Projective unification through duality

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Abstract

Unification problems can be formulated and investigated in an algebraic setting, by identifying substitutions to modal algebra homomorphisms. This opens the door to applications of the notorious duality between modal algebras and descriptive frames. Through substantial use of this correspondence, we give a necessary and sufficient condition for modal formulas to be projective. Applying this result to a number of different logics, we then obtain concise and lightweight proofs of their projective – or non-projective – character. In particular, we prove that the projective extensions of **K5** are exactly the extensions of **K45**. This resolves the open question of whether **K5** is projective.

Keywords: Normal modal logics. Elementary unification. Projective formulas. Duality theory.

1 Introduction

In a propositional language, substitutions can be defined as functions mapping variables to formulas. For reasons related to Unification Theory [2, Section 2], it is usually considered that such functions are almost everywhere equal to the identity function. As a result, one can see a substitution as a function $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ where $\mathcal{L}_P$ (resp. $\mathcal{L}_Q$) is the set of all formulas with variables in a finite set P (resp. Q), and satisfying ($\blacklozenge$) $\sigma(\circ(\varphi_1, \ldots, \varphi_n)) = \circ(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ for all n-ary connectives $\circ$ of the language and all formulas $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_P$. According to this point of view, which is the one usually considered within the context of modal logics [3,7,10], two substitutions $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ and $\tau : \mathcal{L}_P \to \mathcal{L}_{Q'}$ are said to be equivalent with respect to a propositional logic $\mathbf{L}$ (in symbols $\sigma \simeq_{\mathbf{L}} \tau$) if for all $p \in P$, the formulas $\sigma(p)$ and $\tau(p)$ are $\mathbf{L}$-equivalent.

A formula $\varphi \in \mathcal{L}_P$ is $\mathbf{L}$-unifiable if $\mathbf{L}$ contains instances of φ. In that case, any substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ such that $\sigma(\varphi) \in \mathbf{L}$ counts as a $\mathbf{L}$-unifier of φ. A $\mathbf{L}$-unifiable formula $\varphi \in \mathcal{L}$ is $\mathbf{L}$-projective if it possesses a projective $\mathbf{L}$-unifier, that is to say a $\mathbf{L}$-unifier σ such that $\varphi \vdash_{\mathbf{L}} \sigma(p) \leftrightarrow p$ holds for all $p \in P$. Such unifiers are interesting because they constitute by themselves minimal complete

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sets of unifiers [3,7,10]. For this reason, it is of the utmost importance to be able to determine if a given formula is **L**-projective.

Now, condition (⧫) may evoke homomorphism properties. Following this observation, Unification Theory was also formalized and studied in an algebraic setting [9,20]. Indeed, let us consider the Lindenbaum-Tarski algebra [3] $\mathbf{A}_P$ obtained by taking the quotient of $\mathcal{L}_P$ modulo the relation $\equiv_{\mathbf{L}}$ of **L**-equivalence. One can associate to a substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ the map $\sigma^{\triangleright} : \mathbf{A}_P \to \mathbf{A}_Q$ by setting $\sigma^{\triangleright}([\varphi]_{\mathbf{L}}) := [\sigma(\varphi)]_{\mathbf{L}}$ for any formula $\varphi \in \mathcal{L}_P$, whose equivalence class modulo $\equiv_{\mathbf{L}}$ is denoted by $[\varphi]_{\mathbf{L}}$. In this perspective, condition (⧫) then truly expresses the homomorphic character of $\sigma^{\triangleright}$. Obviously, this association between substitutions and homomorphisms of Lindenbaum algebras is one-to-one modulo $\simeq_{\mathbf{L}}$: substitutions associated to the same homomorphism are equivalent modulo $\simeq_{\mathbf{L}}$. Then various properties of substitutions, such as being a **L**-unifier of a formula, admit an algebraic counterpart too.

In this paper, we combine this correspondence with a more traditional one, provided by *Duality Theory*. For any set P of variables, there is indeed a tight connection between the Lindenbaum algebra $\mathbf{A}_P$ and the canonical frame $\mathfrak{F}_P$ of **L** over P, determined by the set of all ultrafilters [4] on $\mathbf{A}_P$. Homomorphisms between Lindenbaum algebras are then in correspondence with bounded morphisms between canonical frames. See [4, Chapter 5], [5, Chapter 7] and [16, Chapter 4] for a general introduction to this subject. Given a finite set P of variables, we make essential use of this duality to construct a necessary and sufficient condition for $\varphi \in \mathcal{L}_P$ to be **L**-projective: the existence of a bounded morphism $f : \mathfrak{F}_P \to \mathfrak{F}_P$ such that the image of f is contained in $\widehat{\varphi}^{\infty}$ and all elements of $\widehat{\varphi}^{\infty}$ are fixpoints of f, where $\widehat{\varphi}^{\infty}$ denotes the set of all points in $\mathfrak{F}_P$ containing $[\Box^n\varphi]_{\mathbf{L}}$ for all $n \in \mathbb{N}$.

This paper is structured as follows. In Section 2, we introduce some basics of modal logic [5] and Unification Theory [6]. Section 3 introduces modal algebras and in particular Lindenbaum algebras, and explains how to 'algebraize' unification problems. In Section 4, we develop some basics of Duality Theory in modal logics, concentrating on the bijective correspondence between bounded morphisms of canonical frames and homomorphisms of Lindenbaum algebras. We then apply these tools to establish the above-mentioned necessary and sufficient condition for a formula to be projective. In Section 5, we use this characterization to investigate the projective character of the extensions

[3] Or Lindenbaum algebra for short.

[4] The points of $\mathfrak{F}_P$ are usually defined as maximal **L**-consistent sets of formulas instead of ultrafilters, but as explained in Section 5, this makes no difference.

[5] We follow the same conventions as in [4,5,16] for talking about modal logics: **KT** is the least modal logic containing the formula usually denoted (**T**), **S4** is the least modal logic containing the formulas usually denoted (**T**) and (**4**), etc.

[6] We usually distinguish between elementary unification and unification with parameters. In elementary unification, all variables are likely to be replaced by formulas when one applies a substitution [1]. In unification with parameters, some variables — called parameters — remain unchanged [19, Chapter 6]. In this paper, we only interest in elementary unification.

of three selected logics: $\mathbf{K4}_n\mathbf{B}_k$, **K4D1**, and **K5**.

2 Background

2.1 Some functional vocabulary

Let $f : X \to Y$ be a function. If $A \subseteq X$, we write $f[A] := \{f(x) \mid x \in A\}$. If $B \subseteq Y$, we write $f^{-1}[B] := \{x \in X \mid f(x) \in B\}$. We denote by $\mathrm{Im}\ f := f[X]$ the *image* of f. We denote by $\mathrm{fp}\ f := \{x \in X \mid f(x) = x\}$ the set of *fixpoints* of f. Given two functions $f : X \to Y$ and $g : Y \to Z$ we denote by $gf : X \to Z$ the *composition* of f and g, defined by $gf : x \mapsto g(f(x))$.

2.2 Modal logics

Let Prop be an infinite countable set of propositional variables. If $P \subseteq \mathsf{Prop}$ we define the modal language $\mathcal{L}_P$ over P by the following grammar:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

where $p \in P$. We write $\mathcal{L} := \mathcal{L}_{\mathsf{Prop}}$. The abbreviations $\top, \vee, \to, \leftrightarrow, \Diamond$ are defined as usual. Given $\varphi \in \mathcal{L}$ we denote by $\mathsf{var}(\varphi)$ the set of variables occurring in φ. If $n \in \mathbb{N}$ we define inductively $\Box^n\varphi$ and $\Box^{\leq n}\varphi$ by:

- $\Box^0\varphi := \varphi$ and $\Box^{\leq 0}\varphi := \varphi$,
- for all $n \in \mathbb{N}$, $\Box^{n+1}\varphi := \Box\Box^n\varphi$ and $\Box^{\leq n+1}\varphi := \Box\Box^n\varphi \wedge \Box^{\leq n}\varphi$.

We then define $\Diamond^n\varphi := \neg\Box^n\neg\varphi$ and $\Diamond^{\leq n}\varphi := \neg\Box^{\leq n}\neg\varphi$.

Definition 2.1 A *normal modal logic* is a set $\mathbf{L}$ of formulas such that:

- $\mathbf{L}$ is closed under *uniform substitution* (for all formulas φ, ψ, if $\varphi \in \mathbf{L}$ and ψ is obtained from φ by uniformly replacing variables in φ by arbitrary formulas then $\psi \in \mathbf{L}$),
- $\mathbf{L}$ contains all propositional tautologies,
- $\mathbf{L}$ is closed under *modus ponens* (for all formulas φ, ψ, if $\varphi \in \mathbf{L}$ and $\varphi \to \psi \in \mathbf{L}$ then $\psi \in \mathbf{L}$),
- $\mathbf{L}$ contains all formulas of the form $\Box(p \to q) \to (\Box p \to \Box q)$,
- $\mathbf{L}$ is closed under *generalization* (for all formulas φ, if $\varphi \in \mathbf{L}$ then $\Box\varphi \in \mathbf{L}$).

From now on we fix a normal modal logic $\mathbf{L}$. Instead of $\varphi \in \mathbf{L}$ we may also write $\vdash_{\mathbf{L}} \varphi$. Given $\varphi, \psi \in \mathcal{L}$, we write $\varphi \equiv_{\mathbf{L}} \psi$ in case $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$. Then $\equiv_{\mathbf{L}}$ is an equivalence relation, and we denote by $[\varphi]_{\mathbf{L}}$ the equivalence class of φ modulo $\equiv_{\mathbf{L}}$. We call $\mathbf{L}$ *locally tabular* if for all finite sets $P \subseteq \mathsf{Prop}$, there are only finitely many equivalence classes modulo $\equiv_{\mathbf{L}}$ [7]. A set Σ of formulas is $\mathbf{L}$-*consistent* if there are no formulas $\varphi_1, \ldots, \varphi_n \in \Sigma$ such that $\vdash_{\mathbf{L}} \neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$.

Given $\varphi, \psi \in \mathcal{L}$, we call a *derivation of ψ from φ in* $\mathbf{L}$ a sequence of formulas $\chi_0, \ldots, \chi_n \in \mathcal{L}$ such that $\chi_n = \psi$ and for all $i \in \{0, \ldots, n\}$, at least one of the following conditions holds:

[7] Locally tabular modal logics possess interesting properties, in particular when it comes to decidability [17].

- $\chi_i \in \mathbf{L}$,
- $\chi_i = \varphi$,
- there exists $j, k \in \{0, \ldots, n\}$ such that $i > j, k$ and χ_i is obtained from χ_j and χ_k by modus ponens,
- there exists $j \in \{0, \ldots, n\}$ such that $i > j$ and χ_i is obtained from χ_j by generalization.

If there exists a derivation of ψ from φ in $\mathbf{L}$, we shall say that ψ is *deducible from φ in* $\mathbf{L}$, and write $\varphi \vdash_{\mathbf{L}} \psi$. A more concise characterization of derivable formulas is given by the following result:

Proposition 2.2 *The following conditions are equivalent:*[8]

(i) $\varphi \vdash_{\mathbf{L}} \psi$,

(ii) *there exists* $k \in \mathbb{N}$ *such that* $\vdash_{\mathbf{L}} \Box^{\leq k}\varphi \to \psi$.

We call an *extension* of $\mathbf{L}$ any normal modal logic $\mathbf{L}'$ such that $\mathbf{L} \subseteq \mathbf{L}'$. If $\mathbf{L}$ is a normal modal logic and $\Sigma \subseteq \mathcal{L}$, we denote by $\mathbf{L} + \Sigma$ the smallest extension of $\mathbf{L}$ containing Σ.

2.3 Unification

A *substitution* is a function $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ with $P, Q \subseteq \mathsf{Prop}$ finite and such that for all $\varphi, \psi \in \mathcal{L}_P$ we have $\sigma(\bot) = \bot$, $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$, and $\sigma(\Box\varphi) = \Box\sigma(\varphi)$. Let $\mathcal{S}$ be the set of all substitutions. The equivalence relation $\simeq_{\mathbf{L}}$ on $\mathcal{S}$ is defined by

$$\sigma \simeq_{\mathbf{L}} \tau \text{ if and only if } \sigma(p) \equiv_{\mathbf{L}} \tau(p) \text{ for all } p \in P$$

where $\sigma, \tau : \mathcal{L}_P \to \mathcal{L}_Q$. Then, we define the preorder $\preccurlyeq_{\mathbf{L}}$ on $\mathcal{S}$ by

$$\sigma \preccurlyeq_{\mathbf{L}} \tau \text{ iff there exists a substitution } \mu : \mathcal{L}_Q \to \mathcal{L}_R \text{ such that } \mu\sigma \simeq_{\mathbf{L}} \tau$$

where $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ and $\tau : \mathcal{L}_P \to \mathcal{L}_R$. Given $\varphi \in \mathcal{L}_P$ we say that $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ is a $\mathbf{L}$-*unifier* of φ if we have $\vdash_{\mathbf{L}} \sigma(\varphi)$. A $\mathbf{L}$-*unifier* $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ of φ is called *concise* if $P = \mathsf{var}(\varphi)$. The formula φ is $\mathbf{L}$-*unifiable* if there exists a $\mathbf{L}$-unifier of φ. A set $\mathcal{T}$ of concise $\mathbf{L}$-unifiers of φ is said to be *complete* if for all concise $\mathbf{L}$-unifiers σ of φ, there exists $\tau \in \mathcal{T}$ such that $\tau \preccurlyeq_{\mathbf{L}} \sigma$. In addition, we call $\mathcal{T}$ a *basis* if $\sigma, \tau \in \mathcal{T}$ and $\sigma \preccurlyeq_{\mathbf{L}} \tau$ implies $\sigma = \tau$.

Proposition 2.3 *For all* $\varphi \in \mathcal{L}$, *if* φ *is* $\mathbf{L}$-*unifiable then for all bases* $\mathcal{T}, \mathcal{U}$ *of concise* $\mathbf{L}$-*unifiers of* φ, *we have* $\mathtt{Card}(\mathcal{T}) = \mathtt{Card}(\mathcal{U})$ [9, Section 2].

A central problem in unification theory is whether a $\mathbf{L}$-unifiable formula possesses a basis of concise $\mathbf{L}$-unifiers. When this is the case, Proposition 2.3 raises a more refined question, that is, how large is such a basis? This gives rise to a full classification of logics based on the cardinality of these bases.

[8] This is a simplified version of the so-called Deduction Theorem in modal logics. When $\mathbf{L} = \mathbf{K}$, see [5, Theorem 3.51] for a proof of it, which can be easily adapted to the general case. See [11] for an interesting discussion of this theorem.

Definition 2.4 For all **L**-unifiable $\varphi \in \mathcal{L}$:

- φ is **L**-*nullary* if there exists no basis of concise **L**-unifiers of φ,
- φ is **L**-*infinitary* if there exists an infinite basis of concise **L**-unifiers of φ,
- φ is **L**-*finitary* if there exists a basis of concise **L**-unifiers of φ with finite cardinality ≥ 2,
- φ is **L**-*unitary* if there exists a basis of concise **L**-unifiers of φ with cardinality 1.

We shall say that:

- **L** is *nullary* if there exists a **L**-nullary **L**-unifiable formula,
- **L** is *infinitary* if every **L**-unifiable formula is either **L**-infinitary, or **L**-finitary, or **L**-unitary and there exists a **L**-infinitary **L**-unifiable formula,
- **L** is *finitary* if every **L**-unifiable formula is either **L**-finitary, or **L**-unitary and there exists a **L**-finitary **L**-unifiable formula,
- **L** is *unitary* if every **L**-unifiable formula is **L**-unitary.

A special case that deserves our attention is that of projective unifiers. Let $\varphi \in \mathcal{L}_P$ with $P \subseteq \mathsf{Prop}$ finite. We call a φ-*projective substitution* a substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ such that for all $p \in P$ we have $\varphi \vdash_{\mathbf{L}} \sigma(p) \leftrightarrow p$. We then say that φ is **L**-*projective* if there exists a φ-projective **L**-unifier of φ. Finally, we shall say that **L** is projective if every **L**-unifiable formula is **L**-projective. For an introduction to projective unification, see [3,7,10]. The logic **K** has been shown to be nullary by Jeřábek [13] who has proved that the unifiable formula $p \to \Box p$ has no basis of unifiers in **K**. The logic **S5**, however, is known to be unitary since many years [1,6]. The truth is that if $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ is a **S5**-unifier of a formula φ then the substitution $\epsilon : \mathcal{L}_P \to \mathcal{L}_P$ defined by $\epsilon(p) = (\Box\varphi \wedge p) \vee (\Diamond\neg\varphi \wedge \sigma(p))$ is a projective unifier of φ in **S5**. Projective unifiers have been used by Ghilardi [10] within the context of a transitive modal logic **L** like **K4**, **S4**, etc. In fact, Ghilardi has shown that if a formula φ possesses a unifier then it possesses a finite basis of unifers, this basis being the set of projective unifiers of a finite set of projective formulas of modal degree at most equal to the modal degree of φ, having the same propositional variables as φ and implying φ in **L**. See also [12] for a syntactic approach to unification in transitive modal logics. Recently, Dzik and Wojtylak [8] have proved that the projective extensions of **S4** are exactly the extensions of **S4.3**, a result improved by Kost [14] who has demonstrated that the projective extensions of **K4** are exactly the extensions of **K4D1**. Finally, Kostrzycka [15] has considered the modal logics of the form $\mathbf{K4}_n\mathbf{B}_k$ (see Section 5) and has proved that they are projective. For the proofs of Propositions 2.5–2.8 below, see [10, Section 2].

Proposition 2.5 *Let $\varphi \in \mathcal{L}_P$, and $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ be a substitution. Then σ is φ-projective if and only if for all formulas $\psi \in \mathcal{L}_P$, we have $\varphi \vdash_{\mathbf{L}} \sigma(\psi) \leftrightarrow \psi$.*

Proposition 2.6 *For all formulas $\varphi \in \mathcal{L}_P$, for all φ-projective substitutions $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ and for all **L**-unifiers $\tau : \mathcal{L}_P \to \mathcal{L}_Q$ of φ, we have $\sigma \preccurlyeq_{\mathbf{L}} \tau$.*

Proposition 2.7 *For all* $\mathbf{L}$*-unifiable* $\varphi \in \mathcal{L}$*, if* φ *is* $\mathbf{L}$*-projective then* φ *is* $\mathbf{L}$*-unitary.*

Proposition 2.8 *If* $\mathbf{L}$ *is projective then* $\mathbf{L}$ *is unitary.*

As a side note, the extension of a projective logic is, up to our knowledge, not necessarily projective.

Remark 2.9 If $\varphi \in \mathcal{L}$, there are obviously infinitely many subsets P of Prop such that $\varphi \in \mathcal{L}_P$. For this reason, many authors require a $\mathbf{L}$-unifier of φ to be of the form $\sigma : \mathcal{L}_{\mathsf{var}(\varphi)} \to \mathcal{L}_Q$ (or 'concise' in our terminology). In our setting, we also allow one to talk about the unifying or φ-projective character of any substitution of the form $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ with $\mathsf{var}(\varphi) \subseteq P$. This offers more flexibility, which will be helpful in Section 5.

As a result, our definition of $\mathbf{L}$-unifiable and $\mathbf{L}$-projective formulas are non-standard[9], but this is harmless. Indeed, if $\mathsf{var}(\varphi) \subseteq P \subseteq P'$ and $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ is a substitution, one can define the substitution $\sigma' : \mathcal{L}_{P'} \to \mathcal{L}_Q$ by setting $\sigma'(p) := \sigma(p)$ for all $p \in P$ and $\sigma'(p) := p$ for all $p \in P' \setminus P$, and it is clear that σ is a $\mathbf{L}$-unifier of φ (resp. is φ-projective) if and only if σ' is a $\mathbf{L}$-unifier of φ (resp. is φ-projective). Likewise, if $\mathsf{var}(\varphi) \subseteq P \subseteq P'$ and $\sigma : \mathcal{L}_{P'} \to \mathcal{L}_Q$ is a substitution, let us denote by $\sigma' := \sigma_{|\mathcal{L}_P}$ the restriction of σ to $\mathcal{L}_P$. Then σ is a $\mathbf{L}$-unifier of φ (resp. is φ-projective) if and only if σ' is a $\mathbf{L}$-unifier of φ (resp. is φ-projective).

2.4 General frames

A *general frame* is a pair $\mathfrak{F} = (X, R, \mathcal{A})$ with X a set of possible worlds, R a binary relation on X, and $\mathcal{A} \subseteq \mathcal{P}(X)$ such that:

- $\emptyset \in \mathcal{A}$,
- $A \in \mathcal{A}$ implies $X \setminus A \in \mathcal{A}$,
- $A, B \in \mathcal{A}$ implies $A \cap B \in \mathcal{A}$,
- $A \in \mathcal{A}$ implies $\Box_R A \in \mathcal{A}$, with $\Box_R A := \{x \in X \mid \forall y \in X, xRy \Rightarrow y \in A\}$.

Further, we call $\mathfrak{F}$ *differentiated* if for all $x, y \in X$ such that $x \neq y$, there exists $A \in \mathcal{A}$ such that $x \in \mathcal{A}$ and $y \notin \mathcal{A}$. We call $\mathfrak{F}$ *tight* if for all $x, y \in X$ such that *not* xRy, there exists $A \in \mathcal{A}$ such that $x \in \Box_R A$ and $y \notin A$. We call $\mathfrak{F}$ *compact* if for all $\mathcal{B} \subseteq \mathcal{A}$, if $\bigcap \mathcal{B}' \neq \emptyset$ whenever $\mathcal{B}' \subseteq \mathcal{B}$ is finite, then $\bigcap \mathcal{B} \neq \emptyset$. Finally, we say that $\mathfrak{F}$ is a *descriptive frame* if $\mathfrak{F}$ is tight, compact and differentiated.

If $\mathfrak{F} = (X, R, \mathcal{A})$ and $\mathfrak{F}' = (X', R', \mathcal{A}')$ are two general frames, we call a *bounded morphism* from $\mathfrak{F} = (X, R, \mathcal{A})$ to $\mathfrak{F}' = (X', R', \mathcal{A}')$ a map $f : X \to X'$ such that:

- if xRy then $f(x)Rf(y)$,
- if $f(x)R'y'$ then there exists $y \in X$ such that $f(y) = y'$ and xRy,
- if $A' \in \mathcal{A}'$ then $f^{-1}[A'] \in \mathcal{A}$.

[9] Note that the unification type *does* remain standard, since it is defined with respect to concise unifiers only.

3 An algebraic perspective

The algebraic aspects of unification have already been investigated in e.g. [9,20]. Here we give a lightweight, self-sufficient account of them. The goal of this section is essentially to state Proposition 3.5, a modest but inspiring starting point.

Definition 3.1 A *modal algebra* is a structure $\mathbf{A} = (A, 0, \neg, \wedge, \Box)$ with $(A, 0, \neg, \wedge)$ a Boolean algebra and $\Box : A \to A$ an operator satisfying $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in \mathbf{A}$. For convenience we will identify $\mathbf{A}$ to its underlying set A.

Definition 3.2 A *homomorphism* from a modal algebra $\mathbf{A}$ to a modal algebra $\mathbf{B}$ is a map $\alpha : \mathbf{A} \to \mathbf{B}$ such that for all $a, b \in \mathbf{A}$ we have $\alpha(0) = 0$, $\alpha(\neg a) = \neg\alpha(a)$, $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$, and $\alpha(\Box a) = \Box\alpha(a)$. We denote by $\mathrm{Ker}\ \alpha := \{(a, b) \in \mathbf{A}^2 \mid \alpha(a) = \alpha(b)\}$ the *kernel* of α.

Definition 3.3 Let $\mathbf{A}$ be a modal algebra. An equivalence relation $\sim$ on $\mathbf{A}$ is called a *congruence* on $\mathbf{A}$ if for all $a, a', b, b' \in \mathbf{A}$:

- $a \sim a'$ implies $\neg a \sim \neg a'$,
- $a \sim a'$ and $b \sim b'$ implies $(a \wedge b) \sim (a' \wedge b')$,
- $a \sim a'$ implies $\Box a \sim \Box a'$.

Then $\sim$ induces a *quotient algebra* $\mathbf{A}_{/\sim}$ over the set of all equivalence classes of $\sim$. If for all $a \in \mathbf{A}$ we denote by $\pi(a)$ the equivalence class of a modulo $\sim$, we obtain a surjective homomorphism $\pi : \mathbf{A} \to \mathbf{A}_{/\sim}$ [16, Section 1.2].

If $P \subseteq \mathsf{Prop}$, a particularly interesting modal algebra is the *Lindenbaum algebra* of $\mathbf{L}$ over P, defined as $\mathbf{A}_P := (\mathcal{L}_{P/\equiv_{\mathbf{L}}}, 0, \neg, \wedge, \Box)$ with

- $0 := [\bot]_{\mathbf{L}}$,
- $\neg\, [\varphi]_{\mathbf{L}} := [\neg\varphi]_{\mathbf{L}}$,
- $[\varphi]_{\mathbf{L}} \wedge [\psi]_{\mathbf{L}} := [\varphi \wedge \psi]_{\mathbf{L}}$,
- $[\Box\varphi]_{\mathbf{L}} := \Box\, [\varphi]_{\mathbf{L}}$.

Notice that if $\mathbf{L}$ is locally tabular and P is finite, then $\mathbf{A}_P$ is finite. A substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ then naturally induces a homomorphism $\sigma^{\triangleright} : \mathbf{A}_P \to \mathbf{A}_Q$ defined by $\sigma^{\triangleright}([\varphi]_{\mathbf{L}}) := [\sigma(\varphi)]_{\mathbf{L}}$. Conversely, one can recover σ from $\sigma^{\triangleright}$ (up to equivalence modulo $\simeq_{\mathbf{L}}$) since we have $\sigma(\varphi) \equiv_{\mathbf{L}} \psi$ for any $\psi \in \sigma^{\triangleright}([\varphi]_{\mathbf{L}})$. There is thus a one-to-one correspondence between homomorphisms and substitutions (up to equivalence modulo $\simeq_{\mathbf{L}}$). For convenience, we will then identify the two: the symbol σ will indifferently denote the substitution σ and the homomorphism $\sigma^{\triangleright}$.

Properties of substitutions can also be expressed algebraically. Given $\varphi \in \mathcal{L}_P$, let $\equiv_\varphi$ be the least congruence on $\mathbf{A}_P$ such that $[\varphi]_{\mathbf{L}} \equiv_\varphi 1$. We then denote by $\pi_\varphi : \mathbf{A}_P \to \mathbf{A}_{P/\equiv_\varphi}$ the homomorphism associated to $\equiv_\varphi$ (as introduced in definition 3.3). Obviously the kernel of π_φ is $\equiv_\varphi$ itself.

Proposition 3.4 *Given $\psi, \theta \in \mathcal{L}_P$, the following are equivalent:*

(i) $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$,

(ii) $[\psi]_{\mathbf{L}} \equiv_{\varphi} [\theta]_{\mathbf{L}}$.

Proof. (**i**) $\Rightarrow$ (**ii**): Suppose that $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$. Then by Proposition 2.2 there exists $n \in \mathbb{N}$ such that $\vdash_{\mathbf{L}} \Box^{\leq n}\varphi \to (\psi \leftrightarrow \theta)$. Thus $\left[\Box^{\leq n}\varphi\right]_{\mathbf{L}} \leq [\psi \leftrightarrow \theta]_{\mathbf{L}}$, and it follows that $\pi_{\varphi}(\left[\Box^{\leq n}\varphi\right]_{\mathbf{L}}) \leq \pi_{\varphi}([\psi \leftrightarrow \theta]_{\mathbf{L}})$. It is easily proved by induction on n that $\pi_{\varphi}(\left[\Box^{\leq n}\varphi\right]_{\mathbf{L}}) = 1$, and therefore $\pi_{\varphi}([\psi \leftrightarrow \theta]_{\mathbf{L}}) = 1$ too. Hence $\pi_{\varphi}([\psi]_{\mathbf{L}}) = \pi_{\varphi}([\theta]_{\mathbf{L}})$, or equivalently $[\psi]_{\mathbf{L}} \equiv_{\varphi} [\theta]_{\mathbf{L}}$.
(**ii**) $\Rightarrow$ (**i**): Let us write $[\psi]_{\mathbf{L}} \sim [\theta]_{\mathbf{L}}$ whenever $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$. It is easily verified that $\sim$ is a congruence on $\mathbf{A}_P$, and that $[\varphi]_{\mathbf{L}} \sim 1$. By construction, $\equiv_{\varphi}$ is then included in $\sim$, and this proves the claim. $\Box$

We may now connect the projective or unifying character of a substitution to its algebraic properties.

Proposition 3.5 *Let $P \subseteq \mathsf{Prop}$ finite and $\varphi \in \mathcal{L}_P$. Then:*

(i) *A substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ is a* **L***-unifier of φ iff* $\mathrm{Ker}\ \pi_{\varphi} \subseteq \mathrm{Ker}\ \sigma$.

(ii) *A substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_P$ is φ-projective iff $\pi_{\varphi}\sigma = \pi_{\varphi}$.*

Proof.

(i) Suppose that σ is a **L**-unifier of φ. Then $\vdash_{\mathbf{L}} \sigma(\varphi)$, or equivalently $\sigma([\varphi]_{\mathbf{L}}) = 1$. Thus Ker σ is a congruence on $\mathbf{A}_P$ containing $([\varphi]_{\mathbf{L}}, 1)$, so by construction it contains $\equiv_{\varphi}$. Conversely, if Ker $\pi_{\varphi} \subseteq$ Ker σ then in particular $([\varphi]_{\mathbf{L}}, 1) \in$ Ker σ and therefore $\vdash_{\mathbf{L}} \sigma(\varphi)$.

(ii) We have

	σ is φ-projective	
iff	$\forall \psi \in \mathcal{L}_P,\ \varphi \vdash_{\mathbf{L}} \sigma(\psi) \leftrightarrow \psi$	by Proposition 2.5
iff	$\forall \psi \in \mathcal{L}_P,\ [\sigma(\psi)]_{\mathbf{L}} \equiv_{\varphi} [\psi]_{\mathbf{L}}$	by Proposition 3.4
iff	$\forall \psi \in \mathcal{L}_P,\ \pi_{\varphi}\sigma([\psi]_{\mathbf{L}}) = \pi_{\varphi}([\psi]_{\mathbf{L}})$	
iff	$\pi_{\varphi}\sigma = \pi_{\varphi}$	

$\Box$

4 Duality

Now it is time to let duality play its role. In this section we introduce some rudiments of Duality Theory, and apply them to our setting. For more details we refer to [4, Chapter 5], [5, Chapter 7] and [16, Chapter 4]. This investigation will ultimately lead to theorem 4.5. First, let **A** be a modal algebra. Given a set $F \subseteq \mathbf{A}$, we call F an *ultrafilter* on **A** if it satisfies the following conditions:

- $0 \notin F$,
- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$,
- for all $a \in \mathbf{A}$, either $a \in F$ or $\neg a \in F$.

Then the *dual* of **A** is the general frame $\mathbf{A}^* := (X_{\mathbf{A}}, R_{\mathbf{A}}, \mathcal{A}_{\mathbf{A}})$ with:

- $X_{\mathbf{A}} := \{F \subseteq \mathbf{A} \mid F \text{ is an ultrafilter}\}$,

- $R_{\mathbf{A}} := \{(F, F') \in X^2 \mid \forall a \in \mathbf{A}, \Box a \in F \Rightarrow a \in F'\}$,
- $\mathcal{A}_{\mathbf{A}} := \{\{F \in X \mid a \in F\} \mid a \in \mathbf{A}\}$.

The frame $\mathbf{A}^*$ is, in fact, a descriptive frame (see Section 2.4). Further, if $\alpha : \mathbf{A} \to \mathbf{B}$ is a homomorphism, we define a bounded morphism $\alpha^* : \mathbf{B}^* \to \mathbf{A}^*$ by

$$\alpha^*(F) := \alpha^{-1}[F].$$

If α and β are two appropriate homomorphisms, the identity

$$(\alpha\beta)^* = \beta^*\alpha^*$$

is easily verified. One can prove that for all descriptive frames $\mathfrak{F}$ there exists a unique modal algebra $\mathbf{A}$ (up to isomorphism) such that $\mathfrak{F}$ and $\mathbf{A}^*$ are isomorphic. Likewise, if $f : \mathbf{B}^* \to \mathbf{A}^*$ is a bounded morphism, there exists a unique homomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ such that $\alpha^* = f$ [10]. In what follows we are going to make extensive use of this correspondence.

Naturally, the dual of the Lindenbaum algebra of $\mathbf{L}$ is of central interest to us. So if $\mathbf{A} = \mathbf{A}_P$ for some $P \subseteq \mathsf{Prop}$, we write $(X_{\mathbf{A}}, R_{\mathbf{A}}, \mathcal{A}_{\mathbf{A}}) = (X_P, R_P, \mathcal{A}_P)$ for simplicity. Given $\varphi \in \mathcal{L}_P$, we also write $\widehat{\varphi} := \{F \in X_P \mid [\varphi]_{\mathbf{L}} \in F\}$, and we then see that $\mathcal{A}_P = \{\widehat{\varphi} \mid \varphi \in \mathcal{L}_P\}$. Also note that if $\mathbf{L}$ is locally tabular and P is finite, then X_P is finite and $\mathcal{A}_P = \mathcal{P}(X_P)$.

Remark 4.1 The tight similarity between $\mathcal{L}_P$ and $\mathbf{A}_P$ materializes itself in a one-to-one correspondence between the *maximal consistent subsets* of $\mathcal{L}_P$ [4, Section 4.2] and the ultrafilters of $\mathbf{A}_P$, realized by the mapping

$$\Gamma \mapsto \{[\varphi]_{\mathbf{L}} \mid \varphi \in \Gamma\}$$

where $\Gamma \subseteq \mathcal{L}_P$ is maximal consistent. In fact, this correspondence induces an isomorphism between the frame $\mathfrak{F}_P := (X_P, R_P)$ and the *canonical frame* of $\mathbf{L}$ over P (as pointed out in [4, Section 5.3]).

Now assume that P is finite and let $\varphi \in \mathcal{L}_P$. In order to characterize the unifiable or projective character of φ, it is crucial to understand the behaviour of $\pi_\varphi^* : (\mathbf{A}_{P/\equiv_\varphi})^* \to \mathbf{A}_P^*$. In the algebraic setting, the relevant information was contained in the kernel of π_φ [11]. In the dual setting, this information turns out to be carried by the image of π_φ^*, which we proceed to describe. We prove that this image coincides with the set $\widehat{\varphi}^\infty := \bigcap_{n \in \mathbb{N}} \widehat{\Box^n \varphi}$.

Lemma 4.2 *Let $F \in X_P$. Then the following are equivalent:*

(i) *F is closed under $\equiv_\varphi$;*

(ii) *$F \in \mathrm{Im}\ \pi_\varphi^*$.*

---

[10] In categorical terms, we thus say that $(\cdot)^*$ is a *dual equivalence* between the category of modal algebras with homomorphisms and the category of descriptive frames with bounded morphisms.

[11] In Proposition 3.5, the kernel of π_φ is not mentioned in item (ii), but it still appears implicitly since $\pi_\varphi\sigma = \pi_\varphi$ can also be phrased as $\{(\sigma(a), a) \mid a \in \mathbf{A}_P\} \subseteq \mathrm{Ker}\ \pi_\varphi$.

Proof.
(**i**) $\Rightarrow$ (**ii**): Suppose that F is closed under $\equiv_\varphi$. We introduce $G := \pi_\varphi[F]$ and prove that G is an ultrafilter on $\mathbf{A}_{P/\equiv_\varphi}$.

- Suppose that $0 \in G$. Then $0 = \pi_\varphi(a)$ for some $a \in F$. Hence $\pi_\varphi(0) = 0 = \pi_\varphi(a)$, which entails $0 \equiv_\varphi a$, and thus $0 \in F$ by assumption, contradicting the fact that F is an ultrafilter. Therefore $0 \notin G$.
- Suppose that $a' \in G$ and $a' \leq b'$. Then $a' = \pi_\varphi(a)$ for some $a \in F$. In addition, since π_φ is surjective, we have $b' = \pi_\varphi(b)$ for some $b \in \mathbf{A}_P$. From $a' \leq b'$ we obtain $a' = a' \wedge b'$, and thus $\pi_\varphi(a) = \pi_\varphi(a) \wedge \pi_\varphi(b) = \pi_\varphi(a \wedge b)$. Hence $a \equiv_\varphi a \wedge b$, and since $a \in F$ our assumption entails $a \wedge b \in F$. From $a \wedge b \leq b$ we obtain $b \in F$. Therefore $b' \in G$.
- Let $a', b' \in G$. We have $a' = \pi_\varphi(a)$ and $b' = \pi_\varphi(b)$ with $a, b \in F$. Then $a' \wedge b' = \pi_\varphi(a \wedge b)$ with $a \wedge b \in F$. Therefore $a' \wedge b' \in G$.
- Let $a' \in \mathbf{A}_{P/\equiv_\varphi}$. Then $a' = \pi_\varphi(a)$ for some $a \in \mathbf{A}_P$. Since F is an ultrafilter we have either $a \in F$ or $\neg a \in F$. Therefore, we have either $a' \in G$ or $\neg a' \in G$.

To prove that $F \in \mathrm{Im}\ \pi_\varphi^*$ we then show that $\pi_\varphi^*(G) = F$, that is, $\pi_\varphi^{-1}[\pi_\varphi[F]] = F$. The inclusion from right to left is trivial. From left to right, suppose that $a \in \pi_\varphi^{-1}[\pi_\varphi[F]]$. Then $\pi_\varphi(a) \in \pi_\varphi[F]$, that is, $\pi_\varphi(a) = \pi_\varphi(b)$ for some $b \in F$. Since F is closed under $\equiv_\varphi$ we obtain $a \in F$ and we are done.
(**ii**) $\Rightarrow$ (**i**): Let $F \in \mathrm{Im}\ \pi_\varphi^*$. Then there exists an ultrafilter $G \in (\mathbf{A}_{P/\equiv_\varphi})^*$ such that $F = \pi_\varphi^*(G) = \pi_\varphi^{-1}[G]$. If $a \in F$ and $a \equiv_\varphi b$ then $\pi_\varphi(b) = \pi_\varphi(a) \in G$, and therefore $b \in F$. This proves that F is closed under $\equiv_\varphi$. $\square$

Proposition 4.3 *We have* $\mathrm{Im}\ \pi_\varphi^* = \widehat{\varphi}^\infty$.

Proof. Let $F \in \mathrm{Im}\ \pi_\varphi^*$. Given $n \in \mathbb{N}$, we have $[\Box^n\varphi]_{\mathbf{L}} \equiv_\varphi \Box^n 1 \equiv_\varphi 1$ with $1 \in F$, so by Lemma 4.2 we obtain $[\Box^n\varphi]_{\mathbf{L}} \in F$ and thus $F \in \widehat{\Box^n\varphi}$.

Conversely, let $F \in \widehat{\varphi}^\infty$. By Lemma 4.2, it suffices to prove that F is closed under $\equiv_\varphi$. So assume $[\psi]_{\mathbf{L}} \in F$ and $[\psi]_{\mathbf{L}} \equiv_\varphi [\theta]_{\mathbf{L}}$. By Proposition 3.4 we obtain $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \theta$, and then by Proposition 2.2 there exists $n \in \mathbb{N}$ such that $\vdash_{\mathbf{L}} \Box^{\leq n}\varphi \to (\psi \leftrightarrow \theta)$. Since $F \in \widehat{\Box^{\leq n}\varphi}$ we obtain $[\psi \leftrightarrow \theta]_{\mathbf{L}} \in F$, and since $[\psi]_{\mathbf{L}} \in F$ we conclude that $[\theta]_{\mathbf{L}} \in F$. $\square$

With this result, we are then ready to transition from the algebraic setting to the dual setting.

Proposition 4.4 *Let* $P \subseteq \mathsf{Prop}$ *finite and* $\varphi \in \mathcal{L}_P$. *Then:*

(i) *for any homomorphism* $\sigma : \mathbf{A}_P \to \mathbf{A}_Q$ *we have* $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \sigma$ *iff* $\mathrm{Im}\ \sigma^* \subseteq \widehat{\varphi}^\infty$ *iff* $\mathrm{Im}\ \sigma^* \subseteq \widehat{\varphi}$;

(ii) *for any homomorphism* $\sigma : \mathbf{A}_P \to \mathbf{A}_P$ *we have* $\pi_\varphi\sigma = \pi_\varphi$ *iff* $\widehat{\varphi}^\infty \subseteq \mathrm{fp}\ \sigma^*$.

Proof. Given $\sigma : \mathbf{A}_P \to \mathbf{A}_Q$, recall that we have $\sigma^* : X_Q \to X_P$.

(i) Suppose that $\mathrm{Ker}\ \pi_\varphi \subseteq \mathrm{Ker}\ \sigma$. Let $F \in \mathrm{Im}\ \sigma^*$. Then $F = \sigma^{-1}[G]$ for some $G \in X_Q$. Let $n \in \mathbb{N}$. We have $\pi_\varphi([\Box^n\varphi]_{\mathbf{L}}) = 1 = \pi_\varphi(1)$, and thus

$\sigma([\Box^n\varphi]_{\mathbf{L}}) = \sigma(1)$ by assumption. Hence $\sigma([\Box^n\varphi]_{\mathbf{L}}) \in G$, and therefore $[\Box^n\varphi]_{\mathbf{L}} \in F$. This proves that $F \in \widehat{\varphi}^\infty$.

That $\mathrm{Im}\ \sigma^* \subseteq \widehat{\varphi}^\infty$ implies $\mathrm{Im}\ \sigma^* \subseteq \widehat{\varphi}$ is immediate. Now suppose that $\mathrm{Im}\ \sigma^* \subseteq \widehat{\varphi}$. First, we prove that $\sigma([\varphi]_{\mathbf{L}}) = 1$. If not, then $\sigma([\neg\varphi]_{\mathbf{L}}) \neq 0$, so by the Ultrafilter Theorem [4, Proposition 5.38] there exists an ultrafilter $G \in X_Q$ such that $\sigma([\neg\varphi]_{\mathbf{L}}) \in G$. It follows that $[\neg\varphi]_{\mathbf{L}} \in \sigma^{-1}[G]$, whereas our assumption entails $\sigma^*(G) \in \widehat{\varphi}$ and thus $[\varphi]_{\mathbf{L}} \in \sigma^{-1}[G]$, a contradiction. Hence $\sigma([\varphi]_{\mathbf{L}}) = 1 = \sigma(1)$, which means that Ker σ is a congruence containing $([\varphi]_{\mathbf{L}}, 1)$. By construction, we then obtain Ker $\pi_\varphi \subseteq$ Ker σ.

(ii) Suppose that $\pi_\varphi\sigma = \pi_\varphi$. Then $\sigma^*\pi^*_\varphi = \pi^*_\varphi$. Now let $F \in \widehat{\varphi}^\infty$. By Proposition 4.3 we have $F \in \mathrm{Im}\ \pi^*_\varphi$. Then there exists an ultrafilter $G \in (\mathbf{A}_{P/\equiv_\varphi})^*$ such that $F = \pi^*_\varphi(G)$. Consequently, $\sigma^*(F) = \sigma^*\pi^*_\varphi(G) = \pi^*_\varphi(G) = F$, and this proves that $F \in \mathrm{fp}\ \sigma^*$.

Conversely, suppose that $\widehat{\varphi}^\infty \subseteq \mathrm{fp}\ \sigma^*$. Given $G \in (\mathbf{A}_{P/\equiv_\varphi})^*$ we have $\pi^*_\varphi(G) \in \widehat{\varphi}^\infty$ by Proposition 4.3 and thus $\pi^*_\varphi(G) \in \mathrm{fp}\ \sigma^*$. Hence $\sigma^*(\pi^*_\varphi(G)) = \pi^*_\varphi(G)$. This proves that $\sigma^*\pi^*_\varphi = \pi^*_\varphi$, and therefore $\pi_\varphi\sigma = \pi_\varphi$.

□

Finally, by combining Proposition 3.5 and Proposition 4.4, we obtain the following characterization.

Theorem 4.5 *Let $\varphi \in \mathcal{L}_P$ with $P \subseteq \mathsf{Prop}$ finite. Then:*

(i) *φ is $\mathbf{L}$-unifiable if and only if there exists a bounded morphism $f : X_Q \to X_P$ with $\mathrm{Im}\ f \subseteq \widehat{\varphi}^\infty$;*

(ii) *φ is $\mathbf{L}$-projective if and only if there exists a bounded morphism $f : X_P \to X_P$ with $\mathrm{Im}\ f \subseteq \widehat{\varphi}^\infty \subseteq \mathrm{fp}\ f$.*

Accordingly, we will call a *dual unifier* of φ any bounded morphism $f : X_Q \to X_P$ such that $\mathrm{Im}\ f \subseteq \widehat{\varphi}^\infty$, and a *projective dual unifier* of φ any bounded morphism $f : X_P \to X_P$ such that $\mathrm{Im}\ f \subseteq \widehat{\varphi}^\infty \subseteq \mathrm{fp}\ f$.

5 Applications

In this section we delve into various applications of theorem 4.5, and turn our attention to the following logics (with $n, k \geq 1$) :

$$\begin{array}{rcl}
\mathbf{K4} & := & \mathbf{K} + (\Diamond\Diamond p \to \Diamond p) \\
\mathbf{K5} & := & \mathbf{K} + (\Diamond p \to \Box\Diamond p) \\
\mathbf{K45} & := & \mathbf{K4} + (\Diamond p \to \Box\Diamond p) \\
\mathbf{K4D1} & := & \mathbf{K4} + \Box(\Box p \to q) \vee \Box(\Box q \to p) \\
\mathbf{K4}_n & := & \mathbf{K} + (\Diamond^{n+1} p \to \Diamond^{\leq n} p) \\
\mathbf{K4}_n\mathbf{D1}_n & := & \mathbf{K4}_n + \Box(\Box^{\leq n} p \to q) \vee \Box(\Box^{\leq n} q \to p) \\
\mathbf{K4}_n\mathbf{B}_k & := & \mathbf{K4}_n + (p \to \Box^{\leq k}\Diamond^{\leq k} p)
\end{array}$$

First, we recall some elementary facts and definitions. Let $P \subseteq \mathsf{Prop}$. From now on we abstract away from the nature of the elements of X_P: we see them as points instead of ultrafilters, and denote them with the letters $x, y, z, \ldots$ Given

$X \subseteq X_P$, we write $R_P X := \{y \in X_P \mid \exists x \in X, xR_P y\}$ and $R_P^{-1} X := \{x \in X_P \mid \exists y \in X, xR_P y\}$. We then call X *upward closed* if $R_P X \subseteq X$, and *downward closed* if $R_P^{-1} X \subseteq X$. By recursion, we also define $R_P^0 := \{(x,x) \mid x \in X_P\}$ and $R_P^{n+1} := \{(x,z) \mid (x,y) \in R_P^n \text{ and } (y,z) \in R_P\}$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we write $R_P^{\leq n} := \bigcup_{k=0}^{n} R_P^k$. The following properties are either well-known, or stem from standard arguments [4,5]:

- if $\mathbf{K4} \subseteq \mathbf{L}$, then $\mathfrak{F}_P$ is transitive;
- if $\mathbf{K5} \subseteq \mathbf{L}$, then $\mathfrak{F}_P$ is *Euclidean*, that is, if $xR_P y$ and $xR_P z$ then $yR_P z$;
- if $\mathbf{K4D1} \subseteq \mathbf{L}$, then $\mathfrak{F}_P$ is transitive and *strongly connected*, that is, if $xR_P y$ and $xR_P z$, then either $yR_P z$ or $zR_P y$;
- if $\mathbf{K4}_n \subseteq \mathbf{L}$, then $\mathfrak{F}_P$ is *n-transitive*, that is, $xR_P^{n+1} y$ implies $xR_P^{\leq n} y$;
- if $\mathbf{K4}_n\mathbf{D1}_n \subseteq \mathbf{L}$, then $\mathfrak{F}_P$ is n-transitive and *strongly n-connected*, that is, if $xR_P y$ and $xR_P z$, then either $yR_P^{\leq n} z$ or $zR_P^{\leq n} y$;
- if $\mathbf{K4}_n\mathbf{B}_k \subseteq \mathbf{L}$ with $n, k \geq 1$, then $\mathfrak{F}_P$ is n-transitive and *k-symmetric*, that is, $xR_P^{\leq k} y$ implies $yR_P^{\leq k} x$ [15].

We first address the logic $\mathbf{K4}_n\mathbf{B}_k$. We propose a relatively short proof of its projective character, thus recovering Kostrzycka's result [15].

Theorem 5.1 *Let $n, k \geq 1$. Every extension of $\mathbf{K4}_n\mathbf{B}_k$ is projective.*

Proof. Let $\mathbf{L}$ be an extension of $\mathbf{K4}_n\mathbf{B}_k$. Let $P \subseteq \mathsf{Prop}$ finite and let $\varphi \in \mathcal{L}_P$ be $\mathbf{L}$-unifiable. Then there exists a $\mathbf{L}$-unifier $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ of φ. Obviously we can assume $P \subseteq Q$. Then, as explained in Remark 2.9, we can construct a $\mathbf{L}$-unifier $\sigma' : \mathcal{L}_Q \to \mathcal{L}_Q$ of φ. By Propositions 3.5 and 4.4 we then obtain a dual unifier $f := (\sigma')^*$ of φ.

We argue that $\widehat{\varphi}^\infty$ is both upward and downward closed for R_Q. For suppose $xR_Q y$. If $x \in \widehat{\varphi}^\infty$ then for all $i \in \mathbb{N}$ we have $x \in \widehat{\Box^{i+1}\varphi}$ and thus $y \in \widehat{\Box^i\varphi}$, and it follows that $y \in \widehat{\varphi}^\infty$. Conversely, suppose that $y \in \widehat{\varphi}^\infty$. Then since $\mathbf{K4}_n\mathbf{B}_k \subseteq \mathbf{L}$ and $xR_Q^{\leq k} y$ we have $yR_Q^{\leq k} x$. Since $\widehat{\varphi}^\infty$ is upward closed a straightforward recursion yields $x \in \widehat{\varphi}^\infty$. Now let us define $g : X_Q \to X_Q$ by

$$g(x) := \begin{cases} x & \text{if } x \in \widehat{\varphi}^\infty \\ f(x) & \text{otherwise} \end{cases}$$

for all $x \in X_Q$. We prove that g is a bounded morphism. First, assume that $xR_Q y$. We have seen that $x \in \widehat{\varphi}^\infty$ iff $y \in \widehat{\varphi}^\infty$, and we know that f is a bounded morphism, so $g(x)R_Q g(y)$ is immediate. Now suppose that $x \in X_Q$ and $g(x)R_Q y'$. If $x \in \widehat{\varphi}^\infty$ then $g(x) = x$ and $xR_Q y'$, and thus $y' \in \widehat{\varphi}^\infty$ too, leading to $g(y') = y'$. Otherwise we have $x \notin \widehat{\varphi}^\infty$ and $g(x) = f(x)$ with $f(x)R_Q y'$. Since f is a bounded morphism we obtain the existence of $y \in X_Q$ such that $f(y) = y'$ and $xR_Q y$. Then $y \notin \widehat{\varphi}^\infty$, and thus $g(y) = y'$ as desired.

Now let $\widehat{\psi} \in \mathcal{A}_Q$. We have

$$g^{-1}[\widehat{\psi}] = (\widehat{\psi} \cap \widehat{\varphi}^\infty) \cup (f^{-1}[\widehat{\psi}] \setminus \widehat{\varphi}^\infty).$$

Since $\vdash_{\mathbf{L}} \Box^{\leq n} p \to \Box^{n+1} p$ we have $\widehat{\varphi}^\infty = \widehat{\Box^{\leq n}\varphi} \in \mathcal{A}_Q$, and therefore $g^{-1}[\widehat{\psi}] \in \mathcal{A}_Q$ too. Finally it is immediate that $\mathrm{Im}\ g \subseteq \widehat{\varphi}^\infty \subseteq \mathrm{fp}\ g$. We conclude that φ is **L**-projective. $\Box$

As mentioned in Section 2, Kost [14] showed that an extension of **K4** is projective if and only if it contains **K4D1**. In Theorem 5.2, we reprove a weaker version of the right-to-left implication, limited to locally tabular logics. In Theorem 5.3 we reprove the left-to-right implication.

Theorem 5.2 *Every locally tabular extension of* **K4D1** *is projective.*

Proof. Let **L** be a locally tabular extension of **K4D1**. Let $P \subseteq \mathsf{Prop}$ finite and let $\varphi \in \mathcal{L}_P$ be **L**-unifiable. Reasoning as above, we obtain the existence of a dual unifier $f : X_Q \to X_Q$ of φ with $Q \subseteq \mathsf{Prop}$ finite and $P \subseteq Q$. To define $g : X_Q \to X_Q$, we consider $x \in X_Q$ and proceed as follows:

(i) if $x \in \widehat{\varphi}^\infty$ we set $g(x) := x$;

(ii) otherwise, if $x \in R_Q^{-1}\widehat{\varphi}^\infty$ then we select $g(x)$ in the set $Y := \{y \in \widehat{\varphi}^\infty \mid xR_Q y\}$ so that $g(x)R_Q y$ for all $y \in Y$;

(iii) otherwise, we set $g(x) := f(x)$.

Case (ii) requires some justification. Since $\mathbf{K4D1} \subseteq \mathbf{L}$, the frame $\mathfrak{F}_Q$ is strongly connected, so for all $y, z \in Y$ we have either $yR_Q z$ or $zR_Q y$. In addition, Y is non-empty by assumption, and finite since **L** is locally tabular. This yields the existence of a 'smallest' element $g(x)$ with respect to R_Q – of course such an element is not necessarily unique. We now prove that g is a bounded morphism. First, suppose that $xR_Q y$. We examine each case for x.

(i) If $x \in \widehat{\varphi}^\infty$, then $y \in \widehat{\varphi}^\infty$ too, and thus $g(x)R_Q g(y)$ is immediate.

(ii) Otherwise, suppose that $x \in R_Q^{-1}\widehat{\varphi}^\infty$. Then $xR_Q g(x)$ and $xR_Q y$, and since $\mathfrak{F}_Q$ is strongly connected we obtain either $g(x)R_Q y$ or $yR_Q g(x)$. Since $g(x) \in \widehat{\varphi}^\infty$ it follows respectively that $y \in \widehat{\varphi}^\infty$ or $y \in R_Q^{-1}\widehat{\varphi}^\infty$. In both cases we have $yR_Q^{\leq 1} g(y)$. Since $xR_Q y$ and $\mathfrak{F}_Q$ is transitive we then obtain $xR_Q g(y)$. Since $g(y) \in \widehat{\varphi}^\infty$, it follows by construction of $g(x)$ that $g(x)R_Q^{\leq 1} g(y)$. If $g(x)R_Q g(y)$ we are done. Otherwise $g(x) = g(y)$. Since $xR_Q g(x)$ and $\mathfrak{F}_Q$ is strongly connected we also have $g(x)R_Q g(x)$. Therefore $g(x)R_Q g(y)$ holds as well.

(iii) Otherwise, we have $g(x) = f(x)$. If $y \in R_Q^{-1}\widehat{\varphi}^\infty$, we have $x \in R_Q^{-1}\widehat{\varphi}^\infty$ too by transitivity, a contradiction. Thus $y \notin R_Q^{-1}\widehat{\varphi}^\infty$ and $g(y) = f(y)$. Since $xR_Q y$ and f is a bounded morphism we obtain $f(x)R_Q f(y)$, and therefore $g(x)R_Q g(y)$.

Now suppose that $g(x)R_Q y$. If x falls in case (i) or case (ii) then $xR_Q^{\leq 1} g(x)$, and by transitivity we obtain $xR_Q y$. In addition, $g(x) \in \widehat{\varphi}^\infty$ entails $y \in \widehat{\varphi}^\infty$. Thus $xR_Q y$ with $g(y) = y$, as desired. Otherwise, x falls in case (iii), and we have $g(x) = f(x)$. Then since f is a bounded morphism there exists $z \in X_Q$ such that $f(z) = y$ and $xR_Q z$. Since $x \notin R_Q^{-1}\widehat{\varphi}^\infty$ we obtain $z \notin R_Q^{-1}\widehat{\varphi}^\infty$ too, and therefore $g(z) = f(z) = y$.

Finally, for any $A \in \mathcal{A}_Q$ we have obviously $g^{-1}[A] \in \mathcal{A}_Q$ since $\mathcal{A}_Q = \mathcal{P}(X_Q)$. It is also immediate that $\mathrm{Im}\, g \subseteq \widehat{\varphi}^\infty \subseteq \mathrm{fp}\, g$. We conclude that φ is **L**-projective. □

Theorem 5.3 *Any projective extension of* **K4** *is also an extension of* **K4D1***.*

Proof. Suppose that $\mathbf{K4} \subseteq \mathbf{L}$. By contraposition suppose that $\mathbf{K4D1} \not\subseteq \mathbf{L}$. We prove that $\varphi := \Box(\Box p \to q) \vee \Box(\Box q \to p)$ is not projective. Let $P := \{p, q\}$. First we have $\nvdash_{\mathbf{L}} \neg\varphi$ by assumption. By the Ultrafilter Theorem [4, Proposition 5.38], there exists an ultrafilter $x \in X_P$ such that $x \in \widehat{\Diamond(\Box p \wedge \neg q) \wedge \Diamond(\Box q \wedge \neg p)}$. Then, by the Existence Lemma [4, Lemma 4.20] (together with remark 4.1), there exist $y, z \in X_P$ such that xR_Py, xR_Pz, $y \in \widehat{\Box p \wedge \neg q}$ and $z \in \widehat{\Box q \wedge \neg p}$. Suppose toward a contradiction that there exists a projective dual unifier $f : X_P \to X_P$ of φ.

Since $y \in \widehat{\Box p}$ and $\mathbf{K4} \subseteq \mathbf{L}$, we have $y \in \widehat{\Box^{n+1} p}$ and thus $y \in \widehat{\Box^n \Box(\Box q \to p)}$ for all $n \in \mathbb{N}$. Therefore $y \in \widehat{\varphi}^\infty$ and $f(y) = y$. Since f is a bounded morphism and xR_Py, we obtain $f(x)R_Py$. Likewise, we can prove that $f(x)R_Pz$. Then since $f(x) \in \widehat{\varphi}^\infty$ we have in particular $f(x) \in \widehat{\varphi}$, and thus either $f(x) \in \widehat{\Box(\Box p \to q)}$ or $f(x) \in \widehat{\Box(\Box q \to p)}$. In the former case we obtain $y \in \widehat{\Box p \to q}$, and in the latter we obtain $z \in \widehat{\Box q \to p}$. Both outcomes are contradictions, and this concludes the proof. □

Interestingly, the proof of theorem 5.3 can easily be adapted to derive an analogous new result for extensions of $\mathbf{K4}_n$.

Theorem 5.4 *Any projective extension of* $\mathbf{K4}_n$ *is also an extension of* $\mathbf{K4}_n\mathbf{D1}_n$*.*

Proof. Suppose that $\mathbf{K4}_n \subseteq \mathbf{L}$. By contraposition suppose that $\mathbf{K4}_n\mathbf{D1}_n \not\subseteq \mathbf{L}$. We prove that $\varphi := \Box(\Box^{\leq n} p \to q) \vee \Box(\Box^{\leq n} q \to p)$ is not projective. Let $P := \{p, q\}$. We have $\nvdash_{\mathbf{L}} \neg\varphi$ by assumption, and arguing as above we obtain an ultrafilter $x \in X_P$ such that $x \in \widehat{\Diamond(\Box^{\leq n} p \wedge \neg q) \wedge \Diamond(\Box^{\leq n} q \wedge \neg p)}$. Then there exist $y, z \in X_P$ such that xR_Py, xR_Pz, $y \in \widehat{\Box^{\leq n} p \wedge \neg q}$ and $z \in \widehat{\Box^{\leq n} q \wedge \neg p}$. Suppose that there exists a projective dual unifier $f : X_P \to X_P$ of φ.

Since $y \in \widehat{\Box^{\leq n} p}$ and $\mathbf{K4}_n \subseteq \mathbf{L}$, we have $y \in \widehat{\Box^{k+1} p}$ and thus $y \in \widehat{\Box^k \Box(\Box^{\leq n} q \to p)}$ for all $k \in \mathbb{N}$. Therefore $y \in \widehat{\varphi}^\infty$ and $f(y) = y$. Since f is a bounded morphism and xR_Py, we obtain $f(x)R_Py$. Likewise, we can prove that $f(x)R_Pz$. Then since $f(x) \in \widehat{\varphi}^\infty$ we have either $f(x) \in \widehat{\Box(\Box^{\leq n} p \to q)}$ or $f(x) \in \widehat{\Box(\Box^{\leq n} q \to p)}$. In the former case we obtain $y \in \widehat{\Box^{\leq n} p \to q}$, and in the latter we obtain $z \in \widehat{\Box^{\leq n} q \to p}$. This yields a contradiction. □

Obviously Theorem 5.2 is weaker than Kost's result, but still covers a decent range of logics. In particular, it is enough to conclude that all extensions of **K45** are projective, since **K5** is locally tabular [18, Corollary 5]. We then refine this result by showing that the projective extensions of **K5** are, in fact, *exactly* the extensions of **K45**. We thus obtain a complete description of the landscape of projective logics above **K5**, which was only partially known prior to our work.

Theorem 5.5 *Let* $\mathbf{L}$ *be an extension of* $\mathbf{K5}$*. Then* $\mathbf{L}$ *is projective if and only if* $\mathbf{K45} \subseteq \mathbf{L}$*.*

Proof. We already know that if $\mathbf{K45} \subseteq \mathbf{L}$ then $\mathbf{L}$ is projective. Conversely, suppose that $\mathbf{K45} \not\subseteq \mathbf{L}$. We prove that $\varphi := \Diamond\Diamond p \to \Diamond p$ is not projective. Let $P := \{p\}$. First we have $\nvdash_{\mathbf{L}} \Diamond\Diamond p \to \Diamond p$ by assumption. Arguing as before, we obtain an ultrafilter $x \in X_P$ such that $x \in \widehat{\Diamond\Diamond p \wedge \Box\neg p}$. Then there exists $y \in X_P$ such that $xR_P^2 y$ and $y \in \widehat{p}$. Suppose toward a contradiction that there exists a projective dual unifier $f : X_P \to X_P$ of φ. Then $f(x)R_P^2 f(y)$. Since y has a predecessor, it belongs to a final cluster (see [18] for a comprehensive description of Euclidean frames). Consequently $y \in \widehat{\varphi}^\infty$, and thus $f(y) = y$. Hence $f(x) \in \widehat{\Diamond\Diamond p}$, and since $f(x) \in \widehat{\varphi}^\infty$ we obtain $f(x) \in \widehat{\Diamond p}$. Therefore there exists $z \in X_P$ such that $f(x)R_P z$ and $z \in \widehat{p}$. Since f is a bounded morphism there exists $t \in X_P$ such that $xR_P t$ and $f(t) = z$. Again $t \in \widehat{\varphi}^\infty$ and therefore $f(t) = t$. Hence $xR_P z$, contradicting $x \in \widehat{\Box\neg p}$. This concludes the proof. $\Box$

6 Conclusion

In this paper, through substantial use of the duality between descriptive frames and modal algebras, we have given a necessary and sufficient condition for modal formulas to be projective. Applying it to the extensions of $\mathbf{K4}_n\mathbf{B}_k$ and **K4D1**, we have reproved known results obtained by Kost [14] and Kostrzycka [15]. Applying it to the extensions of **K5**, we have proved the new result saying that the projective extensions of **K5** are exactly the extensions of **K45**. It should be noted that our proofs are fairly lightweight and concise, as opposed to syntactic methods, which often involve all sorts of technical twists. Of course, this is only a first insight of what duality has to offer. Apart from the results about the unification types of modal logics mentioned in Section 5 and the new result about **K5** extensions, very little is known. For example, the unification types of $\mathbf{KD} := \mathbf{K} + \Diamond\top$ and $\mathbf{KT} := \mathbf{K} + \Box p \to p$ are just known to be non-unitary, seeing that the substitutions $\sigma_\top$ and $\sigma_\bot$ on the propositional variable p defined by $\sigma_\top(p) := \top$ and $\sigma_\bot(p) := \bot$ constitute both a basis of concise **KD**-unifiers and a basis of concise **KT**-unifiers of $\Box p \vee \Box\neg p$. This is an immediate consequence of the fact that **KD** and **KT** possess the modal disjunction property saying that for all formulas φ, ψ, if $\Box\varphi \vee \Box\psi$ is in **KD** (resp. **KT**) then either φ or ψ is in **KD** (resp. **KT**). To take another example, the unification types of $\mathbf{DAlt}_1 := \mathbf{KD} + \Diamond p \to \Box p$ and $\mathbf{KB} := \mathbf{K} + p \to \Box\Diamond p$ are not known either. Therefore, much remain to be done and further investigations are needed for obtaining, by means of our duality approach, the unification types of modal logics such as **KD**, **KT**, $\mathbf{DAlt}_1$ and **KB**.

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