The Crossing Number of Semi-Pair-Shellable Drawings of Complete Graphs

Petra Mutzel * Lutz Oettershagen †

Abstract

The Harary-Hill Conjecture states that for $n \geq 3$ every drawing of K_n has at least

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

crossings. In general the problem remains unsolved, however there has been some success in proving the conjecture for restricted classes of drawings. The most recent and most general of these classes is seq-shellability [16]. In this work, we improve these results and introduce the new class of *semi-pair-shellable* drawings. We show that each drawing in this new class has at least $H(n)$ crossings using novel results on k-edges. So far, approaches for proving the Harary-Hill Conjecture for specific classes rely on a fixed reference face. We successfully apply new techniques in order to loosen this restriction, which enables us to select different reference faces when considering subdrawings. Furthermore, we introduce the notion of k-*deviations* as the difference between an optimal and the actual number of k-edges. Using k-deviations, we gain interesting insights into the essence of k-edges, and we further relax the necessity of fixed reference faces.

1 Introduction

The crossing number $cr(G)$ of a graph G is the smallest number of edge crossings over all possible drawings of G. In a drawing D of $G = (V, E)$ every vertex $v \in V$ is represented by a point and every edge $uv \in E$ with $u, v \in V$ is represented by a simple curve connecting the corresponding points of u and v. We call an intersection point of the interior of two edges a crossing. The Harary-Hill Conjecture states the following.

Conjecture 1 (Harary-Hill [10]) *Let K_n be the complete graph with n vertices, then*

$$cr(K_n) = H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

*Department of Computer Science, TU Dortmund University petra.mutzel@tu-dortmund.de

†Department of Computer Science, TU Dortmund University lutz.oettershagen@tu-dortmund.de

There are construction methods for drawings of K_n that lead to exactly $H(n)$ crossings, for example the class of *cylindrical* drawings first described by Hill [11]. However, there is no proof for the lower bound of the conjecture for arbitrary drawings of K_n with $n \geq 13$. The cases for $n \leq 10$ have been shown by Guy [10] and for $n = 11$ by Pan and Richter [17]. Guy [10] argues that $cr(K_{2n+1}) \geq H(2n+1)$ implies $cr(K_{2(n+1)}) \geq H(2(n+1))$, hence $cr(K_{12}) \geq H(12)$. McQuillan et al. [14] showed that $cr(K_{13}) \geq 219$. Ábrego et al. [1] improved the result to $cr(K_{13}) \in \{223, 225\}$.

Beside these results for arbitrary drawings, there has been success in proving the Harary-Hill Conjecture for different classes of drawings. So far, the conjecture has been verified for 2-page-book [3], cylindrical [5], x-monotone [8, 4], x-bounded [5], shellable [5], bishellable [2] and recently seq-shellable drawings [16]. Seq-shellability is the broadest of the beforehand mentioned classes comprising the others. Here, the proof of the Harary-Hill Conjecture makes use of the concept of k-*edges*. Each edge $e \in E$ in a drawing is assigned a specific value between 0 and $\lfloor \frac{n}{2} \rfloor - 1$ with respect to a fixed reference face. The edge e separates the remaining $n-2$ to vertices into two distinct sets, and is assigned the cardinality k of the smaller of the two sets, i.e. is a k-edge (see section 2 for details). We can express the number of crossings in a drawing in terms of the numbers of k-edges for each $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 2\}$. Therefore, having lower bounds on the (cumulated) number of k-edges implies a lower bound on the crossing number of a drawing. After two cumulations, we obtain *double cumulated k-edges*. However, the possibilities of their usage for further improvements to new classes of drawings seem to be limited.

Our contribution and outline In this work, we resolve the limitations of double cumulated k-edges by applying two new ideas. Firstly, instead of double cumulated k-edges we utilize *triple cumulated* k-edges. Balko et al. introduced these in [8]. Secondly, so far all classes, including seq-shellability, depend on a *globally fixed* reference face. We call a reference face globally fixed if we do not allow to select a different one when considering subdrawings, which constitutes a strong limitation in the proofs. In this work, we show that under certain conditions and/or assumptions, we are able to change the reference face *locally* or even without restrictions. Changing the reference face locally means, given a vertex v incident to an initial reference face F, we select a new reference face F', such that F' is also incident to v. Using the new results, we introduce a new class of drawings for which we show that each drawing in this class has at least $H(n)$ crossings; we call drawings belonging to this class *semi-pair-shellable*. There are semi-pair-shellable drawings that are not seq-shellable. But unlike seq-shellability, semi-pair-shellability does not comprise all previously found classes and only contains drawings with an odd number of vertices. However, every $(\lfloor \frac{n}{2} \rfloor - 1)$-seq-shellable drawing with n odd is semi-pair-shellable. Furthermore, we introduce *k-deviations* of a drawing D of K_n. They are the difference between the numbers of cumulated k-edges in D and reference values corresponding to a drawing with exactly $H(n)$ crossings. They allow us to further relax the necessity of a globally fixed reference face.

The outline of this paper is as follows. In Section 2 we introduce the preliminaries, and in particular the necessary background on (cumulated) k-edges and their usage for verifying the lower bound on the number of crossings. In the following Section 3, we present our novel results for triple cumulated k-edges, followed by the introduction of semi-pair-shellable drawings in Section 4. We show that each drawing in this class has at least $H(n)$ crossings, and discuss the distinctive differences to seq-shellability. In Section 5 we use k-deviations to formulate conditions under which we are able to further loosen the need for a globally fixed reference face. We conjecture these conditions to be true in all good drawings. Assuming our conjecture holds, we prove a lower bound of $H(n)$ crossings for another broad class of drawings. Finally, in Section 6 we draw our conclusions and give an outlook to further possible work. Note that due to the space restrictions some proofs had to be omitted. A full version which contains all proofs and additional figures is available [15].

2 Preliminaries

A *drawing* D of a graph G on the plane is an injection ϕ from the vertex set V into the plane, and a mapping of the edge set E into the set of simple curves, such that the curve corresponding to the edge $e = uv$ has endpoints $\phi(u)$ and $\phi(v)$, and contains no other vertices [19]. We call an intersection point of the interior of two edges a crossing; a shared endpoint of two adjacent edges is not considered a crossing. The crossing number $cr(D)$ of a drawing D equals the number of crossings in D and the crossing number $cr(G)$ of a graph G is the minimum crossing number over all its possible drawings. We restrict our discussions to *good* drawings of K_n, and call a drawing *good* if (1) any two of the curves have finitely many points in common, (2) no two curves have a point in common in a tangential way, (3) no three curves cross each other in the same point, (4) any two edges cross at most once and (5) no two adjacent edges cross. It is known that every drawing with a minimum number of crossings is good [18]. Given a drawing D, we call the points also vertices and the curves edges, V denotes the set of vertices (i.e. points), and E denotes the set of edges (i.e. curves) of D. If we subtract the drawing D from the plane, a set of open regions remains. We call $\mathcal{F}(D) := \mathbb{R}^2 \setminus D$ the set of *faces* of the drawing D. If we remove a vertex v and all its incident edges from D, we get the subdrawing $D - v$. We denote by $f(v)$ the unique face in $D - v$ that contains all the faces that are incident to v in D, and call $f(v)$ the *superface* of v. We might consider the drawing to be on the surface of the sphere S^2, which is equivalent to the drawing on the plane due to the homeomorphism between the plane and the sphere minus one point. Next, we introduce *k-edges*; according to [7] the origins of k-edges lie in computational geometry and problems over n-point set, especially problems on halving lines and k-sets. An early definition in the geometric setting goes back to Erdős et al. [9]. Given a set P of n points in general position in the plane, the authors add a directed edge $e = (p_i, p_j)$ between the two distinct points p_i and p_j, and consider the continuation as line that separates the plane into a left and a right half plane. There is a (possibly empty) point set $P_L \subseteq P$ on the left side of e, i.e. in the left half plane. Erdős et al. assign $k := \min(|P_L|, |P \setminus P_L|)$ to e. Later, the name *k-edge* emerged for any edge that is assigned the value k. Lovász et al. [13] used k-edges for determining a lower bound on the crossing number of rectilinear graph drawings. Finally, Ábrego et al. [3] extended the concept of k-edges from rectilinear to topological graph drawings and used the concept to show that the crossing number of 2-page-book drawings is at least $H(n)$. Every edge in a good drawing D of K_n is a k-edge for a specific value of $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$. Let D be on the surface of the sphere S^2, and $e = uv$ be an edge in D and $F \in \mathcal{F}(D)$ be an arbitrary but fixed face; we call F the *reference face*. Together with any vertex $w \in V \setminus \{u, v\}$, the edge e forms a triangle uvw and hence a closed curve that separates the surface of the sphere into two parts. For an arbitrary but fixed orientation of e, one can distinguish between the left part and the right part of the separated surface. If F lies in the left part of the surface, we say the triangle has orientation $+$ else it has orientation $-$. For e there are $n - 2$ possible triangles in total, of which $0 \leq i \leq n - 2$ triangles have orientation $+$ (or $-$) and $n - 2 - i$ triangles have orientation $-$ (or $+$ respectively). We define the *k-value* of e to be the minimum of i and $n - 2 - i$. We say e is an *i-edge* with respect to the reference face F if its k-value equals i. See Figure 1 for an example.

Ábrego et al. [3] showed that the crossing number of a drawing is expressible in terms of the number of k-edges for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ with respect to the reference face. The following definitions of the *cumulated* numbers of k-edges are used for determining lower bounds of the crossing number. The double cumulated number of k-edges has been defined by Ábrego et al. [3], and the triple cumulated number of k-edges has been introduced by Balko et al. [8] in the context of the crossing number of x-monotone drawings.

Definition 1 [3, 8] *Let D be a good drawing and $E_k(D)$ be the number of k-edges in D with respect to a reference face $F \in \mathcal{F}(D)$ and for each $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$. We denote*

$$\bar{E}_k(D) := \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k} (k+1-i) E_i(D)$$

the double cumulated number of k-edges, *and*

$$\hat{E}_k(D) := \sum_{i=0}^{k} \bar{E}_i(D) = \sum_{i=0}^{k} \binom{k+2-i}{2} E_i(D)$$

the triple cumulated number of k-edges.

We also write *double (triple) cumulated k-edges* or *double (triple) cumulated k-value* instead of double (triple) cumulated number of k-edges. We express the crossing number of a drawing using the triple cumulated k-edges.

Theorem 2 [8] *Let D be a good drawing of K_n and $m = \lfloor \frac{n}{2} \rfloor - 2$. With respect to a reference face $F \in \mathcal{F}(D)$ we have for n odd*

$$cr(D) = 2 \cdot \hat{E}_m(D) - \frac{1}{8} n(n-1)(n-3)$$

and for n even

$$cr(D) = \hat{E}_m(D) + \hat{E}_{m-1}(D) - \frac{1}{8} n(n-1)(n-2).$$

It is an important observation, that for n odd the value $\hat{E}_m(D)$ and n even $\hat{E}_m(D) + \hat{E}_{m-1}(D)$ are identical for all faces of D. Note that this does not apply to the double cumulated case, i.e. $\bar{E}_m(D)$ or $\bar{E}_m(D) + \bar{E}_{m-1}(D)$, respectively. Using the following lower bounds, we are able to verify the Harary-Hill Conjecture.

Corollary 3 [8] *Let D be a good drawing of K_n. If n is odd and*

$$\hat{E}_{\frac{n-1}{2}-2}(D) \geq 3 \binom{\frac{n-1}{2}+2}{4}$$

or n is even and with respect to a face $F \in \mathcal{F}(D)$

$$\hat{E}_{\frac{n}{2}-2}(D) \geq 3 \binom{\frac{n}{2}+2}{4} \quad and \quad \hat{E}_{\frac{n}{2}-3}(D) \geq 3 \binom{\frac{n}{2}+1}{4},$$

then $cr(D) \geq H(n)$.

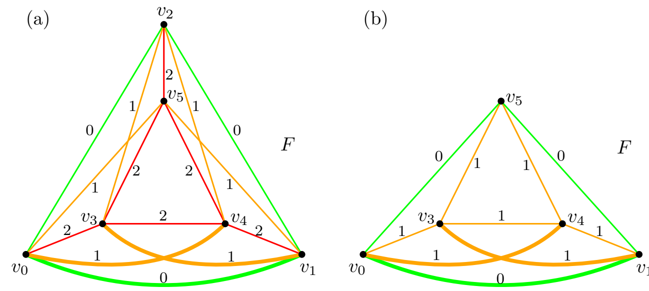

Figure 1: Example (a) shows a crossing optimal drawing D of K_6 with the k-values at the edges. (b) shows the subdrawing $D - v_2$ and its k-values. The fat highlighted edges v_0v_1, v_0v_4 and v_1v_3 are invariant and keep their k-values. The reference face is the outer face F.

If a vertex touches the reference face, it is incident to a certain set of k-edges.

Lemma 4 [3] *Let D be a good drawing of K_n, $F \in \mathcal{F}(D)$ and $v \in V$ be a vertex incident to F. With respect to F, vertex v is incident to two i-edges for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$. Furthermore, if we label the edges incident to v counter clockwise with $e_0, \dots, e_{n-2}$ such that e_0 and e_{n-2} are incident to the face F, then e_i is a k-edge with $k = \min(i, n-2-i)$ for $0 \leq i \leq n-2$.*

The definition of semi-pair-shellability uses seq-shellability, which itself is based on simple sequences.

Definition 5 (Simple sequence) [16] *Let D be a good drawing of K_n, $F \in \mathcal{F}(D)$ and $v \in V$ with v incident to F. Furthermore, let $S_v = (u_0, \dots, u_k)$ with $u_i \in V \setminus \{v\}$ be a sequence of distinct vertices. If u_0 is incident to F and vertex u_i is incident to a face containing F in the subdrawing $D - \{u_0, \dots, u_{i-1}\}$ for all $1 \leq i \leq k$, then we call S_v a* simple sequence *of v.*

Definition 6 (Seq-Shellability) [16] *Let D be a good drawing of K_n. We call D k-*seq-shellable *for $k \geq 0$ if there exists a face $F \in \mathcal{F}(D)$ and a sequence of distinct vertices $a_0, \dots, a_k$ such that a_0 is incident to F, and (1.) for each $i \in \{1, \dots, k\}$, vertex a_i is incident to the face containing F in drawing $D - \{a_0, \dots, a_{i-1}\}$, and (2.) for each $i \in \{0, \dots, k\}$, vertex a_i has a simple sequence $S_i = (u_0, \dots, u_{k-i})$ with $u_j \in V \setminus \{a_0, \dots, a_i\}$ for $0 \leq j \leq k - i$ in drawing $D - \{a_0, \dots, a_{i-1}\}$.*

If a drawing D of K_n is $(\lfloor \frac{n}{2} \rfloor - 2)$-seq-shellable, we omit the $(\lfloor \frac{n}{2} \rfloor - 2)$ part and say that D is seq-shellable. The class of seq-shellable drawings contains all drawings that are $(\lfloor \frac{n}{2} \rfloor - 2)$-seq-shellable.

3 Properties of Triple Cumulated k-Edges

In this section, we present new results for triple cumulated k-edges. First, we introduce the triple cumulated value of edges incident to v. Having a vertex v incident to the reference face F, we know from Lemma 4 that v is incident to two k-edges for each $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 2\}$ and it follows that the triple cumulated number of k-edges incident to v is $\hat{E}_k(D, v) = \sum_{i=0}^{k} \binom{k+2-i}{2} \cdot 2 = 2\binom{k+3}{3}$.

Next, we introduce the *double cumulated invariant* edges. Consider removing a vertex $v \in V$ from a good drawing D of K_n, resulting in the subdrawing $D - v$. By deleting v and its incident edges every remaining edge loses one triangle, i.e. for an edge $uw \in E$ there are only $(n-3)$ triangles uwx with $x \in V \setminus \{u, v, w\}$ (instead of the $(n-2)$ triangles in drawing D). The k-value of any edge $e \in E$ is defined as the minimum number of $+$ or $-$ oriented triangles that contain e. If the lost triangle had the same orientation as the minority of triangles, the k-value of e is reduced by one else it stays the same. Therefore, every k-edge in D with respect to $F \in \mathcal{F}(D)$ is either a k-edge or a $(k-1)$-edge in the subdrawing $D - v$ with respect to $F' \in \mathcal{F}(D - v)$ and $F \subseteq F'$. We call an edge e *invariant* if e has the same k-value with respect to F in D as for F' in D'. See Figure 1 for an example.

For $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ we denote the number of invariant k-edges between D and D' (with respect to F and F' respectively) by $I_k(D, D')$. Furthermore, we define the *double cumulated invariant k-value* as

$$\bar{I}_k(D, D') := \sum_{j=0}^{k} \sum_{i=0}^{j} I_i(D, D') = \sum_{i=0}^{k} (k - i + 1) I_i(D, D').$$

We define $\hat{E}_{-1}(D) := 0$, and introduce the recursive representation for the triple cumulated k-edges.

Lemma 7 *Let D be a good drawing of K_n, $v \in V$ and $F \in \mathcal{F}(D)$. With respect to the reference face F and for all $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 2\}$, we have*

$$\hat{E}_k(D) = \hat{E}_{k-1}(D - v) + \hat{E}_k(D, v) + \bar{I}_k(D, D - v).$$

Using the triple cumulated value, we only have to ensure that $\hat{E}_k(D) \geq 3\binom{k+4}{4}$ for $k = \frac{n-1}{2} - 2$ if n is odd, or for each $k \in \{\frac{n}{2} - 2, \frac{n}{2} - 3\}$ if n is even in order to prove that $cr(D) \geq H(n)$ (Theorem 2). Mutzel and Oettershagen [16] showed that any seq-shellable drawing D of K_n has $\bar{E}_i(D) \geq 3\binom{i+3}{3}$ for all $i \in \{0, \ldots, k\}$ with respect to the reference face F. This implies the following corollary.

Corollary 8 *Let D be a drawing of K_n that is seq-shellable for a reference face $F \in \mathcal{F}(D)$, then $\hat{E}_k(D) \geq 3\binom{k+4}{4}$ for all $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 2\}$ with respect to F.*

The following lemma gives a lower bound on double cumulated invariant edges incident to a vertex that touches the reference face.

Lemma 9 *Let D be a good drawing of K_n with two vertices v and w incident to the reference face $F \in \mathcal{F}(D)$. If v is removed, the double cumulated value of invariant k-edges incident to w with respect to F is at least $\binom{k+2}{2}$ for all $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 2\}$.*

The following lemma is the gist that allows us to locally change the reference face if we have an odd number of vertices.

Lemma 10 *Let D be a good drawing of K_n and $v \in V$. For n odd, the number of double cumulated invariant edges $\bar{I}_{\lfloor \frac{n}{2} \rfloor - 2}(D, D - v)$ is the same with respect to any face incident to v in D and the superface $f(v)$ in $D - v$.*

Proof. Let $m = \lfloor \frac{n}{2} \rfloor - 2$. Lemma 7 implies that with respect to a face incident to v

$$\bar{I}_m(D, D - v) = \hat{E}_m(D) - \hat{E}_{m-1}(D - v) - \hat{E}_m(D, v).$$

$\hat{E}_m(D)$ is the same for all faces of D, the value $\hat{E}_{m-1}(D - v)$ with respect to face $f(v)$ is fixed and for each face incident to v we have $\hat{E}_m(D, v) = 2\binom{m+3}{3}$. Therefore, it follows that also the value of $\bar{I}_m(D, D - v)$ has to be the same for every face incident to v. □

4 Semi-Pair-Shellability

Basis for the new class of semi-pair-shellable drawings are pair-sequences.

Definition 11 (Pair-sequence) *Let D be a good drawing of K_n, $v \in V$ and $P_v = (u_0, \ldots, u_{\lfloor \frac{n}{2} \rfloor - 2})$ be a sequence of distinct vertices $u_i \in V \setminus \{v\}$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$.*

We call P_v a pair-sequence *of v if for $j \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor - 3\}$ and $(n - j)$ odd, the vertex u_j in the drawing $D - \{u_0, \ldots, u_{j-1}\}$ is incident to a face $F' \in \mathcal{F}(D - \{u_0, \ldots, u_{j-1}\})$, where F' is also incident to v, and in the drawing $D - \{u_0, \ldots, u_j\}$ vertex u_{j+1} is incident to face $f(u_j)$, and vertex u_0 is incident to $F \in \mathcal{F}(D)$, where F is also incident to v.*

For example, in Figure 2 vertex v in the drawing of K_{11} has the pair-sequence (u_0, u_1, u_2, u_3). The pair-sequence of vertex v ensures that if we remove v from D, there are enough double cumulated invariant k-edges. Therefore, we are able to guarantee a lower bound on $\hat{E}_{\lfloor \frac{n}{2} \rfloor - 2}(D)$ using Lemma 7.

Lemma 12 *Let D be a good drawing of K_n, $v \in V$ and $(u_0, \ldots, u_{\lfloor \frac{n}{2} \rfloor - 2})$ a pair-sequence of v, then $\bar{I}_{\lfloor \frac{n}{2} \rfloor - 2}(D, D - v) \geq \binom{\lfloor \frac{n}{2} \rfloor + 1}{3}$.*

Proof. Without loss of generality let n be odd and let $m = \frac{n-1}{2} - 2$ (for n even we can proceed similarly and start with $m = \frac{n}{2} - 2$). Lemma 9 states that the double cumulated value of invariant edges incident to u_0 equals $\binom{k+2}{2}$ for $0 \leq k \leq m$ with respect to a face F incident to v and u_0, and the removal of v from D. Likewise,

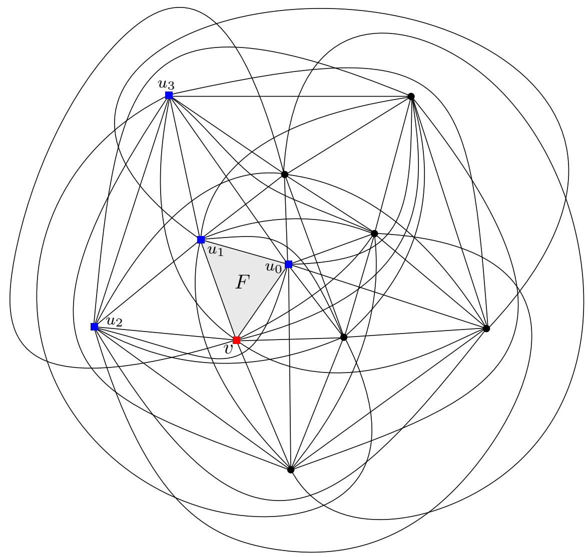

Figure 2: Single-pair-seq-shellable drawing of K_{11}. The initial reference face is F, vertex v has the pair-sequence (u_0, u_1, u_2, u_3).

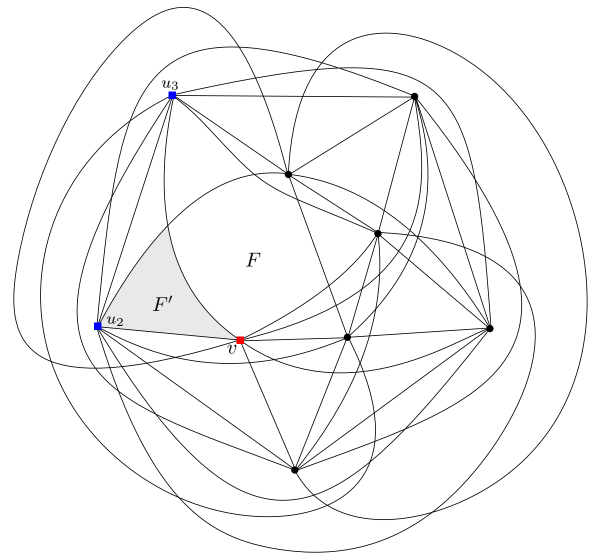

Figure 3: Subdrawing $D-\{u_0, u_1\}$ of the drawing shown in Figure 2. The reference face is now F', which is incident to v and u_2.

the double cumulated value of invariant edges incident to u_1 is at least $\binom{k+2}{2}$ for $0 \leq k \leq m-1$ if we remove v from $D - u_0$ with respect to F. The edge u_0u_1 may be invariant or non-invariant in D with respect to removing v. Now consider the drawing $D - \{u_0, u_1\}$ with $n-2$ vertices and $\frac{n-3}{2} - 2 = \frac{n-1}{2} - 3 = m-1$. Because $n-2$ is odd, we know that for all faces incident to v the value of $\bar{I}_{m-1}(D-\{u_0, u_1\}, D-\{v, u_0, u_1\})$ is the same (Lemma 10). We may select a new reference face F', such that v and u_3 are incident to F', and we can argue again, using Lemma 9, that removing v leads to at least $\binom{k+2}{2}$ for $0 \leq k \leq m-2$ double cumulated value of invariant edges incident to u_2, since u_2 is incident to F'. The double cumulated value of invariant edges incident to u_3 is at least $\binom{k+2}{2}$ for $0 \leq k \leq m-3$ with respect to F' if we remove v from $D - \{u_0, u_1, u_2\}$. Again, the edge u_2u_3 may be invariant or non-invariant in $D - \{u_0, u_1\}$ with respect to removing v.

In general, we are able to change the reference face incident to v if a subdrawing K_r of K_n with $0 < r \leq n$ has an odd number of vertices because the number of double cumulated invariant $(\lfloor \frac{r}{2} \rfloor - 2)$-edges does not change (see Lemma 10). Furthermore, since vertex u_i for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$ is incident to the (current) reference face, u_i contributes at least $\binom{m-i+2}{2}$ to the value of the double cumulated invariant m-value with respect to removing v from D. Thus, $\bar{I}_m(D, D-v) \geq \sum_{i=1}^{m+2} \binom{i}{2} = \binom{m+3}{3}$. $\square$

In Figure 2, both vertices u_0 and u_1 are incident to the initial reference face F. Figure 3 shows the drawing after removing the first pair (i.e. u_0 and u_1). The face F is not incident to any vertex except v. Changing the reference face to F' allows to proceed with u_2 and u_3. Notice that in a drawing D of K_n with n odd, only the value of $\bar{I}_{\lfloor \frac{n}{2} \rfloor - 2}(D, D-v)$ is invariant with respect to changing the reference face. The values $\bar{I}_k(D, D-v)$ for $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 3\}$ may change when selecting a different reference face.

Lemma 13 *Let D be a good drawing of K_n with n odd and $v \in V$. If v has a pair-sequence and for the subdrawing $D-v$ we have $\hat{E}_{\lfloor \frac{n}{2} \rfloor - 3}(D-v) \geq 3\binom{\lfloor \frac{n}{2} \rfloor + 1}{4}$ with respect to $f(v)$, then $cr(D) \geq H(n)$.*

Proof. We have $\hat{E}_{\lfloor \frac{n}{2} \rfloor - 2}(D, v) \geq 2\binom{\lfloor \frac{n}{2} \rfloor + 1}{3}$ for any face that is incident to v in D, and because v has a pair-sequence and due to Lemma 12, it follows that $\bar{I}_{\lfloor \frac{n}{2} \rfloor - 2}(D, D-v) \geq \binom{\lfloor \frac{n}{2} \rfloor + 1}{3}$. Using Lemma 7, it follows for every face incident to v $\hat{E}_{\lfloor \frac{n}{2} \rfloor - 2}(D) \geq 3\binom{\lfloor \frac{n}{2} \rfloor + 2}{4}$. Since n is odd, the result follows with Corollary 3. $\square$

Next, we define semi-pair-shellability.

Definition 14 *Let D be a good drawing of K_n with n odd. If there exists a vertex $v \in V$ that has a pair-sequence and the subdrawing $D-v$ is seq-shellable for $f(v)$, then we call D* semi-pair-shellable.

Using Lemma 13, we show that semi-pair-shellable drawings have at least $H(n)$ crossings. By definition the subdrawing $D-v$ is seq-shellable, hence $\hat{E}_{\lfloor \frac{n}{2} \rfloor - 3}(D - v) \geq 3\binom{\lfloor \frac{n}{2} \rfloor + 1}{4}$ for $f(v)$ (see Corollary 8). Consequently, Theorem 15 follows.

Theorem 15 *If D is a semi-pair-shellable drawing of K_n, then $cr(D) \geq H(n)$.*

The drawing D in Figure 2 is semi-pair-shellable but not seq-shellable. It is impossible to find a vertex sequence and corresponding simple sequences to apply the definition of seq-shellability. However, the subdrawing $D - v$ is seq-shellable for face $f(v)$ and v has a pair-sequence. Consequently, D is semi-pair-shellable.

We are not aware of a crossing optimal semi-pair-shellable drawing that is not seq-shellable. Every $(\lfloor \frac{n}{2} \rfloor - 1)$-seq-shellable drawing D with n odd is also semi-pair-shellable: By definition, D has a vertex sequence $a_0, \dots, a_{\lfloor \frac{n}{2} \rfloor - 1}$, and each a_i has a simple sequence S_i with $i \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$. The first $\lfloor \frac{n}{2} \rfloor - 2$ vertices of S_0 are a pair-sequence for a_0. Moreover, the drawing $D - a_0$ is $(\lfloor \frac{n}{2} \rfloor - 2)$-seq-shellable with the vertex sequence $a_1, \dots, a_{\lfloor \frac{n}{2} \rfloor - 1}$ and its corresponding simple sequences. However, there exist $(\lfloor \frac{n}{2} \rfloor - 2)$-seq-shellable drawings that are not semi-pair-shellable. Thus, semi-pair-shellability is a new distinct class that intersects but does not contain the class of seq-shellable drawings.

5 k-Deviations

In the following, we introduce k-deviations, which we use to represent the difference between (cumulated) k-edges and optimal values; k-deviations allow us to formulate conditions under which we are able to change the reference face even more freely. Note that if for a drawing D of K_n it holds that $E_k(D) = 3(k+1)$ for all $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$, then $cr(D) = H(n)$. We define k-deviations as the difference between this value and the number of k-edges in a drawing.

Definition 16 *Let D be a good drawing of K_n, $F \in \mathcal{F}(D)$ and $E_k(D)$ the number of k-edges for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$ with respect to F. We denote by $\Delta_k(D) \coloneqq E_k(D) - 3(k+1)$ the* k-deviation *of the drawing D for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$ with respect to F. Moreover, we define the cumulated versions of the k-deviation for F as*

$$\bar{\Delta}_k(D) \coloneqq \sum_{i=0}^{k} \sum_{j=0}^{i} \Delta_j(D) = \sum_{i=0}^{k} (k+1-i)\Delta_i(D) \text{ and}$$

$$\hat{\Delta}_k(D) \coloneqq \sum_{i=0}^{k} \bar{\Delta}_i(D) = \sum_{i=0}^{k} \binom{k+2-i}{2} \Delta_i(D).$$

Finally, we define the deviation of the crossing number of D from the Harary-Hill optimal number of crossings as $\Delta_{cr}(D) \coloneqq cr(D) - H(n)$.

We can express k-deviations in the following ways.

Lemma 17 *Let D be a good drawing of K_n. For a reference face $F \in \mathcal{F}(D)$ and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$, we have $\hat{\Delta}_k(D) = \hat{\Delta}_{k-1}(D) + \bar{\Delta}_k(D)$.*

Corollary 18 *Let D be a good drawing of K_n. For n odd we have $\Delta_{cr}(D) = 2\hat{\Delta}_{\frac{n-1}{2}-2}(D)$, and for a reference face $F \in \mathcal{F}(D)$ and n even $\Delta_{cr}(D) = \hat{\Delta}_{\frac{n}{2}-2}(D) + \hat{\Delta}_{\frac{n}{2}-3}(D)$.*

Notice, that Corollary 18 implies Kleitman's parity theorem for complete graphs [12]. The following lemma gives a lower bound on $\hat{\Delta}_{\lfloor \frac{n}{2} \rfloor - 3}(D)$.

Lemma 19 *Let D be a good drawing of K_n with $cr(D) \geq H(n)$. For each $F \in \mathcal{F}(D)$ with $\hat{\Delta}_{\lfloor \frac{n}{2} \rfloor - 2}(D) \geq \bar{\Delta}_{\lfloor \frac{n}{2} \rfloor - 2}(D)$, it holds that $\hat{\Delta}_{\lfloor \frac{n}{2} \rfloor - 3}(D) \geq 0$.*

With the following proposition, we are able to select a new reference face for the subdrawing $D - v$.

Proposition 20 *Let D be a good drawing of K_n with n odd and $v \in V$, such that the subdrawing $D - v$ is seq-shellable for any face $F \in \mathcal{F}(D - v)$. If v has a pair-sequence and in subdrawing $D - v$ for $f(v)$ it holds that $\hat{\Delta}_{\frac{n-1}{2}-2}(D - v) \geq \bar{\Delta}_{\frac{n-1}{2}-2}(D - v)$, then $cr(D) \geq H(n)$.*

So far, for all drawings and all faces we inspected, the condition of Lemma 19 has been fulfilled. We conjecture it to be true for all good drawings of K_n.

Conjecture 2 *Let D be a good drawing of K_n. With respect to any face $F \in \mathcal{F}(D)$, we have*

$$\hat{\Delta}_{\lfloor \frac{n}{2} \rfloor - 2}(D) \geq \bar{\Delta}_{\lfloor \frac{n}{2} \rfloor - 2}(D).$$

Under the assumption that Conjecture 2 holds, we are able to prove the Harary-Hill Conjecture for another new class of drawings. Here, we can select a different reference face for each vertex.

Theorem 21 *Let D be a good drawing of K_n and $v_1, \dots, v_n$ a sequence of the vertices, such that every vertex v_i with $i \in \{1, \dots, n\}$ and i odd has a pair-sequence, and every vertex v_i with $i \in \{1, \dots, n\}$ and i even has a simple sequence. If Conjecture 2 holds, then $cr(D) \geq H(n)$.*

6 Conclusions and Outlook

We introduced semi-pair-shellable drawings of complete graphs and verified that each drawing in this class has at least $H(n)$ crossings. For the first time, we used more than a single globally fixed reference face in order to show lower bounds on the triple cumulated k-edges. Semi-pair-shellability is only defined for drawings of K_n with n odd so far. Extending semi-pair-shellability to drawings of K_n with an even number of vertices is an open problem. Here, it would suffice to show that $\hat{\Delta}_{\lfloor \frac{n}{2} \rfloor - 2}(D) + \hat{\Delta}_{\lfloor \frac{n}{2} \rfloor - 3}(D) \geq 0$ implies $\hat{\Delta}_{\lfloor \frac{n}{2} \rfloor - 3}(D) \geq 0$ in order to generalize our results from semi-pair-shellability to *pair-shellability*, i.e. a version of seq-shellability with pair-sequences instead of simple sequences. Moreover, we introduced k-deviations to formulate conditions under which we are able to select a new reference face in each subdrawing. Proving Conjecture 2 would settle the Harary-Hill Conjecture for a very broad class of drawings, comprising seq- and semi-pair-shellability. Still, there are optimal drawings where each face touches a single vertex only [6], thus no vertex has a simple or pair-sequence.

References

[1] B. Ábrego, O. Aichholzer, S. Fernández-Merchant, T. Hackl, J. Pammer, A. Pilz, P. Ramos, G. Salazar, and B. Vogtenhuber. All good drawings of small complete graphs. In *Proc. 31st European Workshop on Computational Geometry (EuroCG)*, pages 57–60, 2015.

[2] B. Ábrego, O. Aichholzer, S. Fernández-Merchant, D. McQuillan, B. Mohar, P. Mutzel, P. Ramos, R. Richter, and B. Vogtenhuber. Bishellable drawings of K_n. In *Proc. XVII Encuentros de Geometría Computacional (EGC)*, pages 17–20, Alicante, Spain, 2017.

[3] B. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar. The 2-page crossing number of K_n. In *Proceedings of the Twenty-eighth Annual Symposium on Computational Geometry*, SoCG '12, pages 397–404, New York, NY, USA, 2012. ACM.

[4] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar. More on the crossing number of K_n: Monotone drawings. *Electronic Notes in Discrete Mathematics*, 44:411–414, 2013.

[5] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar. Shellable drawings and the cylindrical crossing number of K_n. *Discrete & Computational Geometry*, 52(4):743–753, 2014.

[6] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and B. Vogtenhuber. Non-shellable drawings of K_n with few crossings. In *Proceedings of the 26th Canadian Conference on Computational Geometry, CCCG 2014, Halifax, Nova Scotia, Canada*, 2014.

[7] B. M. Ábrego, M. Cetina, S. Fernández-Merchant, J. Leaños, and G. Salazar. On $\leq k$-edges, crossings, and halving lines of geometric drawings of K_n. *Discrete & Computational Geometry*, 48(1):192–215, 2012.

[8] M. Balko, R. Fulek, and J. Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete & Computational Geometry*, 53(1):107–143, 2015.

[9] P. Erdös, L. Lovász, A. Simmons, and E. G. Straus. Dissection graphs of planar point sets. In *A Survey of Combinatorial Theory*, pages 139–149. Elsevier, 1973.

[10] R. K. Guy. A combinatorial problem. *Nabla, Bulletin of the Malayan Mathematical Society*, 7:68–72, 1960.

[11] F. Harary and A. Hill. On the number of crossings in a complete graph. *Proceedings of the Edinburgh Mathematical Society*, 13(4):333–338, 1963.

[12] D. J. Kleitman. A note on the parity of the number of crossings of a graph. *Journal of Combinatorial Theory, Series B*, 21(1):88–89, 1976.

[13] L. Lovász, K. Vesztergombi, U. Wagner, and E. Welzl. Convex quadrilaterals and k-sets. *Contemporary Mathematics*, 342:139–148, 2004.

[14] D. McQuillan, S. Pan, and R. B. Richter. On the crossing number of K_{13}. *J. Comb. Theory, Ser. B*, 115:224–235, 2015.

[15] P. Mutzel and L. Oettershagen. The crossing number of semi-pair-shellable drawings of complete graphs. *CoRR*, abs/1805.06780, 2018.

[16] P. Mutzel and L. Oettershagen. The crossing number of seq-shellable drawings of complete graphs. In C. S. Iliopoulos, H. W. Leong, and W. Sung, editors, *Combinatorial Algorithms - 29th International Workshop, IWOCA 2018, Singapore, July 16-19, 2018, Proceedings*, volume 10979 of *Lecture Notes in Computer Science*, pages 273–284. Springer, 2018.

[17] S. Pan and R. B. Richter. The crossing number of K_{11} is 100. *Journal of Graph Theory*, 56(2):128–134, 2007.

[18] M. Schaefer. The graph crossing number and its variants: A survey. *The Electronic Journal of Combinatorics (Dec 22, 2017)*, 1000:DS21–May, 2013.

[19] L. A. Székely. A successful concept for measuring nonplanarity of graphs: The crossing number. *Electronic Notes in Discrete Mathematics*, 5:284–287, 2000.