Density or Discrepancy? A VLSI Designer's Dilemma in Hot Spot Analysis

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Abstract

In this era of giga-scale integration, thermal analysis has become one of the hot topics in VLSI chip design. Active thermal sources may be abstracted as a set of weighted points on the chip-floor. The conventional notion of discrepancy that deals with the properties of a set of scattered points may not be able to capture properly all real-life instances in this context. In this paper, we have introduced a new concept, called *density of a region* to study some of the properties of the distribution of these weighted points. We prove several results that help in identifying the regions with maximum and minimum density. We also compare the concept of density with the existing concept of discrepancy.

1 Introduction

Thermal analysis has become one of the hot topics as power density in some regions of a VLSI chip is rising critically with ever-increasing clock-rate and device density. Proper placement of heat sinks and thermal vias have become issues of utmost importance. Some researchers have addressed the problem of hot spot identification [4] and others have proposed alternative placement schemes to cool down a hot chip [5]. Abstracting thermal sources as weighted points on the floor, the problem reduces to identifying the region with the highest concentration of points. Also if the region having minimum concentration of points is identified some source points from regions of higher concentration may be moved there. The motivation for our present research mainly comes from the above discussion.

Distribution of points on a plane has received considerable attention in the literature. It is quite popular in the domain of computational geometry and is equally important for a number of practical applications. For a set S of scattered points on the plane, a question frequently arises - whether a cluster C ($\subseteq S$) is more concentrated or rarified with respect to S. *Discrepancy measure* of a set of points tries to answer the above question [6, 7]. However, in the context of thermal analysis we preferred to introduce a new concept, called *density of the floor*. We prove several results that greatly reduce the search space for identifying the region with maximum (minimum) density. We also propose algorithms for solving some of these problems. Some related work either considers fixed number of points [8] or uses a fixed-size rectangle for calculating pattern density [9].

In this paper, Section 2 introduces the concept of density and states the specific problems that we will solve. Section 3 gives the results that lead to an algorithm for identifying the region with maximum density. Section 4 presents a similar discussion on minimum density. Sections 5 compares density with discrepancy and the final section draws conclusion on this novel discussion.

2 Problem Formulation

Let F be a rectangular floor containing a set S of n points, such that no two points lie on the same horizontal (vertical) line. Each point $p_i \in S$ is attached with a positive real weight w_i. We define *density* of an axis-parallel region R with area $A(R)$ as $\frac{\sum_{p_i \in R} w_i}{A(R)}$. Note that, the weight of a point represents the relative strength of the corresponding thermal source. If all the sources are of uniform strength, the points will become unweighted and the *density* will reduce to $\frac{\#(R)}{A(R)}$, $\#(R)$ being the number of points in R. The normalized density for a set of unweighted points in R is $\delta(R) = \frac{\#(R)}{A(R)} / \frac{n}{Area(F)}$, so that $\delta(F) = 1$. Unless otherwise mentioned, all rectangles referred below are axis-parallel. We will consider the following two problems:

Problem P1 : *Find the cluster of $k (k \geq 2)$ points in S, such that the minimum area rectangle covering them attains the highest density on the floor R.*

Problem P2 : *Find the cluster of $k (k \geq 1)$ points from S, such that the maximum area rectangle covering them attains the lowest density on the bounded floor R.*

3 Problem P1

Note that we do not consider the maximum density of a region covering only one point as it can be made arbitrarily high. First, we prove that the maximum density occurs for a cluster $C \subseteq S$ containing only two points.

Lemma 1 *Let each point $p_i \in S$ be attached with a positive weight w_i and there exist a cluster $S' \subseteq S$ of $k \geq 2$ points such that no two of them lie on the same horizontal (vertical) line. Then there exists a pair of points $p_i, p_j \in S'$ such that the density of the smallest*

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rectangle containing (p_i, p_j) *is greater than the density of the smallest rectangle containing* S'.

Proof. Without loss of generality, we name the points in S' in increasing order of their x-coordinates and let the area of a rectangle R be represented by $A(R)$. Consider the smallest rectangle R' containing the point-set S'. Split R' into $k-1$ vertical strips by drawing vertical lines through each point. The rectangular strip whose vertical sides passes through p_i and p_{i+1} is named as R_i. Now, the density of the entire floor is $\frac{\sum_{i=1}^{k} w_i}{A(R')} = \frac{\sum_{i=1}^{k} w_i}{\sum_{i=1}^{k-1} A(R_i)} \leq \frac{w_1 + 2\sum_{i=2}^{k-1} w_i + w_k}{\sum_{i=1}^{k-1} A(R_i)} = \frac{(w_1+w_2)+(w_2+w_3)+\ldots+(w_{k-1}+w_k)}{A(R_1)+A(R_2)+\ldots+A(R_{k-1})} \leq Max_{i=1}^{k-1} \frac{(w_i + w_{i+1})}{A(R_i)}$ (by Fact 1). Again the smallest rectangle containing the concerned pair may have a higher density, as it may not span the entire strip. Hence the result. □

Fact 1 *If a quantity* $Q = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$, $a_i, b_i > 0$, *then* $Min_{i=1}^{n} \frac{a_i}{b_i} \leq Q \leq Max_{i=1}^{n} \frac{a_i}{b_i}$.

Proof. Let $\frac{a_i}{b_i}$ attain the minimum and maximum value for $i = *$ and $i = \#$ respectively. For $1 \leq i \leq n$,
$\frac{a_*}{b_*} \leq \frac{a_i}{b_i} \leq \frac{a_\#}{b_\#} \Rightarrow \frac{a_*}{b_*} b_i \leq a_i \leq \frac{a_\#}{b_\#} b_i$. (as $b_i > 0$)
Adding we get $\frac{a_*}{b_*} \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{n} a_i \leq \frac{a_\#}{b_\#} \sum_{i=1}^{n} b_i \Rightarrow \frac{a_*}{b_*} \leq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \frac{a_\#}{b_\#}$. Hence the result. □

Note that, by the above lemma in the unweighted case, *P1* gets reduced to the problem of finding the pair of points such that the area of the rectangle having those two points at its diagonally opposite corners is minimum. This can be efficiently solved in polynomial time.

3.1 Algorithm

We now discuss on an algorithm for finding out the smallest area rectangle covering exactly two points from the set S. The simplest way is to consider each pair of points, and compute the density of the rectangle whose diagonal is defined by these two points in $O(n^2)$ time. It works for weighted case also. We show below that the time complexity can be improved for the unweighted case. Note that, a rectangle achieving the maximum density is a *corner rectangle* [2]. Thus, the problem reduces to identifying a corner rectangle having minimum area. An $O(n\log^2 n)$ time and $O(n)$ space algorithm finds out the largest corner rectangle using the technique of monotone matrix searching [2], and we show that it works for our problem also. This algorithm uses a two-level divide and conquer strategy. It first splits the point set into two almost equal halves using a horizontal line H and then by a vertical line V. They intersect at O and split the plane into four quadrants. In each quadrant i, we identify an axis-parallel staircase S_i around the point O (Fig. 1(a)). The convex corners of this staircase contains a member in S and the interior of the axis-parallel polygon bounded by S_i, x-axis and y-axis is empty. Consider the staircases in second and fourth quadrants. A rectangle with top-left and bottom-right corner at the convex corner of these two staircases respectively does not contain any point of second and fourth quadrant. If we consider a matrix M whose rows and columns correspond to the points in second and fourth quadrants respectively, and its each entry correspond to the area of a rectangle defined by one point from each of these staircases at the end of its diagonal, it will be a monotone matrix. This algorithm uses a monotone matrix w.r.t. $\geq$ sign [3], whereas for our case we need a monotone matrix w.r.t. $\leq$ sign.

Definition 1 *A matrix A is said to be* monotone *(with respect to $\leq$ sign) if for every* j, k, j', k' *with* $j < j'$, $k < k'$, *if* $A[j,k] \leq A[j,k']$ *then* $A[j',k] \leq A[j',k']$.

The maximum-valued element in an $n \times n$ monotone matrix can be found in $O(n)$ time [3] and hence in our case we can find the minimum element also in $O(n)$ time. Note that, some of these rectangles may contain point(s) of first and/or third quadrant, needing an $O(n)$ time clean-up procedure [1]. This is followed by two recursive steps one in the vertical direction and the other in the horizontal direction, each contributing a $\log n$ factor to the final complexity of $O(n\log^2 n)$, and it is applicable identically to our problem. In order to search the smallest corner rectangle, we arrange the elements of the matrix such that its rows (U_is) are ordered as in the earlier problem, but columns (V_is) are ordered in the reverse direction as shown in Fig. 1(a). Facts 2

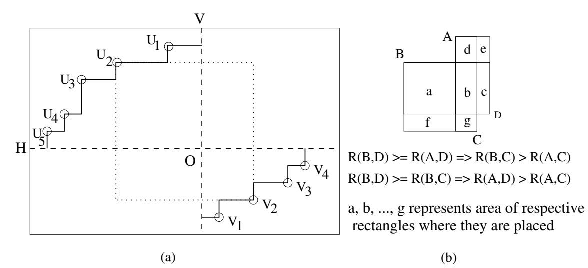


Figure 1: Rectangles with corners in opp. quadrants

and 3 below establish that it is a monotone matrix with respect to $\leq$ sign. Consider Fig. 1(b). Here, a rectangle defined by a pair of points (p, q) as its diagonal, is denoted by $R(p, q)$.

Fact 2 $R(B,D) \geq R(A,D) \Rightarrow R(B,C) > R(A,C)$.

Proof. $R(B,D) \geq R(A,D) \Rightarrow a+b+c \geq b+c+d+e$
$\Rightarrow a+b \geq b+d+e \Rightarrow a+b+f+g \geq b+c+d+e+f+g$
$\Rightarrow a+b+f+g > b+d+g \Rightarrow R(B,C) > R(A,C)$. □

Fact 3 $R(B,D) \geq R(B,C) \Rightarrow R(A,D) > R(A,C)$.

Proof. $R(B,D) \geq R(B,C) \Rightarrow a+b+c \geq a+b+f+g$
$\Rightarrow b+c \geq b+f+g \Rightarrow b+c+d+e \geq b+f+g+d+e$
$\Rightarrow b+c+d+e > b+d+g \Rightarrow R(A,D) > R(A,C)$. □

Hence the complexity of finding the smallest corner rectangle is also $O(n\log^2 n)$.

3.2 Density in General Case

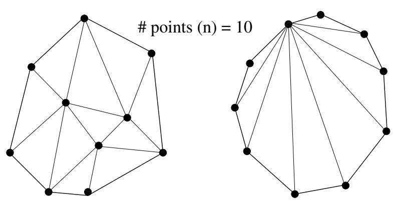
In the isothetic domain, thin rectangles pose problems as their densities become unnecessarily high, which may not properly reflect the real situation. However, similar results are possible beyond the isothetic domain also. For general case, we revise the definition of density for a set of unweighted points as follows:

Definition 2 *Let $S' \subseteq S$ be a subset of points. The density of S' is defined as $\delta(S') = \frac{|S'|}{A(CONV(S'))}$, where $|S'|$ indicates the cardinality of S', and $CONV(S')$ indicates the convex hull of the point set S'.*

Assuming that no three points in S are colinear, Fact 4 says that the maximum density under this definition occurs for a triple of points.

Fact 4 *Let there be a set S of $n > 2$ points on a rectangular floor R such that no three of those points are colinear. There exists a cluster $C \subseteq S$ of three points such that the triangle having those three points as its vertices and covering no other point from the set S achieves the highest density on the floor.*

Proof. Consider a triangulation of any subset $S' \subseteq S$ (Fig. 2(a)). We show that there exists a triangle with higher density (by Def. 2) than that of S'. Let $|S'| = k$.



a) # triangles = 11 > n − 2   b) # triangles = 8 = n − 2

Figure 2: Density in a non-isothetic scenario

The number of triangles in any triangulation of S' is $\geq k-2$, minimum occurs when all the k points lie on the convex hull (Fig. 2(b)). Let the area of the convex hull be A. As average value lies between minimum and maximum, there exists a triangle τ with area $A(\tau) \leq A/(k-2)$. Now, $\delta(S') = \frac{k}{A}$ and $\delta(\tau) \geq \frac{3}{A/(k-2)}$. As $3(k-2) \geq k$, for $k \geq 3$, the result follows. □

4 Problem P2

We now consider the problem of identifying a rectangle on the floor having minimum non-zero density. Given a bounded floor containing a set S of points, the following lemma says that the minimum density occurs for a rectangle containing a single point in S.

Lemma 2 *Let each point $p_i \in S$ be attached with a weight $w_i > 0$, and there exist a cluster $S' \subseteq S$ of $k \geq 1$ point(s). Then there exists a point $p_i \in S'$ such that the density of the largest rectangle that contains only point p_i is less than the density of the largest rectangle R' containing S' and no other points from S.*

Proof. Without loss of generality, consider the points in S' in increasing order of their x-coordinates and let $A(R)$ be the area of rectangle R. The rectangle shown in Fig. 3 represents R' covering k points of S'. It is bounded by points that belong to S but not to S'. Split R' into $k+1$ vertical strips by drawing vertical lines through each point. The vertical sides of strip R_i passes through p_i and p_{i+1}. Now, $\delta(R') = \frac{\sum_{i=1}^k w_i}{A(R')} = \frac{\sum_{i=1}^k w_i}{\sum_{i=0}^k A(R_i)} \leq \frac{\sum_{i=1}^k w_i}{A(R_0)+2\sum_{i=1}^{k-1} A(R_i)+A(R_k)} =$

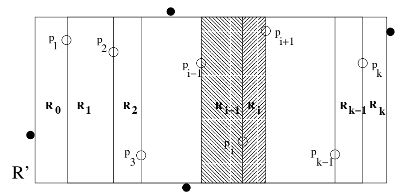


Figure 3: Largest rectangle containing k points

$\frac{w_1+w_2+\ldots+w_k}{(A(R_0)+A(R_1))+(A(R_1)+A(R_2))+\ldots+(A(R_{k-1})+A(R_k))} \geq Min_{i=1}^{k} \frac{w_i}{(A(R_{i-1})+A(R_i))}$ (by Fact 1). Again the density of the largest rectangle containing the concerned point may be less, as it may be possible to extend the shaded portion in Fig. 3 along the vertical direction. □

If $R(p)$ denotes the largest rectangle containing only $p \in S$ and no other point in S, then by the above lemma, for the unweighted case we just need to identify $p_i \in S$ such that $R(p_i)$ is largest among all the points.

5 Density versus Discrepancy

Assuming that the area containing a point set S is a unit square, the *discrepancy measure* of S may be defined as $\Delta = \max_{r \in R} |area(r) - \frac{\#r}{n}|$, where r is an arbitrary rectangular area and $\#r$ indicates the number of points of S lying inside r. For any rectangle r, we introduce a concept of local discrepancy $\Delta_l(r) = |area(r) - \frac{\#r}{n}|$. In Fig. 4, $\Delta_l(r) = |(0.5 * 0.25) - (3/10)| = 0.175$. Let us try to

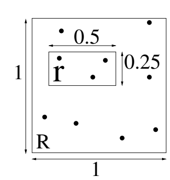


Figure 4: Local discrepancy of a set of points

interpret the main result of Section 3 from the definition of Discrepancy. We choose any three points on the rectangular floor R whose coordinates do not match along

either axis. Let R_3 be the skin-tight rectangle covering only those three points from S. So $\Delta_l(A_3) = |A_3 - 3/n|$, where $A_3 = Area(R_3)$ and density $D_3 = 3/A_3$. From Section 3, we can say that there must exist a pair of points out of these 3 points covered by the skin-tight isothetic rectangle R_2 such that their density $D_2 > D_3$. Now, $D_2 > D_3 \Rightarrow 2/A_2 > 3/A_3 \Rightarrow A_3 > 3/2A_2 \Rightarrow (A_3 - 3/n) > (3/2A_2 - 3/n)$.
Hence $(A_3 - 3/n) > 3/2(A_2 - 2/n)$. This result leads to three possible cases.
Case 1: $A_2 - 2/n$ is +ve $\Rightarrow A_3 - 3/n$ is +ve.
Case 2: $A_3 - 3/n$ is -ve $\Rightarrow A_2 - 2/n$ is -ve.
Case 3: $A_3 - 3/n$ is +ve but $A_2 - 2/n$ is -ve.
Note that out of the three cases only in *Case 1* it is guaranteed that $|(A_3 - 3/n)| > |(A_2 - 2/n)|$, for the other two cases any of those quantities may be greater or smaller. Hence nothing can be concluded by comparing the values of the local discrepancy of three points and that of any pair chosen from those 3 points.

Note that, two set of points having the same density may have their local discrepancy different and conversely. However the main problem is not caused by them. The fact that the rise or fall of density function does not map monotonically to the rise or fall of discrepancy function is the main deterrent in interpreting the results of one domain from the other.

Let us take a look at the expression for discrepancy. Considering an isothetic rectangular region r within R, if we substitute α for area(r)/area(R) and β for $\#r/n$, $\Delta_l(r) = |\alpha - \beta|$, where $0 < \alpha, \beta \leq 1$. The normalized density $\delta(r) = (\#r/area(r))/(n/area(R))$
$= (\#r/n)/(area(r)/area(R)) = \beta/\alpha$.
Now we can look at the differences between density and discrepancy. Consider a case where $n = 10$ points are scattered on a 1x1 rectangular floor such that two different subsets of those points S_1 and S_2 respectively has $\alpha_1 = 0.3$, $\beta_1 = 0.4$ and $\alpha_2 = 0.5$, $\beta_2 = 0.6$. So for S_1, 4 out of the 10 points are scattered in a skin-tight rectangle whose area is 30% of the whole floor and for S_2, 6 out of 10 points are scattered in an area of 0.5 units. The discrepancies of both these sets is same $|0.3-0.4| = |0.5-0.6| = 0.1$. However S_1 has a density of 4/3 and S_2 has a density of 6/5, which are different. Next consider a case with similar set-up where $\alpha_1 = 0.8$, $\beta_1 = 0.6$ and $\alpha_2 = 0.4$, $\beta_2 = 0.3$. Here density of both the sets is 3/4 but the discrepancies are respectively 0.2 and 0.1.

Finally we cite here two examples, where in the first one, increase in density causes discrepancy to rise, and in the second one, decrease in density causes discrepancy to rise. Consider again two set of points S_1 and S_2 with $\alpha_1 = 0.5$, $\beta_1 = 0.6$ and $\alpha_2 = 0.4$, $\beta_2 = 0.6$. For S_1, density = 6/5 and discrepancy is 0.1, whereas for S_2, density = 6/4 (i.e. density increases) and discrepancy = 0.2 (i.e. discrepancy also increases). Now consider two set of points S_3 and S_4 with $\alpha_3 = 0.5$, $\beta_3 = 0.4$ and $\alpha_4 = 0.6$, $\beta_4 = 0.4$. For S_3, density = 4/5 and discrepancy is 0.1, whereas for S_4 ensity = 4/6 (i.e. density decreases) but discrepancy = 0.2 (i.e. discrepancy increases). So by tracking the change in density it is difficult to predict the change in discrepancy and vice versa.

6 Conclusion

We have introduced a new concept for calculating density of points scattered in a plane. We also made some simple but novel observations that considerably reduces the search space for the problems of finding out the zones having maximum or minimum density. Then we have proposed some algorithms based on existing geometric techniques to solve these problems. The whole technique may be applied for facilitating thermal analysis of ULSI chips. We have compared it with an existing concept of discrepancy and found that the results regarding density cannot be easily translated to the domain of discrepancy. Hence the concept of density may have some contribution on its own to interpret the properties of the relative distribution of a set of points scattered on a rectangular floor.

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