Distance-Two Colorings of Barnette Graphs

Tomás Feder* Pavol Hell† Carlos Subi‡

Abstract

Barnette identified two interesting classes of cubic polyhedral graphs for which he conjectured the existence of a Hamiltonian cycle. Goodey proved the conjecture for the intersection of the two classes. We examine these classes from the point of view of distance-two colorings. A distance-two r-coloring of a graph G is an assignment of r colors to the vertices of G so that any two vertices at distance at most two have different colors. Note that a cubic graph needs at least four colors. The distance-two four-coloring problem for cubic planar graphs is known to be NP-complete. We claim the problem remains NP-complete for tri-connected bipartite cubic planar graphs, which we call type-one Barnette graphs, since they are the first class identified by Barnette. By contrast, we claim the problem is polynomial for cubic plane graphs with face sizes $3, 4, 5$, or 6, which we call type-two Barnette graphs, because of their relation to Barnette's second conjecture. We call Goodey graphs those type-two Barnette graphs all of whose faces have size 4 or 6. We fully describe all Goodey graphs that admit a distance-two four-coloring, and characterize the remaining type-two Barnette graphs that admit a distance-two four-coloring according to their face size.

For quartic plane graphs, the analogue of type-two Barnette graphs are graphs with face sizes 3 or 4. For this class, the distance-two four-coloring problem is also polynomial; in fact, we can again fully describe all colorable instances – there are exactly two such graphs.

1 Introduction

Tait conjectured in 1884 [22] that all cubic polyhedral graphs, i.e., all tri-connected cubic planar graphs, have a Hamiltonian cycle; this was disproved by Tutte in 1946 [24], and the study of Hamiltonian cubic planar graphs has been a very active area of research ever since, see for instance [1, 11, 17, 19]. Barnette formulated two conjectures that have been at the centre of much of the effort: (1) that *bipartite* tri-connected cubic planar graphs are Hamiltonian (the case of Tait's conjecture where all face sizes are even) [4], and (2) that tri-connected cubic planar graphs with all face sizes $3, 4, 5$ or 6 are Hamiltonian, cf. [3, 20]. Goodey [12, 13] proved that the conjectures hold on the intersection of the two classes, i.e., that tri-connected cubic planar graphs with all face sizes 4 or 6 are Hamiltonian. When all faces have sizes 5 or 6, this was a longstanding open problem, especially since these graphs (tri-connected cubic planar graphs with all face sizes 5 or 6) are the popular *fullerene graphs* [8]. The second conjecture has now been affirmatively resolved in full [18]. For the first conjecture, two of the present authors have shown in [10] that if the conjecture is false, then the Hamiltonicity problem for tri-connected cubic planar graphs is NP-complete. In view of these results and conjectures, in this paper we call bipartite tri-connected cubic planar graphs *type-one Barnette graphs*; we call cubic plane graphs with all face sizes $3, 4, 5$ or 6 *type-two Barnette graphs*; and finally we call cubic plane graphs with all face sizes 4 or 6 *Goodey graphs*. Note that it would be more logical, and historically accurate, to assume tri-connectivity also for type-two Barnette graphs and for Goodey graphs. However, we prove our positive results without needing tri-connectivity, and hence we do not assume it.

Cubic planar graphs have been also of interest from the point of view of colorings [6, 15]. In particular, they are interesting for distance-two colourings. Let G be a graph with degrees at most d. A *distance-two r-coloring* of G is an assignment of colors from $[r] = \{1, 2, \ldots, r\}$ to the vertices of G such that if a vertex v has degree $d' \leq d$ then the $d' + 1$ colors of v and of all the neighbors of v are all distinct. (Thus a distance-two coloring of G is a classical coloring of G^2.) Clearly a graph with maximum degree d needs at least $d + 1$ colors in any distance-two coloring, since a vertex of degree d and its d neighbours must all receive distinct colors. It was conjectured by Wegner [25] that a planar graph with maximum degree d has a distance-two r-colouring where $r = 7$ for $d = 3$, $r = d + 5$ for $d = 4, 5, 6, 7$, and $r = \lfloor 3d/2 \rfloor + 1$ for all larger d. The case $d = 3$ has been settled in the positive by Hartke, Jahanbekam and Thomas [14], cf. also [23].

For cubic planar graphs in general it was conjectured in [14] that if a cubic planar graph is tri-connected, or has no faces of size five, then it has a distance-two six-coloring. We propose a weaker version of the second case of the conjecture, namely, we conjecture that *a bipartite cubic planar graph can be distance-two six-colored.* We prove this in one special case (Theorem 5), which of course also confirms the conjecture of Hartke, Jahanbekam and Thomas for that case. Heggerness and

*Palo Alto, `tomas@theory.stanford.edu`
†School of Computing Science, SFU, `pavol@sfu.ca`
‡Los Altos Hills, `carlos.subi@hotmail.com`

Telle [16] have shown that the problem of distance-two four-coloring cubic planar graphs is NP-complete. On the other hand, Borodin and Ivanova [5] have shown that subcubic planar graphs of girth at least 22 can be distance-two four-colored. In fact, there has been much attention focused on the relation of distance-two colorings and the girth, especially in the planar context [5, 15].

Our results focus on distance-two colorings of cubic planar graphs, with particular attention on Barnette graphs, of both types. We prove that a cubic plane graph with all face sizes divisible by four can always be distance-two four-colored, and a give a simple condition for when a bi-connected cubic plane graph with all face sizes divisible by three can be distance-two four-colored using only three colors per face. It turns out that the distance-two four-coloring problem for type-one Barnette graphs is NP-complete, while for type-two Barnette graphs it is not only polynomial, but the positive instances can be explicitly described. They include one infinite family of Goodey graphs (cubic plane graphs with all faces of size 4 or 6), and all type-two Barnette graphs which have all faces of size 3 or 6. Interestingly, there is an analogous result for quartic (four-regular) graphs: all quartic planar graphs with faces of only sizes 3 or 4 that have a distance-two five coloring can be explicitly described; there are only two such graphs.

Note that we use the term "plane graph" when the actual embedding is used, e.g., when discussing the faces; on the other hand, when the embedding is unique, as in tri-connected graphs, we stick with writing "planar".

The proofs omitted here can be found in [9].

2 Relations to edge-colorings and face-colorings

Distance-two colorings have a natural connection to edge-colorings.

Theorem 1 *Let G be a graph with degrees at most d that admits a distance-two $(d+1)$-coloring, with d odd. Then G admit an edge-coloring with d colors.*

Proof. The even complete graph K_{d+1} can be edge-colored with d colors by the Walecki construction [2]. We fix one such coloring c, and then consider a distance-two $(d+1)$-coloring of G. If an edge uv of G has colors ab at its endpoints, we color uv in G with the color $c(ab)$. It is easy to see that this yields an edge-coloring of G with d colors. □

We call the resulting edge-coloring of G the *derived edge-coloring* of the original distance-two coloring.

In this paper, we mostly focus on the case $d = 3$ (the *subcubic* case). Thus we use the edge-coloring of K_4 by colors red, blue, green. This corresponds to the unique partition of K_4 into perfect matchings. Note that for every vertex v of K_4 and every edge-color i, there is a unique other vertex u of K_4 adjacent to v in edge-color i. Thus if we have the derived edge-coloring we can efficiently recover the original distance-two coloring. In the subcubic case, in turns out to be sufficient to have just one color class of the edge-coloring of G.

Theorem 2 *Let G be a subcubic graph, and let R be a set of red edges in G. The question of whether there exists a distance-two four-coloring of G for which the derived edge-coloring has R as one of the three color classes can be solved by a polynomial time algorithm. If the answer is positive, the algorithm will identify such a distance-two coloring.*

There is also a relation to face-colorings. It is a folklore fact that the faces of any bipartite cubic plane graph G can be three-colored [21]. This three-face-coloring induces a three-edge-coloring of G by coloring each edge by the color not used on its two incident faces. (It is easy to see that this is in fact an edge-coloring, i.e., that incident edges have distict colors.) We call an edge-coloring that arises this way from some face-coloring of G a *special three-edge-coloring* of G. We first ask when is a special three-edge-coloring of G the derived edge-coloring of a distance-two four-coloring of G.

Theorem 3 *A special three-edge-coloring of G is the derived edge-coloring of some distance-two four-coloring of G if and only if the size of each face is a multiple of 4.*

Proof. The edges around a face f alternate in colors, and the vertices of f can be colored consistently with this alternation if and only if the size of f is a multiple of 4. This proves the "only if" part. For the "if" part, suppose all faces have size multiple of 4. If there is an inconsistency, it will appear along a cycle C in G. If there is only one face inside C, there is no inconsistency. Otherwise we can join some two vertices of C by a path P inside C, and the two sides of P inside C give two regions that are inside two cycles C', C''. The consistency of C then follows from the consistency of each of C', C'' by induction on the number of faces inside the cycle. □

Corollary 4 *Let G be a cubic plane graph in which the size of each face is a multiple of four. Then G can be distance-two four-colored.*

We now prove a special case of the conjecture stated in the introduction, that all bipartite cubic plane graphs can be distance-two six-colored. Recall that the faces of any bipartite cubic plane graph can be three-colored.

Theorem 5 *Suppose the faces of a bipartite cubic plane graph G are three-colored red, blue and green, so that the* *red faces are of arbitrary even size, while the size of each blue and green face is a multiple of* 4*. Then G can be distance-two six-colored.*

Proof. Let G' be the multigraph obtained from G by shrinking each of the red faces. Clearly G' is planar, and since the sizes of blue and green faces in G' are half of what they were in G, they will be even, so G' is also bipartite. Let us label the two sides of the bipartition as A and B respectively. Now consider the special three-edge coloring of G associated with the face coloring of G. Each red edge in this special edge-coloring joins a vertex of A with a vertex of B; we orient all red edges from A to B. Now traversing each red edge in G in the indicated orientation either has a blue face on the left and green face on the right, or a green face on the left and blue face on the right. In the former case we call the edge *class one* in the latter case we call it *class two*. Each vertex of G is incident with exactly one red edge; the vertex inherits the class of its red edge. The vertices around each red face in G are alternatingly in class 1 and class 2. We assign colors $1, 2, 3$ to vertices of class one and colors $4, 5, 6$ to vertices of class two. It remains to decide how to choose from the three colors available for each vertex. A vertex adjacent to red edges in class i has only three vertices within distance two in the same class, namely the vertex across the red edge, and the two vertices at distance two along the red face in either direction. Therefore distance-two coloring for class i corresponds to three-coloring a cubic graph. Since neither class can yield a K_4, such a three-coloring exists by Brooks' theorem [7]. This yields a distance-two six-coloring of G. □

3 Distance-two four-coloring of type-one Barnette graphs is NP-complete

We now state our main intractability result.

Theorem 6 *The distance-two four-coloring problem for tri-connected bipartite cubic planar graphs is NP-complete.*

In this note we only derive the following weaker version of our claim. (See [9] for the proof of the entire claim.)

Theorem 7 *The distance-two four-coloring problem for bipartite planar subcubic graphs is NP-complete.*

Proof. Consider the graph H in Figure 1.

We will reduce the problem of H-coloring planar graphs to the distance-two four-coloring problem for bipartite planar subcubic graphs. In the *H-coloring problem* we are given a planar graph G and and the question is whether we can color the vertices of G with colors that are vertices of H so that adjacent vertices of G obtain adjacent colors. This can be done if and only if G is three-colorable, since the graph H both contains a triangle and is three-colorable itself. (Thus any three-coloring of G is an H-coloring of G, and any H-coloring of G composed with a three-coloring of H is a three-coloring of G.) It is known that the three-coloring problem for planar graphs is NP-complete, hence so is the H-coloring problem.

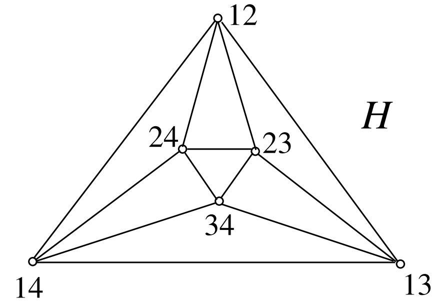

Figure 1: The graph H for the proof of Theorem 7

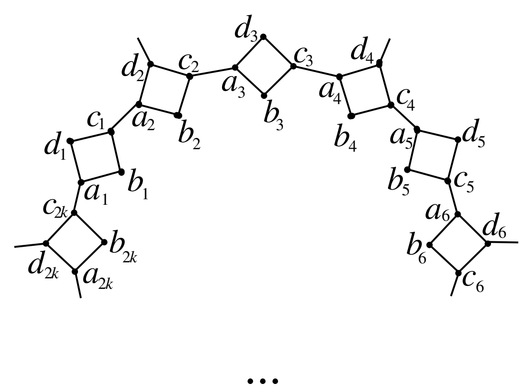

Figure 2: The ring gadget

Thus suppose G is an instance of the H-coloring problem. We form a new graph G' obtained from G by replacing each vertex v of G by a *ring* gadget depicted in Figure 2. If v has degree k, the ring gadget has $2k$ squares. A *link* in the ring is a square $a_ib_ic_id_ia_i$ followed by the edge c_ia_{i+1}. A link is *even* if i is even, and *odd* otherwise. Every even link in the ring will be used for a connection to the rest of the graph G', thus vertex v has k available links. For each edge vw of G we add a new vertex f_{vw} that is adjacent to a vertex d_s in one available link of the ring for v and a vertex d'_t in one available link of the ring for w. (We use primed letters for the corresponding vertices in the ring of w to distinguish them from those in the ring of v.) The actual choice of (the even) subscripts s, t does not matter, as long as each available link is only used once. The resulting graph is clearly subcubic and planar. It is also bipartite, since we can bipartition all its vertices into one independent set A consisting of all the vertices $a_i, c_i, b_{i+1}, di+1$ with odd i in all the rings, and another independent set B consisting of the vertices $a_i, c_i, b_{i+1}, di+1$ with even i in all the rings. Moreover, we place all vertices f_{vw} into the set A. Note that in any distance-two four-coloring of the ring, each link must have four different colors for vertices a_i, b_i, c_i, d_i, and the same color for a_i and a_{i+1}. Thus all a_i have the same color and all c_i have the same color. The pair of colors in b_i, d_i is also the same for all i; we will call it the *characteristic pair of the ring for v*. For any pair ij of colors from $1, 2, 3, 4$, there is a distance-two coloring of the ring that has the characteristic pair ij.

One can prove that G is H-colorable if and only if G' is distance-two four-colorable. □

We remark that (with some additional effort) we can prove that the problem is still NP-complete for the class of tri-connected bipartite cubic planar graphs with no faces of sizes larger than 44.

4 Distance-two four-coloring of Goodey graphs

Recall that Goodey graphs are type-two Barnette graph with all faces of size 4 and 6 [12, 13]. In other words, a *Goodey graph* is a cubic plane graph with all faces having size 4 or 6. By Euler's formula, a Goodey graph has exactly six square faces, while the number of hexagonal faces is arbitrary.

A *cyclic prism* is the graph consisting of two disjoint even cycles $a_1 a_2 \cdots a_{2k} a_1$ and $b_1 b_2 \cdots b_{2k} b_1, k \geq 2$, with the additional edges $a_i b_i$, $1 \leq i \leq 2k$. It is easy to see that cyclic prisms have either no distance-two four-coloring (if k is odd), or a unique distance-two four-coloring (if $k \geq 2$ is even). Only the cyclic prisms with $k = 2, 3$ are Goodey graphs, and thus from Goodey cyclic graphs only the cube (the case of $k = 2$) has a distance-two coloring, which is moreover unique.

In fact, all Goodey graphs that admit distance-two four-coloring can be constructed from the cube as follows. The Goodey graph C_0 is the cube, i.e., the cyclic prism with $k = 2$. The Goodey graph C_1 is depicted in Figure 4. It is obtained from the cube by separating the six square faces and joining them together by a pattern of hexagons, with three hexagons meeting at a vertex tying together the three faces that used to meet in one vertex. The higher numbered Goodey graphs are obtained by making the connecting pattern of hexagons higher and higher. The next Goodey graph C_2 has two hexagons between any two of the six squares, with a central hexagon in the centre of any three of the squares, the following Goodey graph C_3 has three hexagons between any two of the squares and three hexagons in the middle of any three of the squares, and so on. Thus in general we replace every vertex of the cube by a triangular pattern of hexagons whose borders are replacing the edges of the cube. We illustrate the vertex replacement graphs in Figure 3, without giving a formal description. The entire Goodey graph C_1 is depicted in Figure 4.

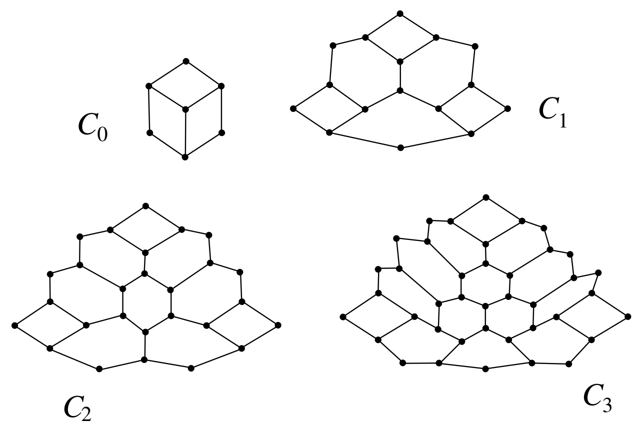

Figure 3: The vertex replacements for Goodey graphs C_0, C_1, C_2, and C_3

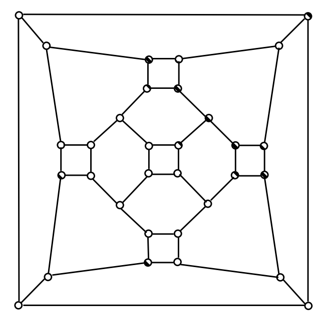

Figure 4: The Goody graph C_1

We have the following results.

Theorem 8 *The Goodey graphs $C_k, k \geq 0$, have a unique distance-two four-coloring, up to permutation of colors.*

Theorem 9 *The Goodey graphs $C_k, k \geq 0$, are the only bipartite cubic planar graphs having a distance-two four-coloring.*

We can therefore conclude the following.

Corollary 10 *The distance-two four-coloring problem for Goodey graphs is solvable in polynomial time.*

Recognizing whether an input Goodey graph is some C_k can be achieved in polynomial time; in the same time bound G can actually be distance-two four-colored.

5 Distance-two four-coloring of type-two Barnette graphs is polynomial

We now return to general type-two Barnette graphs, i.e., cubic plane graphs with face sizes $3, 4, 5$, or 6. As a first step, we analyze when a general cubic plane graph admits a distance-two four-coloring which has three colors on the vertices of every face of G.

Theorem 11 *A cubic plane graph G has a distance-two four-coloring with three colors per face if and only if*

1. *all faces in G have size which is a multiple of* 3,
2. *G is bi-connected, and*
3. *if two faces share more than one edge, the relative positions of the shared edges must be congruent modulo* 3 *in the two faces.*

The last condition means the following: if faces F_1, F_2 meet in edges e, e' and there are n_1 edges between e and e' in (some traversal of) F_1, and n_2 edges between e and e' in (some traversal of) F_2, then $n_1 \equiv n_2 \mod 3$.

Proof. Suppose G has a distance-two four-coloring with three colors in each face. The unique way to distance-two color a cycle with colors $1, 2, 3$ is by repeating them in some order $(123)^*$ along one of the two traversals of the cycle. Therefore the length is a multiple of 3 so (1) holds. Moreover, there can be no bridge in G as that would imply a face that self-intersects and is traversed in opposite directions along any traversal of that face, disagreeing with the order $(123)^*$ in one of them; thus (2) also holds. Finally, (3) holds because the common edges must have the same colors in both faces.

Suppose the conditions hold, and consider the dual G^D of G. (Note that each face of G^D is a triangle.) We find a distance-two coloring of G as follows. Let F be a face in G; according to conditions (1-2), its vertices can be distance-two colored with three colors. That takes care of the vertex F in G^D. Using condition (3), we can extend the coloring of G to any face F' adjacent to F in G^D. Note that we can use the fourth colour, 4, on the two vertices adjacent in F' to the two vertices of a common edge. In this way, we can propagate the distance-two coloring of G along the adjacencies in G^D. If this produces a distance-two coloring of all vertices of G, we are done. Thus it remains to show there is no inconsistency in the propagation. If there is an inconsistency, it will appear along a cycle C in G^D. If there is only one face inside of C, then C is a triangle corresponding to a vertex of G, and there is no inconsistency. Otherwise we can join some two vertices of C by a path P inside C, and the two sides of P inside C give two regions that are inside two cycles C', C''. The consistency of C then follows from the consistency of each of C', C'' by induction on the number of faces inside the cycle. □

It turns out that conditions (1 - 3) are automatically satisfied for cubic plane graphs with faces of sizes 3 or 6.

Corollary 12 *Type-two Barnette graphs with faces of sizes* 3 *or* 6 *are distance-two four-colorable.*

Proof. Such a graph must be bi-connected, i.e., cannot have a bridge, since no triangle or hexagon can self-intersect. Moreover, only two hexagons can have two common edges, and it is easy to check that they must indeed be in relative positions congruent modulo 3 on the two faces. (Since all vertices must have degree three.) Thus the result follows from Theorem 11. □

Theorem 13 *Let G be type-two Barnette graph. Then G is distance-two four-colorable if and only if it is one of the graphs $C_k, k \geq 0$, or all faces of G have sizes* 3 *or* 6.

Proof. If there are faces of size both 3 and 4 (and possibly size 6), then there must be (by Euler's formula) two triangles and three squares, and as in the proof of Theorem 9, the squares must be joined by chains of hexagons, which is not possible with just three squares.

If there is a face of size 5, then there is no distance-two four-coloring since all five vertices of that face would need different colors. □

6 Distance-two coloring of quartic graphs

A *quartic graph* is a regular graph with all vertices of degree four. Thus any distance-two coloring of a quartic graph requires at least five colors. A *four-graph* is a plane quartic graph whose faces have sizes 3 or 4. The argument to view these as analogues of type-two Barnette graphs is as follows. For cubic plane Euler's formula limits the numbers of faces that are triangles, squares, and pentagons, but does not limit the number of hexagon faces. Similarly, for plane quartic graphs, Euler's formula implies that such a graph must have 8 triangle faces, but places no limits on the number of square faces.

We say that two faces are *adjacent* if they share an edge.

Lemma 14 *If a four-graph can be distance-two five-colored, then every square face must be adjacent to a triangle face. Thus G can have at most* 24 *square faces.*

Proof. We view the numbers $1, 2, 3, 4$ modulo 4, and number 5 is separate. Let $u_1u_2u_3u_4$ be a square face that has no adjacent triangle face. (This is depicted in Figure 5 as the square in the middle.) Color u_i by i. Let the adjacent square faces be $u_i u_{i+1} w_{i+1} v_i$. One of v_i, w_i must be colored 5 and the other one $i+2$. Then either all v_i or all w_i are colored 5, say all w_i are colored 5, and all v_i are colored $i+1$. Then $v_i u_i w_i$ cannot be a triangle face, or w_i, w_{i+1} would be both colored 5 at distance two. Therefore $t_i v_i u_i w_i$ must be a square face. (In the figure, this is indicated by the corner vertices being marked by smaller circles; these must exist to avoid a triangle face.) This means that the original square is surrounded by eight square faces for $u_1 u_2 u_3 u_4$, and t_i must have color $i+3$, since u_i, v_{i+3}, v_i, w_i have colors $i, i+1, i+2, 5$.

But then there cannot be a triangle face $x_i v_i w_{i+1}$, since x_i is within distance two of $u_i, u_{i+1}, v_i, t_i, w_{i+1}$ of colors $i, i+1, i+2, i+3, 5$, so each of the adjacent square faces $u_i u_{i+1} w_{i+1} v_i$ for $u_1 u_2 u_3 u_4$ has adjacent square faces as well. This process of moving to adjacent square faces eventually reaches all faces as square faces, contrary to the fact that there are 8 triangle faces. □

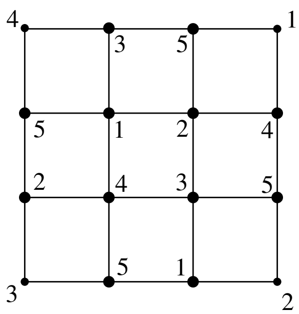

Figure 5: One square without adjacent triangles implies all faces must be squares

It follows that there are only finitely many distance-two five-colorable four-graphs.

Corollary 15 *The distance-two five-coloring problem for four-graphs is polynomial.*

In fact, we can fully describe all four-graphs that are distance-two five-colorable. Consider the four-graphs G_0, G_1 given in Figure 6. The graph G_0 has 8 triangle faces and 4 square faces, the graph G_1 has 8 triangle faces and 24 square faces. Note that G_0 is obtained from the cube by inserting two vertices of degree four in two opposite square faces. Similarly, G_1 is obtained from the cube by replacing each vertex with a triangle and inserting into each face of the cube a suitably connected degree four vertex. (In both figures, these inserted vertices are indicated by smaller size circles.)

Theorem 16 *The only four-graphs G that can be distance-two five-colored are G_0, G_1. These two graphs can be so colored uniquely up to permutation of colors.*

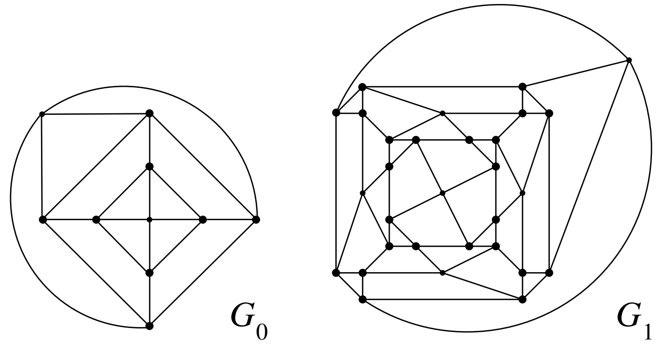

Figure 6: The only four-graphs that admit a distance-two five-coloring

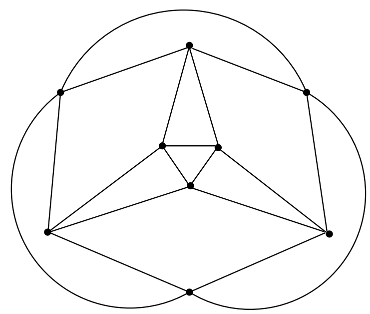

Figure 7: A four-graph requiring nine colors in any distance-two coloring

We close with a few remarks and open problems.

Wegner's conjecture [25] that any planar graph with maximum degree $d = 3$ can be distance-two seven-colored has been proved in [14, 23]. That bound is actually achieved by a type-two Barnette graph, namely the graph obtained from K_4 by subdividing three incident edges. Thus the bound of 7 cannot be lowered even for type-two Barnette graphs.

Wegner's conjecture for $d = 4$ claims that any planar graph with maximum degree four can be distance-two nine-colored. The four-graph in Figure 7 actually requires nine colors in any distance-two coloring. Thus if Wegner's conjecture for $d = 4$ is true, the bound of 9 cannot be lowered, even in the special case of four-graphs. It would be interesting to prove Wegner's conjecture for four-graphs, i.e., to prove that *any four-graph can be distance-two nine-colored.*

Finally, we've conjectured that any bipartite cubic planar graph can be distance-two six-colored (a special case of a conjecture of Hartke, Jahanbekam and Thomas [14]). The hexagonal prism (a cyclic prism with $k = 3$, which is a Goodey graph), actually requires six colors. Hence if our conjecture is true, the bound of 6 cannot be lowered even for Goodey graphs. It would be interesting to prove our conjecture for Goodey graphs, i.e., to prove that *any Goodey graph can be distance-two six-colored.*

References

[1] R.E.L. Aldred, S. Bau, D.A. Holton, and B.D. MacKay. Non-hamiltonian 3-connected cubic planar graphs. *SIAM J. Discrete Math.* 13:25–32, 2000.

[2] B. Alspach. The wonderful Walecki construction. *Bull. Inst. Combin. Appl.* 52:7–20, 2008.

[3] D. Barnette. On generating planar graphs. *Discrete Math.* 7:199–208, 1974.

[4] D. Barnette. Conjecture 5. *Recent Progress in Combinatorics* (Ed. W.T. Tutte), Academic Press, New York 343, 1969.

[5] O.V. Borodin and A.O. Ivanova. 2-distance 4-colorability of planar subcubic graphs with girth at least 22. *Discussiones Math. Graph Theory* 32:141–151, 2012.

[6] O.V. Borodin. Colorings of plane graphs: a survey, *Discrete Math.* 313:517–539, 2013.

[7] R.L. Brooks, On coloring the nodes of a network. *Proc. Cambridge Philos. Soc.* 37:194–197, 1941.

[8] R. Erman, F. Kardoš, J. Miskuf. Long cycles in fullerene graphs. *J. Math. Chemistry* 46:1103–1111, 2009.

[9] T. Feder, P. Hell, and C. Subi. Distance-two colorings of Barnette graphs *arXiv*:1807.01410 [cs.CG]

[10] T. Feder and C. Subi. On Barnette's conjecture. *Electronic Colloquium on Computational Complexity (ECCC)* TR06-015, 2006.

[11] M.R. Garey, D.S. Johnson, and R.E. Tarjan. The planar Hamiltonian circuit problem is NP-complete. *SIAM J. Comput.* 5:704–714, 1976.

[12] P.R. Goodey. Hamiltonian circuits in polytopes with even sided faces. *Israel J. Math.* 22:52–56, 1975.

[13] P.R. Goodey. A class of Hamiltonian polytopes. *J. Graph Theory* 1:181–185, 1977.

[14] S.G. Hartke, S. Jahanbekam, and B. Thomas. The chromatic number of the square of subcubic planar graphs. *arXiv*:1604.06504.

[15] F. Havet. Choosability of the square of planar subcubic graphs with large girth. *Discrete Math.* 309:3353–3563, 2009.

[16] P. Heggernes and J.A. Telle. Partitioning graphs into generalized dominating sets. *Nordic J. Computing* 5:128–143, 1998.

[17] D.A. Holton, B.D. McKay. The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices. *J. Combin. Theory B* 45:305–319, 1988.

[18] F. Kardoš. A computer-assisted proof of Barnette-Goodey conjecture: Not only fullerene graphs are Hamiltonian. *arXiv math*:1409.2440

[19] X. Lu. A note on 3-connected cubic planar graphs. *Discrete Math.* 310:2054–2058, 2010.

[20] J. Malkevitch. Polytopal graphs. *in Selected Topics in Graph Theory* (L. W. Beineke and R. J. Wilson eds.) 3:169–188, 1998.

[21] O. Ore. The four colour problem. *Academic Press*, 1967.

[22] P.G. Tait. Listing's topologie. *Philosophical Magazine*, 5th Series 17:30-46, 1884. Reprinted in Scientific Papers, Vol. II, pp. 85–98.

[23] C. Thomassen. The square of a planar cubic graph is 7-colorable. manuscript 2006.

[24] W.T. Tutte. On hamiltonian circuits. *J. Lond. Math. Soc.* 21:98–101, 1946.

[25] G. Wegner. Graphs with given diameter and a coloring problem. Technical Report, University of Dortmund, Germany, 1977.