Compatible 4-Holes in Point Sets

Ahmad Biniaz* Anil Maheshwari† Michiel Smid†

Abstract

Counting interior-disjoint empty convex polygons in a point set is a typical Erdős-Szekeres-type problem. We study this problem for convex 4-gons. Let P be a set of n points in the plane and in general position. A subset Q of P, with four points, is called a 4-*hole* in P if Q is in convex position and its convex hull does not contain any point of P in its interior. Two 4-holes in P are *compatible* if their interiors are disjoint. We show that P contains at least $\lfloor 5n/11\rfloor - 1$ pairwise compatible 4-holes. This improves the lower bound of $2\lfloor (n-2)/5\rfloor$ which is implied by a result of Sakai and Urrutia (2007).

1 Introduction

Throughout this paper, an *n-set* is a set of n points in the plane and in general position, i.e., no three points are collinear. Let P be an n-set. A *hole* in P is a subset Q of P, with at least three elements, such that Q is in convex position and no element of P lies in the interior of the convex hull of Q. A k-*hole* in P is a hole with k elements. By this definition, a 3-hole in P is an empty triangle with vertices in P, and a 4-hole in P is an empty convex quadrilateral with vertices in P.

The problem of finding and counting holes in point sets has a long history in discrete combinatorial geometry, and has been an active research area since Erdős and Szekeres [14, 15] asked about the existence of k-holes in a point set. In 1931, Esther Klein showed that any 5-set contains a convex quadrilateral [15]; it is easy to see that it also contains a 4-hole. In 1978, Harborth [17] proved that any 10-set contains a 5-hole. In 1983, Horton [18] exhibited arbitrarily large point sets with no 7-hole. The existence of a 6-hole in sufficiently large point sets has been proved by Nicolás [22] and Gerken [16]; a shorter proof of this result is given by Valtr [26].

Two holes Q_1 and Q_2 are *disjoint* if their convex hulls are disjoint, i.e., they do not share any vertex and do not overlap. We say sat Q_1 and Q_2 are *compatible* if the interiors of their convex hulls are disjoint, that is, they can share vertices but do not overlap. A set of holes is called disjoint (resp. compatible) if its elements are pairwise disjoint (resp. compatible). See Figure 1.

*University of Waterloo, Canada. Supported by NSERC Postdoctoral Fellowship. ahmad.biniaz@gmail.com

†Carleton University, Canada. Supported by NSERC. {anil, michiel}@scs.carleton.ca

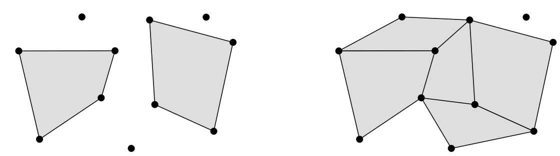

Figure 1: Two disjoint 4-holes (left), and five compatible 4-holes (right).

Since every three points form the vertices of a triangle, by repeatedly creating a triangle with the three leftmost points of an n-set we obtain exactly $\lfloor n/3\rfloor$ disjoint 3-holes. However, this does not generalize to 4-holes, because the four leftmost points may not be in convex position. Obviously, the number of disjoint 4-holes in an n-set is at most $\lfloor n/4\rfloor$. Hosono and Urabe [19] proved that the number of disjoint 4-holes is at least $\lfloor 5n/22\rfloor$; they improved this bound to $(3n-1)/13$ when $n = 13\cdot 2^k - 4$ for some $k \geqslant 0$. A variant of this problem where the 4-holes are vertex-disjoint, but can overlap, is considered in [29]. As for compatible holes, it is easy to verify that the number of compatible 3-holes in any n-set is at least $n-2$ and at most $2n-5$; these bounds are obtained by triangulating the point set: we get $n-2$ triangles, when the point set is in convex position, and $2n-5$ triangles, when the convex hull of the point set is a triangle. Sakai and Urrutia [24] proved among other results that any 7-set contains at least two compatible 4-holes. In this paper we study the problem of finding the maximum number of compatible 4-holes in an n-set.

Devillers *et al.* [13] considered some colored variants of this problem. They proved among other results that any bichromatic n-set has at least $\lceil n/4\rceil - 2$ compatible monochromatic 3-holes; they also provided a matching upper bound. As for 4-holes, they conjectured that a sufficiently large bichromatic point set has a monochromatic 4-hole. Observe that any point set that disproves this conjecture does not have a 7-hole (regardless of colors). For a bichromatic point set $R \cup B$ in the plane, Sakai and Urrutia [24] proved that if $|R| \geqslant 2|B|+5$, then there exists a monochromatic 4-hole. They also studied the problem of blocking 4-holes in a given point set R; the goal in this problem is to find a smallest point set B such that any 4-hole in R has a point of B in its interior. The problem of blocking 5-holes has been studied by Cano *et al.* [12].

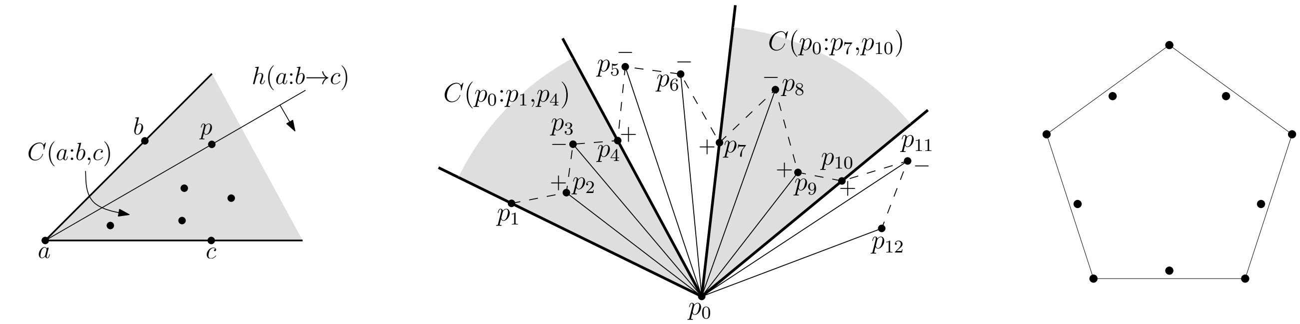

Figure 2: The point p is the attack point of $h(a{:}b\to c)$ (left). The radial ordering of points around p_0 (middle). A 10-set with at most three compatible 4-holes (right).

Aichholzer *et al.* [3] proved that every 11-set contains either a 6-hole, or a 5-hole and a disjoint 4-hole. Bhattacharya and Das [6] proved that every 12-set contains a 5-hole and a disjoint 4-hole. They also proved the existence of two disjoint 5-holes in every 19-set [7]. For more results on the number of k-holes in small point sets and other variations, see the paper by Aichholzer and Krasser [4], a summary of recent results by Aichholzer *et al.* [5], and B. Vogtenhuber's doctoral thesis [27]. Researchers also have studied the problem of counting the number of (not necessarily empty nor compatible) convex quadrilaterals in a point set; see, e.g., [2, 11, 21, 28].

A quadrangulation of a point set P in the plane is a planar subdivision whose vertices are the points of P, whose outer face is the convex hull of P, and every internal face is a quadrilateral; in fact the quadrilaterals are empty and pairwise compatible. Similar to triangulations, quadrangulations have applications in finite element mesh generation, Geographic Information Systems (GIS), scattered data interpolation, etc.; see [9, 10, 23, 25]. Most of these applications look for a quadrangulation that has the maximum number of convex quadrilaterals. To maximize the number of convex quadrilaterals, various heuristics and experimental results are presented in [9, 10]. This raises another motivation to study theoretical aspects of compatible empty convex quadrilaterals in a planar point set.

In this paper we study lower and upper bounds for the number of compatible 4-holes in point sets in the plane. A trivial upper bound is $\lfloor n/2\rfloor - 1$ which comes from n points in convex position. The $\lfloor 5n/22\rfloor$ lower bound on the number of disjoint 4-holes that is proved by Hosono and Urabe [19], simply carries over to the number of compatible 4-holes. Also, as we will see in Section 2, the lower bound of $2\lfloor (n-2)/5\rfloor$ on the number of compatible 4-holes is implied by a result of Sakai and Urrutia [24]. After some new results for small point sets, we prove non-trivial lower bounds on the number of compatible 4-holes in an n-set. We prove that every 9-set (resp. 11-set) contains three (resp. four) compatible 4-holes. Using these results, we prove that every n-set contains at least $\lfloor 5n/11\rfloor - 1$ compatible 4-holes. Our proof of this lower bound is constructive, and immediately yields an $O(n\log^2 n)$-time algorithm for finding this many compatible 4-holes.

Since the initial presentation of this work [8], the problem has attracted further attention. Most prominently, the lower bound on the number of compatible 4-holes has been improved to $\lceil \frac{n-3}{2}\rceil$ by Cravioto-Lagos, González-Martínez, Sakai, and Urrutia [1]. The same bound is claimed in an abstract by Lomeli-Haro, Sakai, and Urrutia in Kyoto International Conference on Computational Geometry and Graph Theory (CGGT2007) [20]. However, this result has not been published yet.

2 Preliminaries

First we introduce some notation from [19]. We define the *convex cone* $C(a{:}b,c)$ to be the region of the angular domain in the plane that is determined by three non-collinear points a, b, and c, where a is the apex, b and c are on the boundary of the domain, and $\angle bac$ is acute (less than $\pi/2$). We denote by $h(a{:}b\to c)$ the rotated half-line that is anchored at a and rotates, in $C(a{:}b,c)$, from the half-line ab to the half-line ac. If the interior of $C(a{:}b,c)$ contains some points of a given point set, then we call the first point that $h(a{:}b\to c)$ meets the *attack point* of $h(a{:}b\to c)$; see Figure 2-left.

Let P be an n-set. We denote by $CH(P)$ the convex hull of P. Let p_0 be the bottommost vertex on $CH(P)$. Without loss of generality assume that p_0 is the origin. Label the other points of P by $p_1,\dots,p_{n-1}$ in clockwise order around p_0, starting from the negative x-axis; see Figure 2-middle. We refer to the sequence $p_1,\dots,p_{n-1}$ as the *radial ordering* of the points of $P\setminus\{p_0\}$ around p_0. We denote by $l_{i,j}$ the straight line through two points with indexed labels p_i and p_j.

It is easy to verify that the number of 4-holes in an n-set in convex position is exactly $\lfloor n/2\rfloor - 1$. Figure 2-right, that is borrowed from [19], shows an example of a 10-set that contains at most three compatible 4-holes; by removing a vertex from the convex hull, we obtain a 9-set with the same number of 4-holes. This example can be extended to larger point sets, and thus, to the following proposition.

Proposition 1 *For every $n \geqslant 3$, there exists an n-set that has at most $\lceil n/2 \rceil - 2$ compatible 4-holes.*

Proposition 2 *The number of compatible 4-holes in an n-set is at most $n-3$.*

Proof. Let P be an n-set. Consider the maximum number of compatible 4-holes in P. The point set P together with an edge set, that is the union of the boundary edges of these 4-holes, introduces a planar graph G. Every 4-hole in P corresponds to a 4-face (a face with four edges) in G, and vice versa. Using Euler formula for planar graphs one can verify that the number of internal 4-faces of G is at most $n-3$. This implies that the number of 4-holes in P is also at most $n-3$. □

Theorem 1 (Klein [15]) *Every 5-set has a 4-hole.*

Theorem 2 (Sakai and Urrutia [24]) *Every 7-set has at least two compatible 4-holes.*

As a warm-up, we show that the number of 4-holes in an n-set P is at least $\lfloor (n-2)/3 \rfloor$. Let p_0 be the bottommost point of P and let $p_1, \dots, p_{n-1}$ be the radial ordering of the other points of P around p_0. Consider $\lfloor (n-2)/3 \rfloor$ cones $C(p_0{:}p_1, p_4), C(p_0{:}p_4, p_7), C(p_0{:}p_7, p_{10}), \dots$ where each cone has three points of P (including p_0) on its boundary and two other points in its interior. See Figure 2-middle. Each cone contains five points (including the three points on its boundary), and by Theorem 1 these five points introduce a 4-hole. Since the interiors of these cones are pairwise disjoint, we get $\lfloor (n-2)/3 \rfloor$ compatible 4-holes in P. We can improve this bound as follows. By defining the cones as $C(p_0{:}p_1, p_6), C(p_0{:}p_6, p_{11}), C(p_0{:}p_{11}, p_{16}), \dots$, we get $\lfloor (n-2)/5 \rfloor$ cones, each of which contains seven points. By Theorem 2, the seven points in each cone introduce two compatible 4-holes, and thus, we get $2 \cdot \lfloor (n-2)/5 \rfloor$ compatible 4-holes in total. Intuitively, any improvement on the lower bound for small point sets carries over to large point sets.

3 Compatible 4-holes in small point sets

In this section we provide lower bounds on the number of compatible 4-holes in 9-sets and 11-sets. In Subsection 3.2 we prove that every 9-set contains at least three compatible 4-holes and every 11-set contains at least four compatible 4-holes. Both of these lower bounds match the upper bounds given in Proposition 1. Due to the nature of this type of problems, our proofs involve case analysis. The case analysis gets more complicated as the number of points increases. To simplify the case analysis, we use two observations and a lemma that are given in Subsection 3.1. To simplify the case analysis further, we prove our claim for 9-sets first, then we use this result to obtain the proof for 11-sets. In this section we may use the term "quadrilateral" instead of 4-hole.

Let P be an n-set. Let p_0 be the bottommost point of P and let $p_1, \dots, p_{n-1}$ be the radial ordering of the other points of P around p_0. For each point p_i, with $i \in \{2, \dots, n-2\}$, we define the *signature* $s(p_i)$ of p_i to be "+" if, in the quadrilateral $p_0 p_{i-1} p_i p_{i+1}$, the inner angle at p_i is greater than π, and "−" otherwise; see Figure 2-middle. We refer to $s(p_2)s(p_3)\dots s(p_{n-2})$ as the *signature sequence* of P with respect to p_0. We refer to $s(p_{n-2})\dots s(p_3)s(p_2)$ as the *reverse* of $s(p_2)s(p_3)\dots s(p_{n-2})$. A *minus subsequence* is a subsequence of − signs in a signature sequence. A *plus subsequence* is defined analogously. For a given signature sequence δ, we denote by $m(\delta)$, the number of minus signs in δ.

3.1 Two observations and a lemma

In this section we introduce two observations and a lemma to simplify some case analysis in our proofs, which come later. Notice that if $s(p_i)\dots s(p_j)$ is a plus subsequence, then the points $p_{i-1}, p_i, \dots, p_j, p_{j+1}$ are in convex position and the interior of their convex hull does not contain any point of P. Also, if $s(p_i)\dots s(p_j)$ is a minus subsequence, then the points $p_0, p_{i-1}, p_i, \dots, p_j, p_{j+1}$ are in convex position and the interior of their convex hull does not contain any point of P. Therefore, the following two observations are valid.

Observation 1 *Let $s(p_i)\dots s(p_j)$ be a plus subsequence of length $2k$, with $k \geqslant 1$. Then, the convex hull of $p_{i-1}, \dots, p_{j+1}$ can be partitioned into k compatible 4-holes. See Figure 3(a).*

Observation 2 *Let $s(p_i)\dots s(p_j)$ be a minus subsequence of length $2k+1$, with $k \geqslant 0$. Then, the convex hull of $p_0, p_{i-1}, \dots, p_{j+1}$ can be partitioned into $k+1$ compatible 4-holes. See Figure 3(b).*

Lemma 3 *Let $s(p_{i+1})s(p_{i+2})\dots s(p_{i+2k})$ be a minus subsequence of length $2k$, with $k \geqslant 1$, and let p_i and p_{i+2k+1} have + signatures. Then, one can find $k+1$ compatible 4-holes in the convex hull of $p_0, p_{i-1}, \dots, p_{i+2k+2}$.*

Proof. Refer to Figures 3(c) and 3(d). For every $j \in \{0, \dots, k\}$ let l_{i+j} be the line through p_{i+j} and $p_{i+2k+1-j}$. These lines might intersect each other, but, for a better understanding of this proof, we visualized them as parallel lines in Figures 3(c) and 3(d).

Notice that the points $p_0, p_i, \dots, p_{i+2k+1}$ are in convex position. If p_{i-1} is below l_i, then we get a 4-hole $p_0 p_{i-1} p_i p_{i+2k+1}$ and k other compatible 4-holes in the convex hull of the points $p_i, \dots, p_{i+2k+1}$; see Figure 3(c). Assume p_{i-1} is above l_i. If p_{i-1} is below some lines in the sequence $l_{i+1}, \dots, l_{i+k}$, then let l_{i+j} be the first one

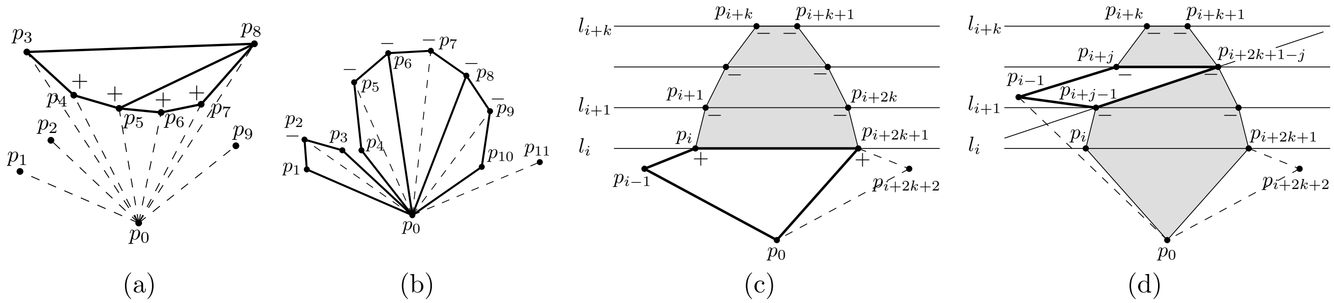

Figure 3: (a) Plus subsequence $s(p_4)s(p_5)s(p_6)s(p_7)$ of length four. (b) Minus subsequences $s(p_2)$ and $s(p_5)\dots s(p_9)$ of lengths one and five. The point p_{i-1} is (c) below l_i, and (d) below l_{i+j} and above all lines $l_i,\dots,l_{i+j-1}$.

in this sequence, that is, p_{i-1} is below l_{i+j} but above all lines $l_i,\dots,l_{i+j-1}$. Notice that in this case p_{i-1} is also above the line through p_{i+j-1} and $p_{i+2k+1-j}$. In this case we get a 4-hole $p_{i-1}p_{i+j}p_{i+2k+1-j}p_{i+j-1}$, and $k-j$ compatible 4-holes in the convex hull of $p_{i+j}\dots,p_{i+2k+1-j}$, and j compatible 4-holes in the convex hull of $p_0,p_i,\dots,p_{i+j-1},p_{i+2k+1-j},\dots,p_{i+2k+1}$; see Figure 3(d). Thus, we get $k+1$ compatible 4-holes in total. Similarly, if p_{i+2k+2} is below one of the lines l_{i+j} for $j\in\{0,\dots,k\}$ we get $k+1$ compatible 4-holes. Thus, assume that both p_{i-1} and p_{i+2k+2} are above all lines $l_i,\dots,l_{i+k}$. In this case we get a 4-hole $p_{i-1}p_{i+2k+2}p_{i+k+1}p_{i+k}$ and k other compatible 4-holes in the convex hull of $p_i,\dots,p_{i+2k+1}$. Thus, we get $k+1$ compatible 4-holes in total. □

Quadrilaterals obtained by Observations 1 and 2 do not overlap because quadrilaterals obtained by Observation 1 lie above the chain $p_1,\dots,p_{n-1}$ while quadrilaterals obtained by Observation 2 lie below this chain. However, the quadrilaterals obtained in the proof of Lemma 3 might lie above and/or below this chain. The quadrilaterals obtained by this lemma can overlap the quadrilaterals obtained by Observations 1 or 2 in the following two cases:

- Consider the first case in the proof of Lemma 3 when p_{i-1} lies below l_i and we create the quadrilateral $p_0p_{i-1}p_ip_{i+2k+1}$. If $s(p_{i-1})$ belongs to a minus subsequence, and we apply Observation 2 on it, then the quadrilateral $p_0p_{i-2}p_{i-1}p_i$ obtained by this observation overlaps the quadrilateral $p_0p_{i-1}p_ip_{i+2k+1}$. Similar issue may arise when $s(p_{i+2k+2})$ belongs to a minus subsequence.

- Consider the last two cases in the proof of Lemma 3 when p_{i-1} lies above l_i. If $s(p_{i-1})$ belongs to a plus subsequence, and we apply Observation 1 on it, then the quadrilaterals obtained by this observation might overlap either the quadrilateral $p_{i-1}p_{i+j}p_{i+2k+1-j}p_{i+j-1}$ or the quadrilateral $p_{i-1}p_{i+2k+2}p_{i+k+1}p_{i+k}$ that is obtained by Lemma 3. Similar issue may arise when $s(p_{i+2k+2})$ belongs to a plus subsequence.

As such, in our proofs, we keep track of the following two assertions when applying Lemma 3 on a subsequence $s(p_{i+1})s(p_{i+2})\dots s(p_{i+2k})$:

Assertion 1. Do not apply Observation 1 on a plus subsequence that contains $s(p_{i-1})$ or $s(p_{i+2k+2})$.

Assertion 2. Do not apply Observation 2 on a minus subsequence that contains $s(p_{i-1})$ or $s(p_{i+2k+2})$.

3.2 Compatible 4-holes in 9-sets and 11-sets

Here we count compatible 4-holes in 9-sets and 11-sets.

Theorem 4 *Every 9-set contains at least three compatible 4-holes.*

Theorem 5 *Every 11-set contains at least four compatible 4-holes.*

In the rest of this section we prove Theorem 4. The proof of Theorem 5, which is given in the full version of our paper [8], has the same structure as of Theorem 4, and make more use of Observations 1-2 and Lemma 3.

Let P be a 9-set. Let p_0 be the bottommost point of P and let $p_1,\dots,p_8$ be the radial ordering of the other points of P around p_0. Let δ be the signature sequence of P with respect to p_0, i.e., $\delta = s(p_2)\dots s(p_6)s(p_7)$. Depending on the value of $m(\delta)$, i.e., the number of minus signs in δ, we consider the following seven cases. Notice that any proof of this theorem for δ carries over to the reverse of δ as well. So, in the proof of this theorem, if we describe a solution for a signature sequence, we skip the description for its reverse.

- $m(\delta)=0$: In this case δ is a plus subsequence of length six. Our result follows by Observation 1.

- $m(\delta)=1$: In this case δ has five plus signs. By Observation 2, we get a quadrilateral by the point with $-$ signature. If four of the plus signs are consecutive, then by Observation 1 we get two more quadrilaterals. Otherwise, δ has two disjoint subsequences of plus signs, each of length at least two. Again, by Observation 1 we get a quadrilateral for each of these subsequences. Therefore, in total we get three 4-holes; these 4-holes are pairwise non-overlapping.

- $m(\delta) = 2$: Notice that δ has a plus subsequence of length at least two. If the two minus signs are non-consecutive, then we get two quadrilaterals by Observation 2 and one by Observation 1. Assume the two minus signs are consecutive. If the four plus signs are consecutive or partitioned into two subsequences of lengths two, then we get two quadrilaterals by Observation 1 and one by Observation 2. The remaining sequences are $+--+++$ and $+++--+$, where the second one is the reverse of the first one. By splitting the first sequence as $+--+|++$ we get two quadrilaterals for the subsequence $+--+$, by Lemma 3. If in this lemma we land up in the last case where both p_{i-1} and p_{i+2k+2} are above l_{i+k}, then we get a third compatible quadrilateral $p_1p_6p_7p_8$, otherwise we get $p_4p_6p_7p_8$. Notice that Assertion 1 holds here.

- $m(\delta) = 3$: If the three minus signs are pairwise non-consecutive, then we get three quadrilaterals by Observation 2. If the three minus signs are consecutive, then δ has a plus subsequence of length at least two. Thus, we get two quadrilaterals by Observation 2 and one by Observation 1. Assume the minus signs are partitioned into two disjoint subsequences of lengths one and two. Then, we get two quadrilaterals for the minus signs. If δ has a plus subsequence of length at least two, then we get a third quadrilateral by this subsequence. The remaining sequences are $+--+-+$ and its reverse.

  We show how to get three compatible 4-holes with the sequence $+--+-+$. See Figure 4. First we look at p_1. If p_1 is below $l_{2,5}$ then the three quadrilaterals $p_0p_1p_2p_5$, $p_2p_3p_4p_5$, and $p_0p_5p_6p_7$ are compatible. Assume p_1 is above $l_{2,5}$. If p_1 is below $l_{3,4}$ then the quadrilaterals $p_1p_3p_4p_2$, $p_0p_2p_4p_5$, and $p_0p_5p_6p_7$ are compatible. Assume p_1 is above $l_{3,4}$. Now, we look at p_6. If p_6 is above $l_{3,4}$ then $p_1p_6p_4p_3$, $p_2p_3p_4p_5$, and $p_0p_5p_6p_7$ are compatible. If p_6 is below $l_{3,4}$ and above $l_{2,5}$ as in Figure 4-left, then $p_0p_2p_3p_5$, $p_3p_4p_6p_5$, and $p_0p_5p_6p_7$ are compatible. Assume p_6 is below $l_{2,5}$ as in Figure 4-right; consequently p_7 is also below $l_{2,5}$ because p_6 has $-$ signature. Since p_5 has $+$ signature, p_4 is above $l_{5,6}$. Now, we look at p_8. If p_8 is above $l_{5,6}$, then $p_4p_8p_6p_5$, $p_2p_3p_4p_5$, and $p_0p_5p_6p_7$ are compatible. If p_8 is below $l_{5,6}$ and above $l_{5,7}$, then $p_5p_6p_8p_7$, $p_2p_3p_4p_5$, and $p_0p_2p_5p_7$ are compatible. Assume p_8 is below $l_{5,7}$ as in Figure 4-right. In this case $p_2p_3p_4p_5$, $p_2p_5p_6p_7$, and $p_0p_2p_7p_8$ are compatible.

- $m(\delta) = 4$: If the two plus signs in δ are consecutive, then we get one quadrilateral by Observation 1 and two by Observation 2. Assume the two plus signs are non-consecutive. If the minus signs are partitioned into three subsequences or two subsequences of lengths one and three, then we get three compatible 4-holes by Observation 2. The remaining sequences are $+----+$, $+--+--$ and its reverse. For the sequence $+----+$ we get three quadrilaterals by Lemma 3. The sequence $+--+--$ can be handled by splitting as $+--+|-|-$, where we get two quadrilaterals for the subsequence $+--+$, by Lemma 3, and one quadrilateral for the last minus sign, by Observation 1. Notice that Assertion 2 holds here as we apply Observation 1 on the last minus sign.

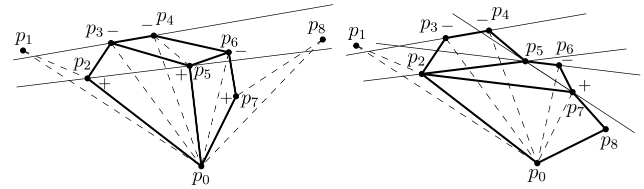

Figure 4: Signature sequence $+--+-+$. Left: p_1 is above $l_{3,4}$, and p_6 is below $l_{3,4}$ and above $l_{2,5}$. Right: p_1 is above $l_{3,4}$, p_6 is below $l_{2,5}$, and p_8 is below $l_{5,7}$.

- $m(\delta) = 5$: If the five minus signs are consecutive, then we get three compatible quadrilaterals by Observation 2. Otherwise, δ has two minus subsequences, one of which has size at least three. By Observation 2 we get three quadrilaterals with these two subsequences.

- $m(\delta) = 6$: The six minus signs are consecutive and our result follows by Observation 2.

4 Compatible 4-holes in n-sets

In this section we prove our main claim for large point sets, that is, every n-set contains at least $\lfloor 5n/11 \rfloor - 1$ compatible 4-holes. As in Section 2, by combining Theorems 4 and 5 with the idea of partitioning the points into some cones with respect to their radial ordering about a point p_0, we can improve the lower bound on the number of compatible 4-holes in an n-set to $3\cdot\lfloor(n-2)/7\rfloor$ and $4\cdot\lfloor(n-2)/9\rfloor$, respectively. In the rest of this section, we first prove a lemma, that can be used to improve these bounds further. We denote by ab the straight-line through two points a and b. We say that a 4-hole Q is *compatible with* a point set A if the interior of Q is disjoint from the interior of the convex hull of A.

Lemma 6 *For every $(r+s)$-set, with $r, s \geqslant 4$, we can divide the plane into two internally disjoint convex regions such that one region contains a set A of at least s points, the other region contains a set B of at least r points, and there exists a 4-hole that is compatible with A and B.*

Before proving this lemma, we note that a similar lemma has been proved by Hosono and Urabe (Lemma 3 in [19]) for disjoint 4-holes, where they obtain a set A' of $s-2$ points, a set B' of $r-2$ points, and a 4-hole Q that is disjoint from A' and B'. However, their lemma does not imply our Lemma 6, because it might not be possible to add two points of Q to A' to obtain a set A of s points such that Q is compatible with A.

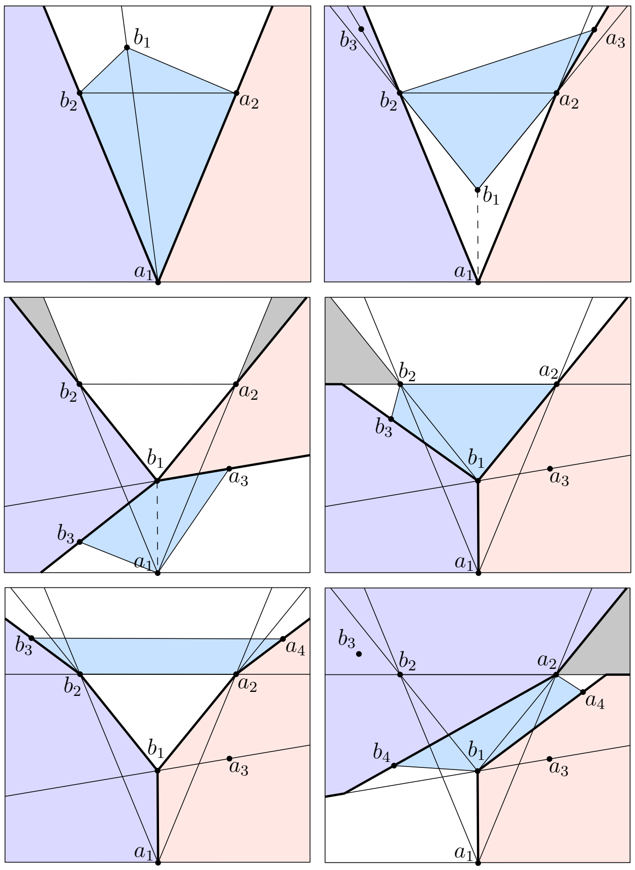

Figure 5: Illustration of Lemma 6. The convex regions with r and s points are shown in light purple and light orange, respectively. The compatible 4-holes with these regions are in blue color. The gray regions are empty.

In the following proof, if there exist two internally disjoint convex regions such that one of them contains a set A of s points, the other contains a set B of r points, and there exists a 4-hole that is compatible with A and B, then we say that A and B are *good*.

Proof of Lemma 6. Consider an $(r+s)$-set. In this proof a "point" refers to a point from this set. Also when we say a convex shape is "empty" we mean that its interior does not contain any point from this set.

Let a_1 be a point on the convex hull of this set, and without loss of generality assume that a_1 is the lowest point. Let a_2 be the point such that $s-2$ points are to the right side of the line a_1a_2. Let A be the set of points that are on or to the right side of a_1a_2, and let B be the set of other points. Notice that A contains s points and B contains r points. Let b_1 be the point of B such that the interior of $C(a_1{:}a_2, b_1)$ does not contain any point. Let b_2 be the point of B such that the interior of $C(a_1{:}a_2, b_2)$ contains only b_1. See Figure 5(top-left).

If b_1 is not in the interior of the triangle $\triangle a_1a_2b_2$, then $a_1a_2b_1b_2$ is a 4-hole that is compatible with A and $(B \setminus \{b_1\}) \cup \{a_1\}$. As shown in Figure 5(top-left), the interiors of the convex hulls of these two sets are disjoint, and thus, these two sets are good. Assume that b_1 is in the interior of $\triangle a_1a_2b_2$. We consider two cases depending on whether or not $C(b_1{:}b_2, a_2)$ is empty.

- $C(b_1{:}b_2, a_2)$ is not empty. If $C(b_1{:}b_2, a_2)$ contains a point of A, then let a_3 be such a point that is the neighbor of a_2 on $CH(A)$; see Figure 5(top-right). Then $b_1b_2a_3a_2$ is a 4-hole, and A and $(B \setminus \{b_1\}) \cup \{a_1\}$ are good. If $C(b_1{:}b_2, a_2)$ contains a point of B, then let b_3 be such a point that is the neighbor of b_2 on $CH(B)$. Then $b_1b_2b_3a_2$ is a 4-hole, and A and $(B \setminus \{b_1\}) \cup \{a_1\}$ are good.

- $C(b_1{:}b_2, a_2)$ is empty. Let a_3 be the attack point of $h(b_1{:}a_1{\to}a_2)$, i.e., the first point that $h(b_1{:}a_1{\to}a_2)$ meets. If the attack point of $h(b_1{:}a_1{\to}b_2)$ is below b_1a_3, then let b_3 be that point; Figure 5(middle-left). In this case $b_1a_3a_1b_3$ is a 4-hole, and $(A \setminus \{a_1\}) \cup \{b_1\}$ and B are good. Assume that the attack point of $h(b_1{:}a_1{\to}b_2)$ is above b_1a_3. We consider the following two cases depending on whether or not there is a point of B above the line a_2b_2.

  – No point of B is above a_2b_2. Let b_3 be the attack point of $h(b_1{:}b_2{\to}a_1)$ as in Figure 5(middle-right). Then $b_1b_3b_2a_2$ is a 4-hole, and $A \cup \{b_1\}$ and $(B \setminus \{b_2\}) \cup \{a_1\}$ are good.

  – Some point of B is above a_2b_2. Let b_3 be such a point that is the neighbor of b_2 on $CH(B)$. If some point of A is above a_2b_2, then let a_4 be such a point that is the neighbor of a_2 on $CH(A)$; see Figure 5(bottom-left). Then $a_2b_2b_3a_4$ is a 4-hole, and $A \cup \{b_1\}$ and $B \cup \{a_1\}$ are good. Assume that no point of A is above a_2b_2. Let a_4 be the attack point of $h(b_1{:}a_2{\to}a_3)$ and b_4 be the attack point of $h(a_2{:}b_1{\to}b_2)$ as in Figure 5(bottom-right). Notice that it might be the case that $b_4 = b_2$. In either case, $b_1b_4a_2a_4$ is a 4-hole, and $(A \setminus \{a_2\}) \cup \{b_1\}$ and $(B \setminus \{b_1\}) \cup \{a_2\}$ are good. □

Theorem 7 *Every n-set contains at least $\lfloor 5n/11 \rfloor - 1$ compatible 4-holes.*

Proof. Let P be an n-set. Our proof is by induction on the number of points in P. The base cases happen when $|P| \leqslant 14$. If $|P| \leqslant 13$, then our claim follows from one of Theorems 1, 2, 4, or 5. If $|P| = 14$, then by applying Lemma 6 on P with $r = s = 7$ we get a 4-hole together with two sets A and B each containing at least 7 points. By Theorem 2 we get two 4-holes in each of A and B. Thus, we get five compatible 4-holes in total. This finishes our proof for the base cases.

Assume that $|P| \geqslant 15$. By applying Lemma 6 on P with $r = n-11$ and $s = 11$ (notice that r is at least four as required by this lemma) we get a 4-hole together with two sets A and B such that the interiors of their convex hulls are disjoint, A contains at least 11 points, and B contains at least $n-11$ points. By Theorem 5 we get four compatible 4-holes in $CH(A)$. By induction, we get $\lfloor 5(n-11)/11 \rfloor - 1$ compatible 4-holes in $CH(B)$. Therefore, in total, we get

$$1 + 4 + \left(\left\lfloor \frac{5(n-11)}{11} \right\rfloor - 1 \right) = \left\lfloor \frac{5n}{11} \right\rfloor - 1$$

compatible 4-holes in P. □

An $O(n \log^2 n)$-time algorithm for computing this many 4-holes follows from the proofs, by using a dynamic convex hull data structure for computing the sets A and B in Lemma 6.

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