Open Problems from CCCG 2017

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The following is a description of the problems presented on July 26th, 2017 at the open-problem session of the *29th Canad. Conf. Computational Geometry* held at Carleton University, in Ottawa.

Near-Delaunay Triangulations
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Let T be a triangulation of a finite point set in the plane. Say that a triangulation is *near-Delaunay* if the opposite angles α and β of each pair of triangles that share an edge sum to at most $\pi + \epsilon$, for $\epsilon > 0$. Note that, if $\epsilon = 0$, then T is Delaunay; see Figure 1. Near-Delaunay triangulations can be constructed by an edge-flipping algorithm.

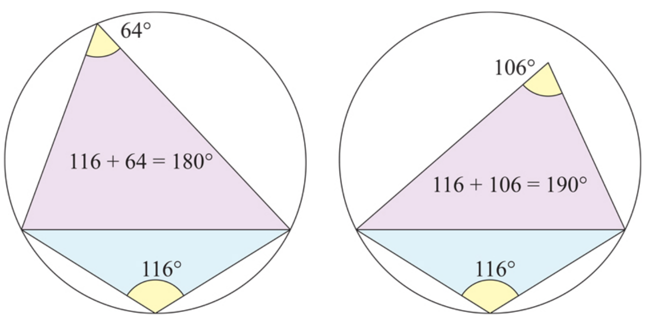

Figure 1: Left: Delaunay triangles. Right: Near-Delaunay triangles.

Have these triangulations been defined previously? Do they have any nice properties?

Update. Scott Mitchell suggested "measuring the signed distance between circumcenters of triangles sharing an edge; for Delaunay triangulations this is simply the dual edge length and non-negative, but for non-DT the circumcenters can be in the wrong order and hence have a negative distance between them. So one could look at the ratio of the dual edge signed-length to the primal edge length (for 2D triangulations) as a continuous measure of how close it is to non-Delaunay." This concept has appeared in the literature on Hodge-optimized triangulations, e.g., [MM+11].

References

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Counting Closed Billiard Paths
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Let a collection of rectangles, all axis-aligned, all enclosed in one rectangle, have a total of n edges. A simple, closed billiard path is a path that is (a) closed, (b) non-self-intersecting, and so forms a simple polygon, (c) never touches a rectangle corner, and (d) all reflections are mirror reflections. Label all rectangle edges, and define the *signature* of a billiard path by the labels of the edges from which it reflects, reducing repeated edge reflections $(ab)^k$ to ab. Thus in Figure 2, the path $12373(56)^2 4$ has signature 12373564, reducing $(56)^2$ to 56.

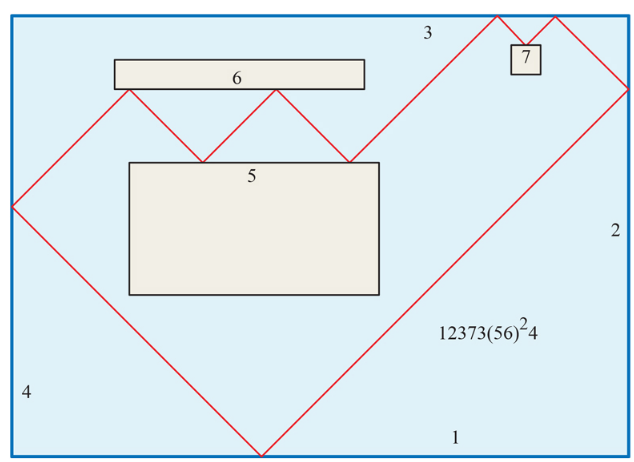

Figure 2: A billiard path of signature length 8.

For simple, closed billiard paths, for any arrangement of rectangles of a total of n edges:

1. What is the maximum length of such a signature?

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2. What is the largest number of distinct signatures achievable for one fixed reflection angle (45° in the figure).
3. What is the largest number of distinct signatures achievable for paths at arbitrary reflection angles?

Tubes in Space

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Let S be a unit-radius sphere in $\mathbb{R}^3$. Place n lines intersecting S to minimize the maximum distance between any two points in S, where distance is measured as follows. Distance off the lines is Euclidean distance, but the distance between any two points on one line is zero. The lines are like very fast transportation tubes. See Figure 3. One line L ($n = 1$)

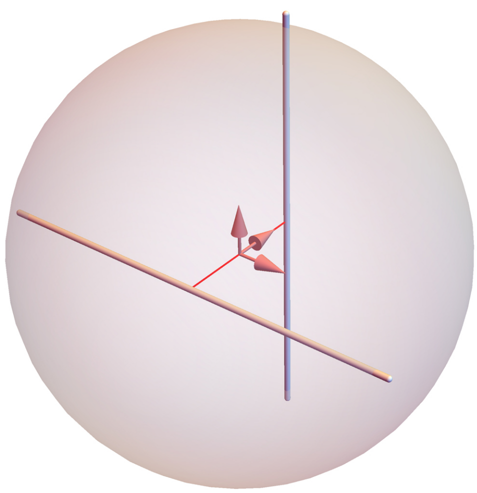

Figure 3: $n = 2$ skew transport tubes.

is useless for pairs of points on antipoldal on the equator formed by the plane perpendicular to L: their distance remains 2. It was observed at the presentation that two lines are also useless: antipodal points on a great circle orthogonal to the two lines are still 2 apart. Three x, y, z-axis lines seem best, apparently reducing the maximum distance to $2\sqrt{\frac{2}{3}} \approx 1.63$. In two dimensions, it seems that equi-angular lines through the center of a circle is the optimal arrangement. All these are conjectures.

Variations:

- Within a unit cube rather than a sphere.
- Assign off-tube speeds 1, and in-tube speeds $s > 1$.
- The same questions in $\mathbb{R}^d$.

General-position subconfigurations

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For the purposes of this problem, a set of points is in *general position* if no line contains three or more of its points. This problem's first two parts concern the d-dimensional point set $\{-1, 0, 1\}^d$ (a grid of size three in each dimension), shown in Figure 4 in projection to the plane.

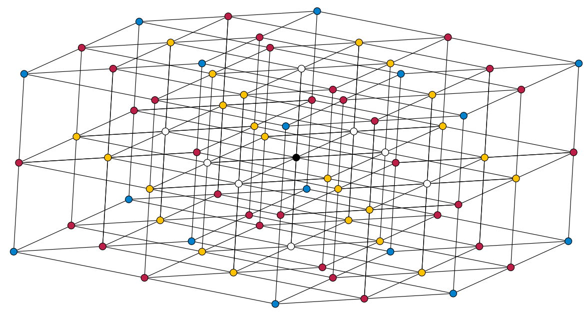

Figure 4: $\{-1, 0, 1\}^d$ for $d = 4$, from [E17].

(a) These points can be partitioned into $d + 1$ subsets, each in general position, by grouping points according to how many coordinates are zero. (The figure's colors show this partition.) The Hales–Jewett theorem [HJ63, S88] implies that any general-position partition has $\omega(1)$ subsets. (This holds even for the weaker condition that no three points form a monotonic line, one in which the three points can be ordered so that all coordinates are increasing or constant.) Can these points be partitioned into fewer than $d+1$ general-position subsets?

(b) The largest subset in this partition (for which the number of zero coordinates of each point is approximately $d/3$) has size $\Theta(3^d/\sqrt{d})$. The density version of the Hales–Jewett theorem implies that all general-position subsets have size $o(3^d)$ [FK89, F91]. Is there a general-position subset with size $\omega(3^d/\sqrt{d})$?

(c) How well can the largest general-position subset and the partition into the fewest general-position subsets be approximated? Is it achievable in polynomial time for arbitrary planar point sets? Both problems are NP-complete and APX-hard, and can be approximated within a factor of $O(\sqrt{n})$ by a simple greedy algorithm that adds each point (in arbitrary order) to the first subset in which it is in general position [E17]. Results of Füredi, Payne and Wood, relating general-position subsets to lines with many points [F91, PW13], suggest that it may be possible to shave a logarithmic factor from this approximation ratio. But is $O(n^{1/2-\epsilon})$ possible, for some $\epsilon > 0$? (It is safe to consider only two dimensions, because points in higher dimensions can be projected to the plane without changing collinearity.)

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Constructing separators for Geometric Graphs

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Given a planar graph $G = (V, E)$, $|V| = n$, the *Planar Separator* theorem of Lipton and Tarjan [LT79] states that there always exists a set of $O(\sqrt{n})$ vertices in V whose removal partitions the graph G into disjoint connected subgraphs, each of size at most $2n/3$. Such a separator can be constructed in $O(n)$ time when the graph is provided.

There are many situations however when a graph is defined implicitly, by a collection of points or of geometric objects, such as for example, the Delaunay triangulation of a set of n points, the edge structure of the convex hull of n points in $\mathbb{R}^3$, or the intersection graph of a collection of disks in the plane where no point is covered by more than two disks. The explicit construction of, e.g., a Delaunay triangulation for n points in the plane requires $O(n \log n)$ time, however it might be possible to construct a separator without having to construct the graph explicitly.

Question 1: Given a set S of n points in $\mathbb{R}^2$, is it possible to find a separator of the Delaunay triangulation of P in time $O(n)$?

Question 2: Given a set S of n points in $\mathbb{R}^3$, is it possible to find a separator of the convex hull of P in time $O(n)$?

For some geometric graphs, e.g., the disk-intersection graph mentioned above, a separator can be found in $O(n)$ time [MTTV97].

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Optimizing Sum of Products

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Given sequences $A = \langle a_0, a_1, ..., a_{n-1}\rangle$ and $B = \langle b_0, b_1, ..., b_{n-1}\rangle$ of real numbers, find a permutation π of A which maximizes

$$\sum_{i=0}^{n-1} a_{\pi(i)} a_{\pi(i+1)} b_i$$

where the indices are taken modulo n. If $b_i = 1$ for all i, then the solution is any cyclic shift of the sequence $\langle a'_0, a'_2, ..., a'_3, a'_1\rangle$ where $a'_0 \geq a'_1 \geq a'_2 \geq a'_3 \geq \cdots$.

This problem is a modified version of an open problem presented in [BD+16]. The paper showed that a variant of the problem allowing the permutation of both A and B can be solved optimally in $O(n \log n)$ time. It is not known if the posed problem is NP−Hard.

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Compatible Triangulations of Labeled Point Sets

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Let P_1, P_2 be a pair of point sets, each containing n points that are labeled from 1 to n. A pair of triangulations T_1 and T_2 of P_1 and P_2 are called *compatible triangulations* or *joint triangulations* if for every face, the clockwise cyclic order of vertices on the boundary is the same, e.g., see Fig. 5(a).

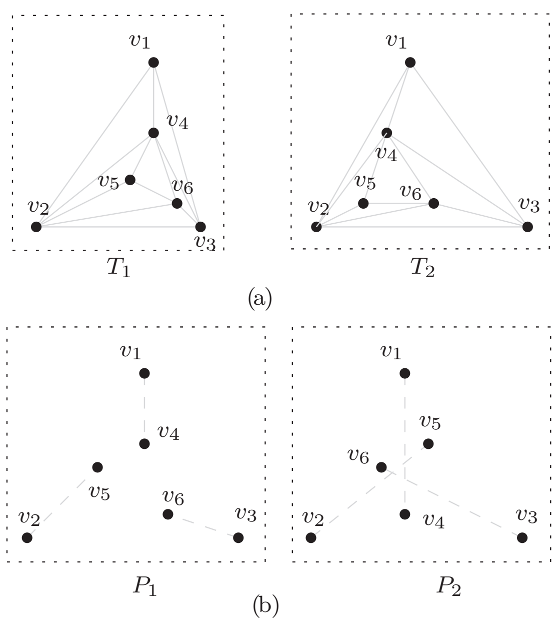

Figure 5: (a) A pair of point sets and their compatible triangulations. (b) A pair of point sets that do not admit compatible triangulations. Any triangulation of P_1 would contain the edges $(v_1, v_4), (v_2, v_5), (v_3, v_6)$, and they intersect in P_2.

Not all pairs of point sets admit compatible triangulations, even when they have the same number of points on the convex-hulls, e.g., see Fig 5(b). Saalfeld in 1987 [S87] proved that, using Steiner points, any pair of point sets with rectangular convex-hull can be triangulated compatibly. In fact, $O(n^2)$ Steiner points suffice for every point set [BSW97], and $\Omega(n^2)$ Steiner points are sometimes necessary [PSS96]. If we are allowed to choose the labels, then such compatible triangulations are conjectured to exist without Steiner points (when P_1 and P_2 have the same number of points on the convex hull) [AAHK03].

In this context we ask the following question: Does there exist an algorithm that, given a pair of labeled point sets, can decide in polynomial time whether they admit compatible triangulations without Steiner points?

The analogous question for compatible triangulations of polygons is solvable in polynomial time [ASS93]. The more general question of minimizing the number of Steiner points required for compatible triangulations of polygonal regions is NP-hard [LM17].

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Binary trees in the $\{\diagup, \diagdown, \overline{\ \ }\}$-grid

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The $\{\diagup, \diagdown, \overline{\ \ }\}$-grid consists of the points with integer coordinates, and all horizontal or diagonal lines through such points. Given a binary tree T, we want an embedding of T in the $\{\diagup, \diagdown, \overline{\ \ }\}$-grid, i.e., vertices are mapped to distinct grid-points, and edges are mapped to straight-line segments along the grid in such a way that no two edges cross. The *width* of such a drawing is the maximal x-coordinate (presuming that the minimal x-coordinate is 1). The main question is:

> How much width (relative to the number of vertices n) is sufficient to embed any binary tree in the $\{\diagup, \diagdown, \overline{\ \ }\}$-grid?

The question could also be asked for variations where we want an *upward* drawing (i.e., the tree is rooted and the y-coordinate of the parent of a node v is no smaller than the y-coordinate of v) and/or an *order-preserving* drawing (i.e., the order of edges around each node is fixed and must be respected in the drawing).

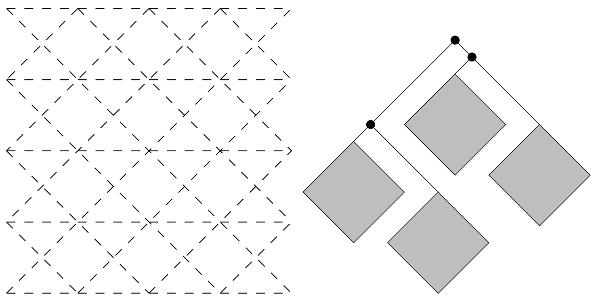

Figure 6: The $\{\diagup, \diagdown, \overline{}\}$-grid, and embedding a complete binary tree in it.

It is known that for the complete binary tree we need width $\Omega(\sqrt{n/\log n})$ [B17]. In the same paper, it was also argued that width $O(\sqrt{n})$ can be achieved for the complete binary tree, by taking an orthogonal construction due to Creszenci et al. [CDP92] and rotating it by 45°. Can we achieve width $O(\sqrt{n})$ for all binary trees?

References

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Gabriel Circle Range Counting
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Suppose that $S = \{x_1, x_2, ..., x_n\} \subseteq \mathbb{R}^2$ and $q \in \mathbb{R}^2$. Can we answer the following question with $O(n)$ storage, $O(n \log n)$ expected preprocessing time, and sub-$O(n^{1/2+\epsilon})$ (optimally $O(\log n)$) query time?
Question: How many x_i fall inside the Gabriel circle $GC(q, x_k)$ for some $1 \le k \le n$? See Figure 7.

The Gabriel circle $GC(a, b)$ is the circle with diameter ab. The characteristic of having a common point q among all Gabriel circles may help to answer the question. A query time $O(n^{1/2+\epsilon})$ is achieved in [AMM13] for general circle range counting.

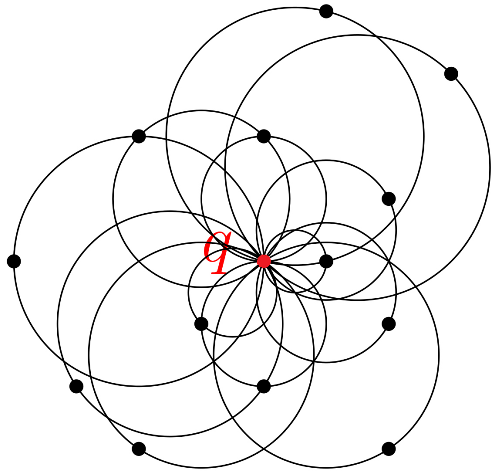

Figure 7: Point q is shared among all Gabriel circles $GC(q, x_k)$.

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Hamiltonian Order-k Delaunay Graphs
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Given a set P of n points in the plane, a pair of points $x, y \in P$ has order k if there exists a disk with x and y on its boundary containing at most k points of P. The edges of a standard Delaunay triangulation have order 0. A graph whose edges consist of every pair of points with order at most k will be referred to as the order-k Delaunay graph. The order-k Gabriel graph, which is a subgraph of the order-k Delaunay graph, is the graph whose edges consist of every pair of points whose Gabriel disk has contains at most k points.

Dillencourt [Dil87] showed that there exist point sets whose Delaunay triangulation is not Hamiltonian. Dillencourt [Dil90] also showed that the Delaunay triangulation is 1-tough, which when n is even implies that it contains a perfect matching. Abellanas et al. [ABG$^+$09] showed that the order-15 Gabriel graph is Hamiltonian. Subsequently, Kaiser et al. [KSC15] showed that the order-10 Gabriel graph is Hamiltonian.

Conjecture: The order-1 Delaunay graph is Hamiltonian.

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