A Paper on Pencils: A Pencil and Paper Puzzle
Pencils is NP-Complete

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Abstract

Pencils is a paper-and-pencil puzzle created by Japanese publisher Nikoli. A puzzle is an m-by-n grid where some squares hold a number or a pencil tip that is pointed in one of the cardinal directions. The goal is to draw 'pencils' that partition the squares of the grid. Each pencil occupies $2k+1$ squares for some $k \geq 1$. A k-pencil has a horizontal or vertical body of length k, a tip pointing away from one end of the body, and a lead that is a path of k squares starting from the tip. In addition, any number inside a body must match the body's size. We show that Pencils is NP-complete even when limited to 1-pencils and 2-pencils.

1 Introduction

This article proves the NP-completeness of a new paper-and-pencil puzzle by Japanese publisher Nikoli. The puzzle is *Pencils* (ペンシルズ) and it was introduced in *Puzzle Communication magazine* Volume 158 [2].

In this article we use *adjacent* and *connected* to mean orthogonally adjacent and orthogonally connected.

1.1 Rules of the Puzzle

Pencils is played on a *board*, which is an m-by-n grid where each square is initially empty or filled with a number or pencil tip pointed in a cardinal direction. A player draws *pencils* which each occupy $2k+1$ connected squares for some $k \geq 1$. A *k-pencil* consists of the following parts:

(P1) The *body* is a horizontal or vertical line of k squares.
(P2) The *tip* is 1 square after one end of the body, and it is pointed away from the body.
(P3) The *lead* is a line through the center of $k+1$ connected squares starting from and including the tip.

The goal of Pencils is to draw pencils on the given grid subject to the following rules [3]:

(P4) The pencils partition the $m \cdot n$ squares of the grid.
(P5) If x is a number on the board, then x must be drawn inside of the body of some x-pencil.

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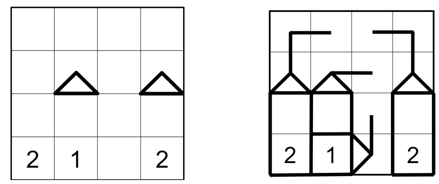
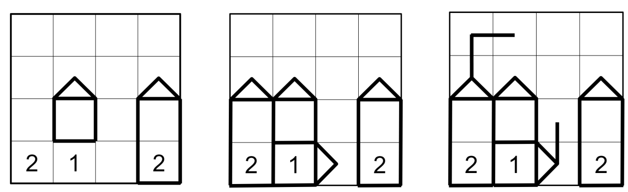
With regard to (P5), an individual x-pencil may have a single x, multiple x's, or no x's inside of it.



(a) Pencils puzzle. (b) Solution.

Figure 1: A 4-by-4 Pencils puzzle that uses 1-pencils and 2-pencils.

A simple puzzle and its solution (originally published in *Puzzle Communication Nikoli* Volume 162 [4]) is displayed in Figure 1, and its solution process is shown in Figure 2. The PENCILS decision problem answers 'yes' or 'no' depending on whether an input board is valid and is solvable based on rules (P1)-(P5).



(a) The tip above the 1-pencil forces a pencil body. The tip above the 2-pencil must be part of the 2-pencil due to the 2-pencil position.

(b) Placement of the 1-pencil and 2-pencil bodies and tips are forced.

(c) Leads for the 1-pencil and the left 2-pencil are drawn in, leaving only one solution for the remaining leads.

Figure 2: Solving the Pencils puzzle in Figure 1.

Notice that a solution to an m-by-n board must fill each of the $m \cdot n$ squares with a finite number of possible symbols. More specifically, a square is covered by a horizontal or vertical body, a tip that points in one of four directions, or by a lead that proceeds horizontally, vertically, or turns 90^o. Therefore, we can guess a possible solution in non-deterministic polynomial-time. Rules (P1)–(P5) can then be checked in polynomial-time.

Remark 1 PENCILS *is in NP.*

1.2 Outline

The central proof of this paper will be done by reduction from a Boolean satisfiability problem. The specific source problem is included in Section 3 along with an outline of the reduction. Section 4 introduces our gadgets, and then Section 5 proves that PENCILS is NP-complete even when restricted to 1-pencils and 2-pencils. Section 6 concludes with open problems. We begin by characterizing the rectangular regions that can be filled with pencils in Section 2.

A preliminary unpublished version of this article was announced on twitter by `@postpostdoc` [5].

2 Empty Rectangles

When solving a pencils puzzle, the solver sometimes faces empty regions of the board that must be completely filled with new pencils. Similarly, we will need to understand how empty space can be filled during our reduction. In this section we provide a full characterization of when rectangular regions can be filled. We formulate this result in terms of solving empty puzzle boards, but we will use the result to solve rectangular "sub-puzzles" inside of larger puzzles.

Define an *empty board* to be an m-by-n grid where each square is empty.

Lemma 1 *Suppose that B is an empty m-by-n board. The decision problem* PENCILS(B) *is* TRUE *if and only if $m \cdot n \notin \{1, 2, 4\}$.*

Proof. We begin by considering the negative cases. Observe that the smallest individual pencil (i.e. a 1-pencil) covers 3 squares. Thus, if B has area 1 or 2, then is too small to be filled with a pencil. Similarly, rectangles of area 4 can only admit a 1-pencil, which then leaves one unfillable square.

Now we consider the remaining positive cases. Since the board B can be rotated 90^o without changing the result of PENCILS(B), we can assume without loss of generality that $m \leq n$. If the area of B is 3, then it must be that $m = 1$ and $n = 3$, and in this case it can be filled with a single 1-pencil. In the remaining cases the area of B is greater than 4, so we can assume that $n \geq 3$.

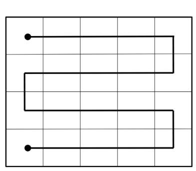


Figure 3: A 5-by-4 grid with a line moving back-and-forth along each row though the centers of the squares in boustrophedon order.

Our strategy is to draw the pencils one after another from end-to-lead in a single line. This line will proceed back-and-forth along each row starting from the top-left square, as illustrated in Figure 3[1]. More specifically, we will primarily draw 1-pencils along the line, since they can turn corners. Since 1-pencils occupy 3 squares, we now proceed in three cases based on the area modulo 3.

- If the area is $3k$, then we draw successive 1-pencils along the line until they fill the entire rectangle.
- If the area $3k + 2$, then recall that our previous assumption that $n \geq 3$. Therefore, we can begin the line with a 2-pencil. This is because the board is wide enough to contain its body and tip, and its lead can bend if $n = 3$ or $n = 4$. Then we fill the remainder of the line with 1-pencils.
- If the area is $3k+1$, then we consider two cases. If $k = 1$, then the area is $3k + 1 = 7$, and it must be that $n = 1$ and $m = 7$. In this case the rectangle can be filled with a single 3-pencil. Otherwise, if $k > 1$, then the area is $3k + 1 \geq 10$. In this case we can draw two 2-pencils along the line, one starting at the beginning of the line and one starting at the end of the line, and then fill in the remainder of the line with 1-pencils.

□

Now we specialize the previous lemma based on 1-pencils and 2-pencils.

Corollary 1 *If B is an empty m-by-n board, then it can be filled entirely with 1-pencils and 2-pencils if and only if $m \cdot n \notin \{1, 2, 4, 7\}$.*

Proof. Observe that the proof of Lemma 1 uses only 1-pencils and 2-pencils, except in the case that $m \cdot n = 7$. Furthermore, 1-pencils and 2-pencils occupy 3 and 5 squares respectively, so it is impossible for them to fill a board of area 7. □

3 Source Problem

Our hardness proof reduces from a satisfiability problem, and in this section we review relevant terminology and results. Then we give a high-level outline of our reduction.

3.1 Rectilinear Planar 1-in-3SAT

A *(Boolean) variable* can be assigned a truth value of TRUE or FALSE. If x_i is a variable, then its *positive literal* is x_i, and its *negative literal* is $\neg x_i$. A (Boolean) formula is in *3 conjunctive normal form (3CNF)* if it is

[1]This back-and-forth order can be described as *boustrophedonic* which is Greek for "as the ox plows".

written $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ where each *clause* C_i has the form $(\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ and each $\ell_{i,j}$ is a literal. A clause is *positive* if every one of its literals is positive, and a 3CNF formula is *positive* if every clause is positive. The 3CNF formula ϕ is a *yes* instance of the 3SAT decision problem if its variables can be assigned so that ϕ evaluates to true; otherwise ϕ is a *no* instance. In other words, 3SAT asks if there is an assignment in which every clause has at least one literal that evaluates to true. The 1-IN-3SAT decision problem instead asks if there is a variable assignment in which exactly one literal evaluates to true. A formula is *planar* if the bipartite incidence graph of variables and clauses is planar. A formula is *rectilinear planar* if the graph can be embedded into a grid in such a way that the vertices can be represented by horizontal line segments and the edges can be drawn as vertical lines.

Theorem 2 (Mulzer and Röte [1]) RECTILINEAR POSITIVE PLANAR 1-IN-3SAT *is NP-Complete.*

We will drop the positive condition from Theorem 2 and instead use RECTILINEAR PLANAR 1-IN-3SAT as our source problem. Since every instance of the former problem is an instance of the latter problem, we can easily conclude that the latter is also NP-complete.

3.2 Reduction Outline

Our reduction constructs a planar graph that represents the 1-in-3 satisfiability (or not) of a logical statement in 3CNF. The graph connects variables to their literals in the statement, with not gates appearing along the connections to negative literals. The reduction will use "variable assignment" gadgets—one for each variable—where the player will be able to select whether a variable has a truth value of TRUE or FALSE. Then, wires will carry these truth values to the corresponding literals in each clause. Because a variable can appear more than once in a statement, we include a gadget to duplicate its truth value onto two different wires, thereby ensuring that the choice is consistent in each clause it appears in. Finally, because the statement is in 3CNF, we will also create a gadget that represents an arbitrary 1-in-3 clause, with wire inputs. Using these gadgets, we will reduce the decision problem, PLANAR 1-IN-3 SAT to PENCILS, by transforming a particular logical statement to a pencils board.

4 Gadgets

In this section we present the various gadgets used in our reduction.

Lemma 3 (Wire) *The gadget shown in Figure 4a transmits a truth value from one part of the puzzle to another as an edge in* PLANAR 1-IN-3 SAT.

Proof. Suppose that we have the left 2-pencil already filled in, pointing into the wire (the direction is forced by the variable assignment gadget, shown later). Then the adjacent 2-pencil must point in the same direction as its neighbor, since there is not room for it to point in the opposite direction. Furthermore, the pencil can neither overlap with its neighbor nor leave a gap of size 1 between itself and its neighbor (as this would be unfillable), so the pencil must have the same position relative to the predrawn 2 as its neighboring 2-pencil. Thus, a 2-pencil/predrawn 2 positioning assigned at the front of the wire gets precisely transmitted to all other parts of the wire. □

Using this lemma, we can establish the formalism that if a wire has its 2-pencils with the number 2 in the square adjacent to the tip, then it carries FALSE, and if the 2 is in the other square, then it carries TRUE.

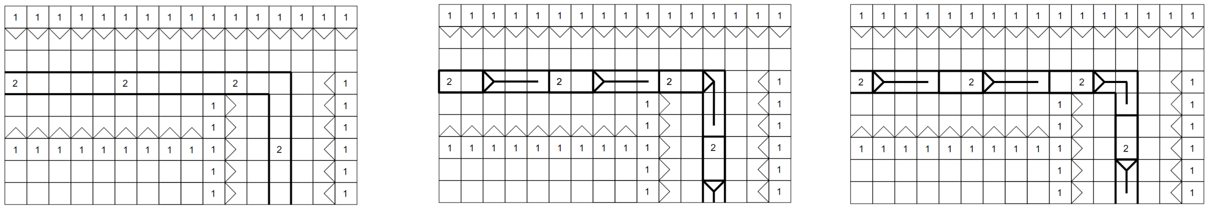
Currently, our wires require that all of our gadgets are a multiple of five squares apart, since the 2's are spaced exactly that far apart in our wire gadget. However, we can deal with this issue with the "modularity switcher" gadget in Figure 5a.

Lemma 4 (Modularity Switch) *The gadget shown in Figure 5a preserves the truth value that a surrounding wire gadget is carrying.*

Proof. Suppose that the incoming truth value is TRUE. Then there will be six unfilled squares between the end of incoming 2's lead and the pair of 2's. Since the 2's on the right are only one square apart from each other, they must both be pointing outward. Thus, the left 2 must have a 2-pencil pointing left, which will occupy either three or four of the empty middle squares. If the 2-pencil fills four middle squares, then there will only be two unfilled middle squares, which cannot be filled by any pencil. Thus, the 2-pencil must fill three middle squares, which must be filled by a 1-pencil. This then forces play on the last 2-pencil, as seen in Figure 5b.

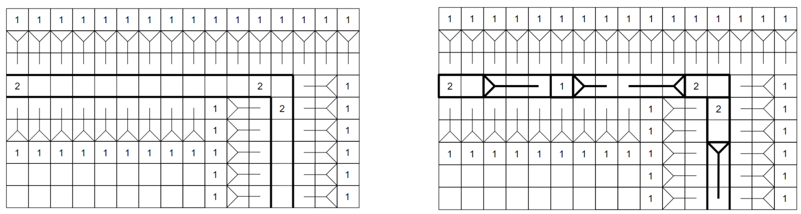
If the incoming truth value is FALSE, then there will seven unfilled squares between the end of the incoming 2's lead and the pair of 2's on the right. Again, the left of the pair of 2-pencils must fill either three or four squares. This 2-pencil cannot occupy three middle squares, for it would leave four squares unoccupied, which cannot be filled. Thus, the 2-pencil must occupy four middle squares, with the remaining three filled by a 1-pencil. This forces the subsequent 2-pencil to play as in Figure 5c. Thus, regardless of the incoming truth value, the modularity switcher does not alter the truth value carried by the wire. □

Lemma 5 (Variable Assignment) *The gadget presented in Figure 6a allows the player to assign a value to a variable that will be transmitted out through the wire on the right.*



(a) The initial wire gadget. (b) The board filled in with True. (c) The board filled in with False.

Figure 4: The initial wire gadget and the manners it can be filled in.



(a) The initial modularity switching gadget. (b) The modularity gadget filled in with value True. (c) The modularity gadget filled in with value False.

Figure 5: The initial modularity gadget and the manners it can be filled in.

Proof. The area in which new pencils may be added is limited to a space of size 9 (there are two possible ways for this to happen, depending on whether the one pencil in the lower right has an upward or leftward pointing lead). Next, the first 2 is located such that its tip must be leftward pointing. If it pointed to the right, there would not be room for the two squares that the line would need to occupy. Thus, the other 2-pencil must be rightward pointing with its body either filling in the square between the 2's or not. Figures 6b and 6c describe how to fill the 3x3 space for True and False variable assignments.

Since the player is not able to play the right 2-pencil another way than the two variable assignments, and the player is able to assign either truth value, our Lemma is proven. □

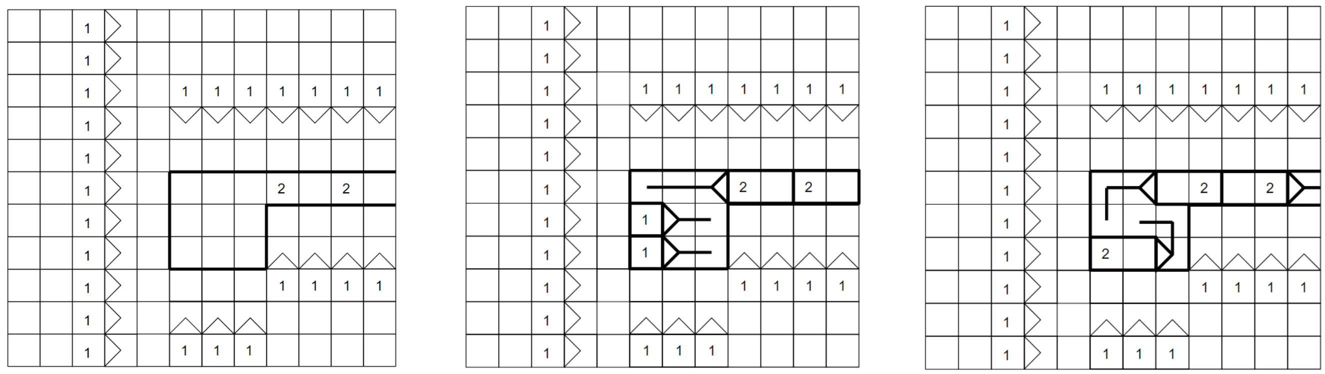
Lemma 6 (Not Gate) *Figure 7a presents a not gate for a leftward facing wire.*

Proof. First, consider the scenario where the initial value of the wire is true. Then, the remaining number of squares up to the next 2 is 7. The next 2 must have its pencil pointing to the left, so it will occupy either three or four of the open spaces. This would leave either three or four consecutive unoccupied spaces. However, we cannot fill four unoccupied spaces by Lemma 1, so we must play the second 2 so that it occupies four of the internal spaces. This forces the last 2 (which must be played to the right) to occupy the empty space between the 2's, making the transmitted value false. On the other hand, if the initial value is false, then there will be eight open squares in the middle. The second 2 can be played so that it occupies three or four spaces. This corresponds to four or five open middle spaces. Of the two options, we can only fill five consecutive middle spaces, so the the second 2 must be played to occupy the empty space between the 2's. This forces the final 2 to be played in the true position. Both of these scenarios are illustrated in Figures 7b and 7c. □

Lemma 7 (Split Gate) *For a given input in the wire on the left, the gadget in Figure 8a assigns truth values to two wires on the right and bottom each carrying the opposite of the given input (to make this a true split gate, we would add a not gate between the input and the gadget or add two more not gates to the ends).*

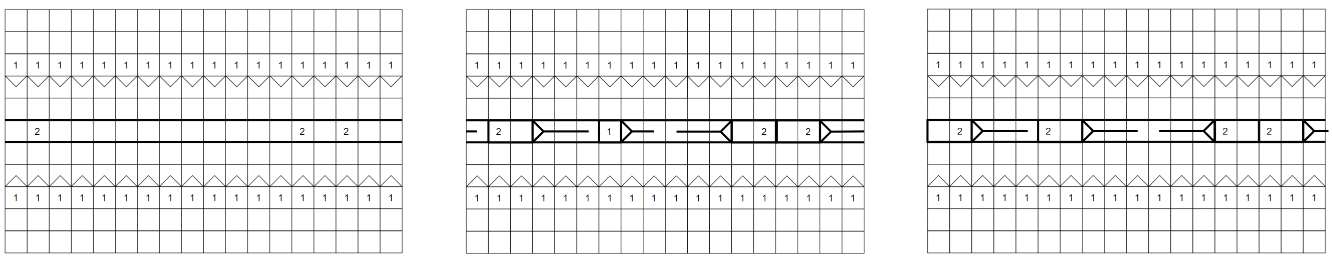
Proof. Of the 2-pencils on the right and on the bottom, the inner pencils of each must be pointing inward; there is not room for them to point outward. If the entering wire is true, then there are 8 remaining spaces within the center of the gadget. The right and bottom inner 2's can occupy either 3 or 4 spaces. If they both occupy 3 squares, then there will be 2 squares unfilled, which cannot be filled by the addition of another pencil. If one occupies 3 squares and the other 4, there will be one square unfilled, which is not fillable by the addition of another pencil. If they both occupy 4 squares, then there are no squares left unfilled, and the gadget is satisfied. This case is forced if the input is true, since there are no other ways to fill the gadget. In this case, the output 2's are both forced to be false, so the input was flipped and placed into two wires, as in the statement of the lemma.

If the entering wire is false, then there are 9 remaining



(a) Unfilled assignment.
(b) Filled with TRUE.
(c) Filled with FALSE.

Figure 6: The assignment gadget unfilled and filled.



(a) An unfilled not gate.
(b) Turning TRUE to FALSE.
(c) Turning FALSE to TRUE.

Figure 7: A not gate unfilled and filled with both truth values.

open squares. If either of the inward pointing 2-pencils occupy 4 squares, then the gadget is unfillable, since there will be either 1 or 2 unfilled squares. Thus, the 2 inward pointing pencils must occupy 3 of the inner squares, leaving 3 open squares, which can be filled with a single 1-pencil. This scenario also forces the output 2's to both be true as desired. □

Lemma 8 (1-in-3 Gate) *The gadget of Figure 9 (which takes in input from three wires) is only fillable if exactly one of the input wires is true.*

Proof. Note that each input wire ends its line at either {RT, LT, BT} if it is true or {RF, LF, BF} if it is false. If all the statements are false, then there are four unoccupied squares, so the gadget is unsolvable if all the wires are carrying false values. If all the the statements are true, then there is only one unoccupied square, so the gadget is unsolvable in this case as well. If two of the statements are true, then there exactly two unfilled squares, so the gadget is unsolvable if two statements are true. If only one statement is true, then there are three connected unoccupied squares, which can be filled with a 1-pencil, so the gadget is solvable if and only if exactly one statement is true. Thus, the gadget serves the purpose of a 1-in-3 Gate. □

5 NP-Completeness of Pencils

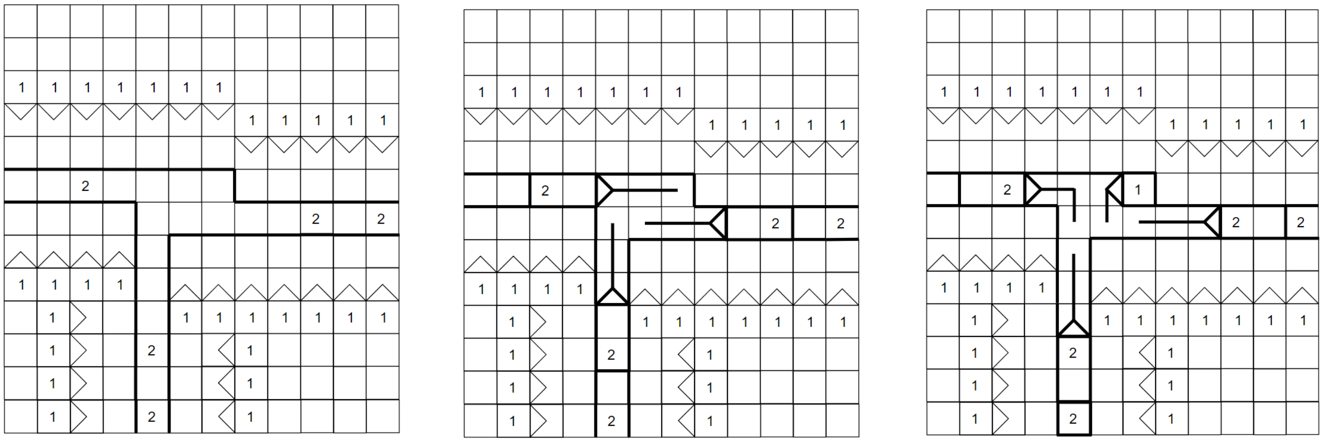
Now we are ready to prove our main result.

Theorem 9 (Pencils is NP-Complete) *For a given board, B, the decision problem* PENCILS(B) *is NP-Complete. Furthermore, this is true when the puzzle designer and solver are restricted to using 1-pencils and 2-pencils.*

Proof. We use Theorem 2 and reduce RECTILINEAR PLANAR 1-IN-3SAT(S) to PENCILS(B). Starting with G_S, the graph corresponding to S, we will encode this graph into a pencils game.

We replace each source variable with the variable assignment gadget and add sufficiently many split and not gates such that each source vector has as many outward going wires as edges leading to literals in the formula. We can then create each clause by leading in the corresponding literals with wires (with not gates if they appear with a ¬ modifier in the formula). This is possible since G_S was planar and we can line up the wires to fit in perfectly by inserting modularity switchers sufficiently many times. Call this pencil board B_S.

By the lemmas for each gadget, if RECTILINEAR PLANAR 1-IN-3SAT(S) is true, then by matching up our variable assignment to that which solves S, we can solve the corresponding pencil board, B_S. Thus, RECTILINEAR



(a) An unfilled split gate. (b) TRUE input. (c) FALSE input.

Figure 8: A split gate unfilled and filled with both truth values.

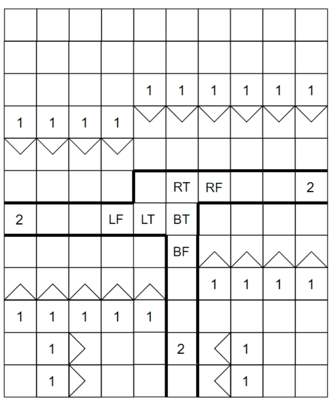


Figure 9: An unfilled 1-in-3 Gate. The non-numeral entries exist to refer to potential pencil endings (and will not affect the actual puzzle).

PLANAR 1-IN-3SAT(S) = TRUE implies PENCILS(B_S) = TRUE.

If PENCILS(B_S) = TRUE, then there must be some assignment of the variable assignment gadgets such that each 1-in-3 gadget was satisfied. However, because this board was derived directly from the graph, it provides a variable assignment for S such that S is true (under 1-in-3 satisfiability rules). Thus, PENCILS(B_S) = TRUE true implies RECTILINEAR PLANAR 1-IN-3SAT(S) = TRUE. So, PENCILS is NP-Hard.

Membership in NP was given in Remark 1. Thus, PENCILS is both NP-Hard and in NP, so it is NP-Complete. □

6 Final Remarks and Open Problems

We proved that a restricted form of the PENCILS decision problem is NP-complete in which only 1-pencils and 2-pencils are used. In this section we provide open problems in several different directions.

Define PENCILS$_S(B)$ as the decision problem in which the puzzle designer and solver are restricted to pencils whose lengths appear in the set S. For example, we proved the NP-completeness of PENCILS$_{\{1,2\}}(B)$, or simply PENCILS$_{1,2}(B)$. This raises the following open problems:

- Is PENCILS$_1(B)$ in P? In other words, is there a polynomial-time algorithm for solving Pencils when only 1-pencils are allowed?
- Is PENCILS$_2(B)$ NP-complete?
- More generally, what is the complexity of PENCILS$_\ell(B)$ for single fixed values of ℓ?

Besides pencil sizes, we could also consider other restrictions to the pencil bodies and leads. For example, we can define a *straight-line pencil* as one in which the lead is a straight line. Similarly, we can define a *horizontal pencil* and a *vertical pencil* based on the orientation of the pencil's body.

- What is the complexity of PENCILS when restricted to straight-line pencils?
- What is the complexity of PENCILS when restricted to horizontal pencils?

Nikoli typically designs individual puzzle instances to have a unique solution. The associated complexity class is *Another Solution Problem* (*ASP*) in which the input is a problem and a solution and the goal is to determine if there is a second solution. This complexity class was popularized by Ueda and Nagao [6]. We pose the question: is PENCILS ASP-hard?

The authors would like to thank the referees whose comments led to many improvements throughout the article.

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