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Distributed Computing

Budget Restricted Market Games with Continuous Spendings

Bachelor's Thesis

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Abstract

Many global markets, such as the one for bubble tea, have been growing at a rapid rate in the recent years. Companies in these markets must develop investment strategies. This paper introduces a game-theoretic model for analyzing location-based investment strategies of players, where locations can have an influence on each other, meaning that investments at one location can generate value at other locations. The influence is specific to each ordered pair of locations. The model is applicable to almost any kind of market where there is some notion of valued locations (abstractly called resources) among which players invest their budget. The main problem consists of finding pure Nash equilbria in this model. This is done by maximizing the payoff function using analytical methods, and numerical methods for the problems where the former are not successful. The pure Nash equilibrium is found for any number of players and resources without influence, and for two players and multiple resources with constant pairwise influence. Given arbitrary pairwise influence factors, the pure Nash equilibrium is found for two players and two resources. An approximation algorithm for the pure Nash equilibrium based on gradient ascent is developed and analyzed for two players and three resources. The yielded results show that the model allows to devise many scenarios where the pure Nash equilibrium is surprising and nontrivial and is effective in modeling complex relationships between various resources.

Contents

Chapter 1

Introduction

The market for bubble tea has enjoyed a strong growth in the past few years, with no end in sight, as many stores continue to open in Europe and around the world outside of the drink's origin, Taiwan. There are countless such examples of rapidly growing markets where companies are opening new locations and must decide where these new locations should be. How should they model this problem and finally decide how much to invest into particular locations?

This papers introduces a game-theoretic model, the *resource competing game*, that provides a way to model the scenario where multiple parties invest into various locations, where each location has some notion of a "value". This model is not confined to any particular market, but enables the analysis of any market which has some abstract concept of multiple locations–physical or virtual–among which investments can be distributed, restricted by a budget. The model used throughout this thesis, which is described in Chapter 3, is a slightly adapted version of the model introduced in [1], a previous thesis on this topic, which focuses primarily on finding pure Nash equilibria with discrete spendings, whereas this thesis considers continuous spendings.

As one can imagine, not every location, e.g. city, has the same potential for opening a new store, since bigger cities attract more people than smaller ones. In fact, bigger cities attract people *from* smaller ones. The model implements this fact using a sense of influence: An investment at one location can generate revenue from a population living at another location.

Figure 1.1 shows a simple way one could represent such a flow of population. For instance, the arrow from Dietikon to Zurich symbolizes the fact that many people who live in Dietikon travel to the city of Zurich regularly, for instance to work. Besides the fact that Zurich is a larger city than Dietikon and therefore an investment in Zurich is of greater value than an investment in Dietikon, Zurich also attracts more people from Dietikon than vice-versa. This means that an investment in Zurich also has an influence on how much revenue can be generated among the population of Dietikon. The model is not limited to influence implied by a flow of population, but admits any other way that one location might have an influence on another.

Figure 1.1: Map of possible flows of population around Zurich. Generated by Google Maps on July 1st, 2021. Graph overlaid by author.

Besides the locations, called *resources*, and their valuation, which is measured as *utility*, the model requires each party, called *player*, to specify a *budget*. Each player can distribute its available budget across the resources. The concept of influence mentioned above makes the model interesting: As the influence between any two resources can differ, it is not straightforward to decide how much to invest in each resource.

The main goal of this thesis is to prove the existence of pure Nash equilibria, states where no player can profit by changing only their own competitive investment strategy, in the resource competing game. Besides proving existence, it aims to provide a way to compute these pure Nash equilibria. The focus lies on the case where there are two players, but the possibility of more players will be discussed as an outlook.

To that end, first, the properties that are directly implied by and intrinsic to the model are explored. This reveals some advantages and disadvantages of the model. The insights gained from this are then used to analyze certain cases of the model, in particular finding pure Nash equilibria for the case of two players. This is divided into the case where the influence is constant and the case where the influence may vary arbitrarily between any pair of resources. In both of these cases, it is further distinguished whether each player has the same budget or if the budget for each player may be different. In the former case the players are called *symmetric*. The model naturally gives rise to the possibility of finding pure Nash equilibria by algorithmic approximation. While utilizing the knowledge gained about the model itself, this is done for the case of three resources, where the analytical approach does not bear any fruit, and forms the end of the analysis.

A GitLab repository has been set up for this thesis. There, all files used to compile this document can be found, including the source code for `Python` and `Mathematica` [2] programs.

Chapter 2

Related Work

In [3], the concept of a *non-cooperative game* is introduced. The model from Chapter 3 is such a game. The main characteristic of this class of games is that the players do not collaborate among each other. In particular, they do not communicate or form coalitions.

Congestion games are a subclass of non-cooperative games, and were first proposed in [4]. In this type of game, there are n players and t resources (called primary factors in the original work), where each player has a strategy consisting of selecting a subset of resources. The cost of a strategy is the sum of the costs of the selected resources. The cost of a resource depends on the number of players that have selected it. In [4], the existence of pure Nash equilibria in congestion games is proven. *Potential games* are introduced in [5], where it is shown that congestion games are isomorphic to *exact* potential games, and therefore always admit pure Nash equilibria.

The concept of *weighted* congestion games was introduced in [6], an extension that includes a weight for each player. This modification alters the cost of a resource to additionally depend on the player, by multiplying the resource's cost by the players weight. The work shows that weighted congestion games do not always possess a pure Nash equilibrium, in contrast to the unweighted versions. Of course, the case where all players' weights are equal to 1 results in an unweighted congestion game as above.

The work in [7] shows that deciding whether a weighted congestion game admits a pure Nash equilibrium is strongly NP-hard. This is the case if the cost functions are non-linear in general. If, however, the cost functions are linear, [8] shows that the game always possesses a pure Nash equilibrium. Additionally, this holds if the cost functions are affine or exponential, as shown in [9].

Many models in literature consider *integer-splittable* weighted congestion games. As mentioned in Chapter 1, this thesis examines the case where players can split their budget arbitrarily, that is, the spending on a resource can be a real number. Such games are called *infinitely splittable* weighted congestion games. It is shown in [10] that if the payoff function is continuous and concave for all players then a pure Nash equilibrium exists.

So far the cost of these games only depended on how many players select a resource and the players' weights. To model the case where the selection of one resource affects the cost of selecting other resources, [11] introduces *local-effect games*. In general, such games do not admit pure Nash equilibria. It is unclear whether there exists an efficient algorithm which can find a pure Nash equilibrium in such games. The effect of resources on each other in local-effect games is similar to the influence α of the model introduced in Chapter 3.

Our model assigns a budget to each player. Similar concepts are present in [12] and [13] as *bandwidth allocation games* and *restricted budget games* respectively. Neither of these two models combine the notion of a budget with that of influence from local-effect games. Our model is a combination of many of the models found in literature, but seems to be unique in some ways, allowing for an interesting examination.

The previous thesis [1] on this topic uses a simplified version of the model in this thesis. It considers only the case where the influence is constant between all pairs of resources. Additionally, it focuses on discrete spendings and shows the existence of pure Nash equilibria for two resources and two symmetric players and for m resources and two symmetric players.

Chapter 3

The Model

Definition 3.1 (Resource competing game)**.** A *resource competing game* is a quintuple $(P, V, (s_i)_{i \in P}, (U_v)_{v \in V}, \alpha)$, where

- P is a set of p players,
- V is a set of m vertices, corresponding to m resources,
- $s_i \in \mathbb{N}_+$ is the budget available to player $i \in P$,
- $U_v \in \mathbb{R}_+$ is the utility of resource $v \in V$, and
- $\alpha : V^2 \to [0, 1]$ is the influence function, denoted as $\alpha(\cdot, \cdot)$[1], where for all $v \in V$ it holds that $\alpha(v, v) = 0$.

  There are $m \cdot (m - 1)$ different α-values. Intuitively, $\alpha(u, v)$ is the influence of resource u on resource v, and is drawn as an edge from u to v. The influence function can also be thought of as a matrix $\boldsymbol{\alpha} \in [0, 1]^{m \times m}$ with a zero diagonal. When the influence function is constant for all pairs of different vertices we simply use $\alpha \in [0, 1]$ as the influence.

A strategy profile $\vec{\sigma} = (\sigma_1, \dots, \sigma_p)$ consists of each players strategy. A player $i \in P$ has strategy $\sigma_i = \{\sigma_{i,v} \in [0, s_i] \mid v \in V\}$, where i can spend no more than s_i, i.e. $\sum_{v \in V} \sigma_{i,v} \leq s_i$. The influence $I_{i,v}$ of player $i \in P$ at resource $v \in V$ is defined as $I_{i,v} = \sigma_{i,v} + \sum_{t \in V} \alpha(t, v) \cdot \sigma_{i,v}$.

Going forward, the word "influence" will be used in the context of the influence α. To refer to the influence of some player at some resource it will always be explicitly stated, for instance the phrase "influence of player i at resource v" for $I_{i,v}$.

Definition 3.2 (Player utility)**.** The utility of player $i \in P$ at resource $v \in V$ is

$$u_{i,v} = U_v \cdot \frac{I_{i,v}}{\sum_{k \in P} I_{k,v}},$$

and the total utility of player i is $u_i = \sum_{v \in V} u_{i,v}$.

[1] We slightly abuse notation and write $\alpha(\cdot, \cdot)$ instead of $\alpha((\cdot, \cdot))$.

The utility of a player is what is typically called the *payoff function*. We will assume that $\sum_{v \in V} \sigma_{i,v} = s_i$, because not spending the full budget is not in the best interest of player $i \in P$ who is trying to maximize their utility.

Definition 3.3 (Symmetric players)**.** A set of players $P' \subseteq P$ is *symmetric* if each player in P' has the same budget s, that is, $s = s_i$ for all $i \in P'$.

When examining the case of two resources, we consider $V = \{v, t\}$ as the set of vertices. For three resources, $V = \{u, v, t\}$, and for the general case of m resources, the set of vertices is $V = \{v_1, \ldots, v_m\}$. For two players we have $P = \{i, j\}$, and for p players $P = \{1, \ldots, p\}$. A resource competing game with $m = 4$ resources and constant influence α is represented as shown in Figure 3.1.

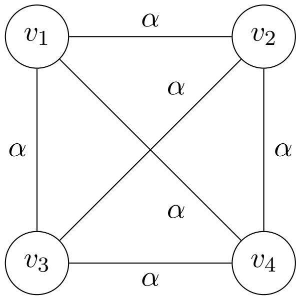

Figure 3.1: Resource competing game with four resources $\{v_1, v_2, v_3, v_4\}$ and constant influence $\alpha \in [0, 1]$.

Next, some of the most basic concepts and notions of game theory that will appear throughout the thesis are established. The following definitions are partly taken from or inspired by [14].

Definition 3.4 (Social utility)**.** The *social utility (SU)* is the sum of all players' utilities:

$$SU := \sum_{i \in P} u_i$$

Definition 3.5 (Social optimum)**.** The *social optimum (SO)* is the strategy profile that maximizes the social utility. The maximized value itself is also called social optimum.

Definition 3.6 (Pure Nash equilibrium)**.** A *pure Nash equilibrium (PNE)* is a strategy profile in which no player can improve their utility by unilaterally changing their strategy.

Chapter 4

Properties of Pure Nash Equilibria

In this chapter we will show that if a pure Nash equilibrium as defined in Definition 3.6 exists, and two players are in such a pure Nash equilibrium, then the ratio of how the two players spend their budget on each resource is equal. All statements of this chapter are proven for the case where the influence α is constant. However, the proofs also hold analogously for the general case of the influence α. Note that we consider only the case where $\alpha \in [0, 1)$, since, as will be proven in Sections 5.2.1 and 5.2.2, for $\alpha = 1$ every strategy profile is a pure Nash equilibrium and thus no meaningful statement about the spending ratio in a pure Nash equilibrium can be made.

4.1 Two Resources with Two Players

First, we consider the case for two resources $V = \{v, t\}$ and two players $P = \{i, j\}$, as depicted in Figure 4.1.

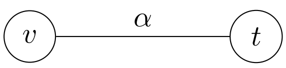

Figure 4.1: Resource competing game with two resources v and t.

Definition 4.1. We define the notation (σ_i, σ_j) for the strategy profile where player i spends σ_i and player j spends σ_j on resource v, and each player spends the rest of their budget on resource t. Further, $u_k(\sigma_i, \sigma_j)$ is defined to be the utility of player $k \in \{i, j\}$ in (σ_i, σ_j).

4.1.1 Symmetric Players

Lemma 4.2. *For any pure Nash equilibrium* (σ', σ'') *it holds that*

$$u_i(\sigma', \sigma'') = u_j(\sigma', \sigma'') = \frac{1}{2}(U_v + U_t).$$

Proof. We first show the equality of the utilities and then prove that they are both equal to $\frac{1}{2}(U_v + U_t)$.

Assume that the equality does not hold, i.e. that (σ', σ'') is a pure Nash equilibrium but $u_i(\sigma', \sigma'') \neq u_j(\sigma', \sigma'')$. We distinguish the two cases

- $u_i(\sigma', \sigma'') < u_j(\sigma', \sigma'')$. By Definition 3.1 it holds that $u_i(\sigma', \sigma'') + u_j(\sigma', \sigma'') = U_v + U_t$. Therefore, we must have

$$u_i(\sigma', \sigma'') < \frac{1}{2}(U_v + U_t) \quad \text{and} \quad u_j(\sigma', \sigma'') > \frac{1}{2}(U_v + U_t)$$

  This means that player i will want to change their strategy to σ'', so that $u_i(\sigma'', \sigma'') = u_i(\sigma'', \sigma'') = \frac{1}{2}(U_v + U_t)$, according to Lemma 4.3.

- $u_i(\sigma', \sigma'') > u_j(\sigma', \sigma'')$. This case is analogous to the one above, where player j will want to change their strategy to σ'.

Since in both cases some player can improve their utility by changing their strategy unilaterally, (σ', σ'') cannot have been a pure Nash equilibrium, and we reach a contradiction.

It follows directly from Definition 3.1 that $u_i(\sigma', \sigma'') + u_j(\sigma', \sigma'') = U_v + U_t$, meaning the utility is equally split, that is,

$$u_i(\sigma', \sigma'') = u_j(\sigma', \sigma'') = \frac{1}{2}(U_v + U_t).$$

□

Lemma 4.3. *For any* $\sigma \in [0, s]$ *it holds that*

$$u_i(\sigma, \sigma) = u_j(\sigma, \sigma) = \frac{1}{2}(U_v + U_t).$$

Proof. If both players follow the exact same strategies, it follows that $I_{i,v} = I_{j,v}$ and $I_{i,t} = I_{j,t}$. The claim then directly follows from Definition 3.1, since $u_{k,u} = \frac{1}{2}U_u$ for $k \in \{i, j\}$ and $u \in \{v, t\}$. □

Theorem 4.4. *If two symmetric players* i *and* j *are in a pure Nash equilibrium and* $\alpha \in [0, 1)$, *their spendings on each resource is equal:*

$$\sigma_{i,v} = \sigma_{j,v} \iff \sigma_{i,t} = \sigma_{j,t}.$$

Proof. Assume that there is a pure Nash equilibrium (σ', σ'') where $\sigma' \neq \sigma''$.

Being in a pure Nash equilibrium, we know that neither player i nor player j can improve their utility by unilaterally changing their strategies. Consequently, given player j plays σ'', it is the optimal choice for player i to play σ'.

Given that player j spends σ'' on v, suppose that player i spends either σ' or σ'' on v. For both cases u_i is the same, i.e. $u_i(\sigma', \sigma'') = u_i(\sigma'', \sigma'')$. To demonstrate this we can consider the two cases in which this is not true, and show why they are not viable:

- $u_i(\sigma', \sigma'') < u_i(\sigma'', \sigma'')$ is not possible, since (σ', σ'') is a pure Nash equilibrium.
- Assuming $u_i(\sigma', \sigma'') > u_i(\sigma'', \sigma'')$, it follows that

$$\frac{1}{2}(U_v + U_t) \overset{(L4.2)}{=} u_i(\sigma', \sigma'') > u_i(\sigma'', \sigma'') \overset{(L4.3)}{=} \frac{1}{2}(U_v + U_t),$$

  reaching a contradiction.

We know that u_i, a function of the spending of i on v given a fixed spending of j on v, is continuous by Definition 3.1. Further, one can see that u_i is not constant, given that $\alpha \neq 1$. Therefore, there must exist a $\sigma^\star$ so that playing $(\sigma^\star, \sigma'')$ yields a better utility for player i. Therefore (σ', σ'') cannot have been a pure Nash equilibrium, and we reach a contradiction. □

4.1.2 Asymmetric Players

Lemma 4.5. *For any pure Nash equilibrium (σ_i, σ_j) it holds that*

$$u_i(\sigma_i, \sigma_j) = \frac{s_i}{s_i + s_j}(U_v + U_t) \quad \textit{and} \quad u_j(\sigma_i, \sigma_j) = \frac{s_j}{s_i + s_j}(U_v + U_t).$$

Proof. Assume that (σ_i, σ_j) is a pure Nash equilibrium, but $u_i(\sigma_i, \sigma_j) \neq \frac{s_i}{s_i+s_j}(U_v + U_t)$. We distinguish the two cases:

- $u_i(\sigma_i, \sigma_j) < \frac{s_i}{s_i+s_j}(U_v + U_t)$. Player i can change their strategy to $\frac{\sigma_j}{s_j}s_i$ in order to improve their utility to $u_i\left(\frac{\sigma_j}{s_j}s_i, \sigma_j\right) = \frac{s_i}{s_i+s_j}(U_v + U_t)$, according to Lemma 4.6.
- $u_i(\sigma_i, \sigma_j) > \frac{s_i}{s_i+s_j}(U_v + U_t) \Leftrightarrow u_j(\sigma_i, \sigma_j) < \frac{s_j}{s_i+s_j}(U_v + U_t)$. This case is analogous to the one above, where player j will want to change their strategy to $\frac{\sigma_i}{s_i}s_j$.

Since in both cases some player can improve their utility by changing their strategy unilaterally, (σ_i, σ_j) cannot have been a pure Nash equilibrium, and we reach a contradiction. □

Lemma 4.6. *For any $\sigma_i \in [0, s_i]$ and $\sigma_j \in [0, s_j]$ where $\frac{\sigma_i}{s_i} = \frac{\sigma_j}{s_j}$ it holds that*

$$u_i(\sigma_i, \sigma_j) = \frac{s_i}{s_i + s_j}(U_v + U_t) \quad \text{and} \quad u_j(\sigma_i, \sigma_j) = \frac{s_j}{s_i + s_j}(U_v + U_t).$$

Proof. For any $\sigma_i \in [0, s_i]$ and $\sigma_j \in [0, s_j]$ where $\frac{\sigma_i}{s_i} = \frac{\sigma_j}{s_j}$, the following holds for the influence of player i at v:

$$\begin{aligned} I_{j,v} &= \sigma_{j,v} + \alpha \cdot \sigma_{j,t} \\ &= \frac{\sigma_{i,v}}{s_i} s_j + \alpha \cdot \frac{\sigma_{i,t}}{s_i} s_j \\ &= \frac{s_j}{s_i}\left(\sigma_{i,v} + \alpha \cdot \sigma_{i,t}\right) \\ &= \frac{s_j}{s_i} \cdot I_{i,v} \end{aligned} \tag{4.1}$$

For the utility of player i at v, we have:

$$\begin{aligned} u_{i,v}(\vec{\sigma_i}, \vec{\sigma_j}) &= U_v \cdot \frac{I_{i,v}}{I_{i,v} + I_{j,v}} \\ &\overset{(4.1)}{=} U_v \cdot \frac{I_{i,v}}{I_{i,v} + \frac{s_j}{s_i} \cdot I_{i,v}} \\ &= U_v \cdot \frac{1}{1 + \frac{s_j}{s_i}} \\ &= U_v \cdot \frac{s_i}{s_i + s_j} \end{aligned}$$

Analogously, it holds that $u_{i,t}(\sigma_i, \sigma_j) = U_t \cdot \frac{s_i}{s_i + s_j}$. Therefore, we have

$$u_i(\sigma_i, \sigma_j) = u_{i,v}(\sigma_i, \sigma_j) + u_{i,t}(\sigma_i, \sigma_j) = \frac{s_i}{s_i + s_j}(U_v + U_t).$$

□

Theorem 4.7. *If two players i and j are in a pure Nash equilibrium and $\alpha \in [0, 1)$, the ratio of how they distribute their respective budget across resources v and t is equal:*

$$\frac{\sigma_{i,v}}{s_i} = \frac{\sigma_{j,v}}{s_j} \quad \text{and} \quad \frac{\sigma_{i,t}}{s_i} = \frac{\sigma_{j,t}}{s_j}.$$

Proof. Assume that there is a pure Nash equilibrium (σ', σ'') where $\frac{\sigma'}{s_i} \neq \frac{\sigma''}{s_j}$.

Being in a pure Nash equilibrium, we know that neither player i nor player j can improve their utility by unilaterally changing their strategies. Consequently, given player j plays σ'', it is the optimal choice for player i to play σ'.

Given that player j spends σ'' on v, suppose that player i spends either σ' or $\frac{\sigma''}{s_j}s_i$ on v. For both cases u_i is the same, i.e. $u_i(\sigma', \sigma'') = u_i\left(\frac{\sigma''}{s_j}s_i, \sigma''\right)$. To demonstrate this, we can consider the two cases in which this is not true, and show why they are not viable:

- $u_i(\sigma', \sigma'') < u_i\left(\frac{\sigma''}{s_j}s_i, \sigma''\right)$ is not possible, since (σ', σ'') is a pure Nash equilibrium.
- Assuming $u_i(\sigma', \sigma'') > u_i\left(\frac{\sigma''}{s_j}s_i, \sigma''\right)$, it follows that

$$\frac{s_i}{s_i + s_j}(U_v + U_t) \overset{(L4.5)}{=} u_i(\sigma', \sigma'') > u_i\left(\frac{\sigma''}{s_j}s_i, \sigma''\right) \overset{(L4.6)}{=} \frac{s_i}{s_i + s_j}(U_v + U_t),$$

  reaching a contradiction.

We know that u_i, a function of the spending of i on v given a fixed spending of j on v, is continuous by Definition 3.1. Further, one can see that u_i is not constant, given that $\alpha \neq 1$. Therefore, there must exist a $\sigma^\star$ so that playing $(\sigma^\star, \sigma'')$ yields a better utility for player i. Therefore (σ', σ'') cannot have been a pure Nash equilibrium, and we reach a contradiction. $\square$

4.2 More Resources with Two Players

We now consider the case of a graph with m resources $V = \{v_1, \ldots, v_m\}$ and two players $P = \{i, j\}$. The neighborhood for resource $v \in V$ is $N(v) = V \setminus \{v\}$, constituting a complete graph.

Definition 4.8. We define the notation $(\vec{\sigma_i}, \vec{\sigma_j})$ with

$$\vec{\sigma_i} = (\sigma_{i,v_1}, \ldots, \sigma_{i,v_{m-1}})^\top \in [0, s_i]^{m-1}$$

and $\vec{\sigma_j} = (\sigma_{i,v_1}, \ldots, \sigma_{i,v_{m-1}})^\top \in [0, s_j]^{m-1}$ for the strategy profile where player $k \in P$ spends $\sigma_{k,v}$ on resource $v \in V$, where $\sigma_{k,v_m} = s_k - \sum_{v \in V \setminus \{v_m\}} \sigma_{k,v}$. Further, $u_k(\vec{\sigma_i}, \vec{\sigma_j})$ is defined to be the utility of player $k \in P$ in $(\vec{\sigma_i}, \vec{\sigma_j})$.

4.2.1 Symmetric Players

Lemma 4.9. *For any pure Nash equilibrium $(\vec{\sigma_i}, \vec{\sigma_j})$ it holds that*

$$u_i(\vec{\sigma_i}, \vec{\sigma_j}) = u_j(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{1}{2}\sum_{v \in V} U_v$$

Proof. We first show the equality of the utilities and then prove that they are both equal to $\frac{1}{2}\sum_{v\in V} U_v$.

Assume that the equality does not hold, i.e. that $(\vec{\sigma_i}, \vec{\sigma_j})$ is a pure Nash equilibrium but $u_i(\vec{\sigma_i}, \vec{\sigma_j}) \neq u_j(\vec{\sigma_i}, \vec{\sigma_j})$. We distinguish the two cases

- $u_i(\vec{\sigma_i}, \vec{\sigma_j}) < u_j(\vec{\sigma_i}, \vec{\sigma_j})$. By Definition 3.1 it holds that $u_i(\vec{\sigma_i}, \vec{\sigma_j}) + u_j(\vec{\sigma_i}, \vec{\sigma_j}) = \sum_{v\in V} U_v$. Therefore, we must have

$$u_i(\vec{\sigma_i}, \vec{\sigma_j}) < \frac{1}{2}\sum_{v\in V} U_v \quad \text{and} \quad u_j(\vec{\sigma_i}, \vec{\sigma_j}) > \frac{1}{2}\sum_{v\in V} U_v$$

  This means that player i will want to change their strategy to $\vec{\sigma_j}$, so that $u_i(\vec{\sigma_i}, \vec{\sigma_j}) = u_j(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{1}{2}\sum_{v\in V} U_v$, according to Lemma 4.10.

- $u_i(\vec{\sigma_i}, \vec{\sigma_j}) > u_j(\vec{\sigma_i}, \vec{\sigma_j})$. This case is analogous to the one above, where player j will want to change their strategy to $\vec{\sigma_i}$.

Since in both cases some player can improve their utility by changing their strategy unilaterally, $(\vec{\sigma_i}, \vec{\sigma_j})$ cannot have been a pure Nash equilibrium, and we reach a contradiction.

It follows directly from Definition 3.1 that $u_i(\vec{\sigma_i}, \vec{\sigma_j}) + u_j(\vec{\sigma_i}, \vec{\sigma_j}) = \sum_{v\in V} U_v$, meaning the utility is equally split, that is,

$$u_i(\vec{\sigma_i}, \vec{\sigma_j}) = u_j(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{1}{2}\sum_{v\in V} U_v.$$

□

Lemma 4.10. *For any $\vec{\sigma} \in [0, s]^{m-1}$ it holds that*

$$u_i(\vec{\sigma}, \vec{\sigma}) = u_j(\vec{\sigma}, \vec{\sigma}) = \frac{1}{2}\sum_{v\in V} U_v.$$

Proof. If both players follow the exact same strategies, it follows that $I_{i,v} = I_{j,v}$ for all $v \in V$. The claim then directly follows from Definition 3.1, since $u_{k,v} = \frac{1}{2}U_v$ for $k \in P$ and $v \in V$. □

Theorem 4.11. *If two symmetric players i and j are in a pure Nash equilibrium $(\vec{\sigma_i}, \vec{\sigma_j})$ and $\alpha \in [0, 1)$, their spending on each resource is equal:*

$$\vec{\sigma_i} = \vec{\sigma_j}.$$

Proof. Assume that there is a pure Nash equilibrium $(\vec{\sigma_i}, \vec{\sigma_j})$ where $\vec{\sigma_i} \neq \vec{\sigma_j}$.

Being in a pure Nash equilibrium, we know that neither player i nor player j can improve their utility by unilaterally changing their strategies. Consequently, given player j plays $\vec{\sigma_j}$, it is the optimal choice for player i to play $\vec{\sigma_i}$.

Given that player j plays $\vec{\sigma_j}$, suppose that player i plays either $\vec{\sigma_i}$ or $\vec{\sigma_j}$. For both cases u_i is the same, i.e. $u_i(\vec{\sigma_i}, \vec{\sigma_j}) = u_i(\vec{\sigma_j}, \vec{\sigma_j})$. To demonstrate this, we can consider the two cases in which this is not true, and show why they are not viable:

- $u_i(\vec{\sigma_i}, \vec{\sigma_j}) < u_i(\vec{\sigma_j}, \vec{\sigma_j})$ is not possible, since $u_i(\vec{\sigma_i}, \vec{\sigma_j})$ is a pure Nash equilibrium.
- Assuming $u_i(\vec{\sigma_i}, \vec{\sigma_j}) > u_i(\vec{\sigma_j}, \vec{\sigma_j})$, it follows that

$$\frac{1}{2}\sum_{v\in V} U_v \overset{(L4.9)}{=} u_i(\vec{\sigma_i}, \vec{\sigma_j}) > u_i(\vec{\sigma_j}, \vec{\sigma_j}) \overset{(L4.10)}{=} \frac{1}{2}\sum_{v\in V} U_v,$$

  reaching a contradiction.

We know that u_i, a function of the spending of i on v given a fixed spending of j on v, is continuous by Definition 3.1. Further, one can see that u_i is not constant, given that $\alpha \neq 1$. Therefore, there must exist a $\vec{\sigma}^\star$ so that playing $(\vec{\sigma}^\star, \vec{\sigma_j})$ yields a better utility for player i. Therefore $(\vec{\sigma_i}, \vec{\sigma_j})$ cannot have been a pure Nash equilibrium, and we reach a contradiction. □

4.2.2 Asymmetric Players

Lemma 4.12. *For any pure Nash equilibrium $(\vec{\sigma_i}, \vec{\sigma_j})$ it holds that*

$$u_i(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{s_i}{s_i + s_j}\sum_{v\in V} U_v \quad \textit{and} \quad u_j(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{s_j}{s_i + s_j}\sum_{v\in V} U_v$$

Proof. Assume that $(\vec{\sigma_i}, \vec{\sigma_j})$ is a pure Nash equilibrium, but $u_i(\vec{\sigma_i}, \vec{\sigma_j}) \neq \frac{s_i}{s_i+s_j}\sum_{v\in V} U_v$. We distinguish the two cases

- $u_i(\vec{\sigma_i}, \vec{\sigma_j}) < \frac{s_i}{s_i+s_j}\sum_{v\in V} U_v$. Player i can change their strategy to $\frac{s_i}{s_j}\vec{\sigma_j}$ in order to improve their utility to $u_i\left(\frac{s_i}{s_j}\vec{\sigma_j}, \vec{\sigma_j}\right) = \frac{s_i}{s_i+s_j}\sum_{v\in V} U_v$, according to Lemma 4.13.
- $u_i(\vec{\sigma_i}, \vec{\sigma_j}) > \frac{s_i}{s_i+s_j}\sum_{v\in V} U_v \Leftrightarrow u_j(\vec{\sigma_i}, \vec{\sigma_j}) < \frac{s_j}{s_i+s_j}\sum_{v\in V} U_v$. This case is analogous to the one above, where player j will want to change their strategy to $\frac{s_j}{s_i}\vec{\sigma_i}$.

Since in both cases some player can improve their utility by changing their strategy unilaterally, $(\vec{\sigma_i}, \vec{\sigma_j})$ cannot have been a pure Nash equilibrium, and we reach a contradiction. □

Lemma 4.13. *For any $\vec{\sigma_i} \in [0, s_i]^{m-1}$ and $\vec{\sigma_j} \in [0, s_j]^{m-1}$ where $\frac{1}{s_i}\vec{\sigma_i} = \frac{1}{s_j}\vec{\sigma_j}$ it holds that*

$$u_i(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{s_i}{s_i + s_j} \sum_{v \in V} U_v \quad \textit{and} \quad u_j(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{s_j}{s_i + s_j} \sum_{v \in V} U_v$$

Proof. Let $\vec{\sigma_i} \in [0, s_i]^{m-1}$ and $\vec{\sigma_j} \in [0, s_j]^{m-1}$ where

$$\frac{1}{s_i}\vec{\sigma_i} = \frac{1}{s_j}\vec{\sigma_j} \iff \forall v \in V : \frac{\sigma_{i,v}}{s_i} = \frac{\sigma_{j,v}}{s_j}$$

Let $v \in V$ be arbitrary. For the influence, it holds that:

$$\begin{aligned} I_{j,v} &= \sigma_{j,v} + \sum_{v' \in V \setminus \{v\}} \alpha \cdot \sigma_{j,v'} \\ &= \frac{\sigma_{i,v}}{s_i} s_j + \sum_{v' \in V \setminus \{v\}} \alpha \cdot \frac{\sigma_{i,v'}}{s_i} s_j \\ &= \frac{s_j}{s_i} \left(\sigma_{i,v} + \sum_{v' \in V \setminus \{v\}} \alpha \cdot \sigma_{i,v'} \right) \\ &= \frac{s_j}{s_i} \cdot I_{i,v} \end{aligned} \tag{4.2}$$

For the utility of player i at v, we have

$$\begin{aligned} u_{i,v}(\vec{\sigma_i}, \vec{\sigma_j}) &= U_v \cdot \frac{I_{i,v}}{I_{i,v} + I_{j,v}} \\ &\overset{(4.2)}{=} U_v \cdot \frac{I_{i,v}}{I_{i,v} + \frac{s_j}{s_i} \cdot I_{i,v}} \\ &= U_v \cdot \frac{1}{1 + \frac{s_j}{s_i}} \\ &= U_v \cdot \frac{s_i}{s_i + s_j} \end{aligned}$$

Therefore, we have

$$u_i(\vec{\sigma_i}, \vec{\sigma_j}) = \sum_{v \in V} u_{i,v}(\vec{\sigma_i}, \vec{\sigma_j}) = \frac{s_i}{s_i + s_j} \sum_{v \in V} U_v.$$

□

Theorem 4.14. *If two players i and j are in a pure Nash equilibrium $(\vec{\sigma_i}, \vec{\sigma_j})$ and $\alpha \in [0, 1)$, the ratio of how they distribute their respective budget across the resources is equal:*

$$\frac{1}{s_i}\vec{\sigma_i} = \frac{1}{s_j}\vec{\sigma_j}.$$

Proof. Assume that there is a pure Nash equilibrium $(\vec{\sigma_i}, \vec{\sigma_j})$ where $\frac{1}{s_i}\vec{\sigma_i} \neq \frac{1}{s_j}\vec{\sigma_j}$.

Being in a pure Nash equilibrium, we know that neither player i nor player j can improve their utility by unilaterally changing their strategies. Consequently, given player j plays $\vec{\sigma_j}$, it is the optimal choice for player i to play $\vec{\sigma_i}$.

Given that player j plays $\vec{\sigma_j}$, suppose that player i plays either $\vec{\sigma_i}$ or $\frac{s_i}{s_j}\vec{\sigma_j}$. For both cases u_i is the same, i.e. $u_i(\vec{\sigma_i}, \vec{\sigma_j}) = u_i\left(\frac{s_i}{s_j}\vec{\sigma_j}, \vec{\sigma_j}\right)$. To demonstrate this, we can consider the two cases in which this is not true, and show why they are not viable:

- $u_i(\vec{\sigma_i}, \vec{\sigma_j}) < u_i\left(\frac{s_i}{s_j}\vec{\sigma_j}, \vec{\sigma_j}\right)$ is not possible, since $u_i(\vec{\sigma_i}, \vec{\sigma_j})$ is a pure Nash equilibrium.
- Assuming $u_i(\vec{\sigma_i}, \vec{\sigma_j}) > u_i\left(\frac{s_i}{s_j}\vec{\sigma_j}, \vec{\sigma_j}\right)$, it follows that

  $$\frac{s_i}{s_i + s_j}\sum_{v\in V} U_v \overset{(L4.12)}{=} u_i(\vec{\sigma_i}, \vec{\sigma_j}) > u_i\left(\frac{s_i}{s_j}\vec{\sigma_j}, \vec{\sigma_j}\right) \overset{(L4.13)}{=} \frac{s_i}{s_i + s_j}\sum_{v\in V} U_v,$$

  reaching a contradiction.

We know that u_i, a function of the spending of i on v given a fixed spending of j on v, is continuous by Definition 3.1. Further, one can see that u_i is not constant, given that $\alpha \neq 1$. Therefore, there must exist a $\vec{\sigma}^\star$ so that playing $(\vec{\sigma}^\star, \vec{\sigma_j})$ yields a better utility for player i. Therefore $(\vec{\sigma_i}, \vec{\sigma_j})$ cannot have been a pure Nash equilibrium, and we reach a contradiction. $\square$

Chapter 5

Constant Influence

In this chapter we consider the influence α to be constant. This means that for all pairs of different resources the influence is equal. The influence α is therefore considered a real number in $[0, 1]$ and not a function.

5.1 Simple Resource Competing Game

For this section we assume both the numbers of resources and the number of players to be arbitrary, that is, we have $V = \{v_1, \dots, v_m\}$ and $P = \{1, \dots, p\}$.

Definition 5.1 (Simple resource competing game)**.** A *simple* resource competing game is a resource competing game with constant influence $\alpha = 0$.

A simple resource competing game is the most basic version of a resource competing game, because it has no influence whatsoever. That is, the spending on a resource has no effect on the utility at any other resource.

Theorem 5.2. *For a simple resource competing game there always exists a pure Nash equilibrium.*

Proof. We will show that the strategy profile where player $i \in P$ spends

$$\sigma_{i,v} = \frac{U_v}{\sum_{t \in V} U_t} \cdot s_i \tag{5.1}$$

on resource $v \in V$ is a pure Nash equilibrium (intuitively, at resource $v \in V$ player $i \in P$ invests the same fraction of their budget as the utility of v compared to the total utility of all resources in the game).

For player $i \in P$ and resource $v \in V$ we have $I_{i,v} = \sigma_{i,v}$ since $\alpha = 0$. The utility of player i is:

$$u_i = \sum_{v \in V} u_{i,v} = \sum_{v \in V} U_v \frac{I_{i,v}}{\sum_{k \in P} I_{k,v}} = \sum_{v \in V} U_v \frac{\sigma_{i,v}}{\sum_{k \in P} \sigma_{k,v}} \tag{5.2}$$

Suppose that all players play the strategy from Equation 5.1. It holds that

$$\frac{\sigma_{i,v}}{\sum_{k\in P}\sigma_{k,v}} = \frac{s_i}{\sum_{k\in P} s_k} \tag{5.3}$$

This is essentially a reiteration of the result shown in Lemma 4.13. Assume that some player $i \in P$ can improve their utility by moving some amount $\delta > 0$ from resource $v \in V$ to resource $t \in V$. It must hold that $U_v > 0$, because otherwise no spending can be moved, and that

$$\delta \le \sigma_{i,v} = \frac{U_v}{\sum_{u\in V} U_u}\cdot s_i, \tag{5.4}$$

because no more than the full current spending on v can be moved. The redistribution of spendings results in a change of utility Δ_i:

$$\begin{aligned}
\Delta_i &:= u_i|_{\sigma_{i,v}\leftarrow\sigma_{i,v}-\delta,\sigma_{i,t}\leftarrow\sigma_{i,t}+\delta} - u_i \\
&= u_{i,v}|_{\sigma_{i,v}\leftarrow\sigma_{i,v}-\delta} + u_{i,t}|_{\sigma_{i,t}\leftarrow\sigma_{i,t}-\delta} - u_{i,v} - u_{i,t} \\
&= U_v\left(\frac{\sigma_{i,v}-\delta}{\left(\sum_{k\in P}\sigma_{k,v}\right)-\delta} - \frac{\sigma_{i,v}}{\sum_{k\in P}\sigma_{k,v}}\right) + U_t\left(\frac{\sigma_{i,t}+\delta}{\left(\sum_{k\in P}\sigma_{k,t}\right)+\delta} - \frac{\sigma_{i,t}}{\sum_{k\in P}\sigma_{k,t}}\right) \\
&\overset{(5.3)}{=} U_v\left(\frac{\sigma_{i,v}-\delta}{\left(\sum_{k\in P}\sigma_{k,v}\right)-\delta} - \frac{s_i}{\sum_{k\in P}\sigma_{k,v}}\right) + U_t\left(\frac{\sigma_{i,t}+\delta}{\left(\sum_{k\in P}\sigma_{k,t}\right)+\delta} - \frac{s_i}{\sum_{k\in P}s_k}\right) \\
&= U_v\frac{\sigma_{i,v}-\delta}{\left(\sum_{k\in P}\sigma_{k,v}\right)-\delta} + U_t\frac{\sigma_{i,t}+\delta}{\left(\sum_{k\in P}\sigma_{k,t}\right)+\delta} - \frac{s_i}{\sum_{k\in P}s_k}(U_v+U_t) \\
&= U_v\frac{\frac{U_v}{\sum_{u\in V}U_u}s_i-\delta}{\left(\sum_{k\in P}\frac{U_v}{\sum_{u\in V}U_u}s_k\right)-\delta} + U_t\frac{\frac{U_t}{\sum_{u\in V}U_u}s_i+\delta}{\left(\sum_{k\in P}\frac{U_t}{\sum_{u\in V}U_u}s_k\right)+\delta} - \frac{s_i}{\sum_{k\in P}s_k}(U_v+U_t) \\
&\overset{\Diamond}{=} \frac{\delta^2\left(s_i-\sum_{k\in P}s_k\right)\left(\sum_{u\in V}U_u\right)(U_v+U_t)}{\left(\sum_{k\in P}s_k\right)\left(\delta\left(\sum_{u\in V}U_u\right)+\left(\sum_{k\in P}s_k\right)U_t\right)\left(-\delta\left(\sum_{u\in V}U_u\right)+\left(\sum_{k\in P}s_k\right)U_v\right)}
\end{aligned} \tag{5.5}$$

The step marked with $\Diamond$ was verified with `Mathematica` [2], and the corresponding program can be found here. In this equation, we can observe the following:

$$\frac{\overbrace{\delta^2}^{>0}\overbrace{\left(s_i-\sum_{k\in P}s_k\right)}^{<0}\overbrace{\left(\sum_{u\in V}U_u\right)}^{>0}\overbrace{(U_t+U_v)}^{>0}}{\underbrace{\left(\sum_{k\in P}s_k\right)}_{>0}\underbrace{\left(\delta\left(\sum_{u\in V}U_u\right)+\left(\sum_{k\in P}s_k\right)U_t\right)}_{>0}\underbrace{\left(-\delta\left(\sum_{u\in V}U_u\right)+\left(\sum_{k\in P}s_k\right)U_v\right)}_{<0\,\Leftrightarrow\,\Delta_i>0}} \tag{5.5 revisited}$$

Therefore it holds that

$$\begin{aligned}\Delta_i > 0 &\Leftrightarrow -\delta\left(\sum_{u\in V} U_u\right) + \left(\sum_{k\in P} s_k\right) U_v < 0 \\ &\Leftrightarrow \delta > \frac{U_v}{\sum_{u\in V} U_u} \sum_{k\in P} s_k > \frac{U_v}{\sum_{u\in V} U_u} \cdot s_i = \sigma_{i,v}\end{aligned} \tag{5.6}$$

which stands in direct contradiction with the constraint in Equation 5.4. Therefore, no player can achieve a better utility by making any unilateral move. This means that, if all players spend their budget according to Equation 5.1, a pure Nash equilibrium is reached, thus proving the claim. □

5.2 Symmetric Players

In this section, some parts of [1] concerning the existence of pure Nash equilibria for symmetric players are presented and proven with a new and simpler approach, by utilizing our results from Chapter 4 analyzing the properties of pure Nash equilibria in our model. Specifically, the idea of resource set reduction with a threshold was introduced in [1].

5.2.1 Pure Nash Equilibrium for Two Resources

We consider the same setup with two resources and two players as in Section 4.1 and Figure 4.1, but with symmetric players, meaning that $s := s_i = s_j$.

The following holds for the influence of player $k \in P$ at resource $u \in V$:

$$I_{k,u} = \sigma_{k,u} + \alpha \cdot \sum_{v\in V\setminus\{u\}} \sigma_{k,v} = \sigma_{k,u} + \alpha \cdot (s - \sigma_{k,u}) = (1-\alpha)\cdot\sigma_{k,u} + \alpha \cdot s \tag{5.7}$$

Theorem 5.3. *For a resource competing game with two resources, two symmetric players and $\alpha \in [0, 1]$ there always exists a pure Nash equilibrium.*

Theorem 5.3 is divided into Lemmas 5.4, 5.5 and 5.6 for the cases $\alpha = 0$, $\alpha = 1$ and $\alpha \in (0, 1)$ respectively.

Lemma 5.4. *For a resource competing game with two resources, two symmetric players and $\alpha = 0$, there always exists a pure Nash equilibrium.*

Proof. This is a simple resource competing game with the special case of $m = 2$ resources and $p = 2$ players shown in Theorem 5.2. □

Lemma 5.5. *For a resource competing game with two resources, two symmetric players and $\alpha = 1$, every strategy profile is a pure Nash equilibrium.*

Proof. This is a special case with $p = 2$ players of Lemma 5.9 which is proven in Section 5.2.2. $\square$

For $\alpha \neq 1$, we define

$$\sigma_v^\star := \left(\frac{U_v}{U_v + U_t} + \frac{\alpha}{1-\alpha} \cdot \frac{U_v - U_t}{U_v + U_t} \right) \cdot s \tag{5.8}$$

Note that $\sigma_t^\star$ is defined analogously.

Lemma 5.6. *For a resource competing game with two resources, two symmetric players and $\alpha \in (0,1)$, there always exists a pure Nash equilibrium.*

Proof. We will prove, that if every player $k \in P$ spends $\sigma_{k,u}$ on resource $u \in V$, where

$$\sigma_{k,u} = \begin{cases} 0 & \text{if } \sigma_u^\star < 0 \\ \sigma_u^\star & \text{if } \sigma_u^\star \in [0,s] \\ s & \text{if } \sigma_u^\star > s \end{cases}, \tag{5.9}$$

we are in a pure Nash equilibrium. In this case, we say player k is playing strategy $\sigma^\star$.

We compute the utility of player i at resource v, giving us

$$\begin{aligned} u_{i,v} &= U_v \cdot \frac{I_{i,v}}{\sum_{k \in P} I_{k,v}} \\ &= U_v \cdot \frac{\sigma_{i,v} + \alpha \cdot \sum_{u \in V \setminus \{v\}} \sigma_{i,u}}{\sum_{k \in P} \sigma_{k,v} + \alpha \cdot \sum_{u \in V \setminus \{v\}} \sigma_{k,u}} \\ &= U_v \cdot \frac{\sigma_{i,v} + \alpha \cdot \sigma_{i,t}}{\sum_{k \in P} \sigma_{k,v} + \alpha \cdot \sigma_{k,t}} \\ &= U_v \cdot \frac{\sigma_{i,v} + \alpha \cdot \sigma_{i,t}}{\sigma_{i,v} + \alpha \cdot \sigma_{i,t} + \sigma_{j,v} + \alpha \cdot \sigma_{j,t}} \\ &= U_v \cdot \frac{\sigma_{i,v} + \alpha \cdot (s - \sigma_{i,v})}{\sigma_{i,v} + \alpha \cdot (s - \sigma_{i,v}) + \sigma_{j,v} + \alpha \cdot (s - \sigma_{j,v})} \end{aligned} \tag{5.10}$$

where we used Definition 3.2 in the first and Definition 3.1 in the second step. Similarly, we compute the utility of player i at resource t, giving us

$$\begin{aligned} u_{i,t} &= U_t \cdot \frac{\sigma_{i,t} + \alpha \cdot (s - \sigma_{i,t})}{\sigma_{i,t} + \alpha \cdot (s - \sigma_{i,t}) + \sigma_{j,t} + \alpha \cdot (s - \sigma_{j,t})} \\ &= U_t \cdot \frac{(s - \sigma_{i,v}) + \alpha \cdot \sigma_{i,v}}{(s - \sigma_{i,v}) + \alpha \cdot \sigma_{i,v} + (s - \sigma_{j,v}) + \alpha \cdot \sigma_{j,v}}. \end{aligned} \tag{5.11}$$

In both derivations, we used $\sigma_{i,t} = s - \sigma_{i,v} \iff s - \sigma_{i,t} = \sigma_{i,v}$. The analogous formulas hold for player j.

We inspect the first and second partial derivatives of u_i with respect to $\sigma_{i,v}$.

$$\begin{aligned}
\frac{\partial u_i}{\partial \sigma_{i,v}} &= \frac{\partial}{\partial \sigma_{i,v}}(u_{i,v} + u_{i,t}) \\
&= \frac{\partial u_{i,v}}{\partial \sigma_{i,v}} + \frac{\partial u_{i,t}}{\partial \sigma_{i,v}} \\
&= \frac{\partial}{\partial \sigma_{i,v}} \left(U_v \cdot \frac{\sigma_{i,v} + \alpha \cdot (s - \sigma_{i,v})}{\sigma_{i,v} + \alpha \cdot (s - \sigma_{i,v}) + \sigma_{j,v} + \alpha \cdot (s - \sigma_{j,v})} \right) \\
&\quad + \frac{\partial}{\partial \sigma_{i,v}} \left(U_t \cdot \frac{(s - \sigma_{i,v}) + \alpha \cdot \sigma_{i,v}}{(s - \sigma_{i,v}) + \alpha \cdot \sigma_{i,v} + (s - \sigma_{j,v}) + \alpha \cdot \sigma_{j,v}} \right) \\
&= (1 - \alpha) \left(U_v \cdot \frac{I_{j,v}}{(I_{i,v} + I_{j,v})^2} - U_t \cdot \frac{I_{j,t}}{(I_{i,t} + I_{j,t})^2} \right) \qquad (5.12)
\end{aligned}$$

$$\frac{\partial^2 u_i}{\partial^2 \sigma_{i,v}} = -2(1 - \alpha)^2 \left(U_v \cdot \frac{I_{j,v}}{(I_{i,v} + I_{j,v})^3} + U_t \cdot \frac{I_{j,t}}{(I_{i,t} + I_{j,t})^3} \right) < 0 \qquad (5.13)$$

The influence of a player $k \in P$ on resource v when playing $\sigma^\star$ is

$$\begin{aligned}
I_v^\star := I_{k,v} &= (1 - \alpha) \cdot \sigma_v^\star + \alpha \cdot s \\
&= (1 + \alpha) \cdot s \cdot \frac{U_v}{U_v + U_t} \\
&= \frac{s' \cdot U_v}{U_v + U_t} \qquad (5.14)
\end{aligned}$$

where $s' := (1 + \alpha) \cdot s$. The analogous formula holds for resource t.

Considering the result of Theorem 4.4, we know that when in a pure Nash equilibrium both players must spend the same amount on each resource. Therefore it suffices to find such a strategy which maximizes the utility for a player. Note that the utility functions are equivalent for both players, and we will continue taking the perspective of player i.

With this in mind, suppose that both players play strategy $\sigma^\star$ as described above. Then we have

$$\begin{aligned}
\frac{\partial u_i}{\partial \sigma_{i,v}} &= (1 - \alpha) \left(U_v \cdot \frac{I_v^\star}{(I_v^\star + I_v^\star)^2} - U_t \cdot \frac{I_t^\star}{(I_t^\star + I_t^\star)^2} \right) \\
&= \frac{(1 - \alpha)}{4} \left(\frac{U_v}{I_v^\star} - \frac{U_t}{I_t^\star} \right) \\
&= \frac{(1 - \alpha)}{4} \left(\frac{U_v}{\frac{s' \cdot U_v}{U_v + U_t}} - \frac{U_t}{\frac{s' \cdot U_t}{U_v + U_t}} \right) \\
&= \frac{(1 - \alpha)}{4} \left(\frac{U_v + U_t}{s'} - \frac{U_v + U_t}{s'} \right) = 0
\end{aligned}$$

which is a maximum due to Equation 5.13 being strictly negative, since $\alpha \neq 1$. This shows that the players are in a pure Nash equilibrium if they both play strategy $\sigma^\star$. □

A verification of the computation of derivatives in Lemma 5.6 can be found here in form of a `Mathematica` [2] program.

Alternatively, in the proof of Lemma 5.6 the following equivalent argument could be made: Suppose player j plays strategy $\sigma^\star$. Now we could consider what happens if player i also plays strategy $\sigma^\star$ in response, and this is the global maximum of the utility function. The opposite holds for player j in response to player i. Therefore, in the strategy profile $\vec{\sigma} = (\sigma^\star, \sigma^\star)$ both players are mutually maximized and no player wants to deviate. This is a pure Nash equilibrium. This holds true also for the following proofs of this kind.

The uniqueness of the pure Nash equilibrium in Lemma 5.6 was shown in [1]. It can also easily be argued using the knowledge about the spending ratio in a pure Nash equilibrium gained from Section 4.1 combined with the fact that the utility function is shown to be strictly concave, as the second derivative in Equation 5.13 is strictly negative.

Note that, going forward, we will often write that in a pure Nash equilibrium player $k \in P$ spends $\sigma_u^\star$ on resource $u \in V$ (or $\sigma_{k,u}^\star$ for asymmetric players introduced in Section 5.3), as defined in many proofs throughout Chapters 5 and 6 for many different setups. In this case it is implicitly assumed that the spending is cut off below 0 and above the players budget, as described in the respective proofs.

5.2.2 Pure Nash Equilibrium for m Resources

We now consider the same setup with m resources and two players as in Section 4.2, where the players are symmetric, hence $s := s_i = s_j$.

The following holds for the influence of player $k \in P$ at resource $u \in V$:

$$I_{k,u} = \sigma_{k,u} + \alpha \cdot \sum_{v \in V \setminus \{u\}} \sigma_{k,v} = \sigma_{k,u} + \alpha \cdot (s - \sigma_{k,u}) = (1-\alpha) \cdot \sigma_{k,u} + \alpha \cdot s \quad (5.15)$$

Theorem 5.7. *For a resource competing game with m resources and two symmetric players there always exists a pure Nash equilibrium.*

Theorem 5.7 is divided into Lemmas 5.8, 5.9 and 5.11 for the cases $\alpha = 0$, $\alpha = 1$ and $\alpha \in (0, 1)$ respectively.

Lemma 5.8. *For a resource competing game with m resources and two symmetric players and $\alpha = 0$, there always exists a pure Nash equilibrium.*

Proof. This is a simple resource competing game with the special case of $p = 2$ players shown in Theorem 5.2. □

Lemma 5.9. *For a resource competing game with m resources and two symmetric players and $\alpha = 1$, every strategy profile is a pure Nash equilibrium.*

Proof. For any $i \in P$ and $u \in V$:

$$I_{i,u} = \sigma_{i,u} + \sum_{v \in V \setminus \{u\}} \alpha \cdot \sigma_{i,v} = \sum_{v \in V} \sigma_{i,v} = s \tag{5.16}$$

It follows that

$$u_{i,u} = \frac{U_u \cdot I_{i,u}}{\sum_{k \in P} I_{k,u}} = \frac{U_u \cdot s}{p \cdot s} = \frac{U_u}{p} \tag{5.17}$$

and therefore

$$u_i = \sum_{u \in V} u_{i,u} = \frac{1}{p} \sum_{u \in V} U_u \tag{5.18}$$

which holds independent of the strategy profile. This means that every strategy profile is a pure Nash equilibrium. □

Note that, in fact, Lemma 5.9 holds for any number of symmetric players.

Lemma 5.10. *Consider a resource competing game with m resources and two players i and j. If player i maximizes their utility by playing σ_i in response to player j playing σ_j, and player j maximizes their utility by playing σ_j in response to player i playing σ_i, then the strategy profile $\vec{\sigma} = (\sigma_i, \sigma_j)$ is a pure Nash equilibrium.*

Proof. If players i and j play strategies σ_i and σ_j respectively as above, both are playing a best response, and are mutually maximized. This means that no player can deviate from their strategy to improve their utility. Therefore, it is a pure Nash equilibrium. □

Note that the special case of Lemma 5.10 with symmetric players and $\sigma' = \sigma_i = \sigma_j$ is very simple: If player i maximizes its utlity by playing σ' in response to player j playing σ', then the same holds immediately for opposite players, since the players have the exact same budget.

Lemma 5.11. *For a resource competing game with m resources and two symmetric players and $\alpha \in (0, 1)$, there always exists a pure Nash equilibrium.*

Proof. Constructing the strategy takes two steps. After explaining these steps, we will proceed to show the correctness of the constructed strategy.

Constructing the strategy The construction consists of building a set V', and then forming strategies consisting of the spendings of a player on the resources in V'.

Step 1: Building the set V' We reduce the set V of resources to a set $V' \subseteq V$ as outlined in Algorithm 5.1. The threshold $T(V')$ used in the algorithm is defined as follows:

$$T\left(V'\right) := \frac{\alpha}{1 + (|V'|-1) \cdot \alpha} \cdot \sum_{t \in V'} U_t \tag{5.19}$$

Algorithm 5.1: Building the set V'

```
V' ← V;
removed ← true;
while removed do
    removed ← false;
    for v ∈ V' do
        if U_v < T(V') then
            V' ← V' \ {v};
            removed ← true;
```

We show that the threshold strictly increases with every resource removed from V'. For this, consider in some step the resource v' being removed from V'.

$$\begin{aligned} T\left(V' \setminus \{v'\}\right) &= \frac{\alpha}{1 + (|V'|-2) \cdot \alpha} \cdot \left(\sum_{t \in V'} U_t - U_{v'} \right) \\ &> \frac{\alpha}{1 + (|V'|-2) \cdot \alpha} \cdot \left(\sum_{t \in V'} U_t - T\left(V'\right) \right) \\ &= \frac{\alpha}{1 + (|V'|-1) \cdot \alpha} \cdot \sum_{t \in V'} U_t = T\left(V'\right) \end{aligned}$$

Thus we see that, *after termination*, the utilities of all removed resources are all strictly smaller than the last threshold: $\forall v \in V \setminus \{V'\} : U_v < T(V')$.

To dismiss the case where it could occur that, *after termination*, $V' = \varnothing$, suppose we are executing the algorithm and have only one resource left in $V' = \{v\}$, that is, we have removed a resource in exactly $m-1$ prior steps. In the next step we remove the last resource v if

$$U_v < T\left(V'\right) = \alpha \cdot U_v, \tag{5.20}$$

which never holds. Thus the algorithm will terminate after at most $m - 1$ steps and the set V' will be non-empty.

Step 2: Strategies We assign a spending to each of the resources $v \in V'$, defined as

$$\sigma_v^\star := \left(\frac{U_v}{\sum_{t \in V'} U_t} + \frac{\alpha}{1-\alpha} \cdot \frac{|V'| \cdot U_v - \sum_{t \in V'} U_t}{\sum_{t \in V'} U_t} \right) \cdot s. \tag{5.21}$$

For the resources not in V', the spending is 0. Together, these spendings form the strategy $\sigma^\star$.

Correctness Proving correctness proceeds in two steps. First, we show that the strategy $\sigma^\star$ fulfills the required properties. Then, we will show that we are in a pure Nash equilibrium if both players play strategy $\sigma^\star$.

Properties of $\sigma^\star$ To ensure that the spendings are valid, it must hold that for any resource $v \in V$, we have $\sigma_v^\star \geq 0$. By construction of $\sigma_v^\star$, the sum $\sum_{v \in V} \sigma_v^\star$ is equal to s.

For the nodes $v \in V \setminus V'$, we have $\sigma_v^\star = 0 \geq 0$. The influence of a player $k \in P$ on resource $v \in V'$ when playing $\sigma^\star$ is

$$I_v^\star := I_{k,v} = \frac{s' \cdot U_v}{\sum_{t \in V'} U_t} \tag{5.22}$$

where $s' = (1 + (m-1) \cdot \alpha) \cdot s$.

We know that the following holds for the threshold from Equation 5.19:

$$\forall v \in V' : U_v \geq T\left(V'\right)$$

Thus, for $v \in V'$ we have

$$\begin{aligned} I_v^\star &= \frac{s' \cdot U_v}{\sum_{t \in V'} U_t} \\ &\geq \frac{s'}{\sum_{t \in V'} U_t} \cdot T\left(V'\right) \\ &= \frac{(1 + (|V'|-1) \cdot \alpha) \cdot s}{\sum_{t \in V'} U_t} \cdot \frac{\alpha}{1 + (|V'|-1) \cdot \alpha} \cdot \sum_{t \in V'} U_t \\ &= \alpha \cdot s \end{aligned}$$

Due to the definition of the influence in Definition 3.1 it follows that $\sigma_v^\star \geq 0$.

Equilibrium From Lemma 5.10 we know that if the best response for a player is to play $\sigma^\star$ if the adversary is playing $\sigma^\star$, then the strategy profile where both players have strategy $\sigma^\star$ is a pure Nash equilibrium. Thus it suffices to show that when both players play $\sigma^\star$ no player wants to change any of their spendings.

Suppose both players are playing the strategy $\sigma^\star$, meaning that $\sigma_{i,v} = \sigma_{j,v} = \sigma^\star_v$ for all $v \in V'$. Assume that player i can increase their utility by moving some amount δ from a resource $v \in V$ to a resource $t \in V$. The following observations can be made analogously for player j. We distinguish four cases:

1. $v \in V \setminus V'$, $t \in V \setminus V'$, which we can immediately dismiss because there is no spending to move away from v,
2. $v \in V \setminus V'$, $t \in V'$, for which the same applies as above,
3. $v \in V'$, $t \in V \setminus V'$, and finally
4. $v \in V'$, $t \in V'$.

We treat cases 3 and 4 together, thus assuming $v \in V'$ and $t \in V$. Moving some amount $\delta > 0$ will result in some change of utility Δ_i. We additionally must assume that

$$\delta \leq \sigma_{i,v} = \sigma^\star_v, \tag{5.23}$$

otherwise we cannot move δ from v to t.

For the change of utility Δ_i it holds that

$$\begin{aligned}
\Delta_i &:= u_i|_{\sigma_{i,v}\leftarrow\sigma_{i,v}-\delta,\sigma_{i,t}\leftarrow\sigma_{i,t}+\delta} - u_i \\
&= \left(\sum_{u\in V} u_{i,u}\right)\Bigg|_{\sigma_{i,v}\leftarrow\sigma_{i,v}-\delta,\sigma_{i,t}\leftarrow\sigma_{i,t}+\delta} - u_i \\
&= U_v \cdot \frac{-\delta(1-\alpha)+I^\star_v}{-\delta(1-\alpha)+2I^\star_v} + U_t \cdot \frac{\delta(1-\alpha)+I^\star_t}{\delta(1-\alpha)+2I^\star_t} \\
&\quad - \sum_{u\in\{v,t\}} U_u \cdot \frac{I^\star_u}{2I^\star_u} \\
&= U_v \cdot \frac{-\delta(1-\alpha)+I^\star_v}{-\delta(1-\alpha)+2I^\star_v} - U_v \cdot \frac{I^\star_v}{2I^\star_v} + U_t \cdot \frac{\delta(1-\alpha)+I^\star_t}{\delta(1-\alpha)+2I^\star_t} - U_t \cdot \frac{I^\star_t}{2I^\star_t} \\
&= \frac{\delta(1-\alpha)}{2}\left(\frac{U_t}{2I^\star_t+\delta(1-\alpha)} - \frac{U_v}{2I^\star_v-\delta(1-\alpha)}\right) \\
&= \frac{\delta(1-\alpha)}{2}\left(\frac{U_t}{\frac{2\cdot s'\cdot U_t}{\sum_{u\in V'} U_u}+\delta(1-\alpha)} - \frac{U_v}{\frac{2\cdot s'\cdot U_v}{\sum_{u\in V'} U_u}-\delta(1-\alpha)}\right) \\
&= \frac{\delta}{2}\left(\frac{1}{\frac{2\cdot s'}{(1-\alpha)\cdot\sum_{u\in V'} U_u}+\frac{\delta}{U_t}} - \frac{1}{\frac{2\cdot s'}{(1-\alpha)\cdot\sum_{u\in V'} U_u}-\frac{\delta}{U_v}}\right) \\
&= \frac{\delta}{2}\left(\frac{1}{c+\frac{\delta}{U_t}} - \frac{1}{c+\frac{-\delta}{U_v}}\right)
\end{aligned}$$

where $c = \frac{2 \cdot s'}{(1-\alpha)\sum_{u \in V'} U_u}$ is an independent constant. Thus, this change is profitable if and only if

$$\begin{aligned} \Delta_i > 0 &\overset{\Diamond}{\iff} c < \frac{\delta}{U_v} \\ &\iff \delta > U_v \cdot c. \end{aligned} \tag{5.24}$$

Additionally, it holds that

$$\delta \overset{(5.23)}{\leq} \sigma_v^\star \overset{\Diamond}{<} U_v \cdot c. \tag{5.25}$$

The two steps above marked with $\Diamond$ were verified with `Mathematica` [2], and the corresponding program with the steps in order can be found here.

The inequality in Equation 5.25 is in direct contradiction with Equation 5.24. Therefore, it never holds that $\Delta_i > 0$, or, in other words, no unilateral change can be made by a player to improve their utility, meaning they are in a pure Nash equilibrium. □

An implementation of the algorithm from Lemma 5.8 in `Python` can be found here.

5.3 Asymmetric Players

This section generalizes on the cases we have examined so far, and allows the two players i and j to have arbitrary budgets s_i and s_j respectively.

5.3.1 Pure Nash Equilibrium for Two Resources

The case we consider now is equivalent to the the one in Section 5.2.1 but with asymmetric players.

The following now holds for the influence of player $k \in P$ at resource $u \in V$:

$$I_{k,u} = (1-\alpha) \cdot \sigma_{k,u} + \alpha \cdot s_k \tag{5.26}$$

The derivation is totally analogous to Equation 5.7.

Theorem 5.12. *For a resource competing game with two resources, two players and $\alpha \in [0,1]$ there always exists a pure Nash equilibrium.*

Theorem 5.12 is divided into Lemmas 5.13, 5.14 and 5.15 for the cases $\alpha = 0$, $\alpha = 1$ and $\alpha \in (0,1)$ respectively.

Lemma 5.13. *For a resource competing game with two resources, two players and $\alpha = 0$, there always exists a pure Nash equilibrium.*

Proof. This is a simple resource competing game with the special case of $m = 2$ resources and $p = 2$ players shown in Theorem 5.2. □

Lemma 5.14. *For a resource competing game with two resources, two players and $\alpha = 1$, there always exists a pure Nash equilibrium.*

Proof. This is a special case with $m = 2$ resources of Lemma 5.18, which will be shown in Section 5.3.2. □

For $\alpha \neq 1$ and player $k \in P$, we define

$$\sigma^\star_{k,v} := \left(\frac{U_v}{U_v + U_t} + \frac{\alpha}{1-\alpha} \cdot \frac{U_v - U_t}{U_v + U_t} \right) \cdot s_k \tag{5.27}$$

Note that $\sigma^\star_{k,t}$ is defined analogously.

Lemma 5.15. *For a resource competing game with two resources, two players and $\alpha \in (0, 1)$, there always exists a pure Nash equilibrium.*

Proof. We will prove, that if every player $k \in P$ spends $\sigma_{i,u}$ on resource $u \in V$, where

$$\sigma_{k,u} = \begin{cases} 0 & \text{if } \sigma^\star_{k,u} < 0 \\ \sigma^\star_{k,u} & \text{if } \sigma^\star_{k,u} \in [0, s_k] \\ s_k & \text{if } \sigma^\star_{k,u} > s_k \end{cases}, \tag{5.28}$$

we are in a pure Nash equilibrium. In this case, we say that player i is playing strategy $\sigma^\star_i$.

As in Equation 5.29, we compute the utility of player i at resource v, giving us

$$u_{i,v} = U_v \cdot \frac{\sigma_{i,v} + \alpha \cdot (s_i - \sigma_{i,v})}{\sigma_{i,v} + \alpha \cdot (s_i - \sigma_{i,v}) + \sigma_{j,v} + \alpha \cdot (s_j - \sigma_{j,v})}, \tag{5.29}$$

and as in Equation 5.11, we compute the utility of player i at resource t, giving us

$$u_{i,t} = U_t \cdot \frac{(s_i - \sigma_{i,v}) + \alpha \cdot \sigma_{i,t}}{(s_i - \sigma_{i,v}) + \alpha \cdot \sigma_{i,v} + (s_j - \sigma_{j,v}) + \alpha \cdot \sigma_{j,v}}. \tag{5.30}$$

Again, we have $\sigma_{i,t} = s_i - \sigma_{i,v} \iff s_i - \sigma_{i,t} = \sigma_{i,v}$. The analogous formulas hold for player j.

We inspect the first and second partial derivatives of u_i with respect to $\sigma_{i,v}$.

$$\frac{\partial u_i}{\partial \sigma_{i,v}} = (1-\alpha) \left(U_v \cdot \frac{I_{j,v}}{(I_{i,v} + I_{j,v})^2} - U_t \cdot \frac{I_{j,t}}{(I_{i,t} + I_{j,t})^2} \right) \tag{5.31}$$

$$\frac{\partial^2 u_i}{\partial^2 \sigma_{i,v}} = -2(1-\alpha)^2 \left(U_v \cdot \frac{I_{j,v}}{(I_{i,v} + I_{j,v})^3} + U_t \cdot \frac{I_{j,t}}{(I_{i,t} + I_{j,t})^3} \right) < 0 \tag{5.32}$$

The computations of these partial derivatives are analogous to Equations 5.12 and 5.13 respectively.

The influence of a player $k \in P$ on resource v when playing $\sigma^\star_{k,v}$ is

$$\begin{aligned} I^\star_{k,v} := I_{k,v} &= (1-\alpha)\cdot\sigma^\star_{i,v} + \alpha\cdot s_k \\ &= (1+\alpha)\cdot s_k \cdot \frac{U_v}{U_v+U_t} \\ &= \frac{(1+\alpha)\cdot s_k\cdot U_v}{U_v+U_t} \end{aligned} \tag{5.33}$$

As in the symmetric case, considering the result of Theorem 4.7, we know that in a pure Nash equilibrium both players must distribute their budgets equally between the resources. If both players play strategies $\sigma^\star_i$ and $\sigma^\star_j$ respectively we know, by construction, that $\frac{\sigma_{i,u}}{s_i} = \frac{\sigma_{j,u}}{s_j}$ for all $u \in V$.

We proceed as in the symmetric case. Suppose that players i and j play strategies $\sigma^\star_i$ and $\sigma^\star_j$ as described above respectively. Then we have

$$\begin{aligned} \frac{\partial u_i}{\partial \sigma_{i,v}} &= (1-\alpha)\left(U_v\cdot\frac{I^\star_{j,v}}{(I^\star_{i,v}+I^\star_{j,v})^2} - U_t\cdot\frac{I^\star_{j,t}}{(I^\star_{i,t}+I^\star_{j,t})^2}\right) \\ &= (1-\alpha)\left(\frac{U_v\cdot\frac{(1+\alpha)\cdot s_j\cdot U_v}{U_v+U_t}}{\left(\frac{(1+\alpha)\cdot s_i\cdot U_v}{U_v+U_t}+\frac{(1+\alpha)\cdot s_j\cdot U_v}{U_v+U_t}\right)^2} - \frac{U_t\cdot\frac{(1+\alpha)\cdot s_j\cdot U_t}{U_v+U_t}}{\left(\frac{(1+\alpha)\cdot s_i\cdot U_t}{U_v+U_t}+\frac{(1+\alpha)\cdot s_j\cdot U_t}{U_v+U_t}\right)^2}\right) \\ &= (1-\alpha)\left(\frac{\frac{(1+\alpha)\cdot s_j\cdot U_v^2}{U_v+U_t}}{\frac{(s_i+s_j)^2(1+\alpha)^2U_v^2}{(U_v+U_t)^2}} - \frac{\frac{(1+\alpha)\cdot s_j\cdot U_t^2}{U_v+U_t}}{\frac{(s_i+s_j)^2(1+\alpha)^2U_t^2}{(U_v+U_t)^2}}\right) \\ &= (1-\alpha)\left(\frac{s_j\cdot(U_v+U_t)\cdot(1+\alpha)}{(s_i+s_j)^2} - \frac{s_j\cdot(U_v+U_t)\cdot(1+\alpha)}{(s_i+s_j)^2}\right) = 0 \end{aligned}$$

which is a maximum due to Equation 5.32 being strictly negative, since $\alpha \neq 1$. This shows that the players i and j are in a pure Nash equilibrium if they play strategies $\sigma^\star_i$ and $\sigma^\star_j$ respectively. □

5.3.2 Pure Nash Equilibrium for m Resources

The case we consider in this section is equivalent to the the one in Section 5.2.2, but with asymmetric players.

The following now holds for the influence of player $k \in P$ at resource $u \in V$:

$$I_{k,u} = (1-\alpha)\cdot\sigma_{k,u} + \alpha\cdot s_k \tag{5.34}$$

The derivation is totally analogous to Equation 5.15.

Theorem 5.16. *For a resource competing game with m resources and two players and $\alpha \in [0,1]$ there always exists a pure Nash equilibrium.*

Theorem 5.16 is divided into Lemmas 5.17, 5.18 and 5.19 for the cases $\alpha = 0$, $\alpha = 1$ and $\alpha \in (0,1)$ respectively.

Lemma 5.17. *For a resource competing game with m resources and two players and $\alpha = 0$, there always exists a pure Nash equilibrium.*

Proof. This is a simple resource competing game with the special case of $p = 2$ players shown in Theorem 5.2. □

Lemma 5.18. *For a resource competing game with m resources and two players and $\alpha = 1$, there always exists a pure Nash equilibrium.*

Proof. For any $i \in P$ and $u \in V$:

$$I_{i,u} = \sigma_{i,u} + \sum_{v \in V \setminus \{u\}} \alpha \cdot \sigma_{i,v} = \sum_{v \in V} \sigma_{i,v} = s_i \tag{5.35}$$

It follows that

$$u_{i,u} = \frac{U_u \cdot I_{i,u}}{\sum_{k \in P} I_{k,u}} = \frac{s_i \cdot U_u}{\sum_{k \in P} s_k} \tag{5.36}$$

and therefore

$$u_i = \sum_{u \in V} u_{i,u} = \frac{s_i}{\sum_{k \in P} s_k} \sum_{u \in V} U_u \tag{5.37}$$

which holds independent of the strategy profile. This means that every strategy profile is a pure Nash equilibrium. □

Note that, in fact, Lemma 5.18 holds for any number of asymmetric players.

Lemma 5.19. *For a resource competing game with m resources and two players and $\alpha \in (0,1)$, there always exists a pure Nash equilibrium.*

Proof. Constructing the strategy takes two steps. After this, we will proceed to show the correctness of the constructed strategy.

Constructing the strategy The first step is almost identical to the first step in the proof of Lemma 5.11.

Step 1: Building the set V' In order to reduce the set of resources, we perform the same procedure, Algorithm 5.1, as in the proof of Lemma 5.6, with the same threshold:

$$T\left(V'\right) := \frac{\alpha}{1 + (|V'| - 1) \cdot \alpha} \cdot \sum_{t \in V'} U_t \qquad \text{(5.19 revisited)}$$

Step 2: Strategies For each player $i \in P$, we assign a spending to each of the resources $v \in V'$, defined as

$$\sigma^{\star}_{i,v} := \left(\frac{U_v}{\sum_{t \in V'} U_t} + \frac{\alpha}{1-\alpha} \cdot \frac{|V'| \cdot U_v - \sum_{t \in V'} U_t}{\sum_{t \in V'} U_t} \right) \cdot s_i. \tag{5.38}$$

For the resources not in V', the spending is 0. Together, these spendings form the strategy $\sigma^{\star}_i$.

Correctness Proving correctness proceeds in two steps. First, we show that the strategy $\sigma^{\star}_i$ for $i \in P$ fulfills the required properties. Then, we will show that we are in a pure Nash equilibrium if both players play $\sigma^{\star}_i$ and $\sigma^{\star}_j$ respectively.

Properties of $\sigma^{\star}_k$ To ensure that the spendings are valid, it must hold that for any player $k \in P$ and resource $v \in V$, we have $\sigma^{\star}_{k,v} \geq 0$. By construction of $\sigma^{\star}_v$, the sum $\sum_{v \in V} \sigma^{\star}_{k,v}$ is equal to s_k.

For the nodes $v \in V \setminus V'$, we have $\sigma^{\star}_{k,v} = 0 \geq 0$. The influence of a player $k \in P$ on resource $v \in V$ when playing $\sigma^{\star}_k$ is

$$I^{\star}_{k,v} := I_{k,v} = \frac{s'_k \cdot U_v}{\sum_{t \in V'} U_t} \tag{5.39}$$

where $s'_k = (1 + (|V'| - 1) \cdot \alpha) \cdot s_k$.

We know that the following holds for the threshold:

$$\forall v \in V' : U_v \geq T\left(V'\right)$$

Thus, for $v \in V'$ we have

$$\begin{aligned}
I^{\star}_{k,v} &= \frac{s'_k \cdot U_v}{\sum_{t \in V'} U_t} \\
&\geq \frac{s'_k}{\sum_{t \in V'} U_t} \cdot T\left(V'\right) \\
&= \frac{s'_k}{\sum_{t \in V'} U_t} \cdot \frac{\alpha}{1 + (|V'| - 1) \cdot \alpha} \cdot \sum_{t \in V'} U_t \\
&= \frac{(1 + (|V'| - 1) \cdot \alpha) \cdot s_k}{\sum_{t \in V'} U_t} \cdot \frac{\alpha}{1 + (|V'| - 1) \cdot \alpha} \cdot \sum_{t \in V'} U_t \\
&= \alpha \cdot s_k
\end{aligned}$$

Due to the definition of the influence in Definition 3.1 it follows that $\sigma^{\star}_{k,v} \geq 0$.

Equilibrium From Lemma 5.10 we know that if the best response for player i is to play $\sigma_i^\star$ if the adversary j is playing $\sigma_j^\star$, and vice-versa, then the strategy profile where players i and j play $\sigma_i^\star$ and $\sigma_j^\star$ respectively is a pure Nash equilibrium. Thus it suffices to show that when the players i and j play these strategies no player wants to change any of their spendings, and thus that both play a best response.

Suppose players i and j play strategies $\sigma_i^\star$ and $\sigma_j^\star$ respectively, meaning that $\sigma_{i,v} = \sigma_{i,v}^\star$ and $\sigma_{j,v} = \sigma_{j,v}^\star$ for all $v \in V'$. Assume that player $i \in P$ can increase their utility by moving some amount δ from a resource $v \in V$ to a resource $t \in V$. The following observations can analogously be made for player j. We distinguish four cases:

1. $v \in V \setminus V'$, $t \in V \setminus V'$, which we can immediately dismiss because there is no spending to move away from v,
2. $v \in V \setminus V'$, $t \in V'$, for which the same applies as above,
3. $v \in V'$, $t \in V \setminus V'$, and finally
4. $v \in V'$, $t \in V'$.

We treat cases 3 and 4 together, thus assuming $v \in V'$ and $t \in V$. Moving some amount $\delta > 0$ will result in some change of utility Δ_i. We additionally must assume that

$$\delta \leq \sigma_{i,v} = \sigma_{i,v}^\star, \tag{5.40}$$

otherwise we cannot move δ from v to t.

Further, for any $u \in V$, we have

$$I_{i,u}^\star + I_{j,u}^\star = \frac{(s_i' + s_j') \cdot U_u}{\sum_{u' \in V} U_{u'}} = \frac{(s_i + s_j) \cdot (1 + (|V'|-1) \cdot \alpha) \cdot U_u}{\sum_{u' \in V} U_{u'}}. \tag{5.41}$$

For the change of utility Δ_i it holds that

$$
\begin{aligned}
\Delta_i &:= u_i\big|_{\sigma_{i,v}\leftarrow\sigma_{i,v}-\delta,\sigma_{i,t}\leftarrow\sigma_{i,t}+\delta} - u_i \\
&= \left(\sum_{u\in V} u_{i,u}\right)\Bigg|_{\sigma_{i,v}\leftarrow\sigma_{i,v}-\delta,\sigma_{i,t}\leftarrow\sigma_{i,t}+\delta} - u_i \\
&= U_v \cdot \frac{-\delta(1-\alpha)+I^\star_{i,v}}{-\delta(1-\alpha)+I^\star_{i,v}+I^\star_{j,v}} + U_t \cdot \frac{\delta(1-\alpha)+I^\star_{i,t}}{\delta(1-\alpha)+I^\star_{i,t}+I^\star_{j,t}} \\
&\qquad - \sum_{u\in\{v,t\}} U_u \cdot \frac{I^\star_{i,u}}{I^\star_{i,u}+I^\star_{j,u}} \\
&= U_v \cdot \frac{-\delta(1-\alpha)+I^\star_{i,v}}{-\delta(1-\alpha)+I^\star_{i,v}+I^\star_{j,v}} - U_v \cdot \frac{I^\star_{i,v}}{I^\star_{i,v}+I^\star_{j,v}} \\
&\qquad + U_t \cdot \frac{\delta(1-\alpha)+I^\star_{i,t}}{\delta(1-\alpha)+I^\star_{i,t}+I^\star_{j,t}} - U_t \cdot \frac{I^\star_{i,t}}{I^\star_{i,t}+I^\star_{j,t}} \\
&= \frac{s_j}{s_i+s_j}\delta(1-\alpha)\left(\frac{U_t}{I^\star_{i,t}+I^\star_{j,t}+\delta(1-\alpha)} - \frac{U_v}{I^\star_{i,v}+I^\star_{j,v}-\delta(1-\alpha)}\right) \\
&= \frac{s_j}{s_i+s_j}\delta\left(\frac{1}{\frac{s'_i+s'_j}{(1-\alpha)\sum_{u\in V}U_u}+\frac{\delta}{U_t}} - \frac{1}{\frac{s'_i+s'_j}{(1-\alpha)\sum_{u\in V}U_u}-\frac{\delta}{U_v}}\right) \\
&= \frac{s_j}{s_i+s_j}\delta\left(\frac{1}{c+\frac{\delta}{U_t}} - \frac{1}{c+\frac{-\delta}{U_v}}\right)
\end{aligned}
$$

where $c = \frac{s'_i+s'_j}{(1-\alpha)\sum_{u\in V'}U_u}$ is an independent constant. Thus, this change is profitable if and only if

$$
\begin{aligned}
\Delta_i > 0 &\overset{\lozenge}{\iff} c < \frac{\delta}{U_v} \\
&\iff \delta > U_v \cdot c.
\end{aligned} \tag{5.42}
$$

Additionally, it holds that

$$
\delta \overset{(5.40)}{\leq} \sigma^\star_v \overset{\lozenge}{<} U_v \cdot c. \tag{5.43}
$$

The two steps above marked with $\lozenge$ were verified with `Mathematica` [2], and the corresponding program with the steps in order can be found here.

The inequality in Equation 5.43 is in direct contradiction with Equation 5.42. Therefore, it never holds that $\Delta_i > 0$, or, in other words, no unilateral change can be made by a player to improve their utility, meaning they are in a pure Nash equilibrium. □

An implementation of the algorithm from Lemma 5.17 in `Python` can be found here.

Chapter 6

Variable Influence

So far we have considered the influence α to be constant. Now we want to examine the case where the influence is not constant, that is, between any pair of resources the influence α can be different.

6.1 Two Resources with Two Symmetric Players

This section covers the generalization of Section 5.2.1, allowing for variable influence.

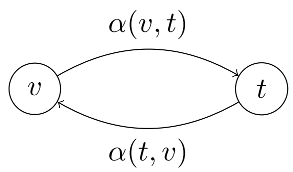

Figure 6.1: Resource competing game with two resources v and t and variable influence.

The following holds for the influence of player i at resource v (analogously for j and for t):

$$I_{i,v} = (1 - \alpha(t,v)) \cdot \sigma_{i,v} + \alpha(t,v) \cdot s \tag{6.1}$$

Theorem 6.1. *In a generalized resource competing game with two resources, two symmetric players and $\alpha : V^2 \to [0,1]$ there always exists a pure Nash equilibrium.*

Proof. Theorem 6.1 is divided into Lemma 6.2, where we require $\alpha : V^2 \to [0,1)$, and Lemma 6.3, where we require $\alpha(v,t) = 1$ or $\alpha(t,v) = 1$. $\square$

For $\alpha(t,v) \neq 1$ and $\alpha(v,t) \neq 1$, we define

$$\sigma_v^\star := \left(\frac{1}{1-\alpha(v,t)} \cdot \frac{U_v}{U_v + U_t} - \frac{\alpha(t,v)}{1-\alpha(t,v)} \cdot \frac{U_t}{U_v + U_t} \right) \cdot s \tag{6.2}$$

Note that $\sigma_t^\star$ is defined analogously.

Lemma 6.2. *In a generalized resource competing game with two resources, two symmetric players and $\alpha : V^2 \to [0, 1)$ there always exists a pure Nash equilibrium.*

Proof. We will prove that if every player $k \in P$ spends $\sigma_{k,u}$ on resource $u \in V$, where

$$\sigma_{k,u} = \begin{cases} 0 & \text{if } \sigma_u^\star < 0 \\ \sigma_u^\star & \text{if } \sigma_u^\star \in [0, s] \\ s & \text{if } \sigma_u^\star > s \end{cases}, \tag{6.3}$$

we are in a pure Nash equilibrium. In this case we say player k is playing strategy $\sigma^\star$.

We compute the utility of player i at resource v, giving us

$$u_{i,v} = U_v \cdot \frac{\sigma_{i,v} + \alpha(t,v) \cdot (s - \sigma_{i,v})}{\sigma_{i,v} + \alpha(t,v) \cdot (s - \sigma_{i,v}) + \sigma_{j,v} + \alpha(t,v) \cdot (s - \sigma_{j,v})} \tag{6.4}$$

where we applied Definitions 3.2 and 3.1. Similarly, we compute the utility of player i at resource t, giving us

$$u_{i,t} = U_t \cdot \frac{(s - \sigma_{i,v}) + \alpha(v,t) \cdot \sigma_{i,v}}{(s - \sigma_{i,v}) + \alpha(v,t) \cdot \sigma_{i,v} + (s - \sigma_{j,v}) + \alpha(v,t) \cdot \sigma_{j,v}}. \tag{6.5}$$

In both derivations we used $\sigma_{i,t} = s - \sigma_{i,v} \iff s - \sigma_{i,t} = \sigma_{i,v}$. The analogous formulas hold for player j.

We inspect the first and second partial derivatives of u_i with respect to $\sigma_{i,v}$.

$$\frac{\partial u_i}{\partial \sigma_{i,v}} = (1 - \alpha(t,v)) \cdot U_v \cdot \frac{I_{j,v}}{(I_{i,v} + I_{j,v})^2} - (1 - \alpha(v,t)) \cdot U_t \cdot \frac{I_{j,t}}{(I_{i,t} + I_{j,t})^2} \tag{6.6}$$

$$\frac{\partial^2 u_i}{\partial^2 \sigma_{i,v}} = -2(1 - \alpha(t,v))^2 \cdot U_v \cdot \frac{I_{j,v}}{(I_{i,v} + I_{j,v})^3} - 2(1 - \alpha(v,t))^2 \cdot U_t \cdot \frac{I_{j,t}}{(I_{i,t} + I_{j,t})^3} < 0 \tag{6.7}$$

The influence of player $k \in P$ on resource v when playing $\sigma^\star$ is

$$I_v^\star := I_{k,v} = \frac{1 - \alpha(t,v) \cdot \alpha(v,t)}{1 - \alpha(v,t)} \cdot \frac{U_v}{U_v + U_t} \cdot s \tag{6.8}$$

Considering the result from Theorem 4.4, we know that when in a pure Nash equilibrium both players must spend the same amount on each resource. Therefore, it suffices to find such a strategy which maximizes the utility for a player. Note that the utility functions are equivalent for both players, and we will continue taking the perspective of player i.

With this in mind, suppose that both players play strategy $\sigma^\star$ as described above. Then we have

$$\begin{aligned}\frac{\partial u_i}{\partial \sigma_{i,v}} &= (1-\alpha(t,v)) \cdot U_v \cdot \frac{I_v^\star}{(I_v^\star + I_v^\star)^2} - (1-\alpha(v,t)) \cdot U_t \cdot \frac{I_t^\star}{(I_t^\star + I_t^\star)^2} \\ &= \frac{1}{4} \cdot (1-\alpha(t,v)) \cdot \frac{U_v}{I_v^\star} - \frac{1}{4} \cdot (1-\alpha(v,t)) \cdot \frac{U_t}{I_t^\star} \\ &= \frac{1}{4} \cdot (1-\alpha(t,v)) \cdot (1-\alpha(v,t)) \cdot s \cdot \frac{(U_v + U_t) - (U_v + U_t)}{1-\alpha(t,v) \cdot \alpha(v,t)} \\ &= 0\end{aligned}$$

which is a maximum due to Equation 6.7 being strictly negative since $\alpha(t,v) \neq 1$ and $\alpha(v,t) \neq 1$. Combined with the result from Theorem 4.4, this shows that the players are in a pure Nash equilibrium if they both play strategy $\sigma^\star$. □

Lemma 6.3. *In a generalized resource competing game with two resources, two symmetric players and $\alpha : V^2 \to [0,1]$, where $\alpha(t,v) = 1$ or $\alpha(v,t) = 1$, there always exists a pure Nash equilibrium.*

Proof. For the case that we have $\alpha(v,t) = 1$ *and* $\alpha(t,v) = 1$ the situation is equivalent to the one described in Theorem 5.2 (simple resource competing game).

To prove the other two cases, suppose that $0 \leq \alpha(v,t) < 1$ and $\alpha(t,v) = 1$. The opposite case then follows analogously.

Player i must decide how to split budget s across resources v and t. Since we have $\alpha(v,t) = 1$, the utility of i at resources v and t are

$$u_{i,v} = U_v \cdot \frac{I_{i,v}}{I_{i,v} + I_{j,v}} = U_v \cdot \frac{\sigma_{i,v} + \sigma_{i,t}}{\sigma_{i,v} + \sigma_{i,t} + \sigma_{j,v} + \sigma_{j,t}} = U_v \cdot \frac{s}{2 \cdot s} = \frac{U_v}{2} \tag{6.9}$$

$$u_{i,t} = U_t \cdot \frac{I_{i,t}}{I_{i,t} + I_{j,t}} = U_t \cdot \frac{\sigma_{i,t} + \alpha(v,t) \cdot \sigma_{i,v}}{\sigma_{i,t} + \alpha(v,t) \cdot \sigma_{i,v} + \sigma_{j,t} + \alpha(v,t) \cdot \sigma_{j,v}} \tag{6.10}$$

Since $u_{i,v}$ does not depend on the strategy, player i will spend all of their budget trying to increase $u_{i,t}$, meaning that $\sigma_{i,v} = 0$ and $\sigma_{i,t} = s$. If both players do this, they will both be mutually maximized in their utility and will thus not want to change any of their spendings. This means we are in a pure Nash equilibrium. □

6.2 Two Resources with Two Asymmetric Players

We will now further generalize and allow asymmetric players. This extension is similar to the one done in Section 5.3.1, namely scaling the spendings with the player-specific budget instead of the common budget.

The following holds for the influence of player i at resource v (analogously for j and for t):

$$I_{i,v} = (1-\alpha(t,v)) \cdot \sigma_{i,u} + \alpha(t,v) \cdot s_i \tag{6.11}$$

Theorem 6.4. *In a generalized resource competing game with two resources, two asymmetric players and $\alpha : V^2 \to [0,1]$ there always exists a pure Nash equilibrium.*

Proof. Theorem 6.4 is divided into Lemma 6.5, where we require $\alpha : V^2 \to [0,1)$, and Lemma 6.6, where we require $\alpha(v,t) = 1$ or $\alpha(t,v) = 1$. □

For $\alpha(t,v) \neq 1$, $\alpha(v,t) \neq 1$ and player $k \in P$, we define

$$\sigma^\star_{k,v} := \left(\frac{1}{1-\alpha(v,t)} \cdot \frac{U_v}{U_v + U_t} - \frac{\alpha(t,v)}{1-\alpha(t,v)} \cdot \frac{U_t}{U_v+U_t} \right) \cdot s_k \tag{6.12}$$

Note that $\sigma^\star_{k,t}$ is defined analogously.

Lemma 6.5. *In a generalized resource competing game with two resources, two asymmetric players and $\alpha : V^2 \to [0,1)$ there always exists a pure Nash equilibrium.*

Proof. We will prove that if every player $k \in P$ spends $\sigma_{k,u}$ on resource $u \in V$, where

$$\sigma_{k,u} = \begin{cases} 0 & \text{if } \sigma^\star_{k,u} < 0 \\ \sigma^\star_{k,u} & \text{if } \sigma^\star_{k,u} \in [0, s_k] \\ s_k & \text{if } \sigma^\star_{k,u} > s_k \end{cases}, \tag{6.13}$$

we are in a pure Nash equilibrium. In this case we say player k is playing strategy $\sigma^\star_k$.

We compute the utility of player i at resource v, giving us

$$u_{i,v} = U_v \cdot \frac{\sigma_{i,v} + \alpha(t,v)\cdot(s-\sigma_{i,v})}{\sigma_{i,v} + \alpha(t,v)\cdot(s-\sigma_{i,v}) + \sigma_{j,v} + \alpha(t,v)\cdot(s-\sigma_{j,v})} \tag{6.14}$$

where we applied Definitions 3.2 and 3.1. Similarly, we compute the utility of player i at resource t, giving us

$$u_{i,t} = U_t \cdot \frac{(s-\sigma_{i,v}) + \alpha(v,t)\cdot\sigma_{i,v}}{(s-\sigma_{i,v}) + \alpha(v,t)\cdot\sigma_{i,v} + (s-\sigma_{j,v}) + \alpha(v,t)\cdot\sigma_{j,v}}. \tag{6.15}$$

In both derivations we used $\sigma_{i,t} = s_i - \sigma_{i,v} \iff s_i - \sigma_{i,t} = \sigma_{i,v}$. The analogous formulas hold for player j.

We inspect the first and second partial derivatives of u_i with respect to $\sigma_{i,v}$.

$$\frac{\partial u_i}{\partial \sigma_{i,v}} = (1-\alpha(t,v)) \cdot U_v \cdot \frac{I_{j,v}}{(I_{i,v}+I_{j,v})^2} - (1-\alpha(v,t)) \cdot U_t \cdot \frac{I_{j,t}}{(I_{i,t}+I_{j,t})^2} \tag{6.16}$$

$$\frac{\partial^2 u_i}{\partial^2 \sigma_{i,v}} = -2(1-\alpha(t,v))^2 \cdot U_v \cdot \frac{I_{j,v}}{(I_{i,v}+I_{j,v})^3} - 2(1-\alpha(v,t))^2 \cdot U_t \cdot \frac{I_{j,t}}{(I_{i,t}+I_{j,t})^3} < 0 \tag{6.17}$$

The influence of a player $k \in P$ on resource v when playing $\sigma^\star_{k,v}$ is

$$I^\star_{k,v} := I_{k,v} = \frac{1-\alpha(t,v)\cdot\alpha(v,t)}{1-\alpha(v,t)} \cdot \frac{U_v}{U_v+U_t} \cdot s_k \tag{6.18}$$

As in the symmetric case, considering the result from Theorem 4.7 we know that in a pure Nash equilibrium both players must distribute their budgets equally between the resources. If both players play strategies $\sigma^\star_i$ and $\sigma^\star_j$ respectively, we have, by construction, $\frac{\sigma_{i,u}}{s_i} = \frac{\sigma_{j,u}}{s_j}$ for all $u \in V$. We proceed as in the symmetric case.

Suppose that players i and j play strategies $\sigma^\star_i$ and $\sigma^\star_i$ as described above respectively. Then we have

$$\begin{aligned}
\frac{\partial u_i}{\partial \sigma_{i,v}} &= (1-\alpha(t,v))\cdot U_v \cdot \frac{I^\star_{j,v}}{(I^\star_{i,v}+I^\star_{j,v})^2} - (1-\alpha(v,t))\cdot U_t \cdot \frac{I^\star_{j,t}}{(I^\star_{i,t}+I^\star_{j,t})^2} \\
&= (1-\alpha(t,v)) \cdot \frac{(1-\alpha(v,t))(U_v+U_t)s_j}{(1-\alpha(t,v)(\alpha(v,t)))(s_i+s_j)^2} \\
&\quad - (1-\alpha(v,t)) \cdot \frac{(1-\alpha(t,v))(U_v+U_t)s_j}{(1-\alpha(t,v)(\alpha(v,t)))(s_i+s_j)^2} \\
&= 0
\end{aligned}$$

which is a maximum due to Equation 6.7 being strictly negative since $\alpha(t,v) \neq 1$ and $\alpha(v,t) \neq 1$. Combined with the result from Theorem 4.7, this shows that the players i and j are in a pure Nash equilibrium if they play strategies $\sigma^\star_i$ and $\sigma^\star_j$ respectively. □

Lemma 6.6. *In a generalized resource competing game with two resources, two symmetric players and $\alpha : V^2 \to [0,1]$, where $\alpha(t,v) = 1$ or $\alpha(v,t) = 1$, there always exists a pure Nash equilibrium.*

Proof. For the case that we have $\alpha(v,t) = 1$ *and* $\alpha(t,v) = 1$ the situation is equivalent to the one described in Theorem 5.2 (simple resource competing game).

To prove the other two cases, suppose that $0 \leq \alpha(v,t) < 1$ and $\alpha(t,v) = 1$. The opposite case then follows analogously.

Player i must decide how to split budget s_i across resources v and t. Since we have $\alpha(v,t) = 1$, the utility of i at resources v and t are

$$u_{i,v} = U_v \cdot \frac{I_{i,v}}{I_{i,v}+I_{j,v}} = U_v \cdot \frac{\sigma_{i,v}+\sigma_{i,t}}{\sigma_{i,v}+\sigma_{i,t}+\sigma_{j,v}+\sigma_{j,t}} = U_v \cdot \frac{s_i}{s_i+s_j} \tag{6.19}$$

$$u_{i,t} = U_t \cdot \frac{I_{i,t}}{I_{i,t}+I_{j,t}} = U_t \cdot \frac{\sigma_{i,t}+\alpha(v,t)\cdot\sigma_{i,v}}{\sigma_{i,t}+\alpha(v,t)\cdot\sigma_{i,v}+\sigma_{j,t}+\alpha(v,t)\cdot\sigma_{j,v}} \tag{6.20}$$

Since $u_{i,v}$ does not depend on the strategy, player i will spend all of their budget trying to increase $u_{i,t}$, meaning that $\sigma_{i,v} = 0$ and $\sigma_{i,t} = s_i$. If both players do this, they will both be mutually maximized in their utility and will thus not want to change any of their spendings. This means we are in a pure Nash equilibrium. $\square$

Chapter 7

Three Resources with Two Symmetric Players

We now consider a generalized resource competing game with three resources $V = \{u, v, t\}$ and two symmetric players $P = \{i, j\}$, as shown in Figure 7.1, allowing the influence α to be different for each pair of nodes. This results in a total of six different values for the influence α. In total this problem has nine parameters: the six influence values mentioned above and a utility for each of the three resources. The budget s is considered a scaling factor of the spending and not a problem parameter, since it has no other effect on the resulting pure Nash equilibrium, as has been shown Chapter 4.

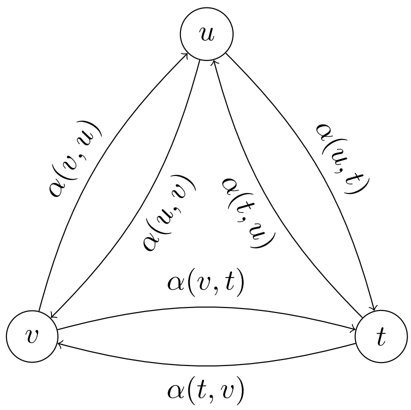

Figure 7.1: Resource competing game with three resources $\{u, v, t\}$ and variable influence.

7.1 Theoretical Considerations

Considering player $i \in P$, there are three variables $\sigma_{i,u}, \sigma_{i,v}$ and $\sigma_{i,t}$. Three cases can be distinguished, namely

1. $\sigma_{i,u}$ and $\sigma_{i,v}$ are variables and $\sigma_{i,t} = s - \sigma_{i,u} - \sigma_{i,v}$,
2. $\sigma_{i,u}$ and $\sigma_{i,t}$ are variables and $\sigma_{i,v} = s - \sigma_{i,u} - \sigma_{i,t}$ and
3. $\sigma_{i,v}$ and $\sigma_{i,t}$ are variables and $\sigma_{i,u} = s - \sigma_{i,v} - \sigma_{i,t}$.

For each of these cases we have two subcases: we can consider the first variable as fixed to some value and the second as a variable that we can adjust and vice versa. This yields a total of six cases called *settings*.

For demonstration purposes, we will consider the first of the three cases, that is, variables $\sigma_{i,u}$ and $\sigma_{i,v}$ with $\sigma_{i,t} = s - \sigma_{i,u} - \sigma_{i,v}$. We compute the first and second derivatives of u_i with respect to $\sigma_{i,u}$ and $\sigma_{i,v}$ respectively. Clearly, $\frac{\partial u_i}{\partial \sigma_{i,t}} = 0$ since $\sigma_{i,t}$ is not considered a variable.

$$\begin{aligned}\frac{\partial u_i}{\partial \sigma_{i,u}} &= U_u(1-\alpha(t,u))\frac{I_{i,u}}{(I_{i,u}+I_{j,u})^2} \\ &+ U_v(\alpha(u,v)-\alpha(t,v))\frac{I_{i,v}}{(I_{i,v}+I_{j,v})^2} \\ &- U_t(1-\alpha(u,t))\frac{I_{i,t}}{(I_{i,t}+I_{j,t})^2}\end{aligned} \tag{7.1}$$

$$\begin{aligned}\frac{\partial^2 u_i}{\partial^2 \sigma_{i,u}} &= -2U_u(1-\alpha(t,u))^2\frac{I_{i,u}}{(I_{i,u}+I_{j,u})^3} \\ &- 2U_v(\alpha(u,v)-\alpha(t,v))^2\frac{I_{i,v}}{(I_{i,v}+I_{j,v})^3} \\ &- 2U_t(1-\alpha(u,t))^2\frac{I_{i,t}}{(I_{i,t}+I_{j,t})^3} < 0\end{aligned} \tag{7.2}$$

$$\begin{aligned}\frac{\partial u_i}{\partial \sigma_{i,v}} &= U_u(\alpha(v,u)-\alpha(t,u))\frac{I_{i,u}}{(I_{i,u}+I_{j,u})^2} \\ &+ U_v(1-\alpha(t,v))\frac{I_{i,v}}{(I_{i,v}+I_{j,v})^2} \\ &- U_t(1-\alpha(v,t))\frac{I_{i,t}}{(I_{i,t}+I_{j,t})^2}\end{aligned} \tag{7.3}$$

$$\begin{aligned}\frac{\partial^2 u_i}{\partial^2 \sigma_{i,v}} &= -2U_u(\alpha(v,u)-\alpha(t,u))^2\frac{I_{i,u}}{(I_{i,u}+I_{j,u})^3} \\ &- 2U_v(1-\alpha(t,v))^2\frac{I_{i,v}}{(I_{i,v}+I_{j,v})^3} \\ &- 2U_t(1-\alpha(v,t))^2\frac{I_{i,t}}{(I_{i,t}+I_{j,t})^3} < 0\end{aligned} \tag{7.4}$$

From $\alpha(u', v') \neq 1$ for all $u', v' \in V$ it follows that $\frac{\partial^2 u_i}{\partial^2 \sigma_{i,u}} < 0$ and $\frac{\partial^2 u_i}{\partial^2 \sigma_{i,v}} < 0$. We also know that

$$\begin{aligned}\frac{\partial^2 u_i}{\partial \sigma_u \partial \sigma_v} = \frac{\partial^2 u_i}{\partial \sigma_v \partial \sigma_u} &= -2U_u(1-\alpha(t,u))(\alpha(v,u)-\alpha(t,u))\frac{I_{i,u}}{(I_{i,u}+I_{j,u})^3} \\ &\quad - 2U_v(\alpha(u,v)-\alpha(t,v))(1-\alpha(t,v))\frac{I_{i,v}}{(I_{i,v}+I_{j,v})^3} \\ &\quad - 2U_t(1-\alpha(u,t))(1-\alpha(v,t))\frac{I_{i,t}}{(I_{i,t}+I_{j,t})^3},\end{aligned} \tag{7.5}$$

meaning that, according to the Sylvester criterion [15], the Hessian

$$\text{Hess}_{u_i} = \begin{bmatrix} \frac{\partial^2 u_i}{\partial^2 \sigma_u} & \frac{\partial^2 u_i}{\partial \sigma_u \partial \sigma_v} \\ \frac{\partial^2 u_i}{\partial \sigma_v \partial \sigma_u} & \frac{\partial^2 u_i}{\partial^2 \sigma_v} \end{bmatrix} \tag{7.6}$$

is negative semidefinite. Thus the function u_i is concave. This fact will aid in developing an approximation algorithm in Section 7.2.1.

Recall, as shown in Theorem 4.11, that the spending of both players in a pure Nash equilibrium is equal on each resource. In this case, also their influence is equal. Let this influence be $I^\star_{v'}$ for resource $v' \in V$. This results in:

$$\begin{aligned}\frac{\partial u_i}{\partial \sigma_{i,u}} &= U_u(1-\alpha(t,u))\frac{I_{i,u}}{(I_{i,u}+I_{j,u})^2} \\ &\quad + U_v(\alpha(u,v)-\alpha(t,v))\frac{I_{i,v}}{(I_{i,v}+I_{j,v})^2} \qquad (7.1 \text{ revisited}) \\ &\quad - U_t(1-\alpha(u,t))\frac{I_{i,t}}{(I_{i,t}+I_{j,t})^2} \\ &= \frac{1}{4}\left(\frac{U_u(1-\alpha(t,u))}{I^\star_u} + \frac{U_v(\alpha(u,v)-\alpha(t,v))}{I^\star_v} - \frac{U_t(1-\alpha(u,t))}{I^\star_t}\right) \qquad (7.7)\end{aligned}$$

$$\begin{aligned}\frac{\partial u_i}{\partial \sigma_{i,v}} &= U_u(\alpha(v,u)-\alpha(t,u))\frac{I_{i,u}}{(I_{i,u}+I_{j,u})^2} \\ &\quad + U_v(1-\alpha(t,v))\frac{I_{i,v}}{(I_{i,v}+I_{j,v})^2} \qquad (7.3 \text{ revisited}) \\ &\quad - U_t(1-\alpha(v,t))\frac{I_{i,t}}{(I_{i,t}+I_{j,t})^2} \\ &= \frac{1}{4}\left(\frac{U_u(\alpha(v,u)-\alpha(t,u))}{I^\star_u} + \frac{U_v(1-\alpha(t,v))}{I^\star_v} - \frac{U_t(1-\alpha(v,t))}{I^\star_t}\right) \qquad (7.8)\end{aligned}$$

Equations 7.7 and 7.8 are hard coded directly into the algorithm presented in Section 7.2.1. The same applies to the other two cases that were not demonstrated.

7.2 Numerical Approximation

It is known that when two players are in a pure Nash equilibrium, their spending ratio with respect to their budgets is equal. In the symmetric case this means their spending is equal. Therefore, from now on, we consider *player-unspecific* spendings, for instance σ_u for the spending of *some* player on resource u.

The approach previously used where the maximum was determined using analytical methods was not fruitful in the case of three resources, neither manually by hand nor with automated solvers of non-linear systems of equations. Therefore, this problem is approached numerically by means of algorithmic approximation.

7.2.1 Approximation Algorithm

Subprocedure

Algorithm 7.1 (`setting_convergence`) attempts to converge to optimal spendings in one given setting. It takes as inputs metaparameters `initial_stepsize`, `max_iterations`, ϵ and as ordinary parameters some current spending $\sigma \in [0, s]^3$, and indices `fix`, `vary` and `other`. These indices are distinct values in $\{0, 1, 2\}$ representing the indices for resources u, v and t respectively. The problem parameters α, U_u, U_v, U_t and s are assumed to be implicitly available. The `initial_stepsize` is equal to $\epsilon \cdot 10^{-2}$ by default.

The algorithm runs for at most `max_iterations` iterations after which it will fail and return `null`, symbolizing that no convergence could be reached in the given number of iterations.

During the complete execution, the value $\sigma_{\texttt{fix}}$ stays fixed, the value $\sigma_{\texttt{vary}}$ is adjusted to maximize the utility, and the value $\sigma_{\texttt{other}}$ is the rest of the budget available:

$$\sigma_{\texttt{other}} = s - \sigma_{\texttt{fix}} - \sigma_{\texttt{vary}} \tag{7.9}$$

This means that increasing $\sigma_{\texttt{vary}}$ decreases $\sigma_{\texttt{other}}$ and decreasing $\sigma_{\texttt{vary}}$ increases $\sigma_{\texttt{other}}$.

Whenever the adjustment of $\sigma_{\texttt{vary}}$ continues to go in the same direction (increasing or decreasing), we will adaptively increase the stepsize by 10% in each step (done by `update_stepsize`). The choice of this adaptive factor is chosen somewhat arbitrarily but explained in Section 7.2.2. Whenever the direction changes, the stepsize is reset to the initial stepsize.

The gradient $\nabla u = \begin{pmatrix} \frac{\partial u}{\partial \sigma_u} & \frac{\partial u}{\partial \sigma_v} & \frac{\partial u}{\partial \sigma_t} \end{pmatrix}^\top$, which depends on the setting, is then computed as shown exemplarily in Equations 7.1 and 7.3 for the first case introduced in Section 7.1. If the absolute difference of the gradient evaluated at `vary` and `other` is smaller than ϵ (line L1), the algorithm terminates and returns the resulting spendings. The reason for this is that in this case changing $\sigma_{\texttt{vary}}$ by some amount results in increased utility which is directly amortized by the fact that changing $\sigma_{\texttt{other}}$ results in decreasing the utility to the same extent. If this is not the case, but the gradient evaluated at `vary` greater than ϵ and there is still some spending that can be moved from $\sigma_{\texttt{other}}$ to $\sigma_{\texttt{vary}}$ (line L2), then $\sigma_{\texttt{vary}}$ is increased and the algorithm continues. If this is also not the case, but the gradient evaluated at `vary` is smaller than $-\epsilon$ and there is still some spending that can be moved from $\sigma_{\texttt{vary}}$ to $\sigma_{\texttt{other}}$ (line L3), then $\sigma_{\texttt{vary}}$ is decreased and the algorithm continues. If none of these cases apply, then the algorithm terminates and returns the resulting spendings, since no changes can be made to increase the utility and therefore convergence is reached.

Algorithm 7.1: `setting_convergence`: Convergence in a setting

Input : `initial_stepsize`, `max_iterations`, ϵ, $\sigma \in [0, s]^3$, `fix`, `vary`, `other`

Output: $\sigma \in [0, s]^3$

stepsize $\leftarrow$ `initial_stepsize`;

`direction` $\leftarrow 0$;

$\texttt{direction}_{\texttt{prev}} \leftarrow 0$;

$j \leftarrow 0$;

while $j <$ `max_iterations` **do**

    $\sigma[\texttt{other}] \leftarrow s - \sigma[\texttt{fix}] - \sigma[\texttt{vary}]$;

    $\texttt{direction}_{\texttt{prev}} \leftarrow \texttt{direction}$;

    ∇u is the gradient in the current setting;

L1   **if** $|\nabla u(\texttt{vary}) - \nabla u(\texttt{other})| < \epsilon$ **then**

        return σ;

L2   **else if** $\nabla u(\texttt{vary}) > \epsilon$ and $\sigma[\texttt{vary}] + \epsilon < s - \sigma[\texttt{fix}]$ **then**

        direction $\leftarrow 1$;

        `update_stepsize()`;

        $\sigma[\texttt{vary}] \leftarrow \sigma[\texttt{vary}] + \min(\sigma[\texttt{other}], \texttt{stepsize})$;

L3   **else if** $\nabla u(\texttt{vary}) < -\epsilon$ and $\sigma[\texttt{vary}] > \epsilon$ **then**

        `direction` $\leftarrow -1$;

        `update_stepsize()`;

        $\sigma[\texttt{vary}] \leftarrow \sigma[\texttt{vary}] - \min(\sigma[\texttt{vary}], \texttt{stepsize})$;

    else

        return σ;

    $j \leftarrow j + 1$;

return `null`;   `// No convergence`

Main procedure

Algorithm 7.2 to find an approximation of the pure Nash equilibrium uses the subprocedure in Algorithm 7.1 once in each of its iterations. It takes as input metaparameters `max_iterations`, ϵ and `starting_fraction` $\in [0,1]$. Again, the problem parameters α, U_u, U_v, U_t and s are assumed to be implicitly available.

As before, the algorithm runs for at most `max_iterations` iterations after which it fails and returns `null`, symbolizing that no convergence could be reached in the given number of iterations.

The initial setting is arbitrary but fixed to be 0 in the algorithm below. Then $\sigma \in [0,s]^3$ is created and initialized as follows:

$$\sigma_{\texttt{fix}} = \texttt{starting\_fraction} \cdot s \quad \text{and} \quad \sigma_{\texttt{vary}} = \sigma_{\texttt{other}} = \frac{1}{2}(s - \sigma_{\texttt{fix}}) \tag{7.10}$$

The `starting_fraction` is set to $1/3$ in all experiments, but could be an arbitrary value in $[0,1]$.

The iterations cycle through the six settings introduced above. Each settings provides the indices `fix`, `vary`, and `other`. With these indices the subprocedure `setting_convergence` is executed which returns the converged spendings in the current setting. If the subprocedure could not converge, i.e. returns `null`, the main procedure also fails to converge and returns `null`. The variable `no_change` is used to keep track of how many consecutive iterations have completed with *no* change of the spendings. As soon as six consecutive iterations with no changes are achieved, the algorithm terminates and returns the result.

For practical purposes, we always use a budget of $s = 1000$. We can use this constant budget since the actual budget does not make a difference since we are only interested in how the budget is split among the resources. The algorithm performs better for higher budgets. For instance, for a budget of $s = 1$ the algorithm will consistently fail to converge, since the stepsize is too large in comparison to the budget.

Note that the parameter `max_iterations` applies to both the subprocedure and the main procedure, meaning that effectively the maximal number of iterations is $(\texttt{max\_iterations})^2$. For the computations and plots presented in this thesis `max_iterations` $\in \{10^3, 10^4\}$ was used. This metaparameter has no effect on the actual result of the algorithm, unless it is exceeded; then, no result is produced.

We denote the approximative pure Nash equilibrium spending generated by the algorithm with $\widehat{\sigma}_{v'}$ for $v' \in V$.

Implementations of Algorithms 7.1 and 7.2 in `Python` can be found here.

Algorithm 7.2: Main procedure: Finding an approximation of a PNE

```
Input  : initial_stepsize, max_iterations, ϵ, starting_fraction ∈ [0, 1]
Output: σ ∈ [0, s]^3
fix, vary, other = get_setting(0);
σ[fix] ← starting_fraction · s;
σ[vary] ← 1/2 (s − σ[fix]);
σ[other] ← 1/2 (s − σ[fix]);
σ_prev ← σ;
no_change ← 0;
j ← 0;
while j < max_iterations do
    fix, vary, other = get_setting(i % 6);
    σ ← setting_convergence(fix,vary,other);
    if σ is null then
        return null;                              // No convergence
    if ||σ_prev − σ||< ϵ then
        no_change ← no_change + 1;
    else
        no_change ← 0;
    if no_change ≥ 6 then
        return σ;
    σ_prev ← σ;
    j ← j + 1;
return null;                                      // No convergence
```

7.2.2 Performance

We denote with $\sigma^\star_{v'}$ the true pure Nash equilibrium spending on resource $v' \in V$. We want to have a measure of how exact our algorithm is. To that end, for a collection of samples of approximated pure Nash equilibrium spendings $\mathcal{S} = (\widehat{\sigma}_{v'})_{1\leq i\leq n}$ for resource $v' \in V$, and corresponding true pure Nash equilibrium spendings $(\sigma^\star_{v'})_{1\leq i\leq n}$, we use the *mean squared error* (MSE):

$$E(\mathcal{S}) = \frac{1}{n}\sum_{i=1}^{n}\left(\sigma^\star_{v',i} - \widehat{\sigma}_{v',i}\right)^2 \tag{7.11}$$

Adaptive Stepsize Factor

A straightforward idea when dealing with stepsizes, which are very common in many types of approximation algorithms, is an adaptive stepsize. The main advantage of this is to increase convergence speed, by increasing the stepsize multiplicatively as long as the steps are going in the same direction. In this section, we analyze this adaptive stepsize factor, determine what value is a sensible choice, and establish if it is necessary.

A comparison of convergence speeds of different adaptive stepsize factors, including a factor of 1.0, meaning that the stepsize is *constant*, is shown in Figure 7.2.

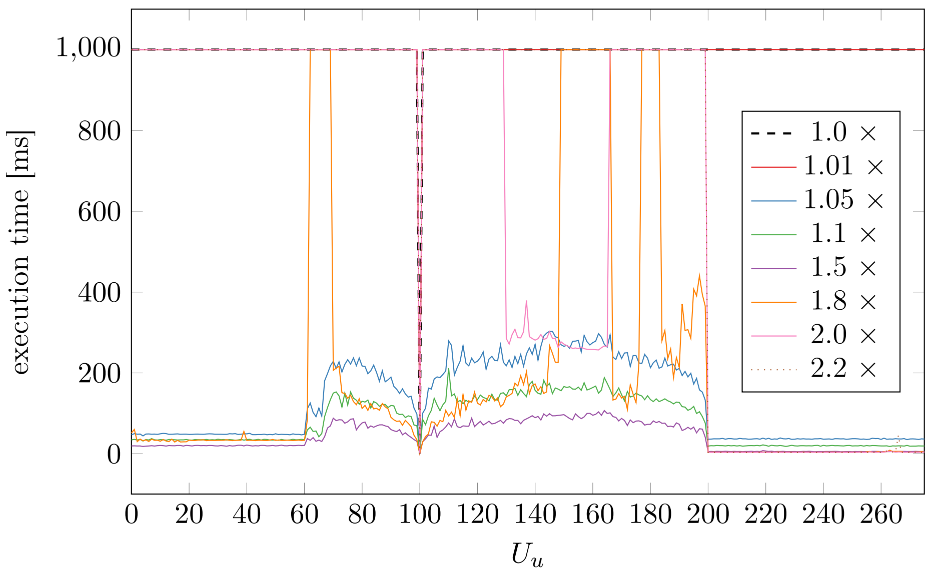

Figure 7.2: Comparison of convergence speeds for different adaptive stepsize factors. $\alpha(u', v') = 0.5$ for all $u', v' \in V$ and $U_v = U_t = 100$.

From the factors chosen, any factor below 1.05 or above 1.5 fails to converge consistently. Any such factor either surpasses the timeout of 1000 ms or does not converge within the given amount of `max_iterations` $= 10^3$. This shows that an adaptive stepsize is necessary in order to achieve reasonable execution times. Attempting to find a middle ground, the factor 1.1 (10%) was chosen for all other experiments. This value was chosen conservatively in order to reduce the likelihood of unforeseen oscillations around the equilibrium, potentially rendering convergence impossible.

The steep dip at $U_u = 100$ is due to the fact that the algorithm is initialized with an equal spending ratio on the resources, and for $U_u = 100$ this happens to be the pure Nash equilibrium.

Despite choosing a specific setup for this experiment, other predetermined values for the influence and utilities confirm these results.

Metaparameter ϵ

Since we are numerically approximating the spending on the resources in a pure Nash equilibrium, we must introduce $\epsilon > 0$ that allows for some unavoidable numerical imprecision. Effectively, this means that if we are closer than ϵ to some value, we consider that we have reached this value. It is clear that, on the one hand, choosing an ϵ that is too large will lead to results that are far away from the true solution. On the other hand, when choosing an ϵ that is too small, the algorithm might take too long to converge to a solution. A comparison of these two aspects with different values for ϵ is provided in order to demonstrate what "too large' and "too small" mean.

First, we consider the effect ϵ has on convergence speed. For a comparison of how fast the algorithm converges for different values of ϵ, see Figure 7.3. Note that this plot does not consider any metric of error minimization, which we deem to be important than convergence speed.

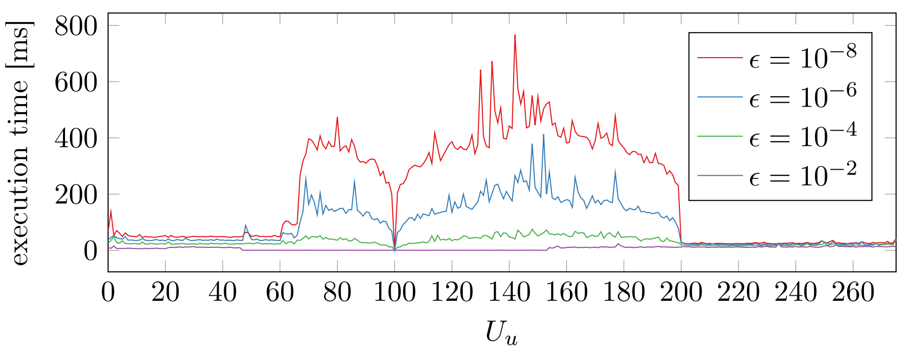

Figure 7.3: Comparison of convergence speeds for different ϵ. $\alpha(u', v') = 0.5$ for all $u', v' \in V$ and $U_v = U_t = 100$.

The steep dip at $U_u = 100$ exists for the same reason as explained previously. As expected, large values for ϵ take almost no time to converge. Having this comparison of convergence speeds for ϵ is useful, but ultimately there must be a tradeoff between convergence speed and error minimization. For this reason, we now consider the effect ϵ has on the error of the approximated solution.

A comparison against the two resource reference solution from Equation 6.2 can be found in Figure 7.4. The algorithm approximates a pure Nash equilibrium for the special case of two resources. This is done by setting $U_t = 0$, $\alpha(t,x) = 0$ and $\alpha(x,t) = 0$ for $x \in \{u, v\}$.

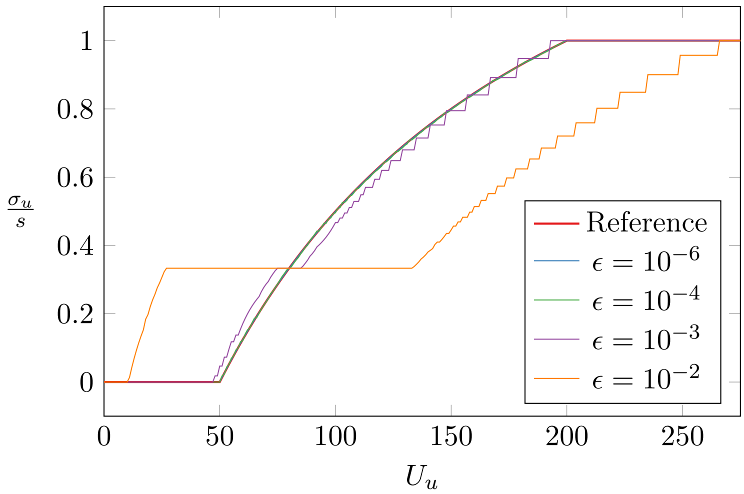

Figure 7.4: Comparison of the approximation algorithm with different values for ϵ against the reference solution from Equation 6.2.

Qualitatively, any ϵ below 10^{-4} provides what one could consider sufficient error minimization for the approximative pure Nash equilibrium spending. The mean squared error for $\epsilon = 10^{-6}$ is approximately $8.61058 \cdot 10^{-4}$. For this reason, and because the runtime constraint is not an issue, $\epsilon = 10^{-6}$ was used throughout this thesis. Note that in order to compute the mean squared error given in Equation 7.11, the spendings ratios presented in Figure 7.4 must be converted back to the spendings by multiplying with the budget $s = 1000$. If this is not done, the mean squared error yields values s^2 times smaller than the true error.

As before, this experiment was performed in other setups which were able to reproduce and confirm these results.

7.2.3 Correctness

In this section, the correctness of the algorithm presented previously is discussed. The algorithm always moves into a direction that improves the utility, achieving the desired maximization property. This is given by the nature of the algorithm, as it performs "gradient ascent", going along the direction of the gradient. Due to the usage of stepsize it is in principle possible that oscillations around the maximum occur, but with the chosen values for the adaptive stepsize factor and ϵ, this was never observed. Even if this problem should occur for some given set of parameters, the subprocedure in Algorithm 7.1 will terminate after `max_iterations` iterations, and therefore return a resulting spending which is still sufficiently close to the true maximum due to the choice of the initial stepsize.

The true analytical solution for a resource competing game with two resources and two symmetric players is known from Equation 6.2. We proceed to compare our algorithm with this solution by setting $U_t = 0$, $\alpha(t,x) = 0$ and $\alpha(x,t) = 0$ for $x \in \{u, v\}$. This comparison can be found in Figure 7.5, where in each plot we vary one of the remaining parameters U_u, U_v, $\alpha(u,v)$ and $\alpha(v,u)$.

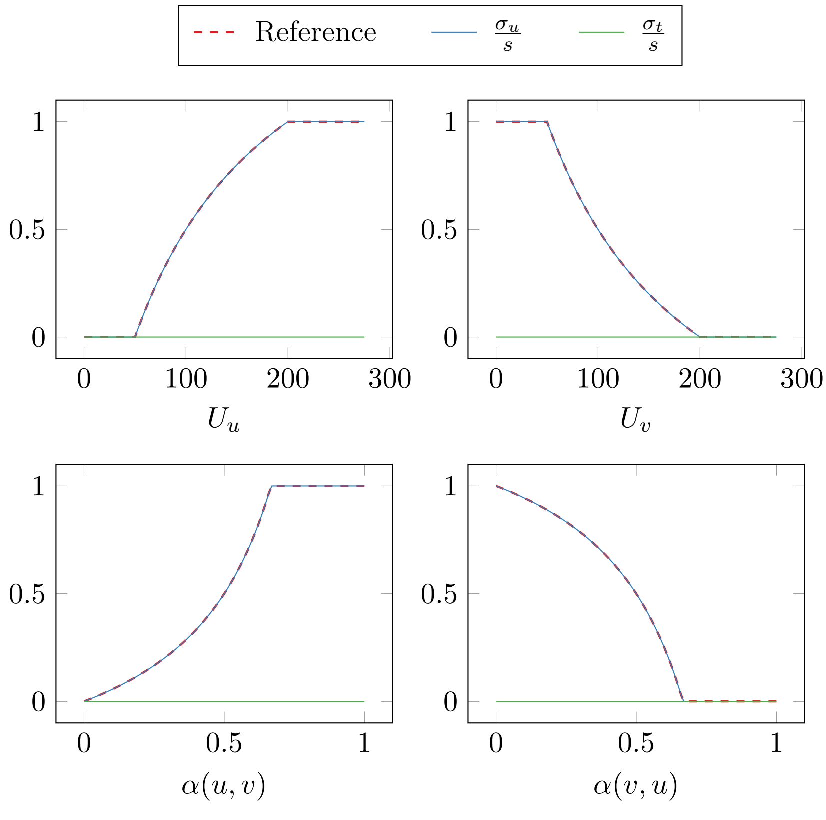

Figure 7.5: Comparison of the reference solution from Equation 6.2 with the solution of the approximation algorithm by settings its parameters accordingly. On the y-axis are the corresponding spending ratios, where the reference solution shows the spending ratio on resource u.

In this experiment, the error of our algorithm is qualitatively non-existent, and numerically vanishingly small, that is, the mean squared error is at most in the order of 10^{-3} for each of the four plots. Note that, the same modification as before must be performed on the values presented in Figure 7.5 in order to compute the mean squared error. While this comparison is not a proof per se, it gives a good indication that the algorithm functions as specified.

As shown in Section 7.1, the utility function we are trying to maximize is concave, meaning that any local maximum is a global maximum, although there might be some sort plateau because u is not *strictly* concave. This is not an issue, since in this case the value of any maxima is unique, and still determines a pure Nash equilibrium, as no player can strictly increase their utility by making a unilateral move.

One must be careful to enjoy this fact, since we are dealing with an optimization problem constrained by the budget s. In our model, this means that the spendings σ_u, σ_v and σ_t must fulfill the constraint

$$\sigma_u + \sigma_v + \sigma_t \leq s \tag{7.12}$$

To visualize how a constraint can cause problems in such an optimization problem, suppose we are trying to maximize the function $f(x) = -(\frac{x}{2}-2)^2+1$, the dimensions of which match those of the utility function for the case of a resource competing game with two resources. Additionally, suppose our constraint is

$$C_f = (-\infty, 1.5] \cup [3, \infty) \tag{7.13}$$

Figure 7.6 shows the function f and the constraints. As is obvious from the plot, the maximum of this function considering the constraints is at $x = 1.5$ and has value $f(1.5) = 2.75$. But what if a stepwise optimization algorithm initializes its first step with $x_0 = 3.5$? Then it will go along the x-axis in negative direction, and, in $i \in \mathbb{N}$ steps, reach the border at $x_i = 3$, and might conclude that $f(3) = 2$ is the maximum. A solution for this problem would be to then check all borders. But, if the problem is multidimensional, that itself is another nontrivial optimization problem.

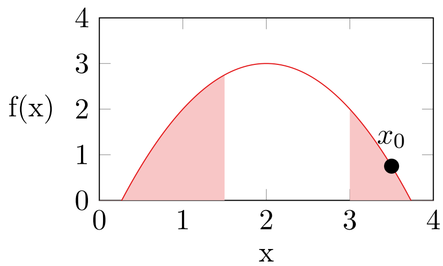

Figure 7.6: Function f and constraint C_f.

The issue arose in this constructed example because of how the constraint in Equation 7.13 is defined. It forms a non-convex set. Luckily, in our model of the resource competing game, the constraints will always form convex sets. Therefore, the algorithm will not run into this problem. Specifically, for the resource competing game with three resources as discussed in this chapter, the constraint in Equation 7.12 forms the convex set

$$C_{u_i} = \{(\sigma_u, \sigma_v) \mid \sigma_u + \sigma_v \leq s\}, \tag{7.14}$$

where we consider σ_u and σ_v to be the variables and then have $\sigma_t = s - \sigma_u - \sigma_v$ as done many times before. For any other setting as introduced in Section 7.1 the constraint is defined analogously. Therefore, this anomaly cannot not happen in our model.

7.3 Toward an Analytical Solution

7.3.1 Varying the Influence

To get an idea of what an analytical solution–if it exists–for the spending on each of the three resources might look like, we conduct an experiment using the algorithm presented in Section 7.2.1, where we vary one of the parameters at a time.

The results of such an experiment is shown in Figure 7.7. At the start of this experiment, all values of α are set to a constant denoted as $\alpha^\dagger$, i.e., $\alpha(u', v') = \alpha^\dagger$ for $u', v' \in V$ and $u' \neq v'$. In Figure 7.7 we have $\alpha^\dagger = 0.5$. Additionally, the utilities of all resources are set to 100, i.e., $U_u = U_v = U_t = 100$. Then, in each of the six plots one of the α-values is varied.

When looking at Figure 7.7, we can observe that in every plot, there are two *turning points* (TPs) that look to be exactly at the same value for the varying α. For this choice of $\alpha^\dagger = 0.5$, these two turning points seem to be located approximately at 0.25 and 0.65. Performing this experiment with $\alpha^\dagger = 0.2$ shows that these turning points are *not* independent of the choice of $\alpha^\dagger$. The existence of these turning points hint toward some kind of piecewise definition of the analytical solution.

To further explore this thought, the experiment from Figure 7.7 is run with many different values for $\alpha^\dagger$. Figure 7.8 shows these two turning points in relation to $\alpha^\dagger$. These turning points were determined using a `Python` program that can be found here.

The next steps toward an analytical solution might consist of examining the functions separately between the turning points, since they seem to have a simpler form there. The definition of the function would then be piecewise, with the turning points as borders.

Quotients of polynomials of degree at most two were fit to the curves from Figure 7.8 using the library `scipy` [16]. This yielded fairly small errors, but it is

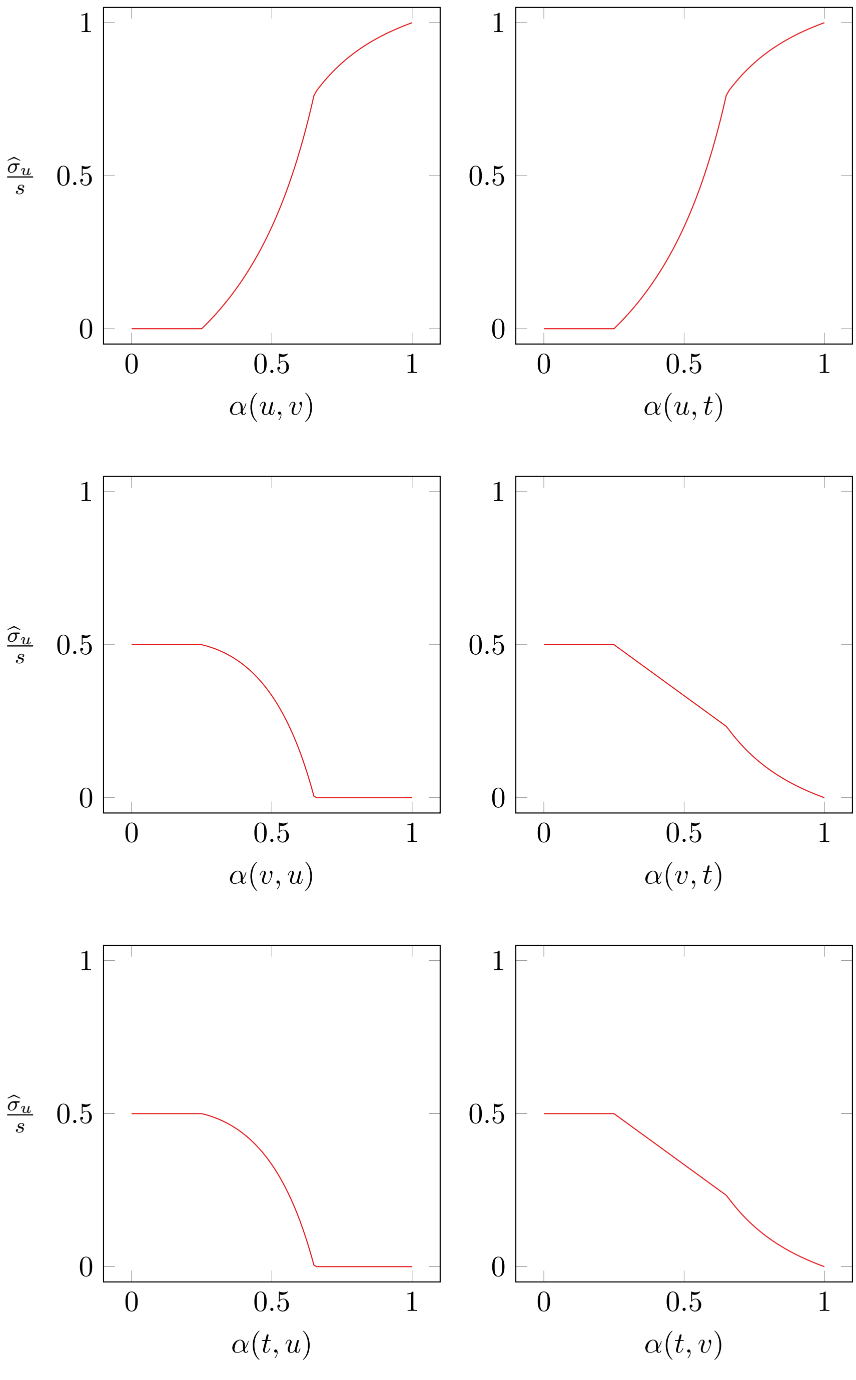

Figure 7.7: In each the plots above the corresponding α is varied from 0 to 1 on the x-axis. All other α values are set to 0.5. The utilities are fixed to $U_u = U_v = U_t = 100$.

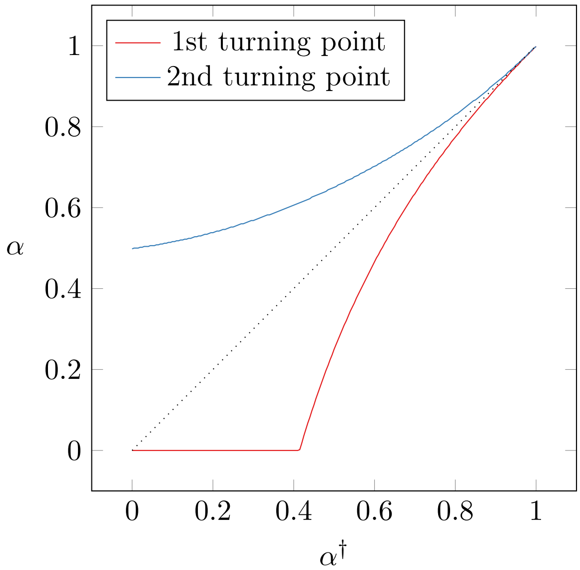

Figure 7.8: Turning points for different values for $\alpha^\dagger$. The y-axis shows the "position" of the turning points as in Figure 7.7.

doubtful whether this is close enough to the true form of the two functions for any meaningful insight to be made. The reason for this is that, while the errors were small, it is possible that the polynomials were of a high enough degree to fit the functions *fairly* well, but not show the true form of the functions.

Chapter 8

Conclusion

8.1 Main Findings

Chapter 4 analyzes the basic properties directly implied by the model. Most importantly, it shows that when two players are in a pure Nash equilibrium then their spending ratio with respect to their respective budgets are equal on each resource.

In Chapter 5 the basic model introduced in Chapter 3 is examined. Section 5.1 considers the case where there is no influence at all, i.e. all resources are totally disconnected. In this case, to be in a pure Nash equilibrium, the players distribute their spendings according to the ratio of the of a resources' utility to the sum of all utilities, as stated in Equation 5.1. In Section 5.2 the existence of a pure Nash equilibrium in a resource competing game with any number of resources and two symmetric players is proven. For the case of two resources, the analytical solution in Equation 5.8 provides the strategy profile directly, and for the general case of m resources an algorithm is presented which allows the computation of the strategy profile after a selection process among the resources. In Section 5.3 the analogous results are proven for the case where, in general, the two players are not symmetric, which, for two resources, culminates in Equation 5.27. The analytical formulas for the symmetric and asymmetric case are analogous, where the former is scaled by the common budget and the latter is scaled by the player-specific budget, supporting the results gained in Chapter 4 about the equal spending ratio.

Chapter 6 allows any pair of resources to have a specific influence on each other. The claims from Sections 5.2.1 and 5.3.1 regarding the case of two resources are proven. The resulting formulas are generalizations of those found in the previous chapter.

Chapter 7 further explores the concept of variable influence by inspecting a resource competing game with three resources and two symmetric players. Since the analytical maximization methods chosen previously are not fruitful, a numerical approach is chosen. An algorithm which uses the gradients in different so-called *settings* to converge to a pure Nash equilibrium is presented in Section 7.2. Some adjustments of this algorithm such as an adaptive stepsize factor and a precision parameter ϵ are analyzed to strike a balance between error minimization and convergence speed, while focusing on error minimization. The correctness is explained with an intuitive argument and backed up with a comparison of the approximation algorithm with the two resource solution found in Chapter 6.

8.2 Outlook

More Resources The next steps of this topic concerning more than two resources would consist of using the approximation algorithm from Chapter 7, which approximates the pure Nash equilibrium for three resources and two symmetric players, to find an analytical solution which directly provides this result given the nine problem parameters, given it exists. This solution would also apply to the case with asymmetric players, as shown in Chapter 4, by scaling the spendings with the players' budgets. This analytical solution might be used to work toward finding a solution for the most general case of the resource competing game with two players, namely for m resources.

More Players It appears likely that the results from Chapter 4 also apply when there are more than two players. Considering this, extending the problems discussed in this thesis to more than two players is straightforward.

Adjustments to the Model While the model is well suited to reflect a sense of influence and makes finding the "ideal" spending for each player nontrivial, it certainly has its drawbacks, which will now be discussed, along with some ways how to improve the model to allow for a more interesting analysis.

The resource competing game introduced in the model is neither directly connected to potential games [5] nor to congestion games [4]. This would allow to profit from previous research in this space and would enable a more advanced analysis, without the need to focus on the simple cases. Since the model was not built on top of some well-known concept but developed from scratch, there is no easy way to modify the model this way.

Due to the definition of the model the social utility is always equal to the sum of all resources no matter the strategy profile. This means that the Price of Anarchy [17, 18] is equal to 1, essentially rendering the concept useless[1]. Therefore, no behavior of the players negatively influences the social utility, meaning the social optimum is always achieved. An interesting way to change the model

[1] Note that this does not hold if we allow only discrete spendings, as in [1].

to allow for a useful concept of the Price of Anarchy would be to introduce some kind of notion of contention penalty on the resources. That is, a resource could become less useful to one player if that player must share it's utility with other players. For example, imagine a resource has utility 100. As of now, every player will get his share of this utility according to his influence on the resource compared to the other players' influences on the resource. Consider two players split this utility equally, both receiving 50. One could imagine that, to some players, only having half of the total influence on some resource is worth *less than* half the utility of the resource. This would result in a social utility smaller than the sum of all resources' utilities and therefore in a Price of Anarchy of greater than 1, and would make for more interesting strategy profiles.

Further, Chapter 4 shows that in a pure Nash equilibrium, each player receives a share of the sum of the resources' utilities according to the proportion of their budget to the sum of all players' budgets. For example, if a player has twice the budget of some other player, they will also receive twice the utility of the resources. While this can certainly be considered "fair", the real world is not always fair. The modification mentioned previously could also have an impact on this, making the scenario more realistic, if needed.

This work shows that, for the cases examined, the utility function is continuous and concave. Therefore, any local maximum is the global maximum, which is unique. One can imagine that this does not nicely reflect the real world, where there is usually not one "best solution". A potential modification would be similar to the one described above.

The players in the resource competing game must all "agree" on the influence between resources. It is entirely possible that the assessment of some player of how the resources influence each other differs from other players' assessments. Unfortunately, there does not seem to be a straightforward solution to this without great modification to the model. That the players have to agree on values for the utilities is also a downside, but this could potentially be solved; one could normalize the utilities for each player so that they sum to the same amount. For this, each player $i \in P$ would have a utility vector $(U_{i,v})_{v \in V}$, where $U_{i,v}$ is the utility that player i considers $v \in V$ to have. This would complicate the finding of pure Nash equilibria which would have to be re-evaluated since the results of Chapter 4 would no longer apply.

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