Logics of Space with Connectedness Predicates: Complete Axiomatizations

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Abstract

In this paper we present a complete quantifier-free axiomatization of several logics on region-based theory of space based on contact relation and connectedness predicates. We prove completeness theorems for the logics in question with respect to three different semantics: algebraic – with respect to several important classes of contact algebras, topological – based on the contact algebras over various classes of topological spaces, and relational semantics with respect to Kripke frames with reflexive and symmetric relations.

Keywords: Spatial logics, mereotopology, contact algebras, connectedness, representation theorems, completeness theorems.

Introduction

This paper is in the field of region-based theory of space (RBTS). The origin of this theory goes back to Whitehead [34] and de Laguna [10] and consists of a radical reconstruction of the classical Euclidean approach to the theory of space, putting on the base of the new approach the more realistic primitive notions, like region as an abstraction of physical body, and some intuitive relations between regions, like part-of, overlap and contact. In this way geometry has been based on mereology – the theory of parts and wholes [30]. While at the beginning this new approach was only on the center of attention of some philosophers and philosophically oriented logicians and mathematicians, now Whitehead's ideas on RBTS flourished and found applications in some areas of computer science: qualitative spatial reasoning (QSR), knowledge representation, geographical information systems, formal ontologies in information systems, image processing, natural language semantics etc. The reason is that the languages of RBTS are quite simple and suitable for description of some qualitative spatial features and properties of space bodies. Recent surveys are [1,4,27,32], surveys concerning various applications are [8,9] (see also special issues of Fundamenta Informaticæ [16] and the Journal of Applied Non-classical Logics [2]). One of the most popular systems among the community of QSR-researchers is the system of Region Connection Calculus (RCC) introduced in [28]. RCC attracted quite intensive research in the field of region-based theory of space and related spatial logics, both on its applied and mathematical aspects. An algebraic reformulation of RCC as a Boolean algebra with an additional relation C called *connection* was presented in [31] (in the subsequent literature *connection* has been renamed with the more convenient name *contact*). Now a common name for various similar systems is the notion of *contact algebra*, the simplest one introduced in [11]. The elements of a given contact algebra are formal counterparts of *regions* and by means of the Boolean operations one can define new regions by means of given ones. Standard models of contact algebras are the algebras of regular closed subsets over some topological spaces with contact aCb holding if the regions a and b share a common point. The relationship of a class of contact algebras and the corresponding class of topological spaces is studied in the topological representation theory of that class, which states that each algebra of the class is representable (in some definite sense) as a contact algebra of regular-closed subsets of a topological space. Representation theory for contact algebras corresponding to RCC was given for the first time in [13], representation theory for contact algebras corresponding to various important classes of topological spaces, was given in [11]. Let us note that contact algebras have also non-topological models based on the notion of adjacency space formalizing some discrete versions of region-based theory of space (see [17], [12], [3]). Note that adjacency spaces can be identified with the standard notion of reflexive graph.

In the present paper we study several spatial logics related to RCC system, containing the connectedness predicates $c(a)$ and $c^{\leq n}(a)$. In topological models the predicate $c(a)$ says that the region a is connected (in a topological sense) and $c^{\leq n}(a)$ says that the region a has at most n connected components. These predicates were studied for the first time in [25,26] (see also [32]). Recently a quite intensive investigation of spatial logics containing $c(a)$ and $c^{\leq n}(a)$ with respect to their expressiveness and complexity has been done in [19] – [23]. In some sense we continue the study started in [19] – [23], but with respect to the question of complete quantifier-free axiomatizations of some of the logics considered in [19]– [23]. Namely we are interested in logics based on the language of contact algebras extended with predicates c and $c^{\leq n}$. Let us note that in contact algebras considered as first-order theories, the predicates c and $c^{\leq n}$ are definable, for instance for c we have the following equivalence:

(#) $c(a) \leftrightarrow (\forall b, d)(b \neq 0 \wedge d \neq 0 \wedge a = b + d \rightarrow bCd)$.

The problem is that the logics considered in [19]– [23] are quantifier-free, and we also want to obtain quantifier-free axiomatizations. The implication from left to the right in (#) is a universal sentence and is equivalent to the following quantifier-free formula, which can be taken as an axiom:

$c(a) \wedge b \neq 0 \wedge d \neq 0 \wedge a = b + d \rightarrow bCd$

The converse implication of (#) is not, however, a universal formula, and we will simulate it axiomatically by a special finitary rule of inference. Quantifier-free axiomatizations of spatial logics based on contact algebras, and some additional finitary rules of inference imitating reasoning with quantifiers were studied for the first time in [3]. In this paper we continue this study in the presence of the new predicates $c^{\leq n}$ (in fact c is just $c^{\leq 1}$).

We introduce also a simplified language in which we replace the predicates $c^{\leq n}$, $(n = 1, 2, \ldots)$, by corresponding sets of nominals $C^{\leq n}$, denoting regions satisfying $c^{\leq n}$.

We introduce three kinds of semantics of the languages in consideration: algebraic – based on contact algebras, topological – corresponding to the main type of point-based models of space, and a Kripke style semantics based on the notion of *adjacency space*. Let us note that the Kripke semantics is a new one for the considered logics and gives a graph sense of the connectedness predicates, for instance $c(a)$ means that a is a connected (in a graph-theoretic sense) set of points. We present axiomatic systems strongly complete in several important classes of topological spaces including all topological spaces, all connected topological spaces, all spaces related to RCC system, all (connected) compact Hausdorff spaces. This makes possible to transfer some of the complexity results obtained in [19] to some new classes of topological spaces. Completeness theorems are based on a special representation theory of contact algebras with predicates $c^{\leq n}$ in topological spaces and separately, the representation theory in adjacency spaces. We show also that with respect to weak completeness some of the additional rules of the logics in question can be eliminated, which implies collapsing of some of the logics. Using a new filtration techniques applicable to axiomatic systems with additional rules of inference, we prove that each of the considered logics is complete in a corresponding class of finite models, from which we derive their decidability.

The rest of the paper is organized as follows. In Section 1 we remind some facts for contact algebras. Section 2 is devoted to contact algebras with predicates c and $c^{\leq n}$ and their representation theory both in topological spaces and in adjacency spaces. In Section 3 we introduce two kinds of spatial logics: one based on contact algebras and predicates c and $c^{\leq n}$, and another one, in which predicates $c^{\leq n}$ are replaced by sets $C^{\leq n}$ of nominals denoting regions satisfying $c^{\leq n}$. We prove here several completeness theorems. Section 4 is for some concluding remarks and future plans.

1 Contact algebras

Definition 1.1 Following [11], by a *contact algebra* we mean any system $\underline{B} = (B, C) = (B, 0, 1, \cdot, +, *, C)$, where $(B, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra, $*$ denotes the Boolean complement, and C is a binary relation in B, called a *contact*, such that

(C1) if xCy, then $x, y \neq 0$,

(C2) $xC(y + z)$ if and only if xCy or xCz,

(C3) if xCy, then yCx,

(C4) if $x \cdot y \neq 0$, then xCy.

$\underline{B}$ is a complete contact algebra if it is a complete Boolean algebra.

Elements of B are called *regions*. The complement of C is denoted by $\overline{C}$. The relation $\ll$ of *nontangential inclusion* is defined as follows: $x \ll y$ if and only if $x\overline{C}y^*$.

Axioms (C2) and (C3) imply the monotonicity of C with respect to $\leq$:

(Mono) if aCb and $a \leq a'$ and $b \leq b'$, then $a'Cb'$.

We consider the standard definitions of subalgebra, isomorphism, embedding, etc. (cf. [7, Ch. V]). A contact subalgebra $\underline{B_1}$ of $\underline{B_2}$ is said to be *dense* if

(Dense) $(\forall a_2 \in B_2)(a_2 \neq 0 \rightarrow (\exists a_1 \in B_1)(a_1 \neq 0 \text{ and } a_1 \leq a_2))$

If h embeds $\underline{B}$ as a dense contact subalgebra, then h is called *dense embedding*.

Consider contact algebras satisfying some of the following axioms:

(Con) if $a \neq 0$ and $a \neq 1$, then aCa^* *connectedness*

(Ext) if $a \neq 1$, then $\exists b \neq 0$ such that $a\overline{C}b$ *extensionality*

(Nor) if $a \ll b$, then $\exists d$ such that $a \ll d \ll b$ *normality*

A contact algebra satisfying axiom (Con) ((Ext) or (Nor)) is said to be *connected* (*extensional* or *normal*).

Contact algebras satisfying axioms (Con) and (Ext) were introduced in [31] under the name *Boolean contact algebras* and were considered as an equivalent formulation of the system RCC [28]. It is proved in [31] that (Ext) is equivalent (under axioms (C1)–(C4)) to each of the following axioms:

(Ext′) $a \leq b$ if and only if $(\forall d \in B)(aCd \rightarrow bCd)$,

(Ext″) $a = b$ if and only if $(\forall d \in B)(aCd \leftrightarrow bCd)$,

(Ext‴) $(\forall b \neq 0)(\exists a \neq 0)(a \ll b)$.

Note that axiom (Con) is equivalent to the axiom

(Con′) if $a \neq 0$, $b \neq 0$, and $a + b = 1$, then aCb.

Similarly, (Nor) is equivalent to the axiom

(Nor′) if $a\overline{C}b$, then $(\exists a'b')(a\overline{C}a'$ and $b\overline{C}b'$ and $a' + b' = 1)$.

Example 1.2 (1) *Topological contact algebras.* Let X be a topological space with operations of closure $Cl(a)$ and interior $Int(a)$. A subset a of X is *regular closed* if $a = Cl(Int(a))$. The set of all regular closed subsets of X is denoted by $RC(X)$. As is known, the regular closed sets with operations $a + b = a \cup b$, $a \cdot b = Cl(Int(a \cap b))$, $a^* = Cl(X \setminus a) = Cl(-a)$, $0 = \varnothing$, and $1 = X$ form a Boolean algebra. Moreover, if we consider the infinite join operation $\sum_{i \in I} a_i = Cl(\bigcup_{i \in I} a_i)$, then the Boolean algebra $RC(X)$ is complete. The contact is defined as follows: $a\, C_X\, b$ if and only if $a \cap b \neq \varnothing$. It satisfies axioms (C1)–(C4) and consequently $RC(X)$ is a contact algebra. All the algebras of the kind $RC(X)$ and all their subalgebras are called *topological contact algebras*.

(2) *Discrete contact algebras.* Let (W, R) be a relational system with a symmetric and

reflexive binary relation R on W. Following Galton [17] we call such systems *adjacency spaces* and the relation R – *adjacency relation.* For $a, b \subseteq W$ define a contact aC_Rb iff $(\exists x \in a)(\exists y \in b)(xRy)$. It can be seen that the Boolean algebra $CA(W,R)$ of all subsets of W with the above relational contact is a contact algebra. Since this is a non-topological example of contact algebras we call $CA(W,R)$ and its subalgebras *discrete contact algebras*. Regions in $CA(W,R)$ are arbitrary subsets of W.

Lemma 1.3 ([12]) *Let $CA(W,R)$ be discrete contact algebra over the adjacency space (W,R). Then:*

(i) $CA(W,R)$ satisfies (Nor) iff R is an equivalence relation.

(ii) $CA(W,R)$ satisfies (Con) iff (W,R), considered as a graph, is a connected graph, i.e. for any two points $x, y \in W$ there is an R-path connecting x and y.

We recall in Appendix A some topological notions used later on.

Theorem 1.4 (Topological representation of contact algebras) *Let $\underline{B} = (B,C)$ be a contact algebra. Then:*

(I) (i) There exists a compact semiregular T_0-space X and a dense embedding h of $\underline{B}$ in the contact algebra of regular closed sets $RC(X)$. Moreover,

(i1) $\underline{B}$ satisfies (Nor) *iff X is κ-normal.*

(i2) $\underline{B}$ satisfies (Con) *iff X is connected.*

(ii) If $\underline{B}$ satisfies axiom (Ext)*, then there exists a compact weakly regular T_1-space X and a dense embedding h of $\underline{B}$ in the contact algebra of regular closed sets $RC(X)$. Moreover,*

(ii1) $\underline{B}$ satisfies (Nor) *iff X is κ-normal.*

(ii2) $\underline{B}$ satisfies (Con*) iff X is connected.*

(iii) If $\underline{B}$ satisfies both axioms (Ext) *and* (Nor)*, then there exists a compact Hausdorff space (X,τ) and a dense embedding h of $\underline{B}$ in the contact algebra of regular closed sets $RC(X)$. Moreover,*

(iii1) $\underline{B}$ satisfies (Con*) iff X is connected.*

(II) If $\underline{B}$ is a complete contact algebra then in all of the above cases h becomes an isomorphism between (B,C) and $(RC(X), C_X)$. In all cases the set $\{h(a) : a \in B\}$ is a base for the closed sets of X.

Remark 1.5 Different parts of Theorem 1.4 have been proved by different authors. In the present form the theorem is taken from [32]. The case (iii) was proved for the first time in [33]. The case (i)+(i1) and (i)+(i1)+(II) was proved in [11, Sec. 5]. The case (ii)+ (ii1) covers RCC system [28]. This case, without compactness, was proved for the first time in [13], and the case with compactness – in [11, Sec. 5]. The fact that in all cases we have compact representation is important, because it will be used in the next section in the representation theory of contact algebras with connectedness predicates.

Definition 1.6 By Theorem 1.4 for each class Σ of contact algebras determined by some of the axioms (Con), (Ext) and (Nor) there exists a class $Top(\Sigma)$ of topological spaces in which the members of Σ are representable. We call the spaces from $Top(\Sigma)$ *corresponding* to the algebras from Σ.

Theorem 1.7 (Discrete representation of contact algebras [12]) *Let* $\underline{B} = (B, C)$ *be a contact algebra. Then there exists an adjacency space* (W, R) *and an embedding* h *into the contact algebra* $CA(W, R)$*. Moreover,* $\underline{B} = (B, C)$ *satisfies* (Nor) *iff* R *is an equivalence relation. If* $\underline{B}$ *is finite then* (W, R) *is also finite and* h *becomes an isomorphism between* $\underline{B}$ *and* $CA(W, R)$*.*

2 Contact algebras with connectedness predicates

We will use the notations of connectedness predicates as in [19,26] – $c(a)$ and $c^{\leq n}(a)$. In topological spaces $c(a)$ means that the region a is connected in a topological sense and $c^{\leq n}(a)$ – that a is a sum of at most n ($n \geq 1$) connected components. Obviously $c(a)$ iff $c^{\leq 1}(a)$. The following lemma is well known for c (see Lemma 4.1 (iii)) and can easily be proved for $c^{\leq n}$ by induction on n.

Lemma 2.1 *Let* X *be a topological space and* $a \in RC(X)$*. Then:*

(i) $c(a)$ *iff* $(\forall b_0 \neq \varnothing, b_1 \neq \varnothing \in RC(X))(a = b_0 \cup b_1 \to b_0 \cap b_1 \neq \varnothing)$,

(ii) $c^{\leq 1}(a)$ *iff* $c(a)$,

$c^{\leq (n+1)}(a)$ *iff* $(\forall b \neq \varnothing, d \neq \varnothing \in RC(X))(a = b \cup d \to c^{\leq n}(b) \vee b \cap d \neq \varnothing)$,

(iii) $c^{\leq n}(a)$ *iff* $(\forall b_0 \neq \varnothing \dots b_n \neq \varnothing \in RC(X))(a = b_0 \cup \dots \cup b_n \to (\exists i, j : 1 \leq i < j \leq n)(b_i \cap b_j \neq \varnothing))$.

Note that Lemma 2.1 (ii) presents an inductive definition for the predicate $c^{\leq n}$ and (iii) – a direct definition for each $n \geq 1$. This suggests to adopt the following abstract definition of predicates $c^{\leq n}$ in arbitrary contact algebras.

Definition 2.2 Let $\underline{B}$ be a contact algebra. We define $c^{\leq n}(a)$ for an arbitrary $a \in B$, $n = 1, 2, \dots$ by induction as follows:

(i) $c(a) \leftrightarrow_{def} (\forall b, d \in B)(b \neq 0 \wedge d \neq 0 \wedge a = b + d \to bCd)$,

(ii) $c^{\leq 1}(a) \leftrightarrow_{def} c(a)$,

(iii) $c^{\leq (n+1)}(a) \leftrightarrow_{def} (\forall b, d \in B)(b \neq 0 \wedge d \neq 0 \wedge a = b + d \to c^{\leq n}(b) \vee bCd)$.

We denote by $C^{\leq n}(B)$ the set of all regions a such that $c^{\leq n}(a)$.

Note that the sets $C^{\leq n}(B)$ are non-empty, because we always have $c^{\leq n}(0)$.

Lemma 2.3 *The following equivalence is true for any* $a \in B$*:*

$c^{\leq n}(a)$ *iff* $(\forall b_0 \dots b_n \in B)(b_0 \neq 0 \wedge \dots \wedge b_n \neq 0 \wedge a = b_0 + \dots + b_n \to (\exists i, j : 0 \leq i < j \leq n)(b_i C b_j))$.

Proof. The proof proceeds by induction on n and the definition of $c^{\leq n}$. □

Now we will see that the abstract definition of connectedness predicates $c^{\leq n}$ in contact algebras $CA(W, R)$ over adjacency spaces coincides with the standard graph-theoretic connectedness (note that each adjacency space (W, R) can be considered as a graph in a standard way).

In order to characterize the predicates $c^{\leq n}$ in discrete contact algebras (see Example 1.2 (2)) we will use the following notations. Let (W, R) be an adjacency space and

$a \subseteq W$. We denote by R_a the restriction of R on a, and R_a^* will denote the reflexive and transitive closure of R_a. Since R is a symmetric and reflexive relation on W, then R_a is the same on the set a and in this case R_a^* is an equivalence relation on a. Having in mind the above remarks, (W, R) is connected in a graph-theoretic sense (path-connected) if for each $x, y \in W$ we have xR^*y. Since for any $a \subseteq W$ the system (a, R_a) is also an adjacency space, a is connected if for each $x, y \in a$ we have xR_a^*y. Since R_a^* is an equivalence relation then a is path-connected iff R_a^* is the universal relation on a. In general R_a^* divides a into equivalence classes called connected components of a. Having the notion of a connected component, $c(a)$ means that a is itself a connected component and $c^{\leq n}(a)$ means that a is a sum of at most n connected components.

Lemma 2.4 *Let (W, R) be adjacency space and $a \subseteq W$. Then:*
(i) $c(a)$ iff $(\forall x, y \in a)(xR_a^*y)$.
(ii) $c^{\leq n}(a)$ iff $(\forall x_0 \ldots x_n \in a)(\exists i, j : 0 \leq i < j \leq n)(x_i R_a^* x_j)$.

Proof. See the appendix B. □

In the next theorem we deal with topological representation theory of contact algebras with connectedness predicates $c^{\leq n}$. Although $c^{\leq n}$ is definable in contact algebras, if we put this predicate among the signature of contact algebra, it changes the notion of embedding – now every embedding must preserve also the new predicate.

Theorem 2.5 *Let $\underline{B} = (B, C)$ be a contact algebra, X be a semiregular and compact topological space and h be an embedding of (B, C) into the contact algebra $RC(X)$ such that the set $\{h(b) : b \in B\}$ forms a base for the closed sets in X. Then for any $a \in B$: $c^{\leq n}(a)$ holds in $\underline{B}$ iff $c^{\leq n}(h(a))$ holds in $RC(X)$.*

Proof. See the Appendix C. □

Let us recall that all embeddings h in Theorem 1.4 are in compact spaces X such that the set $\{h(a) : a \in \underline{B}\}$ forms a base for closed subsets of X. So the assumptions of Theorem 2.5 are fulfilled and hence h preserves the connectedness predicates $c^{\leq n}$. So we have:

Corollary 2.6 *All embeddings from Theorem 1.4 preserve the predicates $c^{\leq n}$.*

In the next theorem we deal with discrete representation of finite contact algebras with connectedness predicates.

Theorem 2.7 *Let $\underline{B} = (B, C)$ be a finite contact algebra. Then there exists a finite adjacency space (W, R) and an isomorphism h between $\underline{B}$ and $CA(W, R)$ such that for every $a \in B$ the following equivalence is true: $c^{\leq n}(a)$ holds in $\underline{B}$ iff $c^{\leq n}(h(a))$ holds in $CA(W, R)$. Moreover,*
(i) $\underline{B}$ satisfies (Con) *iff the graph (W, R) is connected;*
(ii) $\underline{B}$ satisfies (Nor) *iff the relation R is an equivalence relation.*

Proof. The theorem is a direct corollary of Theorem 1.7 and Lemma 2.4. □

3 Spatial logics with connectedness predicates

We will introduce in this section two kinds of spatial logics. The first kind is based on the language of contact algebras extended with connectedness predicates $c^{\leq n}$. The second kind is also based on the language of contact algebras, but instead of the predicates $c^{\leq n}$, we consider for each natural number $n \geq 1$ a special set of nominals denoting regions with the property $c^{\leq n}$.

Definition 3.1 (RCC-like logics) The minimal logic without connectedness predicates and nominals based on the class of all contact algebras, was studied in [3] under the name "Propositional Weak RCC" and denoted by PWRCC. The adjective "Propositional" is used because all logics are quantifier-free, i.e. propositional. We considered in [3] several other logics based on the same language corresponding to the classes of contact algebras satisfying some or all of the axioms (Con), (Ext) and (Nor). We denote these logics putting indices Con, Ext and Nor to the abbreviation PWRCC. The logic $\mathrm{PWRCC}_{Con,Ext}$ is denoted also by PRCC, because it corresponds to the class of all connected and extensional contact algebras, the algebraic equivalent of the RCC system. By the same reason the system $\mathrm{PWRCC}_{Con,Ext,Nor}$ is denoted by PRCC_{Nor}.

In [3] axiomatization of all of these logics was given (see also Appendix D) and strong and weak completeness theorems were proved with respect to topological semantics, corresponding to contact algebras over some classes of topological spaces (namely the topological spaces in which the corresponding contact algebras are representable), and weak completeness with respect to Kripke semantics – corresponding to the contact algebras over some classes of adjacency spaces. It was shown in [3] that with respect to the weak completeness theorem all RCC-like logics collapse into two systems – PWRCC and PWRCC_{Con}. The collapsing classes can be seen in the following diagram.

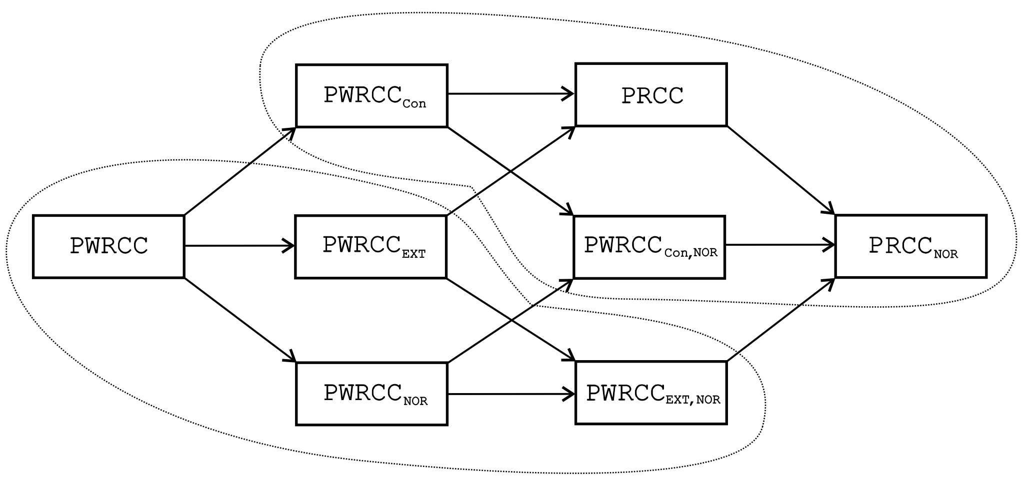

Fig. 1. Propositional RCC type logics

The systems PWRCC and PWRCC_{Con} are equivalent to the systems BRCC-8 introduced by Wolter and Zakharyschev in [35] and interpreted in all topological spaces and in all connected topological spaces. These two systems extended with predicates c and $c^{\leq n}$ are among the important logics studied in [19] – [23].

In this paper we will study all of the logics in the above diagram extended first with the predicates $c^{\leq n}$, and second extended with nominals denoting some regions satisfying $c^{\leq n}$. We will denote the corresponding logics as follows.

Definition 3.2 (Logics with predicates $c^{\leq n}$ and nominals) If $\mathbf{L}$ is the name of one of the above RCC-like logics, its extension with the predicates $c^{\leq n}$, $n \geq 1$, will be denoted by $\mathbf{L}^C$, while extensions with nominals for the regions satisfying $c^{\leq n}$, $n \geq 1$, will be denoted by $\mathbf{L}^{NomC}$. The logics $\mathbf{L}^C$ and $\mathbf{L}^{NomC}$ have the same inclusion diagram as their counterparts $\mathbf{L}$ from Fig 1.

We will present a complete axiomatization of all of these logics, considered in the corresponding classes of contact algebras and prove strong and weak completeness theorems with respect to algebraic, topological and Kripke semantics.

3.1 Syntax

We consider two languages denoted by $L^C = L(\leq, C, c^{\leq n})$ and $L^{NomC} = L(\leq, C, NomC^{\leq n})$. The language $L(\leq, C, c^{\leq n})$ consists of:

• a denumerable set Var of Boolean variables,

• Boolean operations: $+$ (Boolean sum), $\cdot$ (Boolean meet), * (Boolean complement), 0,1 (Boolean constants).

• Relational symbols: $\leq$ (part-of), C (contact), $c^{\leq n}$ for each natural number $n \geq 1$.

• Standard propositional connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ and the propositional constants $\perp, \top$.

• parentheses: (,).

The set of Boolean terms is defined in a standard way from Boolean variables and constants by means of Boolean operations.

Atomic formulas are of the type: $a \leq b$, aCb, $c^{\leq n}(a)$, $\perp$, $\top$, where a, b are Boolean terms. Formulas are defined from atomic formulas by using propositional connectives in a standard way.

Abbreviations: $a = b =_{def} (a \leq b) \wedge (b \leq a)$, $a \neq b =_{def} \neg(a = b)$, $a\overline{C}b =_{def} \neg(aCb)$, $a \ll b =_{def} a\overline{C}b^*$.

The restriction of the language $L(\leq, C, c^{\leq n})$ only for $n = 1$ is denoted by $L^c = L(\leq, C, c)$. Let us note that in [19] the above languages are denoted correspondingly L^C by $\mathcal{C}cc$ and L^c by $\mathcal{C}c$.

The language $L(\leq, C, NomC^{\leq n})$ differs from the language $L(\leq, C, c^{\leq n})$ as follows: instead of predicate symbols $c^{\leq n}$, it has for each natural number $n \geq 1$ a denumerable set $NomC^{\leq n}$ of new propositional letters called nominals ($NomC^{\leq 1}$ is denoted also by $NomC$). If $a \in NomC^{\leq n}$ then the number n is attached to the nominal a and called its characteristic number. Terms now are defined by Boolean variables, Boolean constants and nominals by Boolean operations. Atomic formulas are only of the form $a \leq b$, aCb, $\perp$ and $\top$ where a, b are Boolean terms. The difference between Boolean variables and nominals is that variables can be substituted by arbitrary terms, while nominals can be substituted only by nominals with the same characteristic number.

3.2 Semantics

Both languages are interpreted in contact algebras as follows. Let $\underline{B} = (B, C)$ be a contact algebra with definable predicates $c^{\leq n}$. A mapping $v : Var \rightarrow B$ is called a valuation. Valuations are extended homomorphically to arbitrary Boolean terms in a standard way. A pair $M = (\underline{B}, v)$ is called a model. We define a truth of a formula α in a model $(\underline{B}, v)$, denoted by $(\underline{B}, v) \models \alpha$, inductively as follows:

$(\underline{B}, v) \models a \leq b$ iff $v(a) \leq v(b)$,
$(\underline{B}, v) \models aCb$ iff $v(a)Cv(b)$
$(\underline{B}, v) \models c^{\leq n}(a)$ iff $c^{\leq n}(v(a))$

The interpretation in the language $L(\leq, C, NomC^{\leq n})$ differs as follows. Any valuation v maps Boolean variables into elements of B and if $p \in NomC^{\leq n}$ then $v(p) \in C^{\leq n}(B)$. So nominals from $NomC^{\leq n}$ are mapped into regions satisfying $c^{\leq n}$. The interpretation of formulas is the same.

A formula α (in both languages) is true in the contact algebra (B, C) if for all valuations v we have $(B, C, v) \models \alpha$. If Σ is a class of contact algebras then α is true in Σ if it is true in each member of Σ. The set $\mathbb{L}(\Sigma)$ of all formulas (from the given language) true in Σ is called the logic of Σ. $M = (\underline{B}, v)$ is a model of a set of formulas A if $(\underline{B}, v) \models \alpha$ for every $\alpha \in A$. This is the algebraic semantics of both languages and the models in the form $M = (\underline{B}, v)$ are called algebraic models.

If we consider interpretations in contact algebras of the form $RC(X)$ from a given topological space – this is the topological semantics, and if we consider only interpretations in contact algebras of the form $CA(W, R)$ over some adjacency space (W, R) – this is the Kripke (relational) semantics of the presented languages.

It is shown in [19,21] that the language with predicates $c^{\leq n}$ is equally expressive with the language having only the predicate c, but the translations need exponentially more new variables. Consequently our languages $L(\leq, C, c^{\leq n})$ and $L(\leq, C, c)$ also have equal expressive power.

Although the languages with nominas do not contain the predicates $c^{\leq n}$, we can simulate them by the corresponding nominals from $NomC^{\leq n}$ and some new variables in a way similar to the proof of the above result given in [19,21], showing in this way that the languages with nominals are equally expresive with the languages with predicates $c^{\leq n}$. For instance formulas from the language $L(\leq, C, c)$ can be reduced to formulas from the language $L(\leq, C, NomC)$ as follows. Let α be a formula from $L(\leq, C, c)$, Each positive occurrence of predicates of the form $c(a)$ have to be replaced by $a = q$ with a fresh nominal $q \in NomC$. Each negative occurrence of formulas of the form $c(a)$ have to be replaced by the formula $(a = q_1 + q_2) \wedge (q_1 \neq 0) \wedge (q_2 \neq 0) \wedge q_1\overline{C}q_2$ with fresh variables q_1, q_2. In this way we obtain a formula α' in the language $L(\leq, C, NomC)$ equally satisfiable with the formula α. So, the advantage of logics with the predicates $c^{\leq n}$ is that they allow to speak in a shorter way about connectedness, while the advantage of logics with nominals is that they are simpler, and as we shall see in the next section, they do not require in their axiomatization additional rules.

3.3 Axiomatizations and completeness theorems

Let $\mathbf{L}$ be any of the logics from Definition 3.1. The axiomatization of $\mathbf{L}^C$ (see Definition 3.2) can be obtained from the axiomatization of $\mathbf{L}$ given in [3] (see Appendix D) by adding one additional axiom denoted by (Ax $c^{\leq n}$) for each natural number $n \geq 1$, and by adding one additional rule denoted by (Rule $c^{\leq n}$) for each $n \geq 1$:

$$\text{(Ax } c^{\leq n}\text{)} \quad c^{\leq n}(a) \wedge \bigwedge_{i=0}^{n} p_i \neq 0 \wedge a = \sum_{i=0}^{n} p_i \Rightarrow \bigvee_{0 \leq i < j \leq n} p_i C p_j$$

$$\text{(Rule } c^{\leq n}\text{)} \quad \frac{\alpha \wedge \bigwedge_{i=0}^{n} p_i \neq 0 \wedge a = \sum_{i=0}^{n} p_i \Rightarrow \bigvee_{0 \leq i < j \leq n} p_i C p_j}{\alpha \Rightarrow c^{\leq n}(a)},$$

where α is a formula and $p_0, \ldots, p_n$ are different Boolean variables not occurring in the term a and the formula α (the parameters of the application of the rule).

The axiomatization of $\mathbf{L}^{NomC}$ can be obtained by adding to the axiomatization of $\mathbf{L}$ an additional axiom denoted by (Ax Nom$C^{\leq n}$) for all $n \geq 1$ and for all nominals $a \in C^{\leq n}$:

$$\text{(Ax Nom}C^{\leq n}\text{)} \quad \bigwedge_{i=0}^{n} p_i \neq 0 \wedge a = \sum_{i=0}^{n} p_i \Rightarrow \bigvee_{0 \leq i < j \leq n} p_i C p_j.$$

Theorem 3.3 (Strong Completeness theorem) *Let* $\mathbf{L}$ *be any logic from Definition 3.1 and* $\mathbb{L}$ *be any of the logics* $\mathbf{L}^C$ *and* $\mathbf{L}^{NomC}$. *Denote by* $\Sigma(\mathbf{L})$ *the class of contact algebras corresponding to* $\mathbf{L}$ *and by* $Top(\Sigma(\mathbf{L}))$ *the class of topological spaces corresponding (by Definition 1.6) to* $\Sigma(\mathbf{L})$ *in which the members of* $\Sigma(\mathbf{L})$ *are representable. Then the following conditions are equivalent for any set of formulas* A *in the language of* $\mathbb{L}$*:*

(i) A *is consistent in* $\mathbb{L}$*,*

(ii) A *has an algebraic model in* $\Sigma(\mathbf{L})$*,*

(iii) A *has a topological model in* $Top(\Sigma(\mathbf{L}))$*.*

Proof. See Appendix E. □

Theorem 3.4 (Weak completeness theorem including finite models) *Let* $\mathbf{L}$ *be any logic from Definition 3.1 and* $\mathbb{L}$ *be any of the logics* $\mathbf{L}^C$ *and* $\mathbf{L}^{NomC}$*. Then the following conditions are equivalent for any formula* α *of the language of* $\mathbb{L}$*:*

(i) α *is a theorem of* $\mathbb{L}$*.*

(ii) α *is true in all contact algebras from* $\Sigma(\mathbf{L})$*,*

(iii) α *is true in all contact algebras* $RC(X)$ *from* $Top(\Sigma(\mathbf{L}))$*.*

(iv) α *is true in all finite contact algebras in which all theorems of* $\mathbf{L}$ *are true.*

(v) α *is true in all contact algebras* $CA(W, R)$ *over all finite adjacency spaces* (W, R) *in which all theorems of* $\mathbf{L}$ *are true.*

Proof. The equivalence of (i), (ii) and (iii) follow from Theorem 3.3. The equivalence of (iv) and (v) is a corollary of Theorem 2.7. Obviously (ii) implies (iv). For the implication (iv)$\rightarrow$(i) suppose that α is not theorem of $\mathbf{L}$. Then there is a canonical model (B, C, v) of $\mathbf{L}$ determined by a maximal consistent set Γ (see the proof of Theorem 3.3) which falsifies α. Now by a special filtration construction we will define a finite model (B', C', v')

which also falsifies α. Let $p_1, \ldots, p_k$ be the Boolean variables occurring in α. Let β_i, $i = 1, \ldots, \beta_l$ be the sequence of all sub-formulas of α not belonging to Γ which have a form of the conclusion of some of the special rules. By the properties of Γ for each β_i there exists a finite sequence of Boolean variables making the premise of the corresponding rule not belonging to Γ. For instance if the conclusion $\beta_i = \gamma \Rightarrow c^{\leq n}(a)$ of the rule (Rule $c^{\leq n}$) does not belong to Γ then the premise $\gamma \wedge \bigwedge_{i=0}^{n} r_i \neq 0 \wedge a = \sum_{i=0}^{n} r_i \Rightarrow \bigvee_{0 \leq i < j \leq n} r_i C r_j$ also does not belong to Γ for some Boolean variables $r_0, \ldots, r_n$. Now let $q_1, \ldots, q_m$ be the sequence of all such variables determined by the formulas β_i. Consider the finite Boolean subalgebra B' of B generated by the elements $|p_1|, \ldots, |p_k|, |q_1|, \ldots, |q_m|$ of B. It is itself a contact subalgebra of (B, C) with the same contact C. Let v' be the canonical valuation v but restricted to the variables $p_1, \ldots, p_k, q_1, \ldots, q_m$. It is easy to see that for all Boolean terms a from $p_1, \ldots, p_k, q_1, \ldots, q_m$ we have $v(a) = v'(a)$. The following statement is a kind of *Filtration Lemma*:

Filtration Lemma. The following equivalence is true for all subformulas γ of α: $(B, C, v) \models \gamma$ iff $(B', C, v') \models \gamma$.

The proof goes by induction on the construction of γ. The nontrivial part is for the atomic $\gamma = c^{\leq n}(a)$.

$(\rightarrow)$ Suppose $(B, C, v) \models c^{\leq n}(a)$, and for the sake of contradiction that (B', C, v') $\not\models c^{\leq n}(a)$, i.e. $c^{\leq n}(v'(a))$ is not true in (B', C). Then for some Boolean terms $b_0, \ldots, b_n$ from $p_1, \ldots, p_k, q_1, \ldots, q_m$ we have: $v'(a) = v'(b_0) + \ldots + v'(b_n)$, $v'(b_i) \neq 0$ for $i = 0, \ldots, n$ and for all i, j, $0 \leq i < j \leq n$, we have $v'(b_i)\overline{C}v'(b_j)$. Since $B' \subseteq B$ and $v'(b_i) = v(b_i)$ for all $i = 0, \ldots, n$, we obtain that $c^{\leq n}(v(a))$ does not hold in (B, C, v), which contradicts the assumption.

$(\leftarrow)$. We will reason by contraposition. Suppose that $(B, C, v) \not\models c^{\leq n}(a)$. By the properties of the canonical model this implies that $c^{\leq n}(a) \notin \Gamma$. Then by the properties of the maximal consistent set Γ, the formula $\bigwedge_{i=0}^{n} r_i \neq 0 \wedge a = \sum_{i=0}^{n} r_i \Rightarrow \bigvee_{0 \leq i < j \leq n} r_i C r_j$ also does not belong to Γ for some Boolean variables $r_0, \ldots, r_n$, such that (by the above construction) $|r_0|, \ldots, |r_n|$ are from the generators of B'. This implies that $v'(a) = \sum_{i=0}^{n} v'(r_i)$, $v'(r_i) \neq 0$ for $i = 0 \ldots, n$ and for all i, j, $0 \leq i < j \leq n$, we have $v'(r_i)\overline{C}v'(r_j)$. All this shows that $(B', C, v') \not\models c^{\leq n}(a)$, which finishes the proof of the Filtration lemma.

By the Filtration lemma we obtain that α is falsified in the finite Boolean algebra (B', C).

Now we will show that all theorems of **L** are true in (B', C). To this end we first show that all theorems of **L** are true in the canonical algebra (B, C), which would imply that the same is true for (B', C). This follows from the following observation. By the properties of the canonical model (B, C, v) we have that for any formula β: $(B, C, v) \models \beta$ iff $\beta \in \Gamma$. So, if β is a theorem of **L**, then $\beta \in \Gamma$ and hence $(B, C, v) \models \beta$. Let w be an arbitrary valuation in B. Then w defines a substitution on Boolean variables Sub_w, which can be applied to arbitrary formulas. It is easy to see that the following holds: $(B, C, w) \models \beta$ iff $(B, C, v) \models Sub_w(\beta)$ iff $Sub_w(\beta) \in \Gamma$. But if β is a theorem then $Sub_w(\beta)$ is also a theorem which implies that $(B, C, w) \models \beta$. □

As a consequence of Theorem 3.4 we obtain:

Corollary 3.5 *(i) The logics $PWRCC^C$ and $PWRCC^{NomC}$ are complete in the class of all contact algebras $CA(W,R)$ over arbitrary (finite) adjacency spaces.*

(ii) The logics $PWRCC^C_{Con}$ and $PWRCC^{NomC}_{Con}$ are complete in the class of all contact algebras $CA(W,R)$ over arbitrary (finite) connected adjacency spaces.

Theorem 3.6 (Admisisibility of the rules (EXT) and (NOR)) *(i) The rules* (EXT) *and* (NOR) *are admissible in the logics $PWRCC^C$ and $PWRCC^{NomC}$.*

(ii) The rules (EXT) *and* (NOR) *are admissible in the logics $PWRCC^C_{Con}$ and $PWRCC^{NomC}_{Con}$.*

Proof. We proved in [3] that the rules (EXT) and (NOR) (see Appendix D) are admissible for the logics PWRCC and PWRCC^{Nor}. Inspecting the proof in [3] it can be seen that the same construction can be used for the proof of the logics PWRCC^C and PWRCC^{NomC}. The proof uses Corollary 3.5. So we invite the reader to consult [3]. □

Corollary 3.7 (Elimination of the rules (EXT) and (NOR)) *(i) The logics $PWRCC^C$, $PWRCC^C_{Ext}$, $PWRCC^C_{Nor}$, $PWRCC^C_{Ext,Nor}$ have equal sets of theorems. The logic $PWRCC^C$ is weakly complete in all classes of models in which $PWRCC^C_{Ext}$, $PWRCC^C_{Nor}$, $PWRCC^C_{Ext,Nor}$ are strongly complete.*

(ii) The logics $PWRCC^{NomC}$, $PWRCC^{NomC}_{Ext}$, $PWRCC^{NomC}_{Nor}$, $PWRCC^{NomC}_{Ext,Nor}$ have equal sets of theorems. The logic $PWRCC^{NomC}$ is weakly complete in all classes of models in which $PWRCC^{NomC}_{Ext}$, $PWRCC^{NomC}_{Nor}$, $PWRCC^{NomC}_{Ext,Nor}$ are strongly complete.

(iii) The logics $PWRCC^C_{Con}$, $PWRCC^C_{Ext,Con}$, $PWRCC^C_{Nor,Con}$, $PWRCC^C_{Ext,Nor,Con}$ have equal sets of theorems. The logic $PWRCC^C_{Con}$ is weakly complete in all classes of models in which $PWRCC^C_{Ext,Con}$, $PWRCC^C_{Nor,Con}$, $PWRCC^C_{Ext,Nor,Con}$ are strongly complete.

(iv) The logics $PWRCC^{NomC}_{Con}$, $PWRCC^{NomC}_{Ext,Con}$, $PWRCC^{NomC}_{Nor,Con}$, $PWRCC^{NomC}_{Ext,Nor,Con}$ have equal sets of theorems. The logic $PWRCC^{NomC}_{Con}$ is complete in all classes of models in which $PWRCC^{NomC}_{Ext,Con}$, $PWRCC^{NomC}_{Nor,Con}$, $PWRCC^{NomC}_{Ext,Nor,Con}$ are strongly complete.

(v) All mentioned logics have finite model property, are finitely axiomatizable and hence are decidable.

Let us note that with respect to the weak completeness Corollary 3.5 and Corollary 3.7 show that we have only four interesting logics: PWRCC^C, PWRCC$^C_{Con}$, PWRCC^{NomC} and PWRCC$^{NomC}_{Con}$, which do not contain the rules (Ext) and (Nor). Note that PWRCC^C and PWRCC$^C_{Con}$ are just the logics BRCC extended with the predicates $c^{\leq n}$ studied in [19] – [21] considered respectively in the classes of all topological spaces and all connected topological spaces. The Corollary 3.7 in fact shows that these rules are admissible for these logics, which implies that we may apply the Strong completeness theorem 3.3 to obtain weak completeness for these logics for the classes of models considered for the logics having these rules. For instance Corollary 3.7 says for the logic PWRCC$^C_{Con}$ that it is complete in: (1) all connected topological spaces, (2) all compact connected spaces, (3) all connected weakly regular spaces (spaces for the system RCC), (4) all connected weakly regular and compact T$_1$ spaces, (5) all compact Hausdorff spaces, (6) all models over connected adjacency spaces (W,R). It is shown in [19] that satisfiability for formulas from the language with predicate c is ExpTime complete in the class of all connected topological spaces and NExpTime complete in the same class for formulas from the language containing the predicates $c^{\leq n}$. Now this result can be transferred for all mentioned classes of spaces. Similar transfers are true also for the other three logics. Another remark for these logics is the following open question: is it possible to eliminate the rule (Rule $c^{\leq n}$) replacing it only with some set of axioms. The logics with nominals did not contain this rule and in a sense can be considered as a result of elimination of this rule (Rule $c^{\leq n}$).

4 Concluding remarks

In this paper we have presented complete axiomatizations of several natural spatial logics based on the language of contact algebras with connectedness predicates c and $c^{\leq n}$. Some of these logics were studied semantically with respect to their complexity in [19] – [21]. We proved for the introduced logics strong and weak completeness theorems for various important classes of topological spaces and also completeness theorems with respect to some non-topological spaces – adjacency spaces, which are used for certain discrete models of space. This implies that all these classes have the same complexity of satisfiability and allows to transfer some known results from [19] – [21]. The semantics in adjacency spaces can be considered also as a kind of Kripke semantics over reflexive and symmetric Kripke frames (W, R). This semantics allows to translate our logics in the modal logics over KTB + universal modality, which shows that not only S4, but also KTB has also a spatial meaning (see about this discussion [3, Sec. 1]). In [19] – [21] several other languages containing the predicates c and $c^{\leq n}$ are considered and the problem for their complete axiomatization with respect to the intended semantics remains open. We postpone this for our plans for the future.

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Appendix A. Some topological notions

A topological space X is said to be

• *semiregular* if it has a base $\mathbb{B}$ of regular closed sets; namely, every closed set is the intersection of elements of $\mathbb{B}$,

• *normal* if every pair of closed disjoint sets can be separated by a pair of open sets,

• *κ-normal* (cf. [29]) if every pair of regular closed disjoint sets can be separated by a pair of open sets,

• *weakly regular* (cf. [13]) if it is semiregular and for every nonempty open set a there exits a nonempty open set b such that $\mathrm{Cl}(b) \subseteq a$,

• *connected* if it cannot be represented as the sum of two disjoint nonempty open sets (if $a \subseteq X$, then a is connected if it is connected in the subspace topology),

• a T_0-*space* if for every two different points $x \neq y$ there exists an open set that contains one of them and does not contain the other,

• a T_1-*space* if every one-point set $\{x\}$ is a closed set,

• a *Hausdorff space* (or a T_2-*space*) if every two different points can be separated by a pair of disjoint open sets,

• a *compact space* if it satisfies the following condition: if $\{A_i : i \in I\}$ is a nonempty family of closed sets of X such that for every finite nonempty subset $J \subseteq I$ we have $\bigcap\{A_i : i \in J\} \neq \varnothing$, then $\bigcap\{A_i : i \in I\} \neq \varnothing$.

Lemma 4.1 *The following assertions hold:*

(i) Let X be semiregular. Then X is weakly regular if and only if $\mathrm{RC}(X)$ *satisfies* (Ext) *[13].*

(ii) X is κ-normal if and only if $\mathrm{RC}(X)$ *satisfies* (Nor) *[13].*

(iii) X is connected if and only if $\mathrm{RC}(X)$ *satisfies axiom* (Con) *[5,13].*

(iv) If X is a compact Hausdorff space, then $\mathrm{RC}(X)$ *satisfies* (Ext) *and* (Nor) *[33].*

(iv) If X is a normal Hausdorff space, then $\mathrm{RC}(X)$ *satisfies* (Nor) *[6].*

Appendix B. Proof of Lemma 2.4

Proof. (i) Let $a \subseteq W$. Then (a, R_a) is an adjacency space. Then $CA(a, R_a)$ satisfies axiom (Con') iff $(\forall b, d \subseteq a)(b \neq \varnothing \wedge d \neq \varnothing \wedge a = b \cup d \rightarrow bC_R d)$ iff $c(a)$ iff (by Lemma 1.3 (ii)) a is path-connected, i.e. $(\forall x, y \in a)(xR_a^* y)$.

(ii) We will use the inductive definition of $c^{\leq n}$ and proceed by induction on n. The case $n = 1$ (the base of induction) is just (i). So suppose that the statement is true for n and proceed for $n+1$. We have to prove that the following two conditions are equivalent:

(I) $(\forall b, d \subseteq a)(b \neq \varnothing \wedge d \neq \varnothing \wedge a = b \cup d \rightarrow c^{\leq n}(b) \vee bC_R d)$,

(II) $(\forall x_0, \ldots, x_n, x_{n+1} \in a)(\exists i, j : 0 \leq i < j \leq n+1)(x_i R_a^* x_j)$

$(I) \rightarrow (II)$ Suppose (I) and for the sake of contradiction that (II) does not hold, i.e. $(\exists x_0, \ldots, x_n, x_{n+1} \in a)(\forall i, j : 0 \leq i < j \leq n+1)(x_i \overline{R_a^*} x_j)$. Denote by $|x_i|$ the R_a^*-equivalence class generated by x_i and let $b = \bigcup_{i=0}^{n} |x_i|$, $d = a \smallsetminus b$. Obviously $b \neq \varnothing$, $d \neq \varnothing$ ($x_{n+1} \notin b$ and hence is in d) and $a = b \cup d$. Since for all i, j, $0 \leq i < j \leq n$,

$x_i\overline{R^*}_a x_j$, $x_i \in b$, then by the inductive hypothesis we have $\neg c^{\leq n}(b)$. So by (I) we obtain $bC_R d$, so there exist $y \in b$ and $z \in d$ such that yRz. Since $b = \bigcup_{i=0}^{n} |x_i|$, then there exists $|x_i| \subseteq b$ such that $y \in |x_i|$, hence $x_i R_a^* y$ and by $yR_a z$ we get $x_i R_a^* z$. This implies that $z \in |x_i|$, so $z \in b$, which is impossible ($z \in d$, hence $z \notin b$).

$(II) \to (I)$ Suppose (II) and in order to obtain a contradiction that (I) does not hold, i.e. there exist $b \neq \varnothing$, $d \neq \varnothing$, $a = b \cup d$, $\neg c^{\leq n}(b)$ and $b\overline{C_R}d$. The last condition implies $b \cap d = \varnothing$. Applying the inductive condition to $\neg c^{\leq n}(b)$, we obtain that there exist $x_0, \ldots, x_n \in b$ such that for all i and j, $0 \leq i < j \leq n$, $x_i\overline{R_b^*}x_j$, so $x_i\overline{R_a^*}x_j$. Since $d \neq \varnothing$, there exists $x_{n+1} \in d$. Applying (I) to the sequence $x_0, \ldots x_n, x_{n+1}$ we obtain that there are i and j, $i \neq j$, such that $x_i R_a^* x_j$. The only possibility is $j = n+1$ and $i \leq n$. So $(\exists m)(\exists y_0, \ldots, y_m \in a)(y_0 = x_i \wedge y_m = x_{n+1} \wedge (\forall k < m)(y_k R y_{k+1}))$. Since $y_0 \in b$ and $y_m \in d$ and $b \cap d = \varnothing$, then there exist $y_k \in b$ and $y_{k+1} \in d$, so by $y_k R y_{k+1}$ we obtain $bC_R d$ – a contradiction. □

Appendix C. Proof of Theorem 2.5

Proof. We will use the equivalent definition of $c^{\leq n}$ from Lemma 2.3.

$(\to)$ Suppose $c^{\leq n}(a)$ holds in $\underline{B}$ and for the sake of contradiction that $c^{\leq n}(h(a))$ does not hold in $RC(X)$. Then there are $P_i \in RC(X)$, $i = 0, \ldots, n$ such that:

(1) $P_i \neq \varnothing$, $i = 0, \ldots, n$,

(2) $h(a) = \bigcup_{i=0}^{n} P_i$,

(3) for all i and j, $0 \leq i < j \leq n$, we have $P_i \cap P_j = \varnothing$.

(4) Since all P_i are closed sets each is an intersection of elements from the base $\{h(q) : q \in B\}$, so for every P_i there is a $A_i \subseteq B$ such that $P_i = \bigcap_{p \in A_i} h(p)$. From (2) we have $P_i \subseteq h(a)$ so we may assume that $a \in A_i$ for all i, $i = 0, \ldots, n$.

Now let $i \neq j$, $0 \leq i, j \leq n$ be fixed. Then from (3) and (4) we obtain

(5) $(\bigcap_{p \in A_i} h(p)) \cap (\bigcap_{p \in A_j} h(p)) = \varnothing$.

(6) Applying compactness to (5), there exist a finite subset $A_i^j \subseteq A_i$ and a finite subset $A_j^i \subseteq A_j$ such that

(7) $(\bigcap_{p \in A_i^j} h(p)) \cap (\bigcap_{p \in A_j^i} h(p)) = \varnothing$. Without loss of generality we may assume that $a \in A_i^j$ and $a \in A_j^i$.

(8) Define $A_i' = \bigcup_{j=0, j \neq i}^{n} A_i^j$ for all $i = 0, \ldots, n$. Obviously A_i' is a finite subset of A_i containing a.

(9) It follows from (4), (6), (7) and (8) that $P_i \subseteq \bigcap_{p \in A_i'} h(p) \subseteq \bigcap_{p \in A_i^j} h(p) \subseteq h(a)$, $0 \leq i, j \leq n$ and $i \neq j$.

Now from (9) and (3) we get:

(10) $(\bigcap_{p \in A_i'} h(p)) \cap (\bigcap_{p \in A_j'} h(p)) = \varnothing$, for all $i, j = 0, \ldots, n$ and $i \neq j$.

(11) It follows from (9) that $P_i \subseteq h(p)$ for all $p \in A_i'$, $i, j = 0, \ldots, n$. Since $P_i \in RC(X)$, then by the Boolean product of $RC(X)$ we obtain

(12) $P_i \subseteq \prod_{p \in A_i'} h(p) = h(\prod_{p \in A_i'} p) = h(p_i)$, where $p_i = \prod_{p \in A_i'} p$, $i = 0, \ldots, n$.

(13) It follows from (12) that $p_i \leq p$ for all $p \in A_i'$ especially $p_i \leq a$, because $a \in A_i'$. From here we get

$h(p_i) \subseteq h(p)$ for all $p \in A'_i$ and hence

(14) $h(p_i) \subseteq \bigcap_{p \in A_i} h(p)$, $i = 0, \ldots, n$.

(15) Now from (14) and (10) we get $h(p_i) \cap h(p_j) = \varnothing$ for all i and j, $i \neq j$, $i, j = 0, \ldots, n$, which implies

(16) $p_i \overline{C} p_j$ for all i and j, $i \neq j$, $i, j = 0, \ldots, n$.

(17) It follows from (1) and (12) that $h(p_i) \neq \varnothing$, hence $p_i \neq 0$, $i = 0, \ldots, n$.

(18) Since by (13) $P_i \subseteq h(p_i) \subseteq h(a)$ then $\bigcup_{i=0}^{n} P_i \subseteq \bigcup_{i=0}^{n} h(p_i) \subseteq h(a)$. From here we get $\bigcup_{i=0}^{n} P_i \subseteq h(\bigcup_{i=0}^{n} p_i) \subseteq h(a)$.

(19) From (18) and (2) we get $h(a) \subseteq h(p_0 + \cdots + p_n) \subseteq h(a)$ which implies

(20) $a = p_0 + \cdots + p_n$.

Now (16), (17) and (20) imply $\neg c^{\leq n}(a)$ in $\underline{B}$, which contradicts the assumption.

($\leftarrow$) Suppose the $c^{\leq n}(h(a))$ holds in $RC(X)$ and for the sake of contradiction that $c^{\leq n}(a)$ does not hold in $\underline{B}$. Then there are $p_0, \ldots, p_n \in B$ such that

(21) $p_i \neq 0$, $i = 0, \ldots, n$, $a = p_0 + \cdots + p_n$, and
$p_i \overline{C} p_j$, for all i and j, $i \neq j$, $i, j = 0, \ldots, n$.

¿From (21) we get

(22) $h(p_i) \neq \varnothing$, $i = 0, \ldots, n$, $h(a) = h(p_0) \cup \ldots \cup h(p_n)$, and
$h(p_i) \cap h(p_j) = \varnothing$, for all i and j, $i \neq j$, $i, j = 0, \ldots, n$.

But (22) implies that $c^{\leq n}(h(a))$ does not hold in $RC(X)$. □

Appendix D. Axiomatizations of the RCC-like logics from Definition 3.1

Axioms of PWRCC (see [3, Sec. 6])

I. Axiom schemes of the classical propositional logics.

II. Axioms of Boolean algebra based on $\leq$.

III. Axioms for the contact relation C.

Since all predicate axioms of Boolean algebra and contact algebra are universal sentences, they can be written in our language.

Rules of Inference Modus ponens (MP) $\dfrac{\alpha, \alpha \Rightarrow \beta}{\beta}$

The axiomatizations of the other logics from Definition 3.1 we remind the following two rules from [3]:

For an analog of the first-order axiom (Ext) we introduce the *rule of extensionality*

(EXT) $\dfrac{\alpha \Rightarrow (p = 0 \vee aCp)}{\alpha \Rightarrow (a = 1)}$, where p is a Boolean variable that does not occur in a and α.

For an analog of axiom (Nor) we introduce the following *rule of normality*:

(NOR) $\dfrac{\alpha \Rightarrow (aCp \vee p^{*}Cb)}{\alpha \Rightarrow aCb}$, where p is a Boolean variable that does not occur in a, b, and α.

If **L** is any logic from definition 3.1, then its axiomatization can be obtained from the axiomatization of PWRCC as follows:

• If one wants to axiomatize valid formulas in the class of all connected contact algebras – add the axiom (Con) $a \neq 0 \wedge a \neq 1 \Rightarrow aCa^*$.

• If one wants to axiomatize valid formulas in the class of all contact algebras satisfying the axiom (Ext) – add the rule (EXT).

• If one wants to axiomatize valid formulas in the class of all contact algebras satisfying the axiom (Nor) – add the rule (NOR).

Appendix E. Proof of Theorem 3.3

Proof. We will consider only the case $\mathbb{L} = \mathbf{L}^C$, the other case can be treated similarly. Note that the equivalence (ii)↔(iii) is a corollary from the topological representation Theorem 1.4. The implication (ii)→(i) is obvious and for the implication (i)→(ii) we will use a kind of canonical model construction. This construction is a variant of the Henkin proof of the completeness theorem for the first-order logic adapted for the logics of the considered kind with additional rules. This construction is described in [3, Sec. 7] (see also [32, Sec. 3.3]), so we refer the reader to consult for the details the above references. The main idea is shortly the following.

Each consistent set A can be extended into a maximal consistent set Γ with some special properties depending on the rules of the logic:

(1) Γ contains all theorems of the logic and is closed under the rule modus ponens,

(2) If the conclusion $\alpha \Rightarrow c^{\leq n}(a)$ of the rule (Rule $c^{\leq n}$) does not belong to Γ then the premise $\alpha \wedge \bigwedge_{i=0}^{n} p_i \neq 0 \wedge a = \sum_{i=0}^{n} p_i \Rightarrow \bigvee_{0 \leq i < j \leq n} p_i C p_j$ also does not belong to Γ for some parameters $p_0, \dots, p_n$. Similar conditions are formulated for the other rules.

Then, using Γ, one can construct in a canonical way a contact algebra (B, C) as follows: define in the set of Boolean terms $a \equiv b$ iff $a = b \in \Gamma$. It can be proved that this is a congruence relation with respect to the Boolean operations which makes possible to define a Boolean algebra over the classes $|a|$ modulo this congruence. We define $|a|C|b|$ iff $aCb \in \Gamma$. The axioms of contact guarantee that (B, C) is a contact algebra. Moreover the above properties of Γ and the additional axioms and rules of the logic guarantee that the obtained contact algebra belongs to the class $\Sigma(\mathbf{L})$. For instance the axiom (Ax $c^{\leq n}$) guarantee the implication

$c^{\leq n}(|a|)$ implies $(\forall |p_0|, \dots, |p_n| \in B)(|p_0| \neq |0| \wedge \dots \wedge |p_n| \neq |0| \wedge$ $|a| = |p_0| + \cdots + |p_n| \rightarrow (\exists i, j : 0 \leq i < j \leq n)(|p_i|C|p_j|))$.

The converse implication is guarantied by the property (2) of the set Γ, which shows that the definition of $c^{\leq n}(|a|)$ is fulfilled in (B, C).

By means of Γ one can define a canonical valuation v in B as follows: $v(p) = |p|$ and finally we need to prove the truth lemma saying that $(B, C, v) \models \alpha$ iff $\alpha \in \Gamma$. Then this shows that (B, C, v) is a model of Γ and hence a model of A. □