a meromorphic continuation to the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 - \varepsilon\}$. The continued function is analytic on the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1\}$, except for a double pole at $s = 1$.*

Consequently, there is a constant $c \in \mathbb{R}_{>0}$ such that

$$R_N(\mathrm{SL}_3(\mathcal{O})) = \sum_{n=1}^{N} r_n(\mathrm{SL}_3(\mathcal{O})) \sim c \cdot N(\log N) \qquad \text{as } N \to \infty.$$

A key step in proving this result consists in deriving explicit formulae for the representation zeta function of groups $\mathrm{SL}_3(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring of characteristic 0 and residue field characteristic different from 3. In fact, we also derive similar results for special unitary groups $\mathrm{SU}_3(\mathcal{O}, \mathcal{o})$.

10 New results regarding the conjecture of Larsen and Lubotzky

Very recently, in joint work with Avni, Onn and Voll we prove the following theorem in connection with the conjecture of Larsen and Lubotzky which is stated in Section 8.

Theorem 12 (Avni, Klopsch, Onn and Voll [5]). *Let Φ be an irreducible root system. Then there exists a constant α_Φ such that for every number field k with ring of integers $\mathcal{O}$, every finite set S of places of k and every connected, simply connected absolutely almost simple algebraic group $\mathbf{G}$ over k with absolute root system Φ the following holds.*

If the arithmetic group $\mathbf{G}(\mathcal{O}_S)$ has polynomial representation growth, then $\alpha(\mathbf{G}(\mathcal{O}_S)) = \alpha_\Phi$.

On the one hand, Theorem 12 is weaker than the conjecture of Larsen and Lubotzky, because it does not resolve Serre's conjecture on the Congruence Subgroup Problem. However, Serre's conjecture is known to be true in many cases and we have the following corollary.

Corollary 1. *Serre's conjecture on the Congruence Subgroup Problem implies Larsen and Lubotzky's conjecture on the degrees of representation growth of lattices in higher rank semisimple locally compact groups.*

On the other hand, Theorem 12 is stronger than the conjecture of Larsen and Lubotzky, because it shows that many arithmetic groups with the CSP have the same degree of representation growth, even when they do not embed as lattices into the same semisimple locally compact group. For instance, fixing Φ of type A_{n-1} for some $n \geq 3$, all of the following groups (for which we also display their embeddings as lattices into semisimple locally compact groups) have the same degree of representation growth:

(1) $\mathrm{SL}_n(\mathbb{Z}) \subseteq \mathrm{SL}_n(\mathbb{R})$,

(2) $\mathrm{SL}_n(\mathbb{Z}[\sqrt{2}]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$,

(3) $\mathrm{SL}_n(\mathbb{Z}[i]) \subseteq \mathrm{SL}_n(\mathbb{C})$,

(4) $\mathrm{SL}_n(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$,

(5) $\mathrm{SU}_n(\mathbb{Z}[\sqrt{2}], \mathbb{Z}) \subseteq \mathrm{SL}_n(\mathbb{R})$.

Presently, the only known explicit values of α_Φ are: 2 for Φ of type A_1 (see [13]), and 1 for Φ of type A_2 (see Theorem 10). It remains a challenging problem to find a conceptual interpretation of α_Φ for general Φ.

For the proof of Theorem 12 and further details we refer to the preprint [5].

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Automorphisms of Group Extensions

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Abstract. After a brief survey of the theory of group extensions and, in particular, of automorphisms of group extensions, we describe some recent reduction theorems for the inducibility problem for pairs of automorphisms.

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1 Background from Extension Theory

A *group extension* $\mathbf{e}$ of N by Q is a short exact sequence of groups and homomorphisms

$$\mathbf{e}: \quad N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q,$$

so that $N \simeq \text{Im } \mu = \text{Ker } \varepsilon$, $G/\text{Ker } \varepsilon \simeq Q$. Usually one writes N additively, G and Q multiplicatively.

A *morphism* of extensions is a triple (α, β, γ) of homomorphisms such that the diagram

$$\begin{array}{cccccc} \mathbf{e}_1: & N_1 & \xrightarrow{\lambda_1} & G_1 & \xrightarrow{\mu_1} & Q_1 \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \mathbf{e}_2: & N_2 & \xrightarrow{\lambda_2} & G_2 & \xrightarrow{\mu_2} & Q_2 \end{array}$$

commutes. If α and γ – and hence β – are isomorphisms, then (α, β, γ) is an *isomorphism* of *extensions*. If α, γ are identity maps, it is called an *equivalence*. Let

$$[\mathbf{e}]$$

denote the equivalence class of $\mathbf{e}$ and write

$$\mathcal{E}(Q, N) = \{[\mathbf{e}] \mid \mathbf{e} \text{ an extension of } N \text{ by } Q\}$$

for the category of equivalence classes and morphisms of extensions of N by Q. The main object of extension theory is to describe the set $\mathcal{E}(Q, N)$.

Automorphisms

An isomorphism (α, β, γ) from $\mathbf{e}$ to $\mathbf{e}$ is called an *automorphism* of $\mathbf{e}$,

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \gamma} \\ N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \end{array}$$

The pair $(\alpha, \gamma) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ is then said to be *induced* by β in $\mathbf{e}$. The automorphisms of $\mathbf{e}$ clearly form a group $\mathrm{Aut}(\mathbf{e})$ and

$$\mathrm{Aut}(\mathbf{e}) \simeq N_{\mathrm{Aut}(G)}(\mathrm{Im}\ \mu)\ \leq \mathrm{Aut}(G).$$

We would like to understand the group $\mathrm{Aut}(\mathbf{e})$ and, in particular, to determine which pairs (α, γ) are *inducible in* $\mathbf{e}$.

Couplings and factor sets

Given an extension $\mathbf{e} : N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$, choose a *transversal function*

$$\tau : Q \to G,$$

i.e., a map such that $\tau\varepsilon =$ the identity map on Q. Conjugation in $\mathrm{Im}\ \mu$ by x^τ, $(x \in Q)$, induces an automorphism x^ξ in N,

$$(a^{x^\xi})^\mu = (x^\tau)^{-1} a^\mu x^\tau,\ (a \in N),$$

so we have a function

$$\xi : Q \to \mathrm{Aut}(N).$$

Note that x^ξ depends on the choice of τ, but $x^\xi(\mathrm{Inn}(N))$ does not. Define $x^\chi = x^\xi(\mathrm{Inn}(N)) \in \mathrm{Out}(N)$. Then

$$\chi : Q \to \mathrm{Out}(N)$$

is a homomorphism which is independent of τ. This is the *coupling* of the extension $\mathbf{e}$. Equivalent extensions have the same coupling, so we can form

$$\mathcal{E}_\chi(Q, N),$$

the subcategory of extensions of N by Q with coupling χ.

The function τ is usually not a homomorphism, but

$$x^\tau y^\tau = (xy)^\tau (\varphi(x, y))^\mu$$

where $\varphi(x, y) \in N$. The associative law $(x^\tau y^\tau) z^\tau = x^\tau (y^\tau z^\tau)$ implies that

$$\varphi(x, yz) + \varphi(y, z) = \varphi(xy, z) + \varphi(x, y) \cdot z^\xi \qquad (*)$$

for $x, y, z \in Q$. Such a function $\varphi : Q \times Q \to N$ is called a *factor set*. We may assume that $1_Q^\tau = 1_G$, in which case $\varphi(1, x) = 0 = \varphi(x, 1)$ for all $x \in Q$, and φ is called a *normalized* factor set.

From $x^\tau y^\tau = (xy)^\tau \varphi(x, y)^\mu$ we deduce that

$$x^\xi y^\xi = (xy)^\xi \overline{\varphi(x, y)}, \ (x, y \in Q) \qquad (**)$$

where $\overline{a}$ denotes conjugation by a in N. Call ξ and φ *associated functions* for the extension $\mathbf{e}$.

Constructing extensions

Suppose we are given groups N, Q and functions $\xi : Q \to \mathrm{Aut}(N)$ and $\varphi : Q \times Q \to N$ (normalized), satisfying $(*)$ and $(**)$. Then we can construct an extension

$$\mathbf{e}(\xi, \varphi) \ : \ N \overset{\mu}{\rightarrowtail} G(\xi, \varphi) \overset{\varepsilon}{\twoheadrightarrow} Q,$$

where $G(\xi, \varphi) = Q \times N$, with group operation

$$(x, a)(y, b) = (xy, \ \varphi(x, y) + ay^\xi + b), \quad (x, y \in Q, \ a, b \in N).$$

Also $a^\mu = (1, a)$ and $(x, a)^\varepsilon = x$. Then the transversal function $x \mapsto (x, 0)$ yields associated functions ξ, φ for $\mathbf{e}(\xi, \varphi)$.

If N is abelian, it is a Q-module via the coupling $\xi = \chi : Q \to \mathrm{Out}(N) = \mathrm{Aut}(N)$ and $\varphi \in Z^2(Q, N)$ is a 2-*cocycle*, while there is a bijection

$$\mathcal{E}_\chi(Q, N) \longleftrightarrow H^2(Q, N).$$

2 The Automorphism Group of an Extension

Consider an extension

$$\mathbf{e} : N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$$

with coupling χ. Assume $\mu : N \hookrightarrow G$ is inclusion and $\varepsilon : G \to Q = G/N$ is the canonical map. If $\alpha \in \mathrm{Aut}(\mathbf{e})$, then α induces automorphisms $\alpha|_N$ in N, $\alpha|_Q$ in Q, while $\alpha \mapsto (\alpha|_N, \ \alpha|_Q)$ is a homomorphism,

$$\Psi : \mathrm{Aut}(\mathbf{e}) \to \mathrm{Aut}(N) \times \mathrm{Aut}(Q).$$

If $\alpha \in \mathrm{Ker}\ \Psi$, then α is trivial on N and G/N, so $[G, \alpha] \le A = Z(N)$, while the map $gN \mapsto g^{-1}g^\alpha$, $(g \in G)$, is a *derivation* or 1-cocycle from Q to $Z(N) = A$. In fact $\mathrm{Ker}\ \Psi \simeq Z^1(Q, A)$ and there is an exact sequence

$$0 \to Z^1(Q, A) \to \mathrm{Aut}(\mathbf{e}) \overset{\Psi}{\to} \mathrm{Aut}(N) \times \mathrm{Aut}(Q).$$

It is more difficult to identify $\mathrm{Im}\ \Psi$. This is where the *Wells sequence* comes into play.

Theorem 1. (C. Wells [12]) *Let $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$ be an extension with coupling $\chi : Q \to \mathrm{Out}(N)$ and let $A = Z(N)$. Then there is an exact sequence*

$$0 \to Z^1(Q,A) \to \mathrm{Aut}(\mathbf{e}) \xrightarrow{\Psi} \mathrm{Comp}(\chi) \xrightarrow{\Lambda} H^2(Q,A)$$

where $\mathrm{Comp}(\chi)$ *is the subgroup of* χ*-compatible pairs* $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$*, i.e., pairs satisfying* $\varphi\chi = \chi\overline{\vartheta}$*, with* $\overline{\vartheta}$ *conjugation by* ϑ *in* $\mathrm{Out}(N)$*.*

To see where the compatibility condition comes from, let $\alpha \in \mathrm{Aut}(\mathbf{e})$ induce (ϑ, φ), so that $(\alpha)\Psi = (\vartheta, \varphi)$. From

$$(a^{x^\tau})^\alpha = (a^\alpha)^{(x^\tau)^\alpha}, \quad (a \in N, x \in Q),$$

we get $x^\xi\vartheta \equiv \vartheta(x^\varphi)^\xi \bmod \mathrm{Inn}(N)$. Thus $\vartheta^{-1}x^\chi\vartheta = (x^\varphi)^\chi$ in $\mathrm{Out}(N)$, i.e. $\chi\overline{\vartheta} = \varphi\chi$.

The Wells map Λ

Let $(\vartheta, \varphi) \in \mathrm{Comp}(\Lambda)$. In order to understand where $(\vartheta, \varphi)\Lambda \in H^2(Q,A)$ comes from, we take note of two actions on the set $\mathcal{E}_\chi(Q,N)$.

(i) $H^2(Q,A)$ acts regularly on $\mathcal{E}_\chi(Q,N)$ by adding a fixed 2-cocycle to each factor set.

(ii) $\mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ acts in the natural way on $\mathcal{E}_\chi(Q,N)$.
Hence, given $(\vartheta, \varphi) \in \mathrm{Comp}(\chi)$ and $[\mathbf{e}] \in \mathcal{E}_\chi(Q,N)$, by regularity there is a unique $h \in H^2(Q,A)$ such that $[\mathbf{e}] = ([\mathbf{e}] \cdot (\vartheta, \varphi)) \cdot h$. Define

$$(\vartheta, \varphi)\Lambda = h,$$

so that

$$[\mathbf{e}] = ([\mathbf{e}] \cdot (\vartheta, \varphi)) \cdot (\vartheta, \varphi)\Lambda.$$

Properties of the Wells map

(i) Im $\Psi = $ Ker Λ. (This is a routine calculation.)

For a long time it was believed that Λ, which is clearly not a homomorphism, was merely a set map. Then in 2010 Jin and Liu [4] discovered two very interesting facts about Λ.

(ii) $\Lambda : \mathrm{Comp}(\chi) \to H^2(Q, A)$ is a derivation, so that $\Lambda \in Z^1(\mathrm{Comp}(\chi),\ H^2(Q, A))$ and

$$(UV)\Lambda = (U)\Lambda \cdot V + (V)\Lambda, \quad (U, V \in \mathrm{Comp}(\chi)).$$

(iii) The cohomology class

$$[\Lambda] \in H^1(\mathrm{Comp}(\chi),\ H^2(Q, A))$$

depends on $[\mathbf{e}]$ *only through its coupling* χ, i.e., extensions with the same coupling have cohomologous Wells maps Λ.

Applications of the Wells Sequence

For a given extension $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$ with coupling χ, the *inducibility problem* is to determine when a given pair $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ is induced by some automorphism of $\mathbf{e}$. This happens if and only if $(\vartheta, \varphi) \in \mathrm{Comp}(\chi)$ and $(\vartheta, \varphi)\Lambda = 0$.

We will describe theorems which reduce the inducibility problem to certain subgroups of Q.

Reduction to Sylow subgroups

Consider an extension $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q = G/N$ with coupling χ where Q is finite. Let $\pi(Q) = \{p_1, \ldots, p_k\}$ and choose $P_i \in \mathrm{Syl}_{p_i}(Q)$, say $P_i = R_i/N$. Then we have subextensions

$$\mathbf{e}_i : N \rightarrowtail R_i \twoheadrightarrow P_i$$

with couplings $\chi_i = \chi|_{P_i}$. Let $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$. Then $P_i^{\varphi} \in \mathrm{Syl}_{p_i}(Q)$, so $P_i^{\varphi} = P_i^{g_i^{-1}}$ for some $g_i \in G$. Then $P_i^{\varphi \overline{g_i}} = P_i$, so $\varphi\overline{g}_i|_{P_i} \in \mathrm{Aut}(P_i)$.

Theorem 2. *With the above notation, the pair* (ϑ, φ) *is inducible in* $\mathbf{e}$ *if and only if* $(\vartheta\overline{g}_i,\ \varphi\overline{g}_i|_{P_i})$ *is inducible in* $\mathbf{e}_i$ *for* $i = 1, 2, \ldots, k$.

Proof. Necessity is routine. Assume the condition holds, i.e. $(\vartheta\overline{g}_i,\ \varphi\overline{g}_i|_{P_i})$ is inducible for $i = 1, 2, \ldots, k$. Let $A = Z(N)$.

(i) (ϑ, φ) *is* χ*-compatible.* This is a straightforward calculation.

(ii) (ϑ, φ) *is inducible in* $\mathbf{e}$. To see this, form a subsequence of the Wells sequence for $\mathbf{e}$ by restricting to automorphisms that leave R_i invariant.

$$0 \to Z^1(Q, A) \to N_{\mathrm{Aut}(\mathbf{e})}(R_i) \to C_i \to H^2(Q, A)$$

where $C_i = \{(\lambda, \mu) \in \mathrm{Comp}(\chi) \mid P_i^{\mu} = P_i\}$. Now apply the restriction map for P_i to get the commutative diagram

$$\begin{array}{ccc} C_i & \xrightarrow{\Lambda} & H^2(Q, A) \\ \Big\downarrow \mathrm{res}_{P_i} & & \Big\downarrow \mathrm{res}_{P_i} \\ \mathrm{Comp}(\chi_i) & \xrightarrow{\Lambda_i} & H^2(P_i, A) \end{array}$$

Since (ϑ, φ) and $(\overline{g}_i, \overline{g}_i)$ are χ-compatible, $(\vartheta\overline{g}_i, \varphi\overline{g}_i) \in \mathrm{Comp}(\chi)$. Also

$$(\vartheta\overline{g}_i, \varphi\overline{g}_i)\mathrm{res}_{P_i} \circ \Lambda_i = (\vartheta\overline{g}_i, \varphi\overline{g}_i|_{P_i})\Lambda_i = 0,$$

and $\Lambda \circ \mathrm{res}_{P_i}$ maps $(\vartheta\overline{g}_i, \varphi\overline{g}_i)$ to 0. Since Λ is a derivation,

$$(\vartheta\overline{g}_i, \varphi\overline{g}_i)\Lambda = ((\vartheta, \varphi)(\overline{g}_i, \overline{g}_i))\Lambda = (\vartheta, \varphi)\Lambda \cdot (\overline{g}_i, \overline{g}_i) + (\overline{g}_i, \overline{g}_i)\Lambda = (\vartheta, \varphi)\Lambda.$$

This is because $(\overline{g}_i, \overline{g}_i)$ is obviously inducible and it acts trivially on $H^2(Q, A)$. Thus $((\vartheta, \varphi)\Lambda)\mathrm{res}_{P_i} = 0$ for $i = 1, \ldots, k$.

Apply the corestriction map for P_i, noting that $(\mathrm{res}_{P_i}) \circ (\mathrm{cor}_{P_i})$ is multiplication by $|Q : P_i|$. Also $|Q| \cdot |H^2(Q, A)| = 0$ and $(\vartheta, \varphi)\Lambda$ has order a p_i'-number for all i. Hence $(\vartheta, \varphi)\Lambda = 0$, and (ϑ, φ) is inducible in $\mathbf{e}$. $\boxed{QED}$

Special cases of Theorem 1 have appeared in [3] and [8].

Reduction to finite subgroups

Next consider an extension $\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$ with coupling χ where Q is a *locally finite* group. Choose a *local system* of finite subgroups in Q

$$\{Q_i\}_{i \in I},$$

i.e., every finite subset of Q is contained in some Q_i. Let I be ordered by inclusion, i.e., $i \leq j$ if and only if $Q_i \leq Q_j$. Then $\{Q_i\}$ is a direct system and $Q = \varinjlim \{Q_i\}$. By restricting to Q_i, we form the corresponding subextension

$$\mathbf{e}_i : N \rightarrowtail G_i \twoheadrightarrow Q_i = G_i/N, \quad (i \in I),$$

with coupling $\chi_i = \chi|_{Q_i}$.

Suppose that $(\vartheta, \varphi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(Q)$ is given such that $Q_i^{\varphi} = Q_i$ for all i. (If φ has finite order, such a system $\{Q_i\}$ will always exist). Assume that $(\vartheta, \varphi|_{Q_i})$ is inducible in $\mathbf{e}_i$ for all $i \in I$.

Question: does this imply that (ϑ, φ) is inducible in $\mathbf{e}$?

By restriction form the commutative diagram

$$\begin{array}{ccc}
\mathrm{Comp}(\chi) & \xrightarrow{\Lambda} & H^2(Q,A) \\
\Big\downarrow \mathrm{res}_{Q_i} & & \Big\downarrow \mathrm{res}_{Q_i} \\
\mathrm{Comp}(\chi_i) & \xrightarrow{\Lambda_i} & H^2(Q_i,A)
\end{array}$$

where $A = Z(N)$. Since $(\vartheta, \varphi|_{Q_i})\Lambda_i = 0$, we have $(\vartheta,\varphi)\Lambda \in \mathrm{Ker}(\mathrm{res}_{Q_i})$ for all $i \in I$, and $(\vartheta,\varphi)\Lambda$ belongs to

$$K = \mathrm{Ker}(H^2(Q,A) \to \varprojlim H^2(Q_i,A)) :$$

note here that $\left\{H^2(Q_i,A)\right\}$ is *an inverse system* of abelian groups with restriction maps.

A spectral sequence for $H^n(\varinjlim, -)$

In general cohomology does not interact well with direct limits. However, there is a spectral sequence converging to $H^n(\varinjlim \{Q_i\}, A) = H^n(Q,A)$, namely

$$E_2^{pq} \overset{p+q=n}{\Longrightarrow} H^n(Q,A)$$

where

$$E_2^{pq} = \varprojlim{}^{(p)} \{H^q(Q_i,A)\}$$

and $\varprojlim{}^{(p)}$ is the pth derived functor of $\varprojlim$. (This may be deduced from the Grothendieck spectral sequence – see [6], [9]). Hence when $n = 2$ we obtain a series

$$0 = L_0 \le L_1 \le L_2 \le L_3 = H^2(Q,A)$$

where $L_1 \simeq E_\infty^{20}$, $L_2/L_1 \simeq E_\infty^{11}$ and $L_3/L_2 \simeq E_\infty^{02}$. Thus $L_2 = K$ and in our situation $(\vartheta,\varphi)\Lambda \in L_2$. To prove that $(\vartheta,\varphi)\Lambda = 0$ it suffices to show that

$$E_2^{11} = 0 = E_2^{20}.$$

For this to be true additional conditions must be imposed: for example,

$$\sum_p r_p(A) \ < \ \infty,$$

the sum being for $p = 0$ or a prime, i.e., A has *finite total rank*. In fact this condition implies that

$$\varprojlim{}^{(1)} \left\{H^1(N,A)\right\} = 0 = \varprojlim{}^{(2)} \left\{A^N\right\},$$

(see [2]). Hence $(\vartheta,\varphi)\Lambda = 0$ and (ϑ,φ) is inducible in **e**.

Theorem 3. *With the above notation, assume that $Z(N)$ has finite total rank. Then (ϑ, φ) is inducible in $\mathbf{e}$ if and only if $(\vartheta, \varphi|_{Q_i})$ is inducible in $\mathbf{e}_i$ for all $i \in I$.*

By combining Theorems 1 and 2 we reduce the inducibility problem for Q locally finite to the case of a finite p-group.

Counterexamples

Theorem 3 does not hold without some conditions on $A = Z(N)$. Consider a non-split extension

$$\mathbf{e} : N \rightarrowtail G \twoheadrightarrow Q$$

where G is locally finite, $\pi(N) \cap \pi(Q) = \emptyset$, $2 \notin \pi(N)$ and N is abelian. In fact there are many such extensions – see for example [5], [11]. Let $Q_i \leq Q$ be finite. Then $H^n(Q_i, N) = 0$ for all $n \geq 1$ by Schur's theorem, so that
$\mathbf{e}_i : N \rightarrowtail G_i \twoheadrightarrow Q_i = G_i/N$ splits. Let $\vartheta \in \mathrm{Aut}(N)$ be the inversion automorphism. Then $(\vartheta, 1)$ is inducible in $\mathbf{e}_i$ for every i since $\mathbf{e}_i$ is a split extension. However, $(\vartheta, 1)$ is *not* inducible in $\mathbf{e}$: for if it were, the cohomology class Δ of e would satisfy $\Delta = \Delta\vartheta_* = -\Delta$ and hence $\Delta = 0$ since $H^2(Q, N)$ has no elements of order 2. This is a contradiction.

Remark. Full details of the proofs may be found in [10].

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$*$-group identities on units of group rings

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Abstract. Analogous to $*$-polynomial identities in rings, we introduce the concept of $*$-group identities in groups. When F is an infinite field of characteristic different from 2, we classify the torsion groups with involution G so that the unit group of FG satisfies a $*$-group identity. The history and motivations will be given for such an investigation.

Keywords: Group Identities, Involutions, Group algebras.

MSC 2000 classification: 16U60, 16W10, 16S34.

1 Introduction and motivations

The motivation for the study of this topic is from two sides:

(a) Hartley's conjecture on group identities of units of group rings,

(b) Amitsur's Theorem on $*$-polynomial identities in rings.

Let F be a field and G a group. Write $\mathcal{U}(FG)$ for the unit group of the group algebra FG. We say that a subset S of $\mathcal{U}(FG)$ satisfies a *group identity* if there exists a non-trivial word $w(x_1, \ldots, x_n)$ in the free group on a countable set of generators $\langle x_1, x_2, \ldots \rangle$ such that $w(u_1, \ldots, u_n) = 1$ for all $u_1, \ldots, u_n \in S$.

Brian Hartley in the 80s conjectured that when F is infinite and G is torsion, *if $\mathcal{U}(FG)$ satisfies a group identity then FG satisfies a polynomial identity.* We recall that a subset H of an F-algebra A satisfies a *polynomial identity* if there exists a non-zero polynomial $f(x_1, \ldots, x_n)$ in the free associative algebra on non-commuting variables $x_1, x_2, \ldots$ over F, $F\{x_1, x_2, \ldots\}$, such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in H$ (in this case we shall write also that H is PI).

Hartley's conjecture was solved affirmatively by Giambruno, Jespers and Valenti [3] in the semiprime case (hence, in particular, for fields of characteristic zero) and by Giambruno, Sehgal and Valenti [7] in the general case. Its solution was at the basis of the work of Passman [18] who characterized group algebras whose units satisfy a group identity. Recall that, for any prime p, a group G is said to be p-abelian if its commutator subgroup G' is a finite p-group, and 0-abelian means abelian.

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Theorem 1. *Let F be an infinite field of characteristic $p > 0$ and G a torsion group. The following statements are equivalent:*

(i) $\mathcal{U}(FG)$ satisfies a group identity;

(ii) $\mathcal{U}(FG)$ satisfies the group identity $(x, y)^{p^r} = 1$, for some $r \geq 0$;

(iii) G has a normal p-abelian subgroup of finite index and G' is a p-group of bounded exponent.

In the characteristic zero case, when G is torsion, $\mathcal{U}(FG)$ satisfies a group identity if, and only if, G is abelian. In particular, the fact that G contains a normal p-abelian subgroup of finite index (condition (iii) of the theorem) is equivalent to saying that FG must satisfy a polynomial identity, as was established earlier by Isaacs and Passman (see Corollaries 5.3.8 and 5.3.20 of [17]). More recently, the above results have been extended to the more general context of finite fields in [15] and [16] and arbitrary groups in [9].

Along this line, a natural question of interest is to ask whether group identities satisfied by some special subset of the unit group of FG can be lifted to $\mathcal{U}(FG)$ or force FG to satisfy a polynomial identity. In this framework, the *symmetric units* have been the subject of a good deal of attention.

Assume that F has characteristic different from 2. The linear extension to FG of the map $*$ on G such that $g^* = g^{-1}$ for all $g \in G$ is an *involution* of FG, namely an antiautomorphism of order 2 of FG, called the *classical* involution. An element $\alpha \in FG$ is said to be *symmetric with respect to* $*$ if $\alpha^* = \alpha$. We write FG^+ for the set of symmetric elements, which are easily seen to be the linear combinations of the terms $g + g^{-1}$, $g \in G$. Let $\mathcal{U}^+(FG)$ denote the set of symmetric units. Giambruno, Sehgal and Valenti [8] confirmed a stronger version of Hartley's Conjecture by proving

Theorem 2. *Let FG be the group algebra of a torsion group G over an infinite field F of characteristic different from 2 endowed with the classical involution. If $\mathcal{U}^+(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

Under the same restrictions as in the above theorem, they also obtained necessary and sufficient conditions for $\mathcal{U}^+(FG)$ to satisfy a group identity. Obviously, group identities on $\mathcal{U}^+(FG)$ do not force group identities on $\mathcal{U}(FG)$. To see this it is sufficient to observe that if Q_8 is the quaternion group of order 8, for any infinite field F of characteristic $p > 2$ FQ_8^+ is commutative, hence $\mathcal{U}^+(FQ_8)$ satisfies a group identity but, according to Theorem 1, $\mathcal{U}(FQ_8)$ does not satisfy a group identity. For a complete overview of these and related results we refer to the monograph [13].

Recently, there has been a considerable amount of work on involutions of FG obtained as F-linear extensions of arbitrary group involutions on G (namely antiautomorphisms of order 2 of G) other than the classical one. The final outcome has been the complete classification of the torsion groups G such that the units of FG which are symmetric under the given involution satisfy a group identity (see [5]).

Here we discuss a more general problem, that of $*$-*group identities* on $\mathcal{U}(FG)$. We can define an involution on the free group $\langle x_1, x_2, \ldots\rangle$ via $x_{2i-1}^* := x_{2i}$ for all $i \geq 1$. Renumbering, we obtain the free group with involution $\mathcal{F} := \langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$. We say the unit group $\mathcal{U}(FG)$ satisfies a $*$-group identity if there exists a non-trivial word $w(x_1, x_1^*, \ldots, x_n, x_n^*) \in \mathcal{F}$ such that

$$w(u_1, u_1^*, \ldots, u_n, u_n^*) = 1$$

for all $u_1, \ldots, u_n \in \mathcal{U}(FG)$. Obviously, if $\mathcal{U}^+(FG)$ satisfies the group identity $v(x_1, \ldots, x_r)$, then $\mathcal{U}(FG)$ satisfies the $*$-group identity $v(x_1x_1^*, \ldots, x_rx_r^*)$. It seems of interest to understand the behaviour of the symmetric units when the group of units satisfies a $*$-group identity. The main motivation for this investigation dates back to the classical result of Amitsur on $*$-polynomial identities satisfied by an algebra with involution. Let A be an F-algebra having an involution $*$. We can define an involution on the free algebra $F\{x_1, x_2, \ldots\}$ via $x_{2i-1}^* := x_{2i}$ for all $i \geq 1$. As in the free group case, renumbering we obtain the free algebra with involution $F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$. We say that A satisfies a $*$-polynomial identity (or A is $*$-PI) if there exists a non-zero element $f(x_1, x_1^*, \ldots, x_n, x_n^*) \in F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$ such that $f(a_1, a_1^*, \ldots, a_n, a_n^*) = 0$ for all $a_1, \ldots, a_n \in A$. It is obvious that if the symmetric elements of A satisfy the polynomial identity $g(x_1, \ldots, x_r)$ then A satisfies the $*$-polynomial identity $g(x_1+x_1^*, \ldots, x_r+x_r^*)$. It is more difficult to see that if A satisfies a $*$-polynomial identity, then A^+ satisfies a polynomial identity. The deep result of Amitsur [2] shows that this is the case, by proving that if A satisfies a $*$-polynomial identity, then A satisfies a polynomial identity.

The surprising result we obtain is just a group-theoretical analogue of Amitsur's theorem for the unit groups of torsion group rings endowed with the linear extension of an arbitrary group involution. The original results were established in [6]. Recently a long and detailed survey on the subject by Lee [14] has appeared.

2 $*$-group identities on units of torsion group algebras

Let $\langle X \rangle$ be the free group of countable rank on a set $X := \{x_1, x_2, \ldots\}$. We can regard it as a group with involution by setting, for every $i \geq 1$, $x_{2i-1}^* = x_{2i}$ and extending $*$ to an involution of $\langle X \rangle$ in the obvious way. Write $X_1 := \{x_{2i-1} \,|\, i \geq 1\}$ and $X_2 := \{x_{2i} \,|\, i \geq 1\}$. The group above, we call $\mathcal{F}$, has the following universal property: if H is a group with involution, any map $X_1 \longrightarrow H$ can be uniquely extended to a group homomorphism $f : \mathcal{F} \longrightarrow H$ commuting with the involution.

Let $1 \neq w(x_1, x_1^*, \ldots, x_n, x_n^*) \in \mathcal{F}$ and let H be a group with involution $*$. The word w is said to be a *$*$-group identity* (or $*$-GI) of H if w is equal to 1 for any evaluation $\varphi(x_i) = u_i \in H$, $\varphi(x_i^*) = u_i^* \in H$ with $1 \leq i \leq n$. Clearly a group identity is a $*$-GI. Moreover, since for any $x \in X$ xx^* is symmetric, a group identity on symmetric elements of H yields a $*$-group identity of H. We focus our attention on the *converse* problem, namely the possibility of a $*$-group identity of H to force a group identity on the symmetric elements of H itself when H is the unit group of a group algebra.

One of the key ingredients is the following result dealing with finite-dimensional semisimple algebras with involution over an infinite field.

Lemma 1. *Let A be a finite-dimensional semisimple algebra with involution over an infinite field of characteristic different from 2. If its unit group $\mathcal{U}(A)$ satisfies a $*$-GI, then A is a direct sum of finitely many simple algebras of dimension at most 4 over their centre. Moreover A^+ is central in A.*

Proof. See Lemma 5 of [6]. $\boxed{QED}$

The conclusions of the above lemma are not a novelty in the setting of algebras with involution. For instance the same happens when one considers finite-dimensional semisimple algebras with involution whose symmetric elements are Lie nilpotent (see [4]).

In the framework of group algebras, this gives crucial information on the structure of the basis group. In fact, assume that F is an infinite field of characteristic $p > 2$ and G a finite group with an involution $*$ and let FG have the induced involution. Write $P := \{x \,|\, x \in G,\ x \text{ is a } p\text{-element}\}$. Suppose that $\mathcal{U}(FG)$ satisfies a $*$-group identity w. The Jacobson radical J of the group algebra FG is nilpotent and $*$-invariant. This is sufficient to conclude that $\mathcal{U}(FG/J)$ also satisfies w. But FG/J is finite-dimensional and semisimple. By applying Lemma 1, the simple components of its Wedderburn decomposition are all of dimension at most 4 over their centres. But Lemma 2.6 of [4] or Lemma 3 of [12] show that this forces P to be a (normal and $*$-invariant) subgroup of G.

We can summarize all these deductions in the following

Lemma 2. *Let F be an infinite field of characteristic $p > 2$ and G a finite group with involution and let FG have the induced involution. If $\mathcal{U}(FG)$ satisfies a $*$-group identity, then the p-elements of G form a subgroup.*

It is trivial to see that the conclusion holds for locally finite groups G as well.

Now, let F and G be as in the lemma. We know that if $\mathcal{U}(FG)$ satisfies a $*$-GI, then P is a subgroup, $F(G/P)$ has an induced involution and $\mathcal{U}(F(G/P))$ still satisfies a $*$-GI. By Lemma 1 $F(G/P)^+$ is central in $F(G/P)$. In particular, $F(G/P)^+$ must be commutative. Therefore it is of interest to classify group algebras with linear extensions of arbitrary group involutions whose symmetric elements commute. In order to state this, a definition is required.

We recall that a group G is said to be an LC-group (that is, it has the "lack of commutativity" property) if it is not abelian, but if $g, h \in G$, and $gh = hg$, then at least one of g, h and gh must be central. These groups were introduced by Goodaire. By Proposition III.3.6 of [10], a group G is an LC-group with a unique non-identity commutator (which must, obviously, have order 2) if and only if $G/\zeta(G) \cong C_2 \times C_2$. Here, $\zeta(G)$ denotes the centre of G.

Definition 1. A group G endowed with an involution $*$ is said to be a special LC-group, or SLC-group, if it is an LC-group, it has a unique non-identity commutator z, and for all $g \in G$, we have $g^* = g$ if $g \in \zeta(G)$, and otherwise, $g^* = zg$.

The SLC-groups arise naturally in the following result proved by Jespers and Ruiz Marin [11] for an arbitrary involution on G.

Theorem 3. *Let R be a commutative ring of characteristic different from 2, G a non-abelian group with an involution $*$ which is extended linearly to RG. The following statements are equivalent:*

(i) RG^+ is commutative;

(ii) RG^+ is the centre of RG;

(iii) G is an SLC-group.

We recall that in [1] Amitsur proved that *if R is a ring with involution and R^+ is PI, then R is PI.* Later the same arguments were used by him to prove that *if R is $*$-PI, then R is PI.* In particular, if R is $*$-PI then R^+ is PI. The developments for us were similar. In fact, by using exactly the same arguments as in [5] (Section 3 for the semiprime case and Sections 4 and 5 for the general case) we provide the following result which is the core of [6].

Theorem 4. *Let F be an infinite field of characteristic $p \neq 2$, G a torsion group with an involution $*$ which is extended linearly to FG. The following statements are equivalent:*

(i) the symmetric units of FG satisfy a group identity;

(ii) the units of FG satisfy a $*$-group identity;

(iii) one of the following conditions holds:

(a) FG is semiprime and G is abelian or an SLC-group;

(b) FG is not semiprime, the p-elements of G form a (normal) subgroup P, G has a p-abelian normal subgroup of finite index, and either

(1) G' is a p-group of bounded exponent, or

(2) G/P is an SLC-group and G contains a normal $*$-invariant p-subgroup B of bounded exponent, such that P/B is central in G/B and the induced involution acts as the identity on P/B.

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Commutator width in Chevalley groups

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Abstract. The present paper is the [slightly expanded] text of our talk at the Conference "Advances in Group Theory and Applications" at Porto Cesareo in June 2011. Our main results assert that [elementary] Chevalley groups very rarely have finite commutator width. The reason is that they have very few commutators, in fact, commutators have finite width in elementary generators. We discuss also the background, bounded elementary generation, methods of proof, relative analogues of these results, some positive results, and possible generalisations.

Keywords: Chevalley groups, elementary subgroups, elementary generators, commutator width, relative groups, bounded generation, standard commutator formulas, unitriangular factorisations

MSC 2000 classification: 20G35, 20F12, 20F05

1 Introduction

In the present note we concentrate on the recent results on the commutator width of Chevalley groups, the width of commutators in elementary generators,

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and the corresponding relative results. In fact, localisation methods used in the proof of these results have many further applications, both actual and potential: relative commutator formulas, multiple commutator formulas, nilpotency of K_1, description of subnormal subgroups, description of various classes of overgroups, connection with excision kernels, etc. We refer to our surveys [36, 31, 32] and to our papers [29, 35, 7, 40, 37, 38, 41, 76, 39, 33, 34] for these and further applications and many further related references.

2 Preliminaries

2.1 Length and width

Let G be a group and X be a set of its generators. Usually one considers symmetric sets, for which $X^{-1} = X$.

- The **length** $l_X(g)$ of an element $g \in G$ with respect to X is the minimal k such that g can be expressed as the product $g = x_1 \dots x_k$, $x_i \in X$.
- The **width** $w_X(G)$ of G with respect to X is the supremum of $l_X(g)$ over all $g \in G$. In the case when $w_X(G) = \infty$, one says that G does not have bounded word length with respect to X.

The problem of calculating or estimating $w_X(G)$ has attracted a lot of attention, especially when G is one of the classical-like groups over skew-fields. There are *hundreds* of papers which address this problem in the case when X is either

- the set of elementary transvections
- the set of all transvections or ESD-transvections,
- the set of all unipotents,
- the set of all reflections or pseudo-reflections,
- other sets of small-dimensional transformations,
- a class of matrices determined by their eigenvalues, such as the set of all involutions,
- a non-central conjugacy class,
- the set of all commutators,

etc., etc. Many further exotic generating sets have been considered, such as matrices distinct from the identity matrix in one column, symmetric matrices, etc., etc., etc. We do not make any attempt to list all such papers, there are simply far too many, and vast majority of them produce sharp bounds for classes of rings, which are trivial from our prospective, such as fields, or semi-local rings.

2.2 Chevalley groups

Let us fix basic notation. This notation is explained in [1, 4, 60, 74, 75, 2, 3, 92, 95, 93], where one can also find many further references.

- Φ is a reduced irreducible root system;
- Fix an order on Φ, let Φ^+, Φ^- and $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ are the sets of positive, negative and fundamental roots, respectively.
- Let $Q(\Phi)$ be the root lattice of Φ, $P(\Phi)$ be the weight lattice of Φ and P be any lattice such that $Q(\Phi) \leq P \leq P(\Phi)$;
- R is a commutative ring with 1;
- $G = G_P(\Phi, R)$ is the Chevalley group of type (Φ, P) over R;
- In most cases P does not play essential role and we simply write $G = G(\Phi, R)$ for any Chevalley group of type Φ over R;
- However, when the answer depends on P we usually write $G_{\mathrm{sc}}(\Phi, R)$ for the simply connected group, for which $P = P(\Phi)$ and $G_{\mathrm{ad}}(\Phi, R)$ for the adjoint group, for which $P = Q(\Phi)$;
- $T = T(\Phi, R)$ is a split maximal torus of G;
- $x_\alpha(\xi)$, where $\alpha \in \Phi$, $\xi \in R$, denote root unipotents G elementary with respect to T;
- $E(\Phi, R)$ is the [absolute] elementary subgroup of $G(\Phi, R)$, generated by all root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$;
- $E^L(\Phi, R)$ is the subset (not a subgroup!) of $E(\Phi, R)$, consisting of products of $\leq L$ root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$;
- $H = H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R)$ is the elementary part of the split maximal torus;
- $U^\pm(\Phi, R)$ is the unipotent radical of the standard Borel subgroup $B(\Phi, R)$ or its opposite $B^-(\Phi, R)$. By definition

$$U(\Phi, R) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi^+,\ \xi \in R \big\rangle.$$
$$U^-(\Phi, R) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi^-,\ \xi \in R \big\rangle.$$

2.3 Chevalley groups versus elementary subgroups

Many authors not familiar with algebraic groups or algebraic K-theory do not distinguish Chevalley groups and their elementary subgroups. Actually, these groups are defined dually.

- Chevalley groups $G(\Phi, R)$ are [the groups of R-points of] algebraic groups. In other words, $G(\Phi, R)$ is defined as

$$G(\Phi, R) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], R),$$

where $\mathbb{Z}[G]$ is the affine algebra of G. By definition $G(\Phi, R)$ consists of solutions in R of certain algebraic equations.

- As opposed to that, *elementary* Chevalley groups $E(\Phi, R)$ are generated by elementary generators

$$E(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R \rangle.$$

When $R = K$ is a field, one knows relations among these elementary generators, so that $E(\Phi, R)$ can be defined by generators and relations. However, in general, the elementary generators are described by their action in certain representations.

By the very construction of these groups $E(\Phi, R) \le G(\Phi, R)$ but, as we shall see, in general $E(\Phi, R)$ can be strictly smaller than $G(\Phi, R)$ even for fields. The following two facts might explain, why some authors confuse $E(\Phi, R)$ and $G(\Phi, R)$:

- Let $R = K$ be *any* field. Then $G_{\mathrm{sc}}(\Phi, K) = E_{\mathrm{sc}}(\Phi, K)$.
- Let $R = K$ be an *algebraically closed* field. Then $G_{\mathrm{ad}}(\Phi, K) = E_{\mathrm{ad}}(\Phi, K)$.

However, for a field K that is not algebraically closed one usually has strict inclusion $E_{\mathrm{ad}}(\Phi, K) < G_{\mathrm{ad}}(\Phi, K)$. Also, as we shall see, even for principal ideal domains $E_{\mathrm{sc}}(\Phi, R) < G_{\mathrm{sc}}(\Phi, R)$, in general.

2.4 Elementary generators

By the very construction Chevalley groups occur as subgroups of the general linear group $\mathrm{GL}(n, R)$. Let e be the identity matrix and e_{ij}, $1 \le i, j \le n$, be a matrix unit, which has 1 in position (i, j) and zeros elsewhere. Below we list what the elementary root unipotents, also known as elementary generators, look like for classical groups.

- In the case $\Phi = \mathbb{A}_l$ one has $n = l + 1$. Root unipotents of $\mathrm{SL}(n, R)$ are [elementary] transvections

$$t_{ij}(\xi) = e + \xi e_{ij}, \qquad 1 \le i \ne j \le n, \quad \xi \in R.$$

- In the case $\Phi = \mathrm{D}_l$ one has $n = 2l$. We number rows and columns of matrices from $\mathrm{GL}(n, R)$ as follows: $1, \dots, l, -l, \dots, -1$. Then root unipotents of $\mathrm{SO}(2l, R)$ are [elementary] orthogonal transvections

$$T_{ij}(\xi) = e + \xi e_{ij} - \xi e_{-j,-i}, \qquad 1 \leq i, j \leq -1,\ i \neq \pm j, \quad \xi \in R.$$

- In the case $\Phi = \mathbf{C}_l$ also $n = 2l$ and we use the same numbering of rows and columns as in the even orthogonal case. Moreover, we denote ε_i the sign of i, which is equal to $+1$ for $i = 1, \dots, l$ and to -1 for $i = -1, \dots, -1$. In $\mathbf{C}_l$ there are two root lengths. Accordingly, root unipotents of $\mathrm{Sp}(2l, R)$ come in two stocks. Long root unipotents are the usual linear transvections $t_{i,-i}(\xi)$, $1 \leq i \leq -1$, $\xi \in R$, while short root unipotents are [elementary] symplectic transvections

$$T_{ij}(\xi) = e + \xi e_{ij} - \varepsilon_i \varepsilon_j \xi e_{-j,-i}, \qquad 1 \leq i, j \leq -1,\ i \neq \pm j, \quad \xi \in R.$$

- Finally, for $\Phi = \mathrm{B}_l$ one has $n = 2l+1$ and we number rows and columns of matrices from $\mathrm{GL}(n, R)$ as follows: $1, \dots, l, 0, -l, \dots, -1$. Here too there are two root lengths. The long root elements of the odd orthogonal group $\mathrm{SO}(2l+1, R)$ are precisely the root elements of the even orthogonal groups, $T_{ij}(\xi)$, $i \neq \pm j$, $i, j \neq 0$, $\xi \in R$. The short root elements have the form

$$T_{i0}(\xi) = e + \xi e_{i0} - 2\xi e_{-i,0} - \xi^2 e_{i,-1}, \qquad i \neq 0, \quad \xi \in R.$$

It would be only marginally more complicated to specify root elements of spin groups and exceptional groups, in their minimal faithful representations, see [93, 94].

2.5 Classical cases

Actually, most of our results are already new for classical groups. Recall identification of Chevalley groups and elementary Chevalley groups for the classical cases. The second column of the following table lists traditional notation of classical groups, according to types: $\mathbb{A}_l$ the special linear group, B_l the odd orthogonal group, $\mathbf{C}_l$ the symplectic group, and D_l the even orthogonal group. These groups are defined by algebraic equations. Orthogonal groups are not simply connected, the corresponding simply connected groups are the spin groups. The last column lists the names of their elementary subgroups, generated by the elementary generators listed in the preceding subsection.

Φ	$G(\Phi, R)$	$E(\Phi, R)$
$\mathbb{A}_l$	$\mathrm{SL}(l+1, R)$	$E(l+1, R)$
B_l	$\mathrm{Spin}(2l+1, R)$	$\mathrm{Epin}(2l+1, R)$
	$\mathrm{SO}(2l+1, R)$	$\mathrm{EO}(2l+1, R)$
$\mathbf{C}_l$	$\mathrm{Sp}(2l, R)$	$\mathrm{Ep}(2l, R)$
D_l	$\mathrm{Spin}(2l, R)$	$\mathrm{Epin}(2l, R)$
	$\mathrm{SO}(2l, R)$	$\mathrm{EO}(2l, R)$

Orthogonal groups [and spin groups] in this table are the *split* orthogonal groups. *Split* means that they preserve a bilinear/quadratic form of maximal Witt index. In the case of a field the group $\mathrm{EO}(n, K)$ was traditionally denoted by $\Omega(n, K)$ and called the kernel of spinor norm. Since the group $\mathrm{SO}(n, K)$ is not simply connected, in general $\Omega(n, K)$ is a proper subgroup of $\mathrm{SO}(n, K)$.

2.6 Dimension of a ring

Usually, dimension of a ring R is defined as the length d of the longest strictly ascending chain of ideals $I_0 < I_1 < \ldots < I_d$ of a certain class.

- The most widely known one is the Krull dimension $\dim(R)$ defined in terms of chains of prime ideals of R. Dually, it can be defined as the combinatorial dimension of $\mathrm{Spec}(R)$, considered as a topological space with Zariski topology.

Recall, that the combinatorial dimension $\dim(X)$ of a topological space X is the length of the longest *descending* chain of its *irreducible* subspaces $X_0 > X_1 > \ldots > X_d$. Thus, by definition,

$$\dim(R) = \dim(\mathrm{Spec}(R)).$$

However, we mostly use the following more accurate notions of dimension.

- The Jacobson dimension $\text{j-dim}(R)$ of R is defined in terms of j-ideals, in other words, those prime ideals, which are intersections of maximal ideals. Clearly, $\text{j-dim}(R)$ coincides with the combinatorial dimension of the *maximal* spectrum of the ring R, by definition, $\text{j-dim}(R) = \dim(\mathrm{Max}(R))$

Define dimension $\delta(X)$ of a topological space X as the smallest integer d such that X can be expressed as a *finite* union of Noetherian topological spaces of dimension $\leq d$. The trick is that these spaces do not have to be closed subsets of X.

- The Bass—Serre dimension of a ring R is defined as the dimension of its maximal spectrum, $\delta(R) = \delta(\mathrm{Max}(R))$.

Bass—Serre dimension has many nice properties, which make it better adapted to the study of problems we consider. For instance, a ring is semilocal iff $\delta(R) = 0$ (recall that a commutative ring R is called semilocal if it has finitely many maximal ideals).

2.7 Stability conditions

Mostly, stability conditions are defined in terms of stability of rows, or columns. In this note we only refer to Bass' stable rank, first defined in [9]. We will denote the [left] R-module of rows of length n by ${}^nR$, to distinguish it from the [right] R-module R^n of columns of height n.

A row $(a_1, \ldots, a_n) \in {}^nR$ is called *unimodular*, if its components $a_1, \ldots, a_n$ generate R as a right ideal,

$$a_1R + \ldots + a_nR = R.$$

or, what is the same, if there exist such $b_1, \ldots, b_n \in R$ that

$$a_1b_1 + \ldots + a_nb_n = 1.$$

The *stable rank* $\mathrm{sr}(R)$ of the ring R is the smallest such n that every unimodular row $(a_1, \ldots, a_{n+1})$ of length $n+1$ is *stable*. In other words, there exist elements $b_1, \ldots b_n \in R$ such that the row

$$(a_1 + a_{n+1}b_1, a_2 + a_{n+1}b_2, \ldots, a_n + a_{n+1}b_n)$$

of length n is unimodular. If no such n exists, one writes $\mathrm{sr}(R) = \infty$.

In fact, stable rank is a more precise notion of dimension of a ring, based on linear algebra, rather than chains of ideals. It is shifted by 1 with respect to the classical notions of dimension. The basic estimate of stable rank is Bass' theorem, asserting that $\mathrm{sr}(R) \leq \delta(R) + 1$.

Especially important in the sequel is the condition $\mathrm{sr}(R) = 1$. A ring R has stable rank 1 if for any $x, y \in R$ such that $xR + yR = R$ there exists a $z \in R$ such that $(x + yz)R = R$. In fact, rings of stable rank 1 are weakly finite (one-sided inverses are automatically two-sided), so that this last condition is equivalent to invertibility of $x + yz$. Rings of stable rank 1 should be considered as a class of 0-dimensional rings, in particular, all semilocal rings have stable rank 1. See [87] for many further examples and references.

2.8 Localisation

Let, as usual, R be a commutative ring with 1, S be a multiplicative system in R and $S^{-1}R$ be the corresponding localisation. We will mostly use localisation with respect to the two following types of multiplicative systems.

- Principal localisation: the multiplicative system S is generated by a non-nilpotent element $s \in R$, viz. $S = \langle s \rangle = \{1, s, s^2, \dots\}$. In this case we usually write $\langle s \rangle^{-1} R = R_s$.
- Maximal localisation: the multiplicative system S equals $S = R \setminus \mathfrak{m}$, where $\mathfrak{m} \in \mathrm{Max}(R)$ is a maximal ideal in R. In this case we usually write $(R \setminus \mathfrak{m})^{-1} R = R_{\mathfrak{m}}$.

We denote by $F_S : R \longrightarrow S^{-1}R$ the canonical ring homomorphism called the localisation homomorphism. For the two special cases mentioned above, we write $F_s : R \longrightarrow R_s$ and $F_{\mathfrak{m}} : R \longrightarrow R_{\mathfrak{m}}$, respectively.

Both $G(\Phi, \_)$ and $E(\Phi, \_)$ commute with direct limits. In other words, if $R = \varinjlim R_i$, where $\{R_i\}_{i \in I}$ is an inductive system of rings, then $G(\Phi, \varinjlim R_i) = \varinjlim G(\Phi, R_i)$ and the same holds for $E(\Phi, R)$. Our proofs crucially depend on this property, which is mostly used in the two following situations.

- First, let R_i be the inductive system of all finitely generated subrings of R with respect to inclusion. Then $X = \varinjlim X(\Phi, R_i)$, which reduces most of the proofs to the case of Noetherian rings.
- Second, let S be a multiplicative system in R and R_s, $s \in S$, the inductive system with respect to the localisation homomorphisms: $F_t : R_s \longrightarrow R_{st}$. Then $X(\Phi, S^{-1}R) = \varinjlim X(\Phi, R_s)$, which allows to reduce localisation with respect to any multiplicative system to principal localisations.

2.9 K_1-functor

The starting point of the theory we consider is the following result, first obtained by Andrei Suslin [80] for $\mathrm{SL}(n, R)$, by Vyacheslav Kopeiko [48] for symplectic groups, by Suslin and Kopeiko [81] for even orthogonal groups and by Giovanni Taddei [83] in general.

Theorem 1. *Let Φ be a reduced irreducible root system such that $\mathrm{rk}(\Phi) \geq 2$. Then for any commutative ring R one has $E(\Phi, R) \trianglelefteq G(\Phi, R)$.*

In particular, the quotient

$$K_1(\Phi, R) = G_{\mathrm{sc}}(\Phi, R) / E_{\mathrm{sc}}(\Phi, R)$$

is not just a pointed set, it is a group. It is called K_1-functor.

The groups $G(\Phi, R)$ and $E(\Phi, R)$ behave functorially with respect to both R and Φ. In particular, to an embedding of root systems $\Delta \subseteq \Phi$ there corresponds the map $\varphi : G(\Delta, R) \longrightarrow G(\Phi, R)$ of the corresponding [simply connected] groups, such that $\varphi(E(\Delta, R)) \leq E(\Phi, R)$. By homomorphism theorem it defines the stability map $\varphi : K_1(\Delta, R) \longrightarrow K_1(\Phi, R)$.

In the case $\Phi = \mathbb{A}_l$ this K_1-functor specialises to the functor

$$\mathrm{SK}_1(n, R) = \mathrm{SL}(n, R)/E(n, R),$$

rather than the usual linear K_1-functor $K_1(n, R) = \mathrm{GL}(n, R)/E(n, R)$. In examples below we also mention the corresponding stable K_1-functors, which are defined as limits of $K_1(n, R)$ and $\mathrm{SK}_1(n, R)$ under stability embeddings, as n tends to infinity:

$$\mathrm{SK}_1(R) = \varinjlim \mathrm{SK}_1(n, R), \qquad K_1(R) = \varinjlim K_1(n, R).$$

Another basic tool are stability theorems, which assert that under some assumptions on Δ, Φ and R stability maps are surjective or/and injective. We do not try to precisely state stability theorems for Chevalley groups, since they depend on various analogues and higher versions of stable rank, see in particular [75, 64, 65, 66].

However, to give some feel, we state two classical results pertaining to the case of $\mathrm{SL}(n, R)$. These results, which are due to Bass and Bass—Vaserstein, respectively, are known as surjective stability of K_1 and injective stability of K_1. In many cases they allow to reduce problems about groups of higher ranks, to similar problems for groups of smaller rank.

Theorem 2. *For any $n \geq \mathrm{sr}(R)$ the stability map*

$$K_1(n, R) \longrightarrow K_1(n + 1, R)$$

is surjective. In other words,

$$\mathrm{SL}(n + 1, R) = \mathrm{SL}(n, R)E(n + 1, R).$$

Theorem 3. *For any $n \geq \mathrm{sr}(R) + 1$ the stability map*

$$K_1(n, R) \longrightarrow K_1(n + 1, R)$$

is injective. In other words,

$$\mathrm{SL}(n, R) \cap E(n + 1, R) = E(n, R).$$

2.10 K$_1$-functor: trivial or non-trivial

Usually, K_1-functor is non-trivial. But in some important cases it is trivial. Let us start with some obvious examples.

- $R = K$ is a field.
- More generally, R is semilocal
- R is Euclidean
- It is much less obvious that K_1 does not have to be trivial even for principal ideal rings. Let us cite two easy examples discovered by Ischebeck [43] and by Grayson and Lenstra [26], respectively.
- Let K be a field of algebraic functions of one variable with a perfect field of constants k. Then the ring $R = K \otimes_k k(x_1, \dots, x_m)$ is a principal ideal ring. If, moreover, $m \geq 2$, and the genus of K is distinct from 0, then $\mathrm{SK}_1(R) \neq 1$.
- Let $R = \mathbb{Z}[x]$, and $S \subseteq R$ be the multiplicative subsystem of R generated by cyclotomic polynomials Φ_n, $n \in \mathbb{N}$. Then $S^{-1}R$ is a principal ideal ring such that $\mathrm{SK}_1(S^{-1}R) \neq 1$.

This is precisely why over a Euclidean ring it is somewhat easier to find Smith form of a matrix, than over a principal ideal ring.

However, there are some further examples, when K_1 is trivial. Usually, they are very deep. The first example below is part of the [almost] positive solution of the congruence subgroup problem by Bass—Milnor—Serre and Matsumoto [10, 60]. The second one is the solution of K_1-analogue of Serre's problem by Suslin [80].

- $R = \mathcal{O}_S$ is a Hasse domain.
- $R = K[x_1, \dots, x_m]$ is a polynomial ring over a field.

2.11 K$_1$-functor, abelian or non-abelian

Actually, $K_1(\Phi, R)$ is not only non-trivial. Oftentimes, it is even non-abelian. The first such examples were constructed by Wilberd van der Kallen [45] and Anthony Bak [6]. In both cases proofs are of topological nature and use homotopy theory.

- Wilberd van der Kallen [45] constructs a number of examples, where $K_1(n, R)$ is non-abelian. For instance,

$$R = \mathbb{R}[x_1, x_2, y_1, y_2, y_3, y_4]/(x_1^2 + x_2^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1)$$

is a 4-dimensional ring for which $[\mathrm{SL}(4,R),\mathrm{SL}(4,R)] \not\leq E(4,R)$. In fact, in this case even

$$[\mathrm{SL}(2,R),\mathrm{SL}(4,R)] \not\leq \mathrm{GL}(3,R)E(4,R).$$

- Anthony Bak [6] constructs examples of [finite dimensional] subrings R in the rings of continuous functions $\mathbb{R}^X$ and $\mathbb{C}^X$ on certain topological spaces X, for which not only $K_1(n,R)$, $n \geq 3$, is non-abelian, but even its nilpotency class can be arbitrarily large.

The question arises, as to how non-abelian $K_1(\Phi,R)$ may be. For finite dimensional rings this question was answered by Anthony Bak [6] for $\mathrm{SL}(n,R)$, for other even classical groups by the first author [29] and for all Chevalley groups by the first and the third authors [35].

Theorem 4. *Let Φ be a reduced irreducible root system such that $\mathrm{rk}(\Phi) \geq 2$. Further let R be a commutative ring of Bass—Serre dimension $\delta(R) = d < \infty$. Then $K_1(\Phi,R)$ is nilpotent of class $\leq d+1$.*

This theorem relies on a version of localisation method which Bak called localisation-completion [6]. This method turned out to be crucial for the proof of results we discuss in the present paper, see [36, 31] for more historical background and an introduction to this method in non-technical terms.

3 Main problems

3.1 Statement of the main problems

In this paper we discuss the following problem.

Problem 1. *Estimate the width of $E(\Phi,R)$ with respect to the set of elementary commutators*

$$X = \big\{[x,y] = xyx^{-1}y^{-1},\ x \in G(\Phi,R),\ y \in E(\Phi,R)\big\}.$$

Observe, that one could not have taken the set

$$X = \big\{[x,y] = xyx^{-1}y^{-1},\ x,y \in G(\Phi,R)\big\}$$

here, since $K_1(\Phi,R)$ maybe non-abelian.

It turns out that this problem is closely related to the following problem.

Problem 2. *Estimate the width of $E(\Phi,R)$ with respect to the set of elementary generators*

$$X = \big\{x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R\big\}.$$

The answer in general will be highly unexpected, so we start with discussion of classical situations.

3.2 The group $\mathrm{SL}(2, R)$

Let us mention one assumption that is essential in what follows.

When R is Euclidean, expressions of matrices in $\mathrm{SL}(2, R)$ as products of elementary transvections correspond to continued fractions. Division chains in $\mathbb{Z}$ can be arbitrarily long, it is classically known that two consecutive Fibonacci numbers provide such an example. Thus, we get.

Fact 1. $\mathrm{SL}(2, \mathbb{Z})$ *does not have bounded length with respect to the elementary generators.*

Actually, behavious of the group $\mathrm{SL}(2, R)$ is exceptional in more than one respect. Thus, the groups $E(n, R)$, $n \geq 3$ are perfect. The group $E(2, R)$ is usually not.

Fact 2. $[\mathrm{SL}(2, \mathbb{Z}), \mathrm{SL}(2, \mathbb{Z})]$ *has index* 12 *in* $\mathrm{SL}(2, \mathbb{Z})$.

- Thus, in the sequel we always assume that $\mathrm{rk}(\Phi) \geq 2$.
- In fact, it is material for most of our results that the group $E(\Phi, R)$ is perfect. It usually is, the only counter-examples in rank ≥ 2 stemming from the fact that $\mathrm{Sp}(4, \mathrm{GF}\,2)$ and $G(\mathrm{G}_2, \mathrm{GF}\,2)$ are not perfect. Thus, in most cases one should add proviso that $E(\Phi, R)$ is actually perfect, which amounts to saying that R does not have residue field $\mathrm{GF}\,2$ for $\Phi = \mathrm{B}_2, \mathrm{G}_2$.

The reader may take these two points as standing assumptions for the rest of the note.

3.3 The answers for fields

The following result easily follows from Bruhat decomposition.

Theorem 5. *The width of* $G_{\mathrm{sc}}(\Phi, K)$ *with respect to the set of elementary generators is* $\leq 2|\Phi^+| + 4\,\mathrm{rk}(\Phi)$.

Rimhak Ree [67] observed that the commutator width of semisimple algebraic groups over an algebraically closed fields equals 1. For fields containing ≥ 8 elements the following theorems were established by Erich Ellers and Nikolai Gordeev [21] using Gauss decomposition with prescribed semi-simple part [16]. On the other hand, for very small fields these theorems were recently proven by Martin Liebeck, Eamonn O'Brien, Aner Shalev, and Pham Huu Tiep [51, 52], using explicit information about maximal subgroups and very delicate character estimates.

Actually, the first of these theorems in particular completes the answer to Ore conjecture, whether any element of a [non-abelian] finite simple group is a single commutator.

Theorem 6. *The width of* $E_{\mathrm{ad}}(\Phi, K)$ *with respect to commutators is* 1.

Theorem 7. *The width of* $G_{\mathrm{sc}}(\Phi, K)$ *with respect to commutators is* ≤ 2.

3.4 The answers for semilocal rings

The following results were recently published by Andrei Smolensky, Sury and the third author [73, 96]. Actually, their proofs are easy combinations of Bass' surjective stability [9] and Tavgen's rank reduction theorem [84]. The second of these decompositions, the celebrated Gauss decomposition, was known for semilocal rings, the first one was known for $\mathrm{SL}(n, R)$, see [20], but not in general.

Theorem 8. *Let* $\mathrm{sr}(R) = 1$. *Then the*

$$E(\Phi, R) = U^{+}(\Phi, R)U^{-}(\Phi, R)U^{+}(\Phi, R)U^{-}(\Phi, R).$$

Corollary 1. *Let* $\mathrm{sr}(R) = 1$. *Then the width of* $E(\Phi, R)$ *with respect to the set of elementary generators is at most* $M = 4|\Phi^{+}|$.

Theorem 9. *Let* $\mathrm{sr}(R) = 1$. *Then the*

$$E(\Phi, R) = U^{+}(\Phi, R)U^{-}(\Phi, R)H(\Phi, R)U(\Phi, R).$$

Corollary 2. *Let* $\mathrm{sr}(R) = 1$. *Then the width of* $E(\Phi, R)$ *with respect to the set of elementary generators is at most* $M = 3|\Phi^{+}| + 4\,\mathrm{rk}(\Phi)$.

In particular, the width of $E(\Phi, R)$ over a ring with $\mathrm{sr}(R) = 1$ with respect to commutators is always bounded, but its explicit calculation is a non-trivial task. Let us limit ourselves with the following result by Leonid Vaserstein and Ethel Wheland [90, 91].

Theorem 10. *Let* $\mathrm{sr}(R) = 1$. *Then the width of* $E(n, R)$, $n \geq 3$, *with respect to commutators is* ≤ 2.

There are also similar results by You Hong, Frank Arlinghaus and Leonid Vaserstein [101, 5] for other classical groups, but they usually assert that the commutator width is ≤ 3 or ≤ 4, and sometimes impose stronger stability conditions such as $\mathrm{asr}(R) = 1$, $\Lambda\,\mathrm{sr}(R) = 1$, etc.

The works by Nikolai Gordeev and You Hong, where similar results are established for exceptional groups over local rings [subject to some mild restrictions on their residue fields] are still not published.

3.5 Bounded generation

Another nice class of rings, for which one may expect positive answers to the above problems, are Dedekind rings of arithmetic type.

Let K be an algebraic number field, i.e. either a finite algebraic extension of $\mathbb{Q}$, and further let S be a finite set of (non-equivalent) valuations of K, which contains all Archimedean valuations. For a non-Archimedean valuation $\mathfrak{p}$ of the field K we denote by $v_{\mathfrak{p}}$ the corresponding exponent. As usual, $R = \mathcal{O}_S$ denotes the ring, consisting of $x \in K$ such that $v_{\mathfrak{p}}(x) \geq 0$ for all valuations $\mathfrak{p}$ of K, which do not belong to S. Such a ring $\mathcal{O}_S$ is known as the Dedekind ring of arithmetic type, determined by the set of valuations S of the field K. Such rings are also called Hasse domains, see, for instance, [10]. Sometimes one has to require that $|S| \geq 2$, or, what is the same, that the multiplicative group $\mathcal{O}_S^*$ of the ring $\mathcal{O}_S$ is infinite.

Bounded generation of $\mathrm{SL}(n, \mathcal{O}_S)$, $n \geq 3$, was established by David Carter and Gordon Keller in [11, 12, 13, 14, 15], see also the survey by Dave Witte Morris [61] for a modern exposition. The general case was solved by Oleg Tavgen [84, 85]. The result by Oleg Tavgen can be stated in the following form due to the [almost] positive solution of the congruence subgroup problem [10, 60].

Theorem 11. *Let $\mathcal{O}_S$ be a Dedekind ring of arithmetic type,* $\mathrm{rk}(\Phi) \geq 2$. *Then the elementary Chevalley group $G(\Phi, \mathcal{O}_S)$ has bounded length with respect to the elementary generators.*

In Section 6 we discuss what this implies for the commutator width.

See also the recent works by Edward Hinson [42], Loukanidis and Murty [55, 62], Sury [79], Igor Erovenko and Andrei Rapinchuk [23, 24, 25], for different proofs, generalisations and many further references, concerning bounded generation.

3.6 van der Kallen's counter-example

However, all hopes for positive answers in general are completely abolished by the following remarkable result due to Wilberd van der Kallen [44].

Theorem 12. *The group* $\mathrm{SL}(3, \mathbb{C}[t])$ *does not have bounded word length with respect to the elementary generators.*

It is an amazing result, since $\mathbb{C}[t]$ is Euclidean. Since $\mathrm{sr}(\mathbb{C}[t]) = 2$ we get the following corollary

Corollary 3. *None of the groups* $\mathrm{SL}(n, \mathbb{C}[t])$, $n \geq 3$, *has bounded word length with respect to the elementary generators.*

See also [22] for a slightly easier proof of a slightly stronger result. Later Dennis and Vaserstein [20] improved van der Kallen's result to the following.

Theorem 13. *The group* $\mathrm{SL}(3, \mathbb{C}[t])$ *does not have bounded word length with respect to the commutators.*

Since for $n \geq 3$ every elementary matrix is a commutator, this is indeed stronger, than the previous theorem.

4 Absolute commutator width

Here we establish an amazing relation between Problems 1 and 2.

4.1 Commutator width in $\mathrm{SL}(n, R)$

The following result by Alexander Sivatsky and the second author [72] was a major breakthrough.

Theorem 14. *Suppose that $n \geq 3$ and let R be a Noetherian ring such that $\dim \mathrm{Max}(R) = d < \infty$. Then there exists a natural number $N = N(n, d)$ depending only on n and d such that each commutator $[x, y]$ of elements $x \in E(n, R)$ and $y \in \mathrm{SL}(n, R)$ is a product of at most N elementary transvections.*

Actually, from the proof in [72] one can derive an efficient upper bound on N, which is a *polynomial* with the leading term $48n^6d$.

It is interesting to observe that it is already non-trivial to replace here an element of $\mathrm{SL}(n, R)$ by an element of $\mathrm{GL}(n, R)$. Recall, that a *ring of geometric origin* is a localisation of an affine algebra over a field.

Theorem 15. *Let $n \geq 3$ and let R be a ring of geometric origin. Then there exists a natural number N depending only on n and R such that each commutator $[x, y]$ of elements $x \in E(n, R)$ and $y \in \mathrm{GL}(n, R)$ is a product of at most N elementary transvections.*

Let us state another interesting variant of the Theorem 14, which may be considered as its stable version. Its proof crucially depends on the Suslin—Tulenbaev proof of the Bass—Vaserstein theorem, see [82].

Theorem 16. *Let $n \geq \mathrm{sr}(R) + 1$. Then there exists a natural number N depending only on n such that each commutator $[x, y]$ of elements $x, y \in \mathrm{GL}(n, R)$ is a product of at most N elementary transvections.*

Actually, [72] contains many further interesting results, such as, for example, analogues for the Steinberg groups $\mathrm{St}(n, R)$, $n \geq 5$. However, since this result depends on the centrality of $K_2(n, R)$ at present there is no hope to generalise it to other groups.

4.2 Decomposition of unipotents

The proof of Theorem 14 in [72] was based on a combination of localisation and decomposition of unipotents [77]. Essentially, in the simplest form decomposition of unipotents gives finite polynomial expressions of the conjugates

$$gx_{\alpha}(\xi)g^{-1}, \qquad \alpha \in \Phi,\ \xi \in R,\ g \in G(\Phi, R),$$

as products of factors sitting in proper parabolic subgroups, and, in the final count, as products of elementary generators.

Roughly speaking, decomposition of unipotents allows to plug in explicit polynomial formulas as the induction base — which is the most difficult part of all localisation proofs! — instead of messing around with the length estimates in the conjugation calculus.

To give some feel of what it is all about, let us state an immediate corollary of the Theme of [77]. Actually, [77] provides explicit polynomial expressions of the elementary factors, rather than just the length estimate.

Fact 3. *Let R be a commutative ring and $n \geq 3$. Then any transvection of the form $gt_{ij}(\xi)g^{-1}$, $1 \leq i \neq j \leq n$, $\xi \in R$, $g \in \mathrm{GL}(n, R)$ is a product of at most $4n(n-1)$ elementary transvections.*

It is instructive to compare this bound with the bound resulting from Suslin's proof of Suslin's normality theorem [80]. Actually, Suslin's direct factorisation method is more general, in that it yields elementary factorisations of a broader class of transvections. On the other hand, it is less precise, both factorisations coincide for $n = 3$, but asymptotically factorisation in Fact 3 is better.

Fact 4. *Let R be a commutative ring and $n \geq 3$. Assume that $u \in R^n$ is a unimodular column and $v \in {}^nR$ be any row such that $vu = 0$. Then the transvection $e + uv$ is a product of at most $n(n-1)(n+2)$ elementary transvections.*

Let us state a counterpart of the Theorem 14 that results from the Fact 3 alone, *without* the use of localisation. This estimate works for *arbitrary* commutative rings, but depends on the length of the elementary factor. Just wait until subsection 4.5!

Theorem 17. *Let $n \geq 3$ and let R be a commutative ring. Then there exists a natural number $N = N(n, M)$ depending only on n and M such that each commutator $[x, y]$ of elements $x \in E^M(n, R)$ and $y \in \mathrm{SL}(n, R)$ is a product of at most N elementary transvections.*

It suffices to expand a commutator $[x_1 \ldots x_M, y]$, where x_i are elementary transvections, with the help of the commutator identity $[xz, y] = {}^x[z, y] \cdot [x, y]$, and take the upper bound $4n(n-1) + 1$ for each of the resulting commutators $[x_i, y]$. One thus gets $N \leq M^2 + 4n(n-1)M$.

However, such explicit formulas are only available for linear and orthogonal groups, and for exceptional groups of types E_6 and E_7. Let us state the estimate resulting from the proof of [93, Theorems 4 and 5].

Fact 5. *Let R be a commutative ring and $\Phi = \mathrm{E}_6, \mathrm{E}_7$. Then any root element of the form $gx_\alpha(\xi)g^{-1}$, $\alpha \in \Phi$, $\xi \in R$, $g \in G(\Phi, R)$ is a product of at most $4 \cdot 16 \cdot 27 = 1728$ elementary root unipotents in the case of $\Phi = \mathrm{E}_6$ and of at most $4 \cdot 27 \cdot 56 = 6048$ elementary root unipotents in the case of $\Phi = \mathrm{E}_7$.*

Even for symplectic groups — not to say for exceptional groups of types $\mathrm{E}_8, \mathrm{F}_4$ and G_2! — it is only known that the elementary groups are generated by root unipotents of certain classes, which afford reduction to smaller ranks, but no explicit polynomial factorisations are known, and even no polynomial length estimates.

This is why generalisation of Theorem 14 to Chevalley groups requires a new idea.

4.3 Commutator width of Chevalley groups

Let us state the main result of [78]. While the main idea of proof comes from the work by Alexander Sivatsky and the second author [72], most of the actual calculations are refinements of conjugation calculus and commutator calculus in Chevalley groups, developed by the first and the third authors in [35].

Theorem 18. *Let $G = G(\Phi, R)$ be a Chevalley group of rank $l \geq 2$ and let R be a ring such that $\dim \mathrm{Max}(R) = d < \infty$. Then there exists a natural number N depending only on Φ and d such that each commutator $[x, y]$ of elements $x \in G(\Phi, R)$ and $y \in E(\Phi, R)$ is a product of at most N elementary root unipotents.*

Here we cannot use decomposition of unipotents. The idea of the second author was to use the *second localisation* instead. As in [72] the proof starts with the following lemma, where M has the same meaning as in Subsection 3.4.

Lemma 1. *Let $d = \dim(\mathrm{Max}(R))$ and $x \in \mathrm{G}(\Phi, R)$. Then there exist $t_0, \dots, t_k \in R$, where $k \leq d$, generating R as an ideal and such that $F_{t_i}(x) \in \mathrm{E}^M(\Phi, R_{t_i})$ for all $i = 0, \dots, k$.*

Since $t_0, \dots, t_k$ are unimodular, their powers also are, so that we can rewrite y as a product of y_i, where each y_i is congruent to e modulo a high power of t_i. In the notation of the next section this means that $y_i \in E(\Phi, R, t_i^m R)$.

When the ring R is Noetherian, $G(\Phi, R, t_i^m R)$ injects into $\mathrm{G}(\Phi, R_{t_i})$ for some high power t_i^m. Thus, it suffices to show that $F_{t_i}([x, y_i])$ is a product of bounded number of elementary factors without denominators in $E(\Phi, R_{t_i})$. This is the first localisation.

The second localisation consists in applying the same argument again, this time in R_{t_i}. Applying Lemma 1 once more we can find $s_0, \dots, s_d$ forming a unimodular row, such that the images of y_i in $E(\Phi, R_{t_i s_j})$ are products of at most M elementary root unipotents with denominators s_j. Decomposing

$F_{s_j}(x) \in E(\Phi, R_{s_j})$ into a product of root unipotents, and repeatedly applying commutator identities, we eventually reduce the proof to proving that the length of each commutator of the form

$$\Big[x_\alpha\Big(\frac{t_i^l}{s_j}a\Big), x_\beta\Big(\frac{s_j^n}{t_i}b\Big)\Big]$$

is bounded.

4.4 Commutator calculus

Conjugation calculus and commutator calculus consists in rewriting conjugates (resp. commutators) with denominators as products of elementary generators *without* denominators.

Let us state a typical technical result, the base of induction of the commutator calculus.

Lemma 2. *Given $s, t \in R$ and $p, q, k, m \in \mathbb{N}$, there exist $l, m \in \mathbb{N}$ and $L = L(\Phi)$ such that*

$$\Big[x_\alpha\Big(\frac{t^l}{s^k}a\Big), x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big] \in E^L(\Phi, s^p t^q R).$$

A naive use of the Chevalley commutator formula gives $L \leq 585$ for simply laced systems, $L \leq 61882$ for doubly laced systems and $L \leq 797647204$ for $\Phi = \mathrm{G}_2$. And this is just the first step of the commutator calculus!

Reading the proof sketched in the previous subsection upwards, and repeatedly using commutator identities, we can eventually produce bounds for the length of commutators, ridiculous as they can be.

Recently in [34] the authors succeeded in producing a similar proof for Bak's unitary groups, see [28, 47, 8, 36] and references there. The situation here is in many aspects more complicated than for Chevalley groups. In fact, Bak's unitary groups are not always algebraic, and all calculations should be inherently carried through in terms of *form ideals*, rather then ideals of the ground ring. Thus, the results of [34] heavily depend on the *unitary* conjugation calculus and commutator calculus, as developed in [29, 37].

4.5 Universal localisation

Now something truly amazing will happen. Some two years ago the second author noticed that the width of commutators is bounded by a universal constant that depends on the type of the group alone, see [76]. Quite remarkably, one can obtain a length bound that does not depend either on the dimension of the ring, or on the length of the elementary factor.

Theorem 19. *Let $G = G(\Phi, R)$ be a Chevalley group of rank $l \geq 2$. Then there exists a natural number $N = N(\Phi)$ depending on Φ alone, such that each commutator $[x, y]$ of elements $x \in G(\Phi, R)$ and $y \in E(\Phi, R)$ is a product of at most N elementary root unipotents.*

What is remarkable here, is that there is no dependence on R whatsoever. In fact, this bound applies even to infinite dimensional rings! Morally, it says that in the groups of points of algebraic groups there are very few commutators.

Here is a very brief explanation of how it works. First of all, Chevalley groups are representable functors, $G(\Phi, R) = \mathrm{Hom}(\mathbb{Z}[G], R)$, so that there is a *universal element* $g \in \mathrm{G}(\Phi, \mathbb{Z}[G])$, corresponding to $\mathrm{id} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]$, which specialises to *any* element of the Chevalley group $G(\Phi, R)$ of the same type over *any* ring.

But the elementary subgroup $E(\Phi, R)$ is not an algebraic group, so where can one find universal elements?

The real know-how proposed by the second author consists in construction of the universal coefficient rings for the principal congruence subgroups $G(\Phi, R, sR)$ (see the next section, for the definition), corresponding to the principal ideals. It turns out that this is enough to carry through the same scheme of the proof, with bounds that do not depend on the ring R.

5 Relative commutator width

In the absolute case the above results on commutator width are mostly published. In this section we state relative analogues of these results which are announced here for the first time.

5.1 Congruence subgroups

Usually, one defines congruence subgroups as follows. An ideal $A \trianglelefteq R$ determines the reduction homomorphism $\rho_A : R \longrightarrow R/A$. Since $G(\Phi, \_)$ is a functor from rings to groups, this homomorphism induces reduction homomorphism $\rho_A : G(\Phi, R) \longrightarrow G(\Phi, R/A)$.

- The kernel of the reduction homomorphism ρ_A modulo A is called the principal congruence subgroup of level A and is denoted by $G(\Phi, R, A)$.
- The full pre-image of the centre of $G(\Phi, R/A)$ with respect to the reduction homomorphism ρ_A modulo A is called the full congruence subgroup of level A, and is denoted by $C(\Phi, R, A)$.

But in fact, without assumption that $2 \in R^*$ for doubly laced systems, and without assumption that $6 \in R^*$ for $\Phi = \mathrm{G}_2$, the genuine congruence subgroups should be defined in terms of admissible pairs of ideals (A, B), introduced by Abe, [1, 4, 2, 3], and in terms of form ideals for symplectic groups. One of these ideals corresponds to short roots and another one corresponds to long roots.

In [30] we introduced a more general notion of congruence subgroups, corresponding to admissible pairs: $G(\Phi, R, A, B)$ and $C(\Phi, R, A, B)$, Not to overburden the note with technical details, we mostly tacitly assume that $2 \in R^*$ for $\Phi = \mathrm{B}_l, \mathbf{C}_l, \mathrm{F}_4$ and $6 \in R^*$ for $\Phi = \mathrm{G}_2$. Under these simplifying assumption one has $A = B$ and $G(\Phi, R, A, B) = G(\Phi, R, A)$ and $C(\Phi, R, A, B) = C(\Phi, R, A)$. Of course, using admissible pairs/form ideals one can obtained similar results without any such assumptions.

5.2 Relative elementary groups

Let A be an additive subgroup of R. Then $E(\Phi, A)$ denotes the subgroup of E generated by all elementary root unipotents $x_\alpha(\xi)$ where $\alpha \in \Phi$ and $\xi \in A$. Further, let L denote a nonnegative integer and let $E^L(\Phi, A)$ denote the *subset* of $E(\Phi, A)$ consisting of all products of L or fewer elementary root unipotents $x_\alpha(\xi)$, where $\alpha \in \Phi$ and $\xi \in A$. In particular, $E^1(\Phi, A)$ is the set of all $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in A$.

In the sequel we are interested in the case where $A = I$ is an ideal of R. In this case we denote by

$$E(\Phi, R, I) = E(\Phi, I)^{E(\Phi, R)}$$

the *relative* elementary subgroup of level I. As a *normal* subgroup of $E(\Phi, R)$ it is generated by $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in A$. The following theorem [74, 86, 88] lists its generators as a subgroup.

Theorem 20. *As a subgroup $E(\Phi, R, I)$ is generated by the elements*

$$z_\alpha(\xi, \zeta) = x_{-\alpha}(\zeta) x_\alpha(\xi) x_{-\alpha}(-\zeta),$$

where $\xi \in I$ for $\alpha \in \Phi$, while $\zeta \in R$.

It is natural to regard these generators as the *elementary generators* of $E(\Phi, R, I)$. For the special linear group $\mathrm{SL}(n, \mathcal{O}_S)$, $n \geq 3$, over a Dedekind ring of arithmetic type Bernhard Liehl [54] has proven bounded generation of the elementary relative subgroups $E(n, \mathcal{O}_S, I)$ in the generators $z_{ij}(\xi, \zeta)$. What is remarkable in his result, is that the bound does not depend on the ideal I. Also, he established similar results for $\mathrm{SL}(2, \mathcal{O}_S)$, provided that $\mathcal{O}_S^*$ is infinite.

5.3 Standard commutator formula

The following result was first proven by Giovanni Taddei [83], Leonid Vaserstein [88] and Eiichi Abe [2, 3].

Theorem 21. *Let Φ be a reduced irreducible root system of rank ≥ 2, R be a commutative ring, $I \trianglelefteq R$ be an ideal of R. In the case, where $\Phi = \mathrm{B}_2$ or $\Phi = \mathrm{G}_2$ assume moreover that R has no residue fields $\mathbb{F}_2$ of 2 elements. Then the following standard commutator formula holds*

$$\big[G(\Phi, R), E(\Phi, R, I)\big] = \big[E(\Phi, R), C(\Phi, R, I)\big] = E(\Phi, R, I).$$

In fact, in [30] we established similar result for relative groups defined in terms of admissible pairs, rather then single ideals. Of course, in all cases, except Chevalley groups of type F_4, it was known before, [8, 63, 18].

With the use of level calculations the following result was established by You Hong [100], by analogy with the Alec Mason and Wilson Stothers [59, 56, 57, 58]. Recently the first, third and fourth authors gave another proof, of this result, in the framework of relative localisation [38], see also [97, 40, 98, 31, 37, 41, 32, 39, 33, 76] for many further analogues and generalisations of such formulas.

Theorem 22. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi) \geq 2$. *Further, let R be a commutative ring, and $A, B \trianglelefteq R$ be two ideals of R. Then*

$$[E(\Phi, R, A), G(\Phi, R, B)] = [E(\Phi, R, A), E(\Phi, R, B)].$$

5.4 Generation of mixed commutator subgroups

It is easy to see that the mixed commutator $[E(\Phi, R, A), E(\Phi, R, B)]$ is a subgroup of level AB, in other words, it sits between the relative elementary subgroup $E(\Phi, R, AB)$ and the corresponding congruence subgroup $G(\Phi, R, AB)$.

Theorem 23. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi) \geq 2$. *Further, let R be a commutative ring, and $A, B \trianglelefteq R$ be two ideals of R. When $\Phi = \mathrm{B}_2, \mathrm{G}_2$, assume that R does not have residue field of 2 elements, and when $\Phi = \mathbf{C}_l$, $l \geq 2$, assume additionally that any $a \in R$ is contained in the ideal $a^2R + 2aR$. Then*

$$E(\Phi, R, AB) \leq [E(\Phi, R, A), E(\Phi, R, B)] \leq \\ [G(\Phi, R, A), G(\Phi, R, B)] \leq G(\Phi, R, AB).$$

It is not too difficult to construct examples showing that in general the mixed commutator subgroup $[E(\Phi, R, A), E(\Phi, R, B)]$ can be strictly larger than $E(\Phi, R, AB)$. The first such examples were constructed by Alec Mason and Wilson Stothers [59, 57] in the ring $R = \mathbb{Z}[i]$ of Gaussian integers.

In this connection, it is very interesting to explicitly list generators of the mixed commutator subgroups $[E(\Phi, R, A), E(\Phi, R, B)]$ *as subgroups*. From Theorem 20 we already know most of these generators. These are $z_\alpha(\xi\zeta, \eta)$, where $\xi \in A$, $\zeta \in B$, $\eta, \vartheta \in R$. But what are the remaining ones?

In fact, using the Chevalley commutator formula it is relatively easy to show that $[E(\Phi,R,A),E(\Phi,R,B)]$ is generated by its intersections with the fundamental SL_2's. Using somewhat more detailed analysis the first and the fourth author established the following result, initially for the case of $\mathrm{GL}(n,R)$, $n\geq 3$, see [41] and then, jointly with the third author, for all other cases, see [32, 39].

Theorem 24. *Let R be a commutative ring with 1 and A, B be two ideals of R. Then the mixed commutator subgroup $\big[E(\Phi,R,A),E(\Phi,R,B)\big]$ is generated as a normal subgroup of $E(n,R)$ by the elements of the form*

- $\big[x_\alpha(\xi),{}^{x_{-\alpha}(\eta)}x_\alpha(\zeta)\big]$,
- $\big[x_\alpha(\xi),x_{-\alpha}(\zeta)\big]$,
- $x_\alpha(\xi\zeta)$,

where $\alpha\in\Phi$, $\xi\in A$, $\zeta\in B$, $\eta\in R$.

Another moderate technical effort allows to make it into a natural candidate for the set of elementary generators of $[E(\Phi,R,A),E(\Phi,R,B)]$.

Theorem 25. *Let R be a commutative ring with 1 and I, J be two ideals of R. Then the mixed commutator subgroup $\big[E(\Phi,R,A),E(\Phi,R,B)\big]$ is generated as a group by the elements of the form*

- $\big[z_\alpha(\xi,\eta),z_\alpha(\zeta,\vartheta)\big]$,
- $\big[z_\alpha(\xi,\eta),z_{-\alpha}(\zeta,\vartheta)\big]$,
- $z_\alpha(\xi\zeta,\eta)$,

where $\alpha\in\Phi$, $\xi\in A$, $\zeta\in B$, $\eta,\vartheta\in R$.

5.5 Relative commutator width

Now we are all set to address relative versions of the main problem. The two following results were recently obtained by the second author, with his method of universal localisation [76], but they depend on the construction of generators in Theorems 20 and 25. Mostly, the preceding results were either published or prepublished in some form, and announced at various conferences. These two theorems are stated here for the first time.

Theorem 26. *Let R be a commutative ring with 1 and let $I\trianglelefteq R$, be an ideal of R. Then there exists a natural number $N=N(\Phi)$ depending on Φ alone, such that any commutator $[x,y]$, where*

$$x\in G(\Phi,R,I),\quad y\in E(\Phi,R)\qquad \textit{or}\qquad x\in G(\Phi,R),\quad y\in E(\Phi,R,I)$$

is a product of not more that N elementary generators $z_\alpha(\xi,\zeta)$, $\alpha\in\Phi$, $\xi\in I$, $\zeta\in R$.

Theorem 27. *Let R be a commutative ring with 1 and let $A,B\trianglelefteq R$, be ideals of R. there exists a natural number $N=N(\Phi)$ depending on Φ alone, such that any commutator*

$$[x,y],\qquad x\in G(\Phi,R,A),\quad y\in E(\Phi,R,B)$$

is a product of not more that N elementary generators listed in Theorem 25.

Quite remarkably, the bound N in these theorems does not depend either on the ring R, or on the choice of the ideals I,A,B. The proof of these theorems is not particularly long, but it relies on a whole bunch of universal constructions and will be published in **??**. From the proof, it becomes apparent that similar results hold also in other such situations: for any other functorial generating set, for multiple relative commutators [41, 39], etc.

6 Loose ends

Let us mention some positive results on commutator width and possible further generalisations.

6.1 Some positive results

There are some obvious bounds for the commutator width that follow from unitriangular factorisations. For the $\mathrm{SL}(n,R)$ the following result was observed by van der Kallen, Dennis and Vaserstein. The proof in general was proposed by Nikolai Gordeev and You Hong in 2005, but is still not published, as far as we know.

Theorem 28. *Let $\mathrm{rk}(\Phi)\geq 2$. Then for any commutative ring R an element of $U(\Phi,R)$ is a product of not more than two commutators in $E(\Phi,R)$.*

Combining the previous theorem with Theorem 8 we get the following corollary.

Corollary 4. *Let $\mathrm{rk}(\Phi)\geq 2$ and let R be a ring such that $\mathrm{sr}(R)=1$. Then the any element of $E(\Phi,R)$ is a product of ≤ 6 commutators.*

This focuses attention on the following problem.

Problem 3. *Find the shortest factorisation of $E(\Phi,R)$ of the form*

$$E=UU^-UU^-\dots U^\pm.$$

Let us reproduce another result from the paper by Andrei Smolensky, Sury and the third author [96]. It is proven similarly to Theorem 8, but uses Cooke—Weinberger [17] as induction base. Observe that it depends on the Generalised Riemann's Hypothesis, which is used to prove results in the style of Artin's conjecture on primitive roots in arithmetic progressions. Lately, Maxim Vsemirnov succeeded in improving bounds and in some cases eliminating dependence on GRH. In particular, Cooke—Weinberger construct a division chain of length 7 in the non totally imaginary case, the observation that it can be improved to a division chain of length 5 is due to Vsemirnov [99]. Again, in the form below, with $G(\Phi, \mathcal{O}_S)$ rather than $E(\Phi, \mathcal{O}_S)$, it relies on the almost positive solution of the congruence subgroup problem [10, 60].

Theorem 29. *Let $R = \mathcal{O}_S$ be a Dedekind ring of arithmetic type with infinite multiplicative group. Then under the Generalised Riemann Hypothesis the simply connected Chevalley group $G_{\mathrm{sc}}(\Phi, \mathcal{O}_S)$ admits unitriangular factorisation of length 9,*

$$G_{\mathrm{sc}}(\Phi, \mathcal{O}_S) = UU^-UU^-UU^-UU^-U.$$

In the case, where $\mathcal{O}_S$ has a real embedding, it admits unitriangular factorisation of length 5,

$$G_{\mathrm{sc}}(\Phi, \mathcal{O}_S) = UU^-UU^-U.$$

Corollary 5. *Let $\mathrm{rk}(\Phi) \geq 2$ and let $\mathcal{O}_S$ be a Dedekind ring of arithmetic type with infinite multiplicative group. Then the any element of $G_{\mathrm{sc}}(\Phi, \mathcal{O}_S)$ is a product of ≤ 10 commutators. In the case, where $\mathcal{O}_S$ has a real embedding, this estimate can be improved to ≤ 6 commutators.*

6.2 Conjectures concerning commutator width

We believe that solution of the following two problems is now at hand. In Section 2 we have already cited the works of Frank Arlinghaus, Leonid Vaserstein, Ethel Wheland and You Hong [90, 91, 101, 5], where this is essentially done for *classical* groups, over rings subject to $\mathrm{sr}(R) = 1$ or some stronger stability conditions.

Problem 4. *Under assumption $\mathrm{sr}(R) = 1$ prove that any element of elementary group $E_{\mathrm{ad}}(\Phi, R)$ is a product of ≤ 2 commutators in $G_{\mathrm{ad}}(\Phi, R)$.*

Problem 5. *Under assumption $\mathrm{sr}(R) = 1$ prove that any element of elementary group $E(\Phi, R)$ is a product of ≤ 3 commutators in $E(\Phi, R)$.*

It may well be that under this assumption the commutator width of $E(\Phi, R)$ is always ≤ 2, but so far we were unable to control details concerning semisimple factors.

It seems, that one can apply the same argument to higher stable ranks. Solution of the following problem would be a generalisation of [19, Theorem 4].

Problem 6. *If the stable rank* $\mathrm{sr}(R)$ *of* R *is finite, and for some* $m \geq 2$ *the elementary linear group* $E(m, R)$ *has bounded word length with respect to elementary generators, then for all* Φ *of sufficiently large rank any element of* $E(\Phi, R)$ *is a product of* ≤ 4 *commutators in* $E(\Phi, R)$.

Problem 7. *Let* R *be a Dedekind ring of arithmetic type with infinite multiplicative group. Prove that any element of* $E_{\mathrm{ad}}(\Phi, R)$ *is a product of* ≤ 3 *commutators in* $G_{\mathrm{ad}}(\Phi, R)$.

Some of our colleagues expressed belief that any element of $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, is a product of ≤ 2 commutators. However, for Dedekind rings with *finite* multiplicative groups, such as $\mathbb{Z}$, at present we do not envisage any *obvious* possibility to improve the generic bound ≤ 4 even for large values of n. Expressing elements of $\mathrm{SL}(n, \mathbb{Z})$ as products of 2 commutators, if it can be done at all, should require a lot of specific case by case analysis.

6.3 The group $\mathrm{SL}(2, R)$: improved generators

One could also mention the recent paper by Leonid Vaserstein [89] which shows that for the group $\mathrm{SL}(2, R)$ it is natural to consider bounded generation not in terms of the elementary generators, but rather in terms of the generators of the pre-stability kernel $\tilde{E}(2, R)$. In other words, one should also consider matrices of the form $(e + xy)(e + yx)^{-1}$.

Theorem 30. *The group* $\mathrm{SL}(2, \mathbb{Z})$ *admits polynomial parametrisation of total degree* ≤ 78 *with* 46 *parameters.*

The idea is remarkably simple. Namely, Vaserstein observes that $\mathrm{SL}(2, \mathbb{Z})$ coincides with the pre-stability kernel $\tilde{E}(2, \mathbb{Z})$. All generators of the group $\tilde{E}(2, \mathbb{Z})$, not just the elementary ones, admit polynomial parametrisation. The additional generators require 5 parameters each.

It only remains to verify that each element of $\mathrm{SL}(2, \mathbb{Z})$ has a small length, with respect to this new set of generators. A specific formula in [89] expresses an element of $\mathrm{SL}(2, \mathbb{Z})$ as a product of 26 elementary generators and 4 additional generators, which gives $26 + 4 \cdot 5 = 46$ parameters mentioned in the above theorem.

6.4 Bounded generation and Kazhdan property

The following result is due to Yehuda Shalom [70], Theorem 8, see also [71, 46].

Theorem 31. *Let R be an m-generated commutative ring, $n \geq 3$. Assume that $E(n, R)$ has bounded width C in elementary generators. Then $E(n, R)$ has property T. In an appropriate generating system S the Kazhdan constant is bounded from below*

$$\mathcal{K}(G, S) \geq \frac{1}{C \cdot 22^{n+1}}.$$

Problem 8. *Does the group* $\mathrm{SL}(n, \mathbb{Z}[x])$, *$n \geq 3$, has bounded width with respect to the set of elementary generators?*

If this problem has positive solution, then by Suslin's theorem and Shalom's theorem the groups $\mathrm{SL}(n, \mathbb{Z}[x])$ have Kazhdan property T. Thus,

Problem 9. *Does the group* $\mathrm{SL}(n, \mathbb{Z}[x])$, *$n \geq 3$, have Kazhdan property T?*

If this is the case, one can give a uniform bound of the Kazhdan constant of the groups $\mathrm{SL}(n, \mathcal{O})$, for the rings if algebraic integers. It is known that these group have Kazhdan property, but the known estimates depend on the discriminant of the ring $\mathcal{O}$.

Problem 10. *Prove that the group* $\mathrm{SL}(n, \mathbb{Q}[x])$ *does not have bounded width with respect to the elementary generators.*

It is natural to try to generalise results of Bernhard Liehl [54] to other Chevalley groups. The first of the following problems was stated by Oleg Tavgen in [84]. As always, we assume that $\mathrm{rk}(\Phi) \geq 2$. Otherwise, Problem 12 is open for the group $\mathrm{SL}(2, \mathcal{O}_S)$, provided that the multiplicative group $\mathcal{O}_S^*$ is infinite.

Problem 11. *Prove that over a Dedekind ring of arithmetic type the relative elementary groups $E(\Phi, \mathcal{O}_S, I)$ have bounded width with respect to the elementary generators $z_\alpha(\xi, \zeta)$, with a bound that does not depend on I.*

Problem 12. *Prove that over a Dedekind ring of arithmetic type the mixed commutator subgroups $[E(\Phi, \mathcal{O}_S, A), E(\Phi, \mathcal{O}_S, B)]$ have bounded width with respect to the elementary generators constructed in Theorem* 25, *with a bound that does not depend on A and B.*

6.5 Not just commutators

It is very challenging to understand, to which extent such behaviour is typical for more general classes of group words. There are a lot of recent results showing that the verbal length of the finite simple groups is strikingly small [68, 69, 49, 50, 53, 27]. In fact, under some natural assumptions for large finite simple groups this verbal length is 2. We do not expect similar results to hold for rings other than the zero-dimensional ones, and some arithmetic rings of dimension 1.

Powers are a class of words in a certain sense opposite to commutators. Alireza Abdollahi suggested that before passing to more general words, we should first look at powers.

Problem 13. *Establish finite width of powers in elementary generators, or lack thereof.*

An answer – in fact, *any* answer! – to this problem would be amazing. However, we would be less surprised if for rings of dimension ≥ 2 the verbal maps in $G(R)$ would have very small images.

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