MV-ALGEBRAS

A short tutorial

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Foreword

An MV-algebra A is an abelian monoid $\langle A, 0, \oplus \rangle$ equipped with an operation $\neg$ such that $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$ and, finally,

(1) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$

An example of an MV-algebra is given by the real unit interval $[0, 1]$ equipped with the operations $\neg x = 1 - x$ and $x \oplus y = \min(1, x + y)$. Valid equations yield new valid equations by substituting equals for equals. Chang's completeness theorem states that in this way one obtains from the above equations every valid equation in the MV-algebra $[0, 1]$.

Boolean algebras stand to boolean logic as MV-algebras stand to Łukasiewicz infinite-valued logic. A variable in boolean propositional logic represents a $\{0, 1\}$-valued observable, and identically transforms the output of this observable into a truth-value. In infinite-valued logic a variable transforms the output of a real-valued bounded observable into a truth-value lying in the unit real interval $[0, 1]$. The completeness theorem for Łukasiewicz logic states that the rules of Modus Ponens and substitution are sufficient to obtain all tautologies (i.e., all equations of the form $\tau = \neg 0$ for τ an MV-term) in the infinite-valued calculus of Łukasiewicz starting from a few basic tautologies (originally due to Łukasiewicz) corresponding to the defining equations of MV-algebras.

The need for infinitely many truth-values naturally arises, e.g., in the Rényi-Ulam game of Twenty Questions where some of the answers may be erroneous. Here answers do not obey classical two-valued logic. As a matter of fact, two equal answers to the same repeated question usually give more information than a single answer. Using Chang completeness theorem, we shall see that the underlying logic of Rényi-Ulam games is Łukasiewicz infinite-valued propositional logic.

Using the completeness theorem we shall also give a short geometric proof of McNaughton theorem, representing free MV-algebras as piecewise linear functions with integer coefficients. The only prerequisite to understand the proofs of these main results is some acquaintance with the rudiments of elementary algebra, topology, and finite-dimensional vector spaces.

In the final sections we shall briefly survey other fundamental results and applications of MV-algebras, including (i) the extension to infinite-valued Łukasiewicz logic of De Finetti's no-Dutch-Book criterion for coherent probability assignments; (ii) the categorical equivalence Γ between MV-algebras and lattice-ordered abelian groups with order-unit; (iii) the relation between countable MV-algebras and approximately finite-dimensional C^*-algebras of operators in Hilbert space (iv) the class of σ-complete MV-algebras. We shall provide adequate references where the interested reader will find complete proofs of all these results.

Chapter 1

Games and MV-algebras

1.1 Antefact: Twenty Questions with Lies

The crucial problem of interpreting n truth-values when $n > 2$ vexed, among others, Łukasiewicz himself.

As shown in this tutorial, a simple interpretation can be given in the framework of *Rényi-Ulam games*, the variant of the game of Twenty Questions where $n-2$ lies, or errors, are allowed in the answers. The case $n = 2$ corresponds to the traditional game without lies. The game is described by Rényi's [45, page 47] as a problem of fault-tolerant adaptive search with errors, as follows:

> [...] I made up the following version, which I called "Bar-kochba with lies". Assume that the number of questions which can be asked to figure out the "something" being thought of is fixed and the one who answers is allowed to lie a certain number of times. The questioner, of course, doesn't know which answer is true and which is not. Moreover the one answering is not required to lie as many times as is allowed. For example, when only two things can be thought of and only one lie is allowed, then 3 questions are needed [...] If there are four things to choose from and one lie is allowed, then five questions are needed. If two or more lies are allowed, then the calculation of the minimum number of questions is quite complicated [...] It does seem to be a very profound problem [...]

The minimization problem for the number of questions is also posed by Ulam in his book "Adventures of a Mathematician" [51, p.281]:

> Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half million? then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than $\log_2(1000000)$. Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer?

The Rényi-Ulam game is an interesting variant of the familiar game of Twenty Questions. To fix ideas, let Carole (the responder) and Paul (the questioner) be the two players. From Paul's viewpoint it is immaterial whether wrong answers arise just because Carole is unable to answer correctly, or because she is (moderately) mendacious, or else Carole is always sincere and accurate, but distortion may corrupt up to e of the transmitted bits carrying her yes-no answers. Thus Ulam–Rényi games are part of Berlekamp's communication theory with feedback [1].

Both Rényi and Ulam were interested in the situation where up to e of the answers may be erroneous/mendacious/inaccurate. The problem is to minimize the number q of bits transmitted by Carole, while still guaranteeing that the original message can be recovered by Paul, even if up to e of the bits may have been distorted. In the particular case when all questions are asked at the outset, optimal strategies in Ulam–Rényi games are the same as optimal e-error-correcting codes.

In this tutorial we shall not be interested in the optimization problem posed by Rényi and Ulam. Interested readers may consult the surveys [7] and [42]. Rather, we intend to show that states of knowledge in every game form an MV-algebra, just as states of knowledge in the game without lies form a boolean algebra. Chang completeness theorem will be used to decide when two states of knowledge are the same in any possible game.

1.2 First properties of MV-algebras

Definition 1.2.1 An *MV-algebra* $\langle A, \oplus, \neg, 0\rangle$ is a set A equipped with a binary operation $\oplus$, a unary operation $\neg$ and a distinguished constant 0 satisfying the following equations:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z \tag{1.1}$$

$$x \oplus y = y \oplus x \tag{1.2}$$

$$x \oplus 0 = x \tag{1.3}$$

$$\neg\neg x = x \tag{1.4}$$

$$x \oplus \neg 0 = \neg 0 \tag{1.5}$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x. \tag{1.6}$$

A *subalgebra* of an MV-algebra A is a subset S of A containing the zero element of A, closed under the operations of A—and equipped with the restriction to S of these operations.

Examples. Equip the real unit interval $[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ with the operations $x \oplus y = \min\{1, x+y\}$ and $\neg x = 1 - x$. Then $[0,1] = \langle [0,1], \oplus, \neg, 0\rangle$

is an MV-algebra. For any MV-algebra A and set X, the set A^X of all functions $f\colon X \to A$ becomes an MV-algebra if the operations $\oplus$ and $\neg$ and the element 0 are defined pointwise. Given a boolean algebra $\langle A, \vee, \wedge, -, 0, 1\rangle$, then $\langle A, \vee, -, 0\rangle$ is an MV-algebra, where $\vee$, $-$ and 0 denote, respectively, the join, the complement and the smallest element in A. The rational numbers in $[0,1]$, and, for each integer $n \geq 2$, the n-element set

(1.7) $\quad \mathbf{L}_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$

yield examples of subalgebras of $[0,1]$.

Derived Operations. On any MV-algebra A we define $1 = \neg 0$. Further, the operations $\odot$ and $\ominus$ are defined by

(1.8) $\quad x \odot y = \neg(\neg x \oplus \neg y), \quad \text{and} \quad x \ominus y = x \odot \neg y.$

The following identities are immediate consequences of (1.4):

(1.9) $\quad \neg 1 = 0$

(1.10) $\quad x \oplus y = \neg(\neg x \odot \neg y).$

Setting $y = \neg 0$ in (1.6) we obtain:

(1.11) $\quad x \oplus \neg x = 1.$

Direct inspection shows that in the MV-algebra $[0,1]$, $x \odot y = \max\{0, x+y-1\}$ and $x \ominus y = \max\{0, x-y\}$.

Notation. For notational simplicity, the $\neg$ operation will be assumed to be more binding than any other operation, and the $\odot$ operation will be more binding than both $\oplus$ and $\ominus$.

Exercise 1.2.2 *For any elements x and y in an MV-algebra A the following conditions are equivalent:*

(i) $\neg x \oplus y = 1$,

(ii) $x \odot \neg y = 0$,

(iii) $y = x \oplus (y \ominus x)$,

(iv) There is an element $z \in A$ such that $x \oplus z = y$.

For any $x, y \in A$ let us agree to write $x \leq y$ if x and y satisfy the above equivalent conditions (i)-(iv). It follows that $\leq$ is a partial order, called the *natural order* of A. Indeed, reflexivity is equivalent to (1.11), antisymmetry follows from conditions (ii) and (iii), and transitivity follows from condition (iv). An MV-algebra whose natural order is total is called an *MV-chain*. Note that, by (iv), the natural order of the MV-chain $[0,1]$ coincides with the natural order of the real numbers.

Exercise 1.2.3 *Let A be an MV-algebra. For each $a \in A$, $\neg a$ is the unique solution x of the simultaneous equations:*

$$\begin{cases} a \oplus x &= 1 \\ a \odot x &= 0\,. \end{cases} \tag{1.12}$$

Exercise 1.2.4 *In every MV-algebra A the natural order $\leq$ has the following properties:*

(i) $x \leq y$ *if and only if* $\neg y \leq \neg x$*;*

(ii) If $x \leq y$ *then for each* $z \in A$, $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$*;*

(iii) If $x \odot y \leq z$ *then* $x \leq \neg y \oplus z$*.*

Proposition 1.2.5 *Let A be an MV-algebra. Then the natural order determines a lattice structure over A. The join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by*

$$x \vee y = (x \odot \neg y) \oplus y = (x \ominus y) \oplus y, \tag{1.13}$$

$$x \wedge y = \neg(\neg x \vee \neg y) = x \odot (\neg x \oplus y). \tag{1.14}$$

Proof. We first settle (1.13). From (1.6), (1.11) and 1.2.4(ii) we get $x \leq (x \ominus y) \oplus y$ and $y \leq (x \ominus y) \oplus y$. Suppose $x \leq z$ and $y \leq z$. By (i) and (iii) in 1.2.2, $\neg x \oplus z = 1$ and $z = (z \ominus y) \oplus y$. From (1.6) we now have

$$\neg((x \ominus y) \oplus y) \oplus z = (\neg(x \ominus y) \ominus y) \oplus y \oplus (z \ominus y)$$

$$= (y \ominus \neg(x \ominus y)) \oplus \neg(x \ominus y) \oplus (z \ominus y)$$

$$= (y \ominus \neg(x \ominus y)) \oplus \neg x \oplus y \oplus (z \ominus y) = (y \ominus \neg(x \ominus y)) \oplus \neg x \oplus z = 1.$$

Therefore, $(x \ominus y) \oplus y \leq z$, which settles (1.13). We also get (1.14) from (1.13) and 1.2.4(i). ∎

In the particular case when A is an MV-chain we have

Lemma 1.2.6 *In every MV-chain A we have:*

(i) If $x \oplus y < 1$ *then* $x \odot y = 0$*,*

(ii) If $x \oplus y = x \oplus z$ *and* $x \odot y = x \odot z$ *then* $y = z$*,*

(iii) If $x \oplus y = x \oplus z < 1$ *then* $y = z$*,*

(iv) If $x \odot y = x \odot z > 0$ *then* $y = z$*,*

(v) $x \oplus y = x$ *iff* $x = 1$ *or* $y = 0$*,*

(vi) $x \oplus y = x$ *iff* $\neg x \oplus \neg y = \neg y$,

(vii) *If* $x \oplus y = 1$ *and* $x \oplus z < 1$ *then* $(x \odot y) \oplus z = (x \oplus z) \odot y$.

Proof. (i), (ii) and (iii) are immediate. Condition (iv) follows from (iii) by 1.2.4(i). Condition (v) follows from (iii). From (v) one immediately obtains (vi). Finally, to prove (vii), since by assumption $\neg y \leq x$, we get $\neg y \oplus (x \odot y) \oplus z = (\neg y \vee x) \oplus z = x \oplus z < 1$ and $\neg y \oplus (y \odot (x \oplus z)) = \neg y \vee (x \oplus z) = x \oplus z$, whence (vii) follows from (iii). ■

Turning to the general case, an application of 1.2.4 immediately yields

Proposition 1.2.7 *The following equations hold in every MV-algebra:*

(i) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,

(ii) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

Proposition 1.2.8 *Every MV-algebra satisfies the equation*

(1.15) $(x \ominus y) \wedge (y \ominus x) = 0$.

Proof. Recalling (1.6) and the basic properties of $\oplus$ and $\odot$ we get

$$(x \ominus y) \wedge (y \ominus x) = (x \ominus y) \odot (\neg(x \ominus y) \oplus (y \ominus x)) =$$

$$x \odot \neg y \odot (y \oplus \neg x \oplus (y \ominus x)) = x \odot (\neg x \oplus (y \ominus x)) \odot (\neg(\neg x \oplus (y \ominus x)) \oplus \neg y) =$$

$$(y \ominus x) \odot (\neg(y \ominus x) \oplus x) \odot (\neg(\neg x \oplus (y \ominus x)) \oplus \neg y) =$$

$$y \odot \neg x \odot (\neg(y \ominus x) \oplus x) \odot ((x \odot \neg(y \ominus x)) \oplus \neg y) =$$

$$\neg x \odot (x \oplus \neg(y \ominus x)) \odot y \odot (\neg y \oplus (x \odot (\neg y \oplus x))) =$$

$$\neg x \odot (x \oplus \neg(y \ominus x)) \odot (x \odot (\neg y \oplus x)) \odot (\neg(x \odot (\neg y \oplus x)) \oplus y) = 0,$$

since $\neg x \odot x = 0$. ■

Let A be an MV-algebra. For each $x \in A$, we let $0x = 0$, and for each integer $n \geq 0$, $(n+1)x = nx \oplus x$.

Using 1.2.4 and Proposition 1.2.7 it is easy to prove

Exercise 1.2.9 *Let* x *and* y *be elements of an MV-algebra* A*. If* $x \wedge y = 0$ *then for each integer* $n \geq 0$, $nx \wedge ny = 0$.

Chapter 2

Homomorphisms and ideals

Given MV-algebras A and B we assume the reader knows the definition of homomorphism $h: A \to B$. The *kernel* of a homomorphism $h: A \to B$ is the set $Ker(h) = h^{-1}(0) = \{x \in A \mid h(x) = 0\}$. Also, the reader knows the definition of h being *injective* (equivalently, an *embedding*), and *surjective*. When h is an *isomorphism* of A onto B we write $A \cong B$. An *ideal* of an MV-algebra A is a subset I of A containing 0, closed under minorants and under the $\oplus$ operation. The intersection of any family of ideals of A is still an ideal of A. For every subset $W \subseteq A$, the intersection of all ideals $I \supseteq W$ is said to be the ideal *generated* by W. An ideal I of an MV-algebra A is *proper* if $I \neq A$. A proper ideal I is *prime* if for all $x, y \in A$, either $(x \ominus y) \in I$ or $(y \ominus x) \in I$. Also, I is *maximal* if it is proper and no proper ideal of A strictly contains I. We denote by $\mathcal{I}(A)$, $\mathcal{P}(A)$ and $\mathcal{M}(A)$ the sets of ideals, prime ideals and maximal ideals, of A, respectively.

For later use, we now collect some easily proved relations between ideals and kernels of homomorphisms:

Exercise 2.0.10 *Let A, B be MV-algebras, and $h: A \to B$ a homomorphism. Then the following properties hold:*

(i) For each $J \in \mathcal{I}(B)$, $h^{-1}(J) = \{x \in A \mid h(x) \in J\} \in \mathcal{I}(A)$. Thus in particular, $Ker(h) \in \mathcal{I}(A)$.

(ii) $h(x) \leq h(y)$ iff $x \ominus y \in Ker(h)$.

(iii) h is injective iff $Ker(h) = \{0\}$.

(iv) $Ker(h) \neq A$ iff in B the zero element does not coincide with 1 (for short, B is nontrivial*).*

(v) $Ker(h) \in \mathcal{P}(A)$ iff B is nontrivial and the image $h(A)$, as a subalgebra of B, is an MV-chain. $\square$

Definition 2.0.11 The *distance function* $d: A \times A \longrightarrow A$ is defined by

(2.1) $\quad d(x, y) = (x \ominus y) \oplus (y \ominus x).$

In the MV-algebra $[0, 1]$, $d(x, y) = |x - y|$. In every boolean algebra the distance function coincides with the symmetric difference operation. Recalling (1.2.2)-(1.2.4) we also have

Exercise 2.0.12 *In every MV-algebra A we have:*

(i) $d(x, y) = 0$ *iff* $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$,

(iv) $d(x, y) = d(\neg x, \neg y)$,

(v) $d(x \oplus s, y \oplus t) \leq d(x, y) \oplus d(s, t)$.

Hint for the proof of (iii) and (v). First note the identity $\neg(x \ominus z) \oplus (x \ominus y) \oplus (y \ominus z) = (\neg x \vee \neg y) \oplus (z \vee y) \geq \neg y \oplus y = 1$. Hence, $(x \ominus z) \leq (x \ominus y) \oplus (y \ominus z)$. In a similar way we obtain $(z \ominus x) \leq (y \ominus x) \oplus (z \ominus y)$, whence (iii) follows from the monotonicity of $\oplus$ One similarly proves (v) by observing that $\neg((x \oplus s) \ominus (y \oplus t)) \oplus (x \ominus y) \oplus (s \ominus t) = \neg(x \oplus s) \oplus (x \vee y) \oplus (t \vee s) \geq \neg(x \oplus s) \oplus x \oplus s = 1$.

As an immediate consequence we have

Proposition 2.0.13 *Let I be an ideal of an MV-algebra A. Then the binary relation $\equiv_I$ on A defined by $x \equiv_I y$ iff $d(x, y) \in I$ is a* congruence *relation. (Stated otherwise, $\equiv_I$ is an equivalence relation such that $x \equiv_I s$ and $y \equiv_I t$ imply $\neg x \equiv_I \neg s$ and $x \oplus y \equiv_I s \oplus t$.) Moreover, $I = \{x \in A \mid x \equiv_I 0\}$.*

Conversely, if $\equiv$ is a congruence on A, then $\{x \in A \mid x \equiv 0\}$ is an ideal, and $x \equiv y$ iff $d(x, y) \equiv 0$. Therefore, the correspondence $I \mapsto \equiv_I$ is a bijection from the set of ideals of A onto the set of congruences on A. $\Box$

Given $x \in A$, the equivalence class of x with respect to $\equiv_I$ will be denoted by x/I and the quotient set $A/\equiv_I$ by A/I. Since $\equiv_I$ is a congruence, defining on the set A/I the operations

(2.2) $\quad \neg(x/I) = \neg x/I \quad$ and $\quad x/I \oplus y/I = (x \oplus y)/I,$

the system $\langle A/I, \oplus, \neg, 0/I\rangle$ becomes an MV-algebra, called *the quotient algebra of A by the ideal I.* Moreover, the correspondence $x \mapsto x/I$ defines a homomorphism h_I from A onto the quotient algebra A/I, which is called *the natural homomorphism from A onto A/J.* Note that $Ker(h_I) = I$.

From the identity $Ker(h) = Ker(h_{Ker(h)})$ we immediately get

Proposition 2.0.14 *For any two MV-algebras A and B, and homomorphism h of A onto B, there is an isomorphism $f\colon A/Ker(h) \to B$ such that $f(x/Ker(h)) = h(x)$ for all $x \in A$.*

Proposition 2.0.15 *If A is an MV-chain, then all proper ideals of A are prime.*

The proof immediately follows from 2.0.10(v).

Proposition 2.0.16 *Let A be an MV-algebra and J be an ideal A. Then the map $I \mapsto h_J(I)$ determines an inclusion preserving one-one map from the set of ideals of A containing J onto the set of ideals of the quotient MV-algebra A/J. The inverse map also preserves inclusions, and is given by taking the inverse image $h_J^{-1}(K)$ of each ideal K of A/J.*

Proof. Let I be an ideal of A with $J \subseteq I$. Since h_J is onto A/J and $Ker(h_J) = J \subseteq I$, by 2.0.10 (ii) we have $h_J(I) \in \mathcal{I}(A/J)$ and $h_J^{-1}(h_J(I)) \subseteq I$. We also have $I = h_J^{-1}(h_J(I))$. On the other hand, by 2.0.10 (i), $h_J^{-1}(K) \in \mathcal{I}(A)$ for each $K \in \mathcal{I}(A/J)$. It is now sufficient to note that $J = h_J^{-1}(\{0\}) \subseteq h_J^{-1}(K)$ and $h_J(h_J^{-1}(K)) = K$. ■

Remark: One immediately sees that, when A is an MV-chain the set $\mathcal{I}(A)$ is totally ordered by inclusion.

The next proposition will play an important role in the proof of Chang Subdirect Representation Theorem.

Proposition 2.0.17 *Let A be an MV-algebra. If $a \in A$ and $a \neq 0$ then there is a prime ideal P of A such that $a \notin P$.*

Proof. By hypothesis, $a \notin \{0\}$. Then by Zorn Lemma there is an ideal I of A which is maximal with respect to the property that $a \notin I$. We claim that I is a prime ideal. Let x and y be elements of A, and suppose that both $x \ominus y \notin I$ and $y \ominus x \notin I$ (absurdum hypothesis). Then the ideal generated by I and $x \ominus y$ (i.e., the smallest ideal containing I and $x \ominus y$) must contain the element a; stated otherwise, $a \leq s \oplus p(x \ominus y)$ for some $s \in I$ and some integer $p \geq 1$. Similarly, there is an element $t \in I$ and an integer $q \geq 1$ such that $a \leq t \oplus q(y \ominus x)$. Let $u = s \oplus t$ and $n = \max(p, q)$. Then $u \in I$, $a \leq u \oplus n(x \ominus y)$ and $a \leq u \oplus n(y \ominus x)$. Hence by (1.14) and (1.15), together with Proposition 1.2.7(ii) and 1.2.9, we have $a \leq (u \oplus n(x \ominus y)) \wedge (u \oplus n(y \ominus x)) = u \oplus (n(x \ominus y) \wedge n(y \ominus x)) = u$, whence $a \in I$, a contradiction. ■

2.1 Subdirect products

The *direct product* $\prod_{i \in I} A_i$ of a family $\{A_i\}_{i \in I}$ of MV-algebras is the MV-algebra obtained by endowing the set-theoretical cartesian product of the A_i's with the pointwise MV-operations. The zero element of $\prod_{i \in I} A_i$ is the function $i \in I \mapsto 0_i \in A_i$. For each $i \in I$, the map $\pi_i\colon \prod_{i \in I} A_i \to A_i$ is defined by stipulating that, for all $f \in \prod_{i\in I} A_i$, $\pi_i(f) = f(i)$. Each π_i is a homomorphism onto A_i, called the *ith projection function.* In particular, for each MV-algebra A and each nonempty set X, the MV-algebra A^X is the direct product of the family $\{A_x\}_{x\in X}$, where $A_x = A$ for all $x \in X$. An MV-algebra A is a *subdirect product* of a family $\{A_i\}_{i\in I}$ of MV-algebras if there exists an injective homomorphism $h\colon A \to \prod_{i\in I} A_i$ such that for each $i \in I$, the composite map $\pi_i \circ h$ is a homomorphism *onto* A_i. If A is a subdirect product of the family $\{A_i\}_{i\in I}$, then A is isomorphic to the subalgebra $h(A)$ of $\prod_{i\in I} A_i$; moreover, the restriction of each projection to $h(A)$ must be a surjective mapping.

The following result is a particular case of a theorem of Universal Algebra, due to Birkhoff. We give the proof for the sake of completeness.

Theorem 2.1.1 *An MV-algebra A is a subdirect product of a family $\{A_i\}_{i\in I}$ of MV-algebras if and only if there is a family $\{J_i\}_{i\in I}$ of ideals of A such that*

(i) $A_i \cong A/J_i$ *for each* $i \in I$

and

(ii) $\bigcap_{i\in I} J_i = \{0\}$.

Proof. Let $\{J_i\}_{i\in I}$ be a family of ideals of A satisfying (i) and (ii). Let $h\colon A \to \prod_{i\in I} A_i$ be defined by $(h(x))_i = x/J_i$. It follows from (ii) that $Ker(h) = \{0\}$, whence, by 2.0.10(iii), h is injective. Since for each $i \in I$ and $\alpha \in A/J_i$ there is $a \in A$ such that $\alpha = a/J_i$, it follows that $\pi_i \circ h$ maps A onto A/J_i. Thus, A is a subdirect product of the family $\{A/J_i\}_{i\in I}$, as required.

Conversely, suppose that A is a subdirect product of MV-algebras $\{A_i\}_{i\in I}$. Let $h\colon A \to \prod_{i\in I} A_i$ be the corresponding 1-1 homomorphism, and, for each $i \in I$, let $J_i = Ker(\pi_i \circ h)$. By Proposition 2.0.14, $A_i \cong A/J_i$ for each $i \in I$. If $x \in \bigcap_{i\in I} J_i$, then $\pi_i(h(x)) = 0$ for each $i \in I$. Then $h(x) = 0$, and since h is injective, $x = 0$. In conclusion, $\bigcap_{i\in I} J_i = \{0\}$, and conditions (i) and (ii) hold true. ■

The following result, known as Chang Subdirect Representation Theorem, is a main ingredient in the proof of Chang Completeness Theorem 7.0.18.

Theorem 2.1.2 *Every MV-algebra A is a subdirect product of MV-chains.*

Proof. By Theorem 2.1.1 and 2.0.10(v), A is a subdirect product of MV-chains if there are prime ideals $\{P_i\}_{i\in I}$ of A with $\bigcap_{i\in I} P_i = \{0\}$. Now recall Proposition 2.0.17. ■

Chapter 3

MV-equations

We assume the reader has a definition of *term in the language of MV-algebras,* for short, MV-term. We write $\tau(x_1, \ldots, x_n)$ to mean that the variables occurring in the term τ are included in the set $\{x_1, \ldots, x_n\}$. We shall use the symbols $\odot$, $\ominus$, $\vee$, $\wedge$ and 1 to write MV-terms in abbreviated form, in the light of (1.8)-(1.14).

Let A be an MV-algebra, τ an MV-term in the variables $x_1, \ldots, x_t$, and assume $a_1, \ldots, a_t$ are elements of A. Substituting an element $a_i \in A$ for all occurrences of the variable x_i in τ, for $i = 1, \ldots, t$, and interpreting the symbols 0, $\oplus$ and $\neg$ as the corresponding operations in A, we obtain an element of A, denoted $\tau^A(a_1, \ldots, a_t)$.

An *MV-equation* (for short, an *equation*) in the variables $x_1, \ldots, x_t$ is a pair (τ, σ) of MV-terms in the variables $x_1, \ldots, x_t$. Following tradition, we shall write $\tau = \sigma$ instead of (τ, σ). An MV-algebra A *satisfies* the MV-equation $\tau = \sigma$, in symbols, $A \models \tau = \sigma$, if $\tau^A(a_1, \ldots, a_t) = \sigma^A(a_1, \ldots, a_t)$ for any $a_1, \ldots, a_t \in A$.

The following lemma is a particular case of a general well known fact, to the effect that equations are preserved under subalgebras, quotients and products.

Exercise 3.0.3 *Let A, B, A_i (for all $i \in I$) be MV-algebras. We then have*

(i) If $A \models \tau = \sigma$ then $S \models \tau = \sigma$ for each subalgebra S of A.

(ii) If $h\colon A \to B$ is a homomorphism, then for each MV-term τ in the variables $x_1, \ldots, x_s$ and each s-tuple $(a_1, \ldots, a_s)$ of elements of A we have $\tau^B(h(a_1), \ldots, h(a_s)) = h(\tau^A(a_1, \ldots, a_s))$. In particular, when h maps A onto B, from $A \models \tau = \sigma$ it follows that $B \models \tau = \sigma$.

(iii) If $A_i \models \tau = \sigma$ for each $i \in I$, then $\prod_{i \in I} A_i \models \tau = \sigma$.

Corollary 3.0.4 *Let A be a subdirect product of MV-algebras $\{A_i\}_{i \in I}$; let $\tau = \sigma$ be an MV-equation in the variables $x_1, \ldots, x_s$. Then $A \models \tau = \sigma$ if and only if $A_i \models \tau = \sigma$ for each $i \in I$.*

From Theorem 2.1.1 we obtain:

Corollary 3.0.5 *An MV-equation is satisfied by all MV-algebras if and only if it is satisfied by all MV-chains.*

Corollary 3.0.5 greatly simplifies the proof of many equations, e.g., those in Proposition 3.0.6 below. In the next chapter we will prove the stronger result stating that an equation holds in all MV-algebras if and only if it holds in the algebra $[0, 1]$.

Proposition 3.0.6 *The following equations hold in every MV-algebra A:*

(3.1) $x \oplus y \oplus (x \odot y) = x \oplus y,$

(3.2) $(x \ominus y) \oplus ((x \oplus \neg y) \odot y) = x,$

(3.3) $(x \odot y) \oplus ((x \oplus y) \odot z) = (x \odot z) \oplus ((x \oplus z) \odot y).$

Proof. By Corollary 3.0.5, A may be assumed to be a chain. If $x \oplus y = 1$, then (3.1) follows by (1.5). If $x \oplus y < 1$, then (3.1) follows from Lemma 1.2.6(i). For a proof of (3.2), if $x \leq y$ then $x \ominus y = 0$ and $x = x \wedge y = (x \oplus \neg y) \odot y$; if $y < x$ then $(x \ominus y) \oplus (x \wedge y) = (x \ominus y) \oplus y = x \vee y = x$.

In order to prove (3.3) we first settle the following equation:

(3.4) $(x \odot y) \oplus ((x \oplus y) \odot z) = (x \oplus y) \odot ((x \odot y) \oplus z).$

This equation can be proved arguing by cases: if $x \oplus y = 1$ then both members coincide with $(x \odot y) \oplus z$. If $x \oplus y < 1$ then by Lemma 1.2.6(i) both members coincide with $(x \oplus y) \odot z$. This settles (3.4).

From (1.4) and (1.10) we also get:

(3.5) $\neg((x \odot y) \oplus ((x \oplus y) \odot z)) = (\neg x \odot \neg y) \oplus ((\neg x \oplus \neg y) \odot \neg z).$

We are now in a position to prove (3.3).

Case 1: $x \oplus y \oplus z < 1$.

Then since A is a chain, by Lemma 1.2.6(i), both members of (3.3) are equal to 0.

Case 2: $\neg x \oplus \neg y \oplus \neg z < 1$.

Same as Case 1, recalling (3.5).

There remains to consider

Case 3: $x \oplus y \oplus z = 1$ and $\neg x \oplus \neg y \oplus \neg z = 1$.

Subcase 3.1: $x \oplus y = 1$ and $x \oplus z < 1$, or $x \oplus y < 1$ and $x \oplus z = 1$. It is enough to consider the case $x \oplus y = 1$ and $x \oplus z < 1$. Then $x \odot z = 0$, and (3.3) becomes $(x \odot y) \oplus z = (x \oplus z) \odot y$, which follows from Lemma 1.2.6(vii).

Subcase 3.2: $x \oplus y = x \oplus z = 1$. Then (3.3) becomes

(3.6) $(x \odot y) \oplus z = (x \odot z) \oplus y.$

This equation certainly holds when $x \odot y = 0$ or $x \odot z = 0$. Indeed, suppose $x \odot y = 0$. Since $x \oplus y = 1$, it follows from 1.2.3 that $x = \neg y$, whence from

$y = \neg x \leq z$ we obtain $(x \odot y) \oplus z = z = y \vee z = (\neg y \odot z) \oplus y = (x \odot z) \oplus y$. Similarly, (3.6) holds when $x \odot z = 0$.

We next claim that if one of the members of (3.6) is equal to 1 then so is the other. Assume, for instance, $(x \odot y) \oplus z = 1$. Since $\neg x \oplus \neg y \oplus \neg z = 1$ is equivalent to $x \odot y \odot z = 0$, it follows from 1.2.3 that $z = \neg(x \odot y) = \neg x \oplus \neg y$. Hence, by Proposition 1.2.7, $(x \odot z) \oplus y = (x \odot (\neg x \oplus \neg y)) \oplus y = (x \wedge \neg y) \oplus y = (x \oplus y) \wedge (\neg y \oplus y) = 1$.

To complete our analysis of Subcase 3.2 we may restrict to the case when $(x \odot y) \oplus z < 1$, $(x \odot z) \oplus y < 1$, $x \odot y > 0$, $x \odot z > 0$. Then by Lemma 1.2.6(vii) we obtain $x \odot (z \oplus (x \odot y)) = (x \odot z) \oplus (x \odot y) > 0$, and $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$. This settles (3.6) in the light of Lemma 1.2.6(iv).

Subcase 3.3: $x \oplus y < 1$ and $x \oplus z < 1$.

Then by Lemma 1.2.6(i), $x \odot y = 0$ and $x \odot z = 0$, i.e., $\neg x \oplus \neg y = 1$ and $\neg x \oplus \neg z = 1$. Using (3.5) and arguing as in Subcase 3.2 (with $\neg x$, $\neg y$, $\neg z$ instead of x, y, z) we conclude that (3.3) also holds in this case. ∎

Chapter 4

The role of abelian ℓ-groups

In this chapter we shall give a self-contained proof of Chang completeness theorem [5, 6] stating that if an equation holds in the unit real interval $[0, 1]$, then the equation holds in every MV-algebra. [1] This will give us the opportunity of introducing the $\mathbf{\Gamma}$ functor, a major tool in the study of MV-algebras.

4.1 The $\mathbf{\Gamma}$ functor

A *partially ordered abelian group* is an abelian group $\langle G, +, -, 0\rangle$ endowed with a partial order relation $\leq$ having the following *translation invariance* property

(4.1) if $x \leq y$ then $t + x \leq t + y$,

for all $x, y, t \in G$. The *positive cone* G^+ of G is defined by $G^+ = \{x \in G \mid 0 \leq x\}$. If the order relation is total, (i.e., when $G = G^+ \cup -G^+$), then G is a *totally ordered abelian group.* When the order structure of G determines a lattice structure, G is called a *lattice-ordered abelian group*, abbreviated *ℓ-group*. In any ℓ-group we have

(4.2) $t + (x \vee y) = (t + x) \vee (t + y)$ and $t + (x \wedge y) = (t + x) \wedge (t + y)$.

For each element x of an ℓ-group G, the *positive part* x^+, the *negative part* x^-, and the *absolute value* $|x|$ of x are defined as follows:

(4.3) $x^+ = 0 \vee x, \quad x^- = 0 \vee -x, \quad |x| = x^+ + x^- = x \vee -x.$

An *order-unit* u of G is an element $0 \leq u \in G$ such that for each $x \in G$, there is an integer $n \geq 0$ such that $|x| \leq nu$.

Definition 4.1.1 Let G be an ℓ-group. For any element $u \in G$, $u > 0$ we let

$$[0, u] = \{x \in G \mid 0 \leq x \leq u\},$$

[1] The completeness of the infinite-valued Łukasiewicz calculus was first proved by [48] using heavy syntactical machinery.

and for each $x, y \in [0, u]$,

$$x \oplus y \;=\; u \wedge (x + y), \quad \text{and} \quad \neg x \;=\; u - x.$$

The structure $\langle [0, u], \oplus, \neg, 0\rangle$ is denoted $\mathbf{\Gamma}(G, u)$.

Proposition 4.1.2 $\mathbf{\Gamma}(G, u)$ *is an MV-algebra.*

Proof. We shall only prove that $\mathbf{\Gamma}(G, u)$ satisfies (1.6). For all $x, y \in [0, u]$ we have $\neg(\neg x \oplus y) \oplus y = y \oplus \neg(y \oplus \neg x) = u \wedge (y + (u - (u \wedge (y + u - x)))) = u \wedge (y + u + (-u \vee (-y - u + x))) = u \wedge ((y + u - u) \vee (y + u - y - u + x)) = u \wedge (y \vee x) = y \vee x = x \vee y$. This shows that x and y are interchangeable. ■

Notation. Following common usage, we let $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ denote the additive groups of reals, rationals, integers, with the natural order. In the particular case when $G = \mathbb{R}$, $\mathbf{\Gamma}(\mathbb{R}, 1)$ coincides with the MV-algebra [0,1]. We also have $\mathbb{Q} \cap [0, 1] = \mathbf{\Gamma}(\mathbb{Q}, 1)$. Recalling Definition (1.7), for each integer $n \geq 2$, we have $\mathbf{L}_n = \mathbf{\Gamma}(\mathbb{Z}\frac{1}{n-1}, 1)$, where $\mathbb{Z}\frac{1}{n-1} = \{\frac{z}{n-1} \mid z \in \mathbb{Z}\}$.

Definition 4.1.3 Let G and H be ℓ-groups. A function $h\colon G \to H$ is said to be an *ℓ-group homomorphism* if for each $x, y \in G$, $h(x - y) = h(x) - h(y)$, $h(x \vee y) = h(x) \vee h(y)$ and $h(x \wedge y) = h(x) \wedge h(y)$. Suppose that $0 < u \in G$ and $0 < v \in H$, and let $h\colon G \to H$ be an ℓ-group homomorphism such that $h(u) = v$. Then h is said to be a *unital* ℓ-homomorphism.

Letting $\mathbf{\Gamma}(h)$ be the restriction of h to the unit interval $[0, u]$, then $\mathbf{\Gamma}(h)$ is a homomorphism from $\mathbf{\Gamma}(G, u)$ into $\mathbf{\Gamma}(H, v)$.

As an immediate consequence of the definition we have

Proposition 4.1.4 *Let $\mathcal{A}$ denote the category whose objects are pairs $\langle G, u\rangle$ with G an ℓ-group and u a distinguished order-unit of G, and whose morphisms are unital ℓ-homomorphisms. Then $\mathbf{\Gamma}$ is a functor from $\mathcal{A}$ into the category $\mathcal{MV}$ of MV-algebras.* □

This result will be strengthened below in Theorem 6.0.15 and, finally, in 11.1.1.

4.2 Good sequences

Let A be an MV-algebra. Then a sequence $\mathbf{a} = (a_1, a_2, \ldots)$ of elements of A is called *good* if for each $i = 1, 2 \ldots,$ $a_i \oplus a_{i+1} = a_i$, and there is an integer n such that $a_r = 0$ for all $r > n$. Instead of writing $\mathbf{a} = (a_1, \ldots, a_n, 0, 0, ...)$ we shall often abbreviate $\mathbf{a} = (a_1, \ldots, a_n)$. Thus we have identical good sequences

(4.4) $(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0^m)$,

where 0^m denotes an m-tuple of zeros. For each $a \in A$, the good sequence $(a, 0 \ldots, 0, \ldots)$ will be denoted by (a).

For totally ordered MV-algebras, Lemma 1.2.6(v) yields the following characterization of good sequences:

Proposition 4.2.1 *Each good sequence of an MV-chain A has the form*

(4.5) $(1^p, a)$ *for some integer* $p \geq 0$ *and* $a \in A$.

Lemma 4.2.2 *Suppose that $A \subseteq \prod_i A_i$ is the subdirect product of a family $\{A_i\}_{i \in I}$ of MV-algebras. A sequence $\mathbf{a} = (a_1, \ldots, a_n \ldots)$ of elements of A is a good sequence if and only if for each $i \in I$ the sequence $(\pi_i(a_1), \ldots, \pi_i(a_n), \ldots)$ is a good sequence in A_i, and there is an integer $n_0 \geq 0$ such that whenever $n > n_0$ then for all $i \in I$, $\pi_i(a_n) = 0$.*

Proof. Indeed, $a_n \oplus a_{n+1} = a_n$ is the same as $\pi_i(a_n \oplus a_{n+1}) = \pi_i(a_n)$ for each $i \in I$. ■

Lemma 4.2.3 *Let A be an MV-algebra. If $\mathbf{a} = (a_1, \ldots, a_n, \ldots)$ and $\mathbf{b} = (b_1, \ldots, b_n, \ldots)$ are good sequences of A, then so is $\mathbf{c} = (c_1, \ldots, c_n, \ldots)$ given by $c_n = a_n \vee b_n$ for each n.*

Proof. There is an integer n_0 such that $c_n = 0$ for all $n > n_0$. By Theorem 2.1.2, A is a subdirect product of a family $\{C_i\}_{i \in I}$ of MV-chains. For each $i \in I$ the sequences $\mathbf{a}_i = (\pi_i(a_1), \ldots, \pi_i(a_n), \ldots)$ and $\mathbf{b}_i = (\pi_i(b_1), \ldots, \pi_i(b_n), \ldots)$ are good sequences of C_i. Hence, by Proposition 4.2.1, $\mathbf{a}_i = (1^p, \alpha_i)$ and $\mathbf{b}_i = (1^q, \beta_i)$, where α_i and β_i are in C_i. Therefore, $\pi_i(c_n) = 1$ if $n \leq \max\{p, q\}$ and $\pi_i(c_n) = 0$ if $n > \max\{p, q\} + 1$. For $n = \max\{p, q\} + 1$, we have $\pi_i(c_n) = \alpha_i$ if $p > q$, $\pi_i(c_n) = \beta_i$ if $p < q$ and $\pi_i(c_n) = \max\{\alpha_i, \beta_i\}$ when $p = q$. Consequently, letting $\mathbf{c}_i = (\pi_i(c_1), \ldots, \pi_i(c_n), \ldots)$ it follows that $\mathbf{c}_i$ is a good sequence for each $i \in I$, whence we conclude that $\mathbf{c}$ is a good sequence of A. ■

Example. For every real number $\alpha \geq 0$ let $\lfloor \alpha \rfloor$ denote the greatest integer $\leq \alpha$, and $\langle \alpha \rangle = \alpha - \lfloor \alpha \rfloor$. Then α can be written as

$$\alpha = 1 + \ldots + 1 + \langle \alpha \rangle + 0 + 0 + \ldots$$

with $\lfloor \alpha \rfloor$ many consecutive 1's. Considered as elements of the MV-algebra $[0, 1]$, the above summands $\alpha_1, \alpha_2, \ldots$ of α satisfy $\alpha_i \oplus \alpha_{i+1} = \alpha_i$ for every integer $i \geq 1$. For $0 \leq \beta \in \mathbb{R}$, let similarly

$$\beta = \beta_1 + \ldots + \beta_{m-1} + \langle \beta \rangle + 0 + \ldots,$$

where $\beta_1 = \ldots = \beta_{m-1} = 1 = \alpha_1 = \ldots = \alpha_{n-1}$, $0 = \alpha_{n+1} = \alpha_{n+2} = \ldots$, and $0 = \beta_{m+1} = \beta_{m+2} = \ldots$. Let $\gamma = \alpha + \beta$. Then $\gamma = \gamma_1 + \gamma_2 + \ldots$, where $\gamma_1 = \ldots = \gamma_{n+m-2} = 1$, $\gamma_{n+m-1} = \langle \alpha \rangle \oplus \langle \beta \rangle$, $\gamma_{n+m} = \langle \alpha \rangle \odot \langle \beta \rangle$, and $0 = \gamma_{n+m+1} = \gamma_{n+m+2} = \ldots$. In a more compact notation, for each $i = 1, 2, \ldots$, the summand γ_i is given by

(4.6) $$\gamma_i = \alpha_i \oplus (\alpha_{i-1} \odot \beta_1) \oplus (\alpha_{i-2} \odot \beta_2) \oplus \ldots \oplus (\alpha_2 \odot \beta_{i-2}) \oplus (\alpha_1 \odot \beta_{i-1}) \oplus \beta_i.$$

Equations (4.6) and (4.4) motivate the following

Definition 4.2.4 For any two good sequences

$$\mathbf{a} = (a_1, \ldots, a_n) \text{ and } \mathbf{b} = (b_1, \ldots, b_m),$$

their *sum* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is defined by $\mathbf{c} = (c_1, c_2, \ldots)$, where for all $i = 1, 2, \ldots$

$$(4.7) \quad c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus \ldots \oplus (a_1 \odot b_{i-1}) \oplus b_i.$$

Since $a_p = b_q = 0$ whenever $p > n$ and $q > m$, then c_j identically vanishes for each $j > m + n$. The notation $\mathbf{c} = (c_1, \ldots, c_{n+m}) = (a_1, \ldots, a_n) + (b_1, \ldots, b_m)$ is self-explanatory.

The following immediate consequence of (4.7) will be frequently used to compute the sum of two good sequences in an MV-chain:

$$(4.8) \quad (1^p, a) + (1^q, b) = (1^{p+q},\ a \oplus b,\ a \odot b).$$

Chapter 5

Chang monoid M_A

Since by equation (3.1), $(a \oplus b, a \odot b)$ is a good sequence, applying Theorem 2.1.2 and Lemma 4.2.2 together with (4.8), we immediately get that *the sum of two good sequences is a good sequence.* We denote by M_A the set of good sequences of A equipped with addition.

Proposition 5.0.5 *Let A be an MV-algebra A. Then M_A is an abelian monoid with the following additional properties:*

(i) (cancellation) *For any good sequences* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$*, if* $\mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{c}$ *then* $\mathbf{b} = \mathbf{c}$.

(ii) (zero-law) *If* $\mathbf{a} + \mathbf{b} = (0)$ *then* $\mathbf{a} = \mathbf{b} = (0)$.

Proof. By (4.7), $\mathbf{a} + (0) = \mathbf{a}$, addition is commutative, and the zero-law holds. To prove associativity, by Theorem 2.1.2 we can safely assume A to be totally ordered. By Proposition 4.2.1 and equation (3.3) in Proposition 3.0.6, letting $\mathbf{a} = (1^p, a)$, $\mathbf{b} = (1^q, b)$, and $\mathbf{c} = (1^r, c)$, we have the identities

$$(\mathbf{b} + \mathbf{a}) + \mathbf{c} = (1^{p+q+r}, a \oplus b \oplus c, (a \odot b) \oplus ((a \oplus b) \odot c), a \odot b \odot c)$$

$$= (1^{p+q+r}, a \oplus b \oplus c, (a \odot c) \oplus ((a \oplus c) \odot b), a \odot b \odot c) = \mathbf{b} + (\mathbf{a} + \mathbf{c}).$$

Similarly, to prove cancellation, avoiding trivialities, assume that a, b and c are different from 1. If $q = r$, then by Lemma 1.2.6(ii), $b = c$, and we are done. If $q < r - 1$ then from the identity $(1^{p+q}, a \oplus b, a \odot b) = (1^{p+r}, a \oplus c, a \odot c)$ we get $a \odot b = 1$, i.e., $a = b = 1$, which is a contradiction. If $q = r - 1$ then $a \odot b = a$ and $a \oplus b = 1$, which is impossible because these two equalities imply that $b = 1$. The cases corresponding to $r < q$ are similarly shown to lead to contradiction. ■

Proposition 5.0.6 *Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ be good sequences in an MV-algebra A. Recalling* (4.4) *assume, without loss of generality, $m = n$. Then the following are equivalent:*

(i) There is a good sequence $\mathbf{c}$ *such that* $\mathbf{b}+\mathbf{c}=\mathbf{a}$*;*

(ii) $b_i \le a_i$ *for all* $i=1,\dots,n$.

Proof. (i) $\Rightarrow$ (ii) is immediate from (4.7). (ii) $\Rightarrow$ (i). In the light of Theorem 2.1.2, we can safely assume A to be totally ordered. Using now (ii) and (vi) in Lemma 1.2.6, we see that $(\neg b_n,\dots,\neg b_1)$ is a good sequence. Let us denote by $\mathbf{c}=\mathbf{a}-\mathbf{b}$ the good sequence obtained by dropping the first n terms in $(a_1,\dots,a_n)+(\neg b_n,\dots,\neg b_1)$. We shall prove that $\mathbf{c}+\mathbf{b}=\mathbf{a}$. By (4.5), $\mathbf{a}=(1^p,a)$ and $\mathbf{b}=(1^q,b)$. To avoid trivialities assume both a and b to be different from 0 and from 1. Then $q\le p$. Upon rewriting $\mathbf{b}=(1^q,\ b,\ 0^{p-q})$, from $n=p+1$ we get $(\neg b_n,\dots,\neg b_1)=(1^{p-q},\ \neg b,\ 0^q)$, and hence $\mathbf{c}$ is obtained by dropping the first $p+1$ terms from $(1^{2p-q},\ a\oplus\neg b,\ a\ominus b)$.

In case $b\le a$, we have $a\oplus\neg b=1$, $\mathbf{c}=(1^{p-q},a\ominus b)$ and $\mathbf{c}+\mathbf{b}=(1^p,\ (a\ominus b)\oplus b,\ (a\ominus b)\odot b)=(1^p,\ b\vee a,\ 0)=(1^p,\ a)=\mathbf{a}$.

In case $b>a$, we have $p>q$, $a\ominus b=0$, $\mathbf{c}=(1^{p-q-1},\ a\oplus\neg b)$ and $\mathbf{c}+\mathbf{b}=(1^{p-1},\ a\oplus\neg b\oplus b,(a\oplus\neg b)\odot b)=(1^p,a\wedge b)=(1^p,a)=\mathbf{a}$. ∎

Definition 5.0.7 Given any two good sequences $\mathbf{a}$ and $\mathbf{b}$ of A we write

(5.1) $\mathbf{b}\le\mathbf{a}$ iff $\mathbf{b}$ and $\mathbf{a}$ satisfy the equivalent conditions of 5.0.6.

Proposition 5.0.8 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be good sequences.*

(i) If $\mathbf{b}\le\mathbf{a}$ *then there is a unique good sequence* $\mathbf{c}$ *such that* $\mathbf{b}+\mathbf{c}=\mathbf{a}$*. This* $\mathbf{c}$*, denoted* $\mathbf{a}-\mathbf{b}$*, is given by*

(5.2) $\mathbf{c}=(a_1,\dots,a_n)+(\neg b_n,\dots,\neg b_1)$ *omitting the first* n *terms.*

(ii) In particular, for each $a\in A$ *we have*

(5.3) $(\neg a)=(1)-(a)$.

(iii) The order is translation invariant*, in the sense that* $\mathbf{b}\le\mathbf{a}$ *implies* $\mathbf{b}+\mathbf{d}\le\mathbf{a}+\mathbf{d}$ *for every good sequence* $\mathbf{d}$.

Proof. By an easy adaptation of the proof of Proposition 5.0.6, together with Proposition 5.0.5 (i). ∎

Proposition 5.0.9 *Let* $\mathbf{a}=(a_1,\dots,a_n,\dots)$ *and* $\mathbf{b}=(b_1,\dots,b_n,\dots)$ *be good sequences of an MV-algebra A.*

(i) The sequence $\mathbf{a}\vee\mathbf{b}=(a_1\vee b_1,\dots,a_n\vee b_n,\dots)$ *is good, and is in fact the supremum of* $\mathbf{a}$ *and* $\mathbf{b}$ *with respect to the order defined by (5.1).*

(ii) Analogously, the good sequence $\mathbf{a}\wedge\mathbf{b}=(a_1\wedge b_1,\dots,a_n\wedge b_n,\dots)$ *is the infimum of* $\mathbf{a}$ *and* $\mathbf{b}$.

(iii) For all $a,b,c\in A$ *we have*

(5.4) $((a)+(b))\wedge(1)=(a\oplus b)$.

Proof. By Lemma 4.2.3, together with Proposition 5.0.6(ii) and (4.7). ∎

Chapter 6

Chang ℓ-group G_A

From the cancellative abelian monoid M_A, enriched with the lattice-order of Proposition 5.0.9, one can routinely obtain an ℓ-group G_A such that M_A is isomorphic, both as a monoid and as a lattice, to the positive cone ${G_A}^+$. To this purpose, recalling the construction of $\mathbb{Z}$ from $\mathbb{N}$, two pairs $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}', \mathbf{b}')$ of good sequences are called *equivalent* iff $\mathbf{a} + \mathbf{b}' = \mathbf{a}' + \mathbf{b}$. Transitivity of this relation follows from Proposition 5.0.5(i).

Notation. The equivalence class of the pair $(\mathbf{a}, \mathbf{b})$ shall be denoted by $[\mathbf{a}, \mathbf{b}]$.

Definition 6.0.10 Let $G_A = \langle G_A, 0, +, - \rangle$ be the set of equivalence classes of pairs of good sequences, where the *zero element* 0 is the equivalence class $[(0), (0)]$, *addition* is defined by $[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] = [\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{d}]$, and *subtraction* is defined by $-[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}]$. One immediately sees that G_A is an abelian group. G_A is called the *enveloping group* of A.

We shall now equip G_A with a lattice-order. Let $(\mathbf{a}, \mathbf{b})$ be a pair of good sequences of the MV-algebra A. By Proposition 5.0.6(i), $(\mathbf{a}, \mathbf{b})$ has an equivalent pair of the form $(\mathbf{e}, (0))$ if and only if $\mathbf{a} \geq \mathbf{b}$. Let M'_A be the submonoid of G_A given by the equivalence classes of pairs $(\mathbf{e}, (0))$, for all good sequences $\mathbf{e}$. Since the map $\mathbf{e} \mapsto (\mathbf{e}, (0))$ induces an isomorphism of the monoid M_A onto M'_A, we shall freely identify the two monoids M_A and M'_A.

Definition 6.0.11 Let A be an MV-algebra, and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in M_A$. We say that the equivalence class $[\mathbf{c}, \mathbf{d}]$ *dominates* the equivalence class $[\mathbf{a}, \mathbf{b}]$, in symbols,

$$[\mathbf{a}, \mathbf{b}] \preceq [\mathbf{c}, \mathbf{d}],$$

if $[\mathbf{c}, \mathbf{d}] - [\mathbf{a}, \mathbf{b}] = [\mathbf{e}, (0)]$ for some good sequence $\mathbf{e} \in M_A$. Equivalently, $[\mathbf{a}, \mathbf{b}] \preceq [\mathbf{c}, \mathbf{d}]$ iff $\mathbf{a} + \mathbf{d} \leq \mathbf{c} + \mathbf{b}$, where $\leq$ is the partial order of M_A given by Definition 5.0.7.

Proposition 6.0.12 *Let A be an MV-algebra.*

(i) The relation $\preceq$ is a translation invariant partial order, making G_A into an ℓ-group. Specifically, for any two pairs of good sequences $(\mathbf{a},\mathbf{b})$ and $(\mathbf{c},\mathbf{d})$ the supremum of their equivalence classes in G_A is the equivalence class of $((\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d})$, where $\vee$ is the supremum in M_A given by Proposition 5.0.9. In symbols,

$$[\mathbf{a},\mathbf{b}] \bigvee [\mathbf{c},\mathbf{d}] = [(\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}]. \tag{6.1}$$

(ii) Similarly, the infimum $[\mathbf{a},\mathbf{b}] \bigwedge [\mathbf{c},\mathbf{d}]$ is given by

$$[\mathbf{a},\mathbf{b}] \bigwedge [\mathbf{c},\mathbf{d}] = [(\mathbf{a}+\mathbf{d})\wedge(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}]. \tag{6.2}$$

(iii) The map $\mathbf{a}\in M_A \mapsto [\mathbf{a},(0)]$ is an isomorphism between the monoid M_A, equipped with the lattice-order of Proposition 5.0.9, and the positive cone ${G_A}^+ = \{[\mathbf{c},\mathbf{d}]\in G_A \mid \mathbf{c}\geq\mathbf{d}\}$, with the lattice-order inherited by restriction of $\preceq$.

Proof. (i) The proof that $\preceq$ is a translation invariant partial order on G_A is routine. In order to prove (6.1), first of all, from the inequality $\mathbf{a}+\mathbf{d}\leq(\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b})$ we obtain $[\mathbf{a},\mathbf{b}]\preceq[(\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}]$, and, symmetrically, $[\mathbf{c},\mathbf{d}]\preceq[(\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}]$. Thus, $[(\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}]$ is an upper bound of $[\mathbf{a},\mathbf{b}]$ and $[\mathbf{c},\mathbf{d}]$. To show that this is indeed *the least* upper bound, for any upper bound $[\mathbf{p},\mathbf{q}]$ we must find an element $\mathbf{z}\in M_A$ such that

$$\mathbf{p}+\mathbf{d}+\mathbf{b} = \mathbf{z}+\mathbf{q}+((\mathbf{a}+\mathbf{d})\vee(\mathbf{b}+\mathbf{c})). \tag{6.3}$$

By hypothesis, there are $\mathbf{x},\mathbf{y}\in M_A$ such that $\mathbf{p}+\mathbf{b}=\mathbf{x}+\mathbf{q}+\mathbf{a}$ and $\mathbf{p}+\mathbf{d}=\mathbf{y}+\mathbf{q}+\mathbf{c}$. Let $\mathbf{z}\in M_A$ be such that $\mathbf{x}+\mathbf{y}=\mathbf{z}+(\mathbf{x}\vee\mathbf{y})$; the existence of $\mathbf{z}$ is ensured by the inequality $\mathbf{x}+\mathbf{y}\geq\mathbf{x}\vee\mathbf{y}$, using Propositions 5.0.6 and 4.2.3. One now establishes (6.3) using the cancellation property of M_A, as follows:

$$2\mathbf{p}+\mathbf{b}+\mathbf{d} = 2\mathbf{q}+\mathbf{a}+\mathbf{c}+\mathbf{x}+\mathbf{y} = 2\mathbf{q}+\mathbf{a}+\mathbf{c}+\mathbf{z}+(\mathbf{x}\vee\mathbf{y})$$

$$= \mathbf{z}+\mathbf{q}+((\mathbf{x}+\mathbf{q}+\mathbf{a}+\mathbf{c})\vee(\mathbf{y}+\mathbf{q}+\mathbf{a}+\mathbf{c}))$$

$$= \mathbf{z}+\mathbf{q}+((\mathbf{p}+\mathbf{b}+\mathbf{c})\vee(\mathbf{p}+\mathbf{d}+\mathbf{a})) = \mathbf{p}+\mathbf{z}+\mathbf{q}+((\mathbf{b}+\mathbf{c})\vee(\mathbf{d}+\mathbf{a})).$$

One similarly proves (ii). Finally, (iii) is an immediate consequence of the definitions of the partial orders $\leq$ and $\preceq$. ∎

Definition 6.0.13 The ℓ-group G_A with the above lattice-order is called the *Chang ℓ-group* of the MV-algebra A.[1]

Recalling Definition 6.0.11 we immediately have

Proposition 6.0.14 *The element $[(1),(0)]$ is an order-unit of the ℓ-group G_A.*

[1] Chang [6] only dealt with the totally ordered case.

A crucial property of the ℓ-group G_A is given by the following result:

Theorem 6.0.15 *For every MV-algebra A the correspondence $a \mapsto \varphi_A(a) = [(a),(0)]$ defines an isomorphism from A onto $\mathbf{\Gamma}(G_A,[(1),(0)])$.*

Proof. One first notes that $[(0),(0)] \preceq [\mathbf{a},\mathbf{b}] \preceq [(1),(0)]$ iff there is $c \in A$ such that $(\mathbf{a},\mathbf{b})$ is equivalent to $((c),(0))$. Thus, φ_A maps A onto the unit interval $[[(0),(0)],[(1),(0)]]$ of G_A. It is easy to see that this map is one-one. By (5.4), $\varphi_A(a \oplus b) = (\varphi_A(a) + \varphi_A(b)) \bigwedge [(1),(0)]$, and by (5.3), $\varphi_A(\neg a) = [(1),(0)] - \varphi_A(a)$. Therefore, φ_A is a homomorphism of A onto $\mathbf{\Gamma}(G_A,((1),(0)))$. ∎

Remark: An MV-algebra A is a chain if and only if G_A is totally ordered. Indeed, if A is totally ordered, then it follows from Proposition 5.0.6(i) that M_A is totally ordered, and this implies that G_A is a totally ordered group. The converse is an immediate consequence of Theorem 6.0.15.

Chapter 7

Chang completeness

An *ℓ-group term* in the variables $x_1, \ldots, x_t$ is a string of symbols over the alphabet $\{x_1, \ldots, x_n, 0, -, +, \vee, \wedge, (,)\}$ which is obtained by the same inductive procedure used in Chapter 1.4 to define MV-terms. Let τ be an ℓ-group term in the variables $x_1, \ldots,\ x_t$ and G be an ℓ-group. Substituting an element $a_i \in G$ for all occurrences of the variable x_i in τ, for $i = 1, \ldots, t$, and interpreting the symbols 0, $-$, $+$, $\vee$ and $\wedge$ as the corresponding operations in G, we obtain an element of G, denoted $\tau^G(a_1, \ldots, a_t)$. To each MV-term τ in the n variables $x_1, \ldots, x_n$ we associate an ℓ-group term $\hat{\tau}$ in the $n+1$ variables $(x_1, \ldots, x_n, y)$, according to the following stipulations: (i): $\hat{x}_i = x_i$, for each $i = 1, \ldots, n$; (ii): $\hat{0} = 0$; (iii): $\widehat{\neg\sigma} = (y - \hat{\sigma})$; (iv): $\widehat{(\rho \oplus \sigma)} = (y \wedge (\hat{\rho} + \hat{\sigma}))$. The mapping $\tau \mapsto \hat{\tau}$ is well defined. We then have a purely syntactic counterpart of the mappings $(G, u) \mapsto \mathbf{\Gamma}(G, u)$ and $A \mapsto G_A$, in a sense that is made precise by the following two propositions:

Proposition 7.0.16 *If G is a totally ordered abelian group, $0 < u \in G$, $0 \leq g_1, \ldots g_n \leq u$ and $A = \mathbf{\Gamma}(G, u)$, then for every MV-term $\tau(x_1, \ldots, x_n)$ we have $\tau^A(g_1, \ldots, g_n) = \hat{\tau}^G(g_1, \ldots, g_n, u)$.*

Proof. By a trivial induction on the number of operation symbols in τ. ■

Conversely, upon identifying (via Proposition 6.0.12(iii)) M_A with the positive cone of G_A, we have:

Proposition 7.0.17 *If A is an MV-chain, $a_1, \ldots, a_n \in A$, $G = G_A$ is the Chang ℓ-group of A, and $\tau(x_1, \ldots, x_n)$ is an MV-term, then the one-term good sequence $(\tau^A(a_1, \ldots, a_n)) \in G$ coincides with $\hat{\tau}^G((a_1), \ldots, (a_n), (1))$.*

Proof. By induction on the number of operation symbols in τ, using equations (5.2) and (5.3). ■

Theorem 7.0.18 (Completeness Theorem) *An equation holds in $[0, 1]$ if and only if it holds in every MV-algebra.*

Proof. Suppose an equation fails in an MV-algebra A. By Corollary 3.0.5, A may be assumed to be totally ordered. Using the distance function, we may safely assume that the equation has the form $\tau(x_1,\dots,x_n)=0$. There are elements $a_1,\dots,a_n\in A$ such that $\tau^A(a_1,\dots,a_n)>0$. Letting G_A denote the Chang ℓ-group of A, and again using the identification $M_A=G_A^+$, by Proposition 7.0.17 we have $0<\hat\tau^{G_A}((a_1),\dots,(a_n),(1))\leq(1)$. It is not hard to see that G_A, is torsion-free. Let S be the subgroup of G_A generated by the elements $(a_1),\dots,(a_n),(1)$, with the induced total order. Then S can be identified with the free abelian group $\mathbb{Z}^r$, for some integer $r\geq 1$; its elements $(a_1),\dots,(a_n)$ and (1) are concretely represented as vectors $\mathbf{a}_1,\dots,\mathbf{a}_n,\mathbf{a}_{n+1}\in\mathbb{Z}^r$; the positive cone of S is a submonoid P of $\mathbb{Z}^r$ such that $P\cap -P=\{0\}$ and $P\cup -P=\mathbb{Z}^r$. If $r=1$ we are done. So let's assume $r>1$. For any two vectors $\mathbf{a},\mathbf{b}\in\mathbb{Z}^r$ we write $\mathbf{a}\leq_P\mathbf{b}$ iff $\mathbf{b}-\mathbf{a}\in P$. Let us display the subterms $\sigma_1,\sigma_1,\dots,\sigma_t$ of $\hat\tau$ as follows: $\sigma_1=x_1,\ \dots,\sigma_n=x_n,\quad \sigma_{n+1}=y,\ \sigma_{n+2},\ \sigma_{n+3},\ \dots,\sigma_{t-1},\ \sigma_t=\hat\tau$. We can safely assume that the list contains the zero term. The map $x_1\mapsto\mathbf{a}_1,\dots,x_n\mapsto\mathbf{a}_n,\ y\mapsto\mathbf{a}_{n+1}$ uniquely extends to an interpretation $\sigma_j\mapsto\mathbf{a}_j$ $(j=1,\dots,t)$ of each subterm of $\hat\tau$ into an element of the totally ordered group $T=(\mathbb{Z}^r,\leq_P)$. In particular we have the inequalities

$$0\ \leq_P\ \mathbf{a}_{\forall j}\ \leq_P\mathbf{a}_{n+1},\quad 0\ <_P\ \mathbf{a}_t\ =\ \hat\tau^T(\mathbf{a}_1,\dots,\mathbf{a}_n,\mathbf{a}_{n+1}). \tag{7.1}$$

Let ω be a permutation of $\{1,\dots,t\}$ such that $\mathbf{a}_{\omega(1)}\leq_P\mathbf{a}_{\omega(2)}\leq_P\dots\leq_P\mathbf{a}_{\omega(t)}$. *By a small perturbation we shall replace* $\leq_P$ *by another total order* $\leq_{P'}$ *over* $\mathbb{Z}^r$ *in such a way that the above inequalities still hold with respect to* $\leq_{P'}$, *and the ordered group* $(\mathbb{Z}^r,\leq_{P'})$ *is isomorphic to a subgroup of the additive group* $\mathbb{R}$ *with the natural order.* To this purpose, for each $j=2,\dots,t$, let the vector $d_j\in P$ be defined by $d_j=\mathbf{a}_{\omega(j)}-\mathbf{a}_{\omega(j-1)}$. Embedding $\mathbb{Z}^r$ into $\mathbb{R}^r$, we define the positive and the negative span of the d_j's as follows:

$$P^*=\{\sum_{j=2}^{t}\lambda_j d_j\mid 0\leq\lambda_j\in\mathbb{R}\},\quad N^*=-P^*. \tag{7.2}$$

Then P^* is a closed and convex subset of $\mathbb{R}^r$, and whenever $\mathbf{a}\in P^*$ and $0\leq\alpha\in\mathbb{R}$, then $\alpha\mathbf{a}\in P^*$. It is not hard to see that 0 is an extremal point of P^*; (For otherwise, let I be a minimal subset of $\{1,\dots,t\}$ such that $0=\sum_{j\in I}\lambda_j d_j$ for some $0<\lambda_j\in\mathbb{R}$ and $d_j\neq 0$. Then the tuple $(\lambda_j)_{j\in I}$ is uniquely determined up to multiplication by a constant factor $0<\gamma\in\mathbb{R}$. Since $0\leq_P d_j\in\mathbb{Z}^r$, there are integers $0<n_j$ such that $0=\sum_{j\in I}n_jd_j$. By definition of P, for each $i\in I$ we have $d_i\ \leq_P\ \sum_{j\in I}n_jd_j$, whence $d_i=0$, a contradiction). A similar argument shows that

$$P^*\cap P=P^*\cap\mathbb{Z}^r\quad\text{and}\quad P^*\cap N^*=\{0\}. \tag{7.3}$$

For any $i,j=1,\dots,t$ we then obtain $\mathbf{a}_i\leq_P\mathbf{a}_j$ iff $\mathbf{a}_j-\mathbf{a}_i\in P^*$.

Claim.[1] For some vector $\mathbf{g}\in\mathbb{R}^r$, the hyperplane $\pi_{\mathbf{g}}=\{\mathbf{v}\in\mathbb{R}^r\mid\ \mathbf{g}\cdot\mathbf{v}=0\}$ *separates* P^* and N^*, in the sense that $\pi_{\mathbf{g}}\cap P^*=\{0\}=\pi_{\mathbf{g}}\cap N^*$.

[1] This is a classical result. We include a proof for the sake of completeness. As usual, we denote by $\mathbf{a}\cdot\mathbf{b}$ the scalar product of vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^r$.

The proof is by induction on r. The basis is trivial. For the induction step, assume $r \geq 2$ and let δ be the boundary of the unit ball in $\mathbb{R}^r$. Then for some vector $\mathbf{w} \in \delta$, the line $\rho_{\mathbf{w}} = \{\lambda \mathbf{w} \mid \lambda \in \mathbb{R}\}$ is such that $\rho_{\mathbf{w}} \cap P^* = \{0\}$. (For otherwise, since by (7.3) P^* contains no line, each vector $\mathbf{u} \in \delta$ belongs to exactly one of P^* and N^*; recalling that P^* and N^* are closed, we obtain a partition of δ into disjoint closed sets $\delta \cap P^*$ and $\delta \cap N^*$, thus contradicting the connectedness of δ). Let $P^*_{\mathbf{w}}$ be the projection of P^* into the $(r-1)$-dimensional subspace $\pi_{\mathbf{w}} = \{\mathbf{v} \in \mathbb{R}^r \mid \mathbf{w} \cdot \mathbf{v} = 0\}$. Then $P^*_{\mathbf{w}}$ can be identified with a closed convex subset of $\mathbb{R}^{r-1}$ having 0 as an extremal point, and such that whenever $\mathbf{a} \in P^*_{\mathbf{w}}$ and $0 \leq \alpha \in \mathbb{R}$, then $\alpha \mathbf{a} \in P^*_{\mathbf{w}}$. By induction hypothesis, there is a hyperplane π in $\mathbb{R}^{r-1}$ such that $\pi \cap P^*_{\mathbf{w}} = \{0\}$. It follows that the hyperplane $\pi + \rho_{\mathbf{w}}$ of $\mathbb{R}^r$ intersects P^* only in 0, as required. Picking now a vector $\mathbf{g} = (\gamma_1, \ldots, \gamma_r) \in \mathbb{R}^r$ such that $\pi + \rho_{\mathbf{w}} = \pi_{\mathbf{g}}$, our claim is settled.

As an equivalent reformulation of our claim, in the light of the convexity of P^*, we can safely write $\mathbf{g} \cdot \mathbf{d}_j > 0$ for all nonzero vectors $\mathbf{d}_j$, $j = 2, \ldots, t$. By continuity, and recalling that $r > 1$, $\mathbf{g}$ can be assumed to be *in general position*, in the sense that $\gamma_1, \ldots, \gamma_r$ are linearly independent over $\mathbb{Q}$. Let

$$\pi^+_{\mathbf{g}} = \{(\zeta_1, \ldots, \zeta_r) \in \mathbb{R}^r \mid \sum \gamma_i \zeta_i \geq 0\}, \quad \text{and} \quad P' = \pi^+_{\mathbf{g}} \cap \mathbb{Z}^r.$$

Then from (7.2) it follows that $P^* \subseteq \pi^+_{\mathbf{g}}$ and $N^* \subseteq -\pi^+_{\mathbf{g}}$. Consider now the totally ordered abelian group $T' = (\mathbb{Z}^r, \leq_{P'})$. By (7.3), for all $i, j = 1, \ldots, t$ we have

$$\text{(7.4)} \qquad \mathbf{a}_i \leq_P \mathbf{a}_j \ \text{ iff } \ \mathbf{a}_j - \mathbf{a}_i \in P^* \ \text{ iff } \ \mathbf{a}_j - \mathbf{a}_i \notin N^* \ \text{ iff } \ \mathbf{a}_i \leq_{P'} \mathbf{a}_j.$$

For any vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n, \mathbf{b}_{n+1} \in \mathbb{Z}^r$, the map $x_1 \mapsto \mathbf{b}_1, \ldots, x_n \mapsto \mathbf{b}_n, y \mapsto \mathbf{b}_{n+1}$ uniquely extends to an interpretation $\sigma_j \mapsto \mathbf{b}_j$, $j = 1, \ldots, t$ of all subterms σ_j of $\hat{\tau}$ into elements $\mathbf{b}_j$ of T'. In the particular case when $\mathbf{b}_1 = \mathbf{a}_1, \ldots, \mathbf{b}_{n+1} = \mathbf{a}_{n+1}$, arguing by induction on the number of operation symbols occurring in σ_j, from (7.4) we obtain $\mathbf{b}_j = \mathbf{a}_j$ for all $j = 1, \ldots, t$; indeed, all inequalities in (7.1) are still valid with respect to the new total order relation $\leq_{P'}$ over $\mathbb{Z}^r$. From the independence of the γ's over $\mathbb{Q}$, it follows that the totally ordered group T' is isomorphic to the subgroup $U = \mathbb{Z}\gamma_1 + \ldots + \mathbb{Z}\gamma_r$ of $\mathbb{R}$ generated by $\gamma_1, \ldots, \gamma_r$, with the natural order. An isomorphism is given by the map $\theta \colon \mathbf{b} = (b_1, \ldots, b_r) \in T' \mapsto b \cdot \mathbf{g} = b_1\gamma_1 + \cdots + b_r\gamma_r \in U$. Since the inequalities in (7.1) are preserved under isomorphism, letting $\kappa_1 = \theta(\mathbf{a}_1), \ldots, \kappa_n = \theta(\mathbf{a}_n), \kappa_{n+1} = \theta(\mathbf{a}_{n+1}), \ldots, \kappa_t = \theta(\mathbf{a}_t)$ we get $0 \leq \kappa_1, \ldots, \kappa_n \leq \kappa_{n+1}$ and $0 < \kappa_t \leq \kappa_{n+1}$. Assuming without loss of generality, $\kappa_{n+1} = 1$, we have $\kappa_t = \hat{\tau}^U(\kappa_1, \ldots, \kappa_n, 1) > 0$. By Proposition 7.0.16, in the MV-algebra $B = \mathbf{\Gamma}(U, 1)$ we have $\tau^B(\kappa_1, \ldots, \kappa_n) \neq 0$, whence, a fortiori, the equation $\tau = 0$ fails in the MV-algebra $[0, 1]$. [2] ∎

[2] By continuity, the equation fails in the rational MV-algebra $\mathbb{Q} \cap [0, 1]$. Taking the least common multiple of the denominators of the rational numbers witnessing that the equation fails in $\mathbb{Q} \cap [0, 1]$, we also see that the equation fails in some finite chain $\mathbf{L}_n$.

Chapter 8

Free MV-algebras

Knowledge of free MV-algebras is a useful tool for a full understanding of "consequence" in infinite-valued Łukasiewicz logic (see [10, Section 4, especially 4.6]), and to evaluate the computational complexity of the tautology problem (Theorem 13.1.1 below). Further, the ideal theory of free MV-algebras yields characterization theorems for important classes of MV-algebras. As will be shown in this chapter, free MV-algebras consist of McNaughton functions. The latter are a source of geometric inspiration not only for MV-algebras, but also for algebras arising from other many-valued logics, and even for ℓ-groups.

8.1 McNaughton theorem

Let $\kappa \geq 1$ be a cardinal. For each ordinal $\alpha < \kappa$ let $[0,1]^\kappa$ be the (Tichonov) κ-cube and $\pi_\alpha\colon [0,1]^\kappa \to [0,1]$ be the αth projection (= coordinate, = identity) function. As a routine consequence of the completeness theorem, the free MV-algebra $Free_\kappa$ over κ many free generators is the smallest MV-algebra of $[0,1]$-valued functions defined over $[0,1]^\kappa$ containing all projections π_α, for each ordinal $\alpha < \kappa$, and closed under the pointwise operations. Moreover, the π_α's constitute a free generating set in $Free_\kappa$. To get a concrete realization of the free MV-algebra $Free_\kappa$ we prepare

Definition 8.1.1 Let $n = 1, 2, 3, \ldots$. A function $f\colon [0,1]^n \to [0,1]$ is a *McNaughton function* over $[0,1]^n$ if f is continuous with respect to the natural product topology of $[0,1]^n$, and there are linear polynomials $l_1, \ldots, l_u$ with integer coefficients, such that for each $\mathbf{y} \in [0,1]^n$ there is $j \in \{1, \ldots, u\}$ with $f(\mathbf{y}) = l_j(\mathbf{y})$. For λ an infinite cardinal, we say that $g\colon [0,1]^\lambda \to [0,1]$ is a *McNaughton function* over $[0,1]^\lambda$ if there are ordinals $\alpha(1) < \ldots < \alpha(m) < \lambda$ and a McNaughton function f over $[0,1]^m$ such that for each $\mathbf{x} \in [0,1]^\lambda$, $g(\mathbf{x}) = f(x_{\alpha(1)}, \ldots, x_{\alpha(m)})$.

We next give a short, self-contained and constructive proof of McNaughton representation Theorem [30] for free MV-algebras.

Theorem 8.1.2 *The free MV-algebra* $Free_\kappa$ *coincides with the MV-algebra of McNaughton functions over* $[0,1]^\kappa$ *with pointwise operations.*

Proof. Let $f\colon [0,1]^n \to [0,1]$ be a McNaughton function. Let $l_1, \ldots, l_u$ be the distinct linear pieces of f. Our aim is to show that f is obtainable from the identity functions π_i via a finite number of applications of the operations $\neg$ and $\oplus$. To this purpose, for any permutation σ of the set $\{1, \ldots, u\}$, we define the closed convex polyhedron P_σ by

$$P_\sigma = \{\mathbf{x} \in [0,1]^n \mid l_{\sigma(1)}(\mathbf{x}) \ \le\ l_{\sigma(2)}(\mathbf{x}) \ \le \cdots \le\ l_{\sigma(u)}(\mathbf{x})\}.$$

Let Ω be the set of permutations σ such that P_σ is n-dimensional. A moment's reflection shows that $[0,1]^n = \bigcup_{\sigma\in\Omega} P_\sigma$. Let ξ be an arbitrary permutation in Ω. In the interior int P_ξ of P_ξ the above inequalities are strict, $l_{\xi(1)} < l_{\xi(2)} < \cdots < l_{\xi(u)}$. Therefore, there is a unique index $i = i_\xi$ such that f coincides with $l_{\xi(i_\xi)}$ over int P_ξ; thus, $l_{\xi(i)} > f$ for $i > i_\xi$ and $l_{\xi(i)} < f$ for $i < i_\xi$. Let the function $g^\xi\colon [0,1]^n \to \mathbb{R}$ be defined by $g^\xi = \bigwedge_{i\ge i_\xi} l_{\xi(i)}$.

Claim 1. $g^\xi \le f$ over $[0,1]^n$.

Otherwise (absurdum hypothesis) we have

(8.1) $\quad g^\xi(\mathbf{z}) > f(\mathbf{z})$ for some $\mathbf{z} \in [0,1]^n$.

By continuity, we can assume $\mathbf{z}$ to lie in the interior of P_ζ for some permutation $\zeta \in \Omega$. Pick $\mathbf{x} \in$ int P_ξ, and let $\mathbf{w}$ be the unit vector in $\mathbb{R}^n$ in the direction from $\mathbf{x}$ to $\mathbf{z}$. Let the two points X and Z in $\mathbb{R}^{n+1}$ be given by $X = (\mathbf{x}, f(\mathbf{x}))$ and $Z = (\mathbf{z}, f(\mathbf{z}))$. By (8.1) for all small $\epsilon > 0$ the point $(\mathbf{x}+\epsilon\mathbf{w}, f(\mathbf{x}+\epsilon\mathbf{w}))$ lies above the segment XZ, and the point $(\mathbf{z}-\epsilon\mathbf{w}, f(\mathbf{z}-\epsilon\mathbf{w}))$ lies below the segment XZ. There certainly exists a point $\mathbf{y}$ with $\mathbf{x} < \mathbf{y} < \mathbf{z}$ such that $(\mathbf{y}, f(\mathbf{y}))$ lies on XZ and $(\mathbf{y}+\eta\mathbf{w}, f(\mathbf{y}+\eta\mathbf{w}))$ lies below XZ for all small $\eta > 0$. It follows that f at $\mathbf{y}$ coincides with some l_j such that $l_j(\mathbf{x}) > f$ and $l_j(\mathbf{z}) < f$, thus contradicting (8.1).

Having thus settled our claim we have the identity $f = \bigvee_{\xi\in\Omega} \bigwedge_{i\ge i_\xi} l_{\xi(i)}$ over the whole cube $[0,1]^n$. To complete the proof of the theorem is suffices to settle the following

Claim 2. Let the function $l\colon [0,1]^n \to \mathbb{R}$ be given by $l(\mathbf{x}) = b + m_1x_1 + \cdots + m_nx_n$, with $m_1, \ldots, m_n, b \in \mathbb{Z}$. Let $l^\sharp = (l \vee 0) \wedge 1$. Then $l^\sharp$ is obtainable from the projections π_i via a finite number of applications of the operations $\neg$ and $\oplus$.

We argue by induction on $m = |m_1| + \ldots + |m_n|$. The basis $m = 0$ is trivial. For the induction step, it is no loss of generality to assume that $\max(|m_1|, \ldots, |m_n|) = |m_1|$.

Case 1. $m_1 > 0$.

Then let $h = l - x_1$. By induction the claim holds for both functions $h^\sharp$ and $(h+1)^\sharp$. We shall prove the identity

(8.2) $\quad l^\sharp = (h + x_1)^\sharp = (h^\sharp \oplus x_1) \odot (h+1)^\sharp \quad \forall \mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n.$

Firstly, the identity trivially holds for all $\mathbf{x}$ is such that $h(\mathbf{x}) > 1$, or $h(\mathbf{x}) < -1$. Secondly, if $h(\mathbf{x}) \in [0,1]$, then $h^\sharp(\mathbf{x}) = h(\mathbf{x})$, and $(h(\mathbf{x}) + 1)^\sharp = 1$. Since $0 \leq x_1 \leq 1$, $(h(\mathbf{x}) + x_1)^\sharp = h(\mathbf{x}) \oplus x_1$, which establishes (8.2). Finally, if $h(\mathbf{x}) \in [-1,0]$, then $h^\sharp(\mathbf{x}) = 0$ and $(h(\mathbf{x})+1)^\sharp = h(\mathbf{x})+1$, whence (8.2) follows from the identities $(h(\mathbf{x})+x_1)^\sharp = \max(0, h(\mathbf{x})+x_1) = \max(0, x_1+h(\mathbf{x})+1-1)$ $= x_1 \odot (h(\mathbf{x})+1)$.

Case 2. $m_1 < 0$

Then one simply notes that $\neg(1-g)^\sharp = 1-(1-g)^\sharp = g^\sharp$ and that, by the analysis of Case 1, $(1-g)^\sharp$ is obtainable from the projection functions π_i via a finite number of applications of the operations $\neg$ and $\oplus$. ■

Exercise 8.1.3 Modifying the proof of Proposition 2.0.17, prove the Subdirect Representation Theorem for ℓ-groups. From the proof of Chang completeness theorem extract a proof that if an equation fails in an ℓ-group, then it fails in the additive group $\mathbb{R}$ of real numbers, and also fails in $\mathbb{Z}$. Modify the proof of McNaughton theorem to show that the free n-generator abelian ℓ-group precisely consists of all continuous real-valued piecewise linear homogeneous functions over $\mathbb{R}^n$, where each piece has integer coefficients.

Chapter 9

Logic of Rényi-Ulam games

Closing a circle of ideas, let us inspect a round of Ulam game: Initially, the two players agree to fix a nonempty finite set S of numbers, called the *search space*, and an integer $e \geq 0$. Then Carole chooses a number $x_{secret} \in S$. Paul must find x_{secret} by asking yes-no questions. Carole is allowed at most e errors/lies in her answers. By definition, a *question* is a subset of S: thus for instance, the question ″is x_{secret} odd?″ is nothing else but the set of all odd numbers in S. We shall conveniently identify ourselves with Paul. Carole's *answers* are propositions of either form "yes, (it is even)", or "no, (it is odd)". Our current state of knowledge about x_{secret} is uniquely determined by recording Carole's answers.

9.1 The MV-algebra of states of knowledge

In the familiar game of Twenty Questions, a complete record R of our knowledge of x_{secret} is provided by the current *set* of Carole's answers—two equal answers carrying the same information as one. To see when two records are equivalent, let the function $R^{\#}: S \to \{0,1\}$ tell us the current truth-value of every z in the search space S, as recorded by R. In detail, the truth-value of z is the quantity

(9.1) $\quad 1 \dot{-} |\{\text{answers } \in R \text{ falsified by } z\}|,$

where $\dot{-}$ denotes truncated addition, and $|X|$ denotes the number of elements of X. Stated otherwise, for each $z \in S$, $R^{\#}(z) = 1$ iff z does not falsify any answer; $R^{\#}(z) = 0$ iff z falsifies at least one answer. Then it is immediately seen that two records R and P are equivalent iff $R^{\#} = P^{\#}$. For example, the record $R = \{x_{secret} \text{ is odd}, x_{secret} \text{ is between 4 and 8}\}$ is equivalent to the record $\{x_{secret} \text{ is either 5 or 7}\}$.

Also in the game with $e \geq 1$ lies, our knowledge about x_{secret} is given by the record R of Carole's answers. However, R is now a *multiset*—because *repeated equal answers to the same repeated question carry more information than single* *answers.*[1] Thus each answer A has a multiplicity, telling us how many times it occurs in the multiset R. The *union* $R' \smile R''$ of two records is the record obtained by giving each answer $A \subseteq S$ a multiplicity equal to the sum of its multiplicity in R' plus its multiplicity in R''. With each record R we associate the truth-value function

$$R^{\#}: S \to \{0, \frac{1}{e+1}, \frac{2}{e+1}, \dots, \frac{e}{e+1}, 1\}$$

measuring (in units of $e+1$) the distance of each $z \in S$ from the condition of being discarded as a candidate for x_{secret}. Thus for every $z \in S$ we have

$$(9.2) \qquad R^{\#}(z) \quad = \quad 1 \dot{-} \frac{|\{\text{answers } \in R \text{ falsified by } z\}|}{e+1}.$$

As in the error-free case, *equivalence* of two records R and P is defined by

$$(9.3) \qquad R \equiv P \text{ iff } R^{\#}(x) = P^{\#}(x) \ \ \forall x \in S.$$

For notational simplicity, the equivalence class $[R]$ of R shall be denoted by r.

Definition 9.1.1 A *state of knowledge* in a Rényi-Ulam game over the search space S with e lies is an equivalence class r of records. The *initial* state of knowledge 1 is the equivalence class of the empty record (Paul has received no answers yet.) The *incompatible* state 0 is the equivalence class of the record containing $e+1$ copies of the empty set. We let $\mathcal{K}_{S,e}$ denote the set of states. Given states of knowledge $r = [R]$ and $p = [P]$ we write $r \leq p$ (read: "r is *more restrictive* than p") iff $R^{\#} \leq P^{\#}$.

Direct inspection shows that $\mathcal{K}_{S,e}$ has an interesting algebraic structure:

Proposition 9.1.2

1. *The binary relation $\leq$ is a partial order over the set $K_{S,e}$ of states;*

2. *The union operation $\smile$ of records induces a well defined operation $\odot$ on $K_{S,e}$ with an operation by the stipulation*

$$r \odot p = [R] \odot [P] = [R \smile P].$$

   The $\odot$ operation is commutative, associative, and the initial state 1 is the neutral element for $\odot$; further $r \odot 0 = 0$ for all $r \in K_{S,e}$;

3. *Among all states of knowledge in $K_{S,e}$ that are incompatible with a state r there is a least restrictive one, denoted $\neg r$; thus, $r \odot \neg r = 0$, and whenever a state s satisfies $s \odot r = 0$ then $s \leq \neg r$;*

[1] To see this, let us assume that Carole can only lie at most once. Suppose we ask twice the following question "is the secret number even ?" . If Carole's answer is "yes" in both cases, then x_{secret} must be even. However, after the first answer we are not certain that x_{secret} is even.

4. *The $\neg$ operation equips the abelian monoid $K_{S,e}$ with an involution: $r = \neg\neg r$; further, $\neg 0 = 1$ and $\neg(\neg r \odot s) \odot s = \neg(\neg s \odot r) \odot r$;*

5. *The structure $\langle K_{S,e}, \odot, \neg, 1\rangle$ is an MV-algebra.*[2]

6. *The partial order over $K_{S,e}$ is definable in terms of the operations $\odot$ and $\neg$ by $r \leq p$ iff $r \odot \neg p = 0$.*

9.2 Main results

As we have just seen, the involutive monoidal structure of $\mathcal{K}_{S,e}$ is sufficiently rich to reconstruct the order structure of states of knowledge. One is then naturally led to study the class of MV-algebras $\mathcal{K}_{S,e}$, where S and e range over all possible Rényi-Ulam games.

The following nontrivial consequence of Chang Completeness Theorem gives a natural semantics for the infinite-valued Łukasiewicz calculus [50]:

Theorem 9.2.1 *Given MV-terms $\sigma = \sigma(x_1, \ldots, x_n)$ and $\tau = \tau(x_1, \ldots, x_n)$, the following conditions are equivalent for the equation $\sigma = \tau$:*

(i) For every finite set S and integer $e \geq 0$, the equation is valid in the MV-algebra $\mathcal{K}_{S,e}$ (thus, letting the x_i range over all possible states in the Rényi-Ulam game over search space S with e errors.)

(ii) For every integer $e \geq 0$ and singleton set $\{j\}$, the equation holds in the MV-algebra $\mathcal{K}_{\{j\},e}$.

(iii) The equation holds in the standard MV-algebra over the rational unit interval $\langle \mathbb{Q} \cap [0,1], \odot, \neg, 1\rangle$, where, as usual, $x \odot y = \max(0, x + y - 1)$.

(iv) The equation holds in every MV-algebra.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from 3.0.3(iii), because $\mathcal{K}_{S,e}$ is a product of $|S|$ many copies of the chain $\mathcal{K}_{\{j\},e} = \mathbf{L}_{e+2}$. For the equivalence (ii) $\Leftrightarrow$ (iii) one notes that the final part of the proof of Chang Completeness Theorem 7.0.18 shows that, if an equation fails in $\mathbb{Q} \cap [0,1]$, then it fails in a finite subalgebra of the form $\mathbf{L}_{e+2}$, for some $e = 0, 1, 2, \ldots$. Finally, the equivalence (iii) $\Leftrightarrow$ (iv) is a reformulation of Chang Completeness Theorem. ∎

Examples. The equation $x \odot x = x$, holds in all Rényi-Ulam games with $e = 0$ and does not hold when $e > 0$. The weaker equation $x \odot x \odot x = x \odot x$ precisely holds in all Rényi-Ulam games with one lie. By adding suitable variants of the above equations one can thus formalize Rényi-Ulam games with $e = 2, 3, \ldots$ errors.

[2] The MV-algebra $K_{S,e}$ is here defined in terms of $\neg, \odot$ and 1, rather than on $\neg, \oplus$ and 0. All readers who have followed us thus far will easily reconstruct the appropriate $(\neg, \odot, 1)$-redefinition of MV-algebra, by a suitable dualization of the $(\neg, \oplus, 0)$-axioms (1.1)-(1.6)

Corollary 9.2.2 *There is a Turing machine deciding which equations are valid in the MV-algebra $\mathcal{K}_{S,e}$ for all possible S and e.*

Proof. We shall describe a Turing machine T yielding a decision procedure for the problem. Using the distance function, it is enough to consider equations of the form $\sigma = 1$. T consists of two parts, T_1 and T_2, where T_1 lexicographically enumerates all proofs obtainable from the MV-axioms by repeated application of the familiar rules of equational logic (substitutions of equals by equals) and halts iff a proof of $\sigma = 1$ is obtained. On the other hand, T_2 enumerates all n-tuples $(r_1, \ldots, r_n)$ of rational numbers in [0,1], and halts iff there is an n-tuple $(\overline{r}_1, \ldots, \overline{r}_n)$ such that $\sigma(\overline{r}_1, \ldots, \overline{r}_n) \neq 1$. By the above theorem, precisely one of T_1 and T_2 halts within a finite number of steps. In case T_1 halts, by (iv) above, the equation $\sigma = 1$ is valid in all Rényi-Ulam games; in the other case, by (iii) the equation is not. ■

One can now pose the problem of giving a game semantics to various many-valued logics in terms of suitable variants of the Rényi-Ulam game. One may also ask what is the relation between the above algorithm to test equivalence of two MV-terms—and "MV-algebraic equational logic", i.e., the usual substitution of equals for equals, starting from the defining equations of MV-algebras. From a general result of Birkhoff in Universal Algebra, together with Chang completeness theorem, it follows that MV-algebraic equational logic yields a method to compute all valid equations. As explained in [10, Chapter 4], writing $x \to y$ instead of $\neg x \oplus y$ one may ask which $(\neg, \to)$-terms are tautologies (i.e., are equivalent to 1) in Łukasiewicz infinite-valued propositional logic; then Chang completeness shows that Modus Ponens and Substitution are all we need to compute every tautology.

Chapter 10

Betting on $[0,1]$-events

In the remaining sections of this tutorial we will proceed at a more rapid pace. For complete proofs the interested reader will be addressed to the relevant literature.

A natural framework for introducing probability in classical and non-classical logic is as follows: Suppose two players Ada (the bookmaker) and Blaise (the bettor) wager money on the occurrence of the events described by formulas $\psi_1, \ldots, \psi_n$. Thus, after Ada has assigned a "betting odd" $\beta_i \in [0,1]$ to each ψ_i, Blaise chooses "stakes" $\sigma_1, \ldots, \sigma_n \geq 0$, and pays Ada $\sigma_i\beta_i$, with the stipulation that he will get $\sigma_i V(\psi_i)$ from her in the "possible world" V where the truth-value $V(\psi_i)$ is known. If the ψ_i are two-valued, Blaise will receive the full stake σ_i if V evaluates ψ_i to 1, and otherwise Blaise will receive nothing.

While real bookmakers never accept "reverse bets", Ada is willing to do so: in other words, she also accepts *negative stakes* σ_i, to the effect that *she* must pay Blaise $|\sigma_i|\beta_i$, to receive from him $|\sigma_i|V(\psi_i)$ in the possible world V. No matter the signs of the stakes σ_i, the total balance of Ada's "book" $\{\langle\psi_i, \beta_i\rangle \mid i = 1, \ldots, n\}$ is given by the formula

$$(10.1)\sum_{i=1}^{n} \sigma_i(\beta_i - V(\psi_i)),$$

where money transfers are oriented in such a way that "positive" means "Blaise-to-Ada".

As a matter of practical necessity, Ada will arrange her book in such a way that Blaise cannot choose stakes $\sigma_1, \ldots, \sigma_n$ ensuring him to win money in every possible world V. The non-existence of real numbers $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ ensuring $0 > \sum_{i=1}^{n} \sigma_i(\beta_i - V(\psi_i))$ for every V, is known as De Finetti's (no-Dutch-Book) *coherence* criterion for probability assignments. As shown by De Finetti himself, [11, pp. 311-312], [12, pp. 85-90] this criterion is necessary and sufficient for the β_i to be extendable to a finitely additive measure on the (boolean algebra generated by these) formulas.

De Finetti conceived of the no-Dutch-Book criterion as a tool for dealing with probability without making any assumption on the repeatability of events, and on their logic-algebraic structure. Thus it is quite natural to investigate Dutch Books for non-tarskian semantics. In his paper [41] Paris asks for a generalization of the no-Dutch-Book theorem to Łukasiewicz infinite-valued propositional calculus. The problem is quite interesting, because (i) infinite-valued events, and their possible worlds, can be defined no less precisely in Łukasiewicz logic than yes-no events are definable in two-valued logic, and (ii) we do bet and reason on such events very often.

Thus for instance, the "possible worlds" for the event ψ "Philip will *soon* be appointed Foreign Minister" are all instants V between "today" and "two years from now". In the possible world V_1 = "today", the truth-value $V_1(\psi)$ is 1, in the possible world V_2 = "two years from now" $V_2(\psi) = 0$, and in all intermediate possible worlds the truth-value is given by, say, linear interpolation. The precise definition of how V assigns a truth-value to ψ must be explicitly stated in Ada's book. If Ada's belief of the event ψ is $\beta = 1/10$ and Blaise sets a stake of 1000 euro, then Blaise pays now 100 euro, and he will receive $1000 \times V(\psi)$ euro once $V(\psi)$ is known. Precisely as in the classical yes-no case, the stake is paid proportionally to Ada's belief β and is returned, in the opposite direction, proportionally to the truth-value $V(\psi)$; when Blaise bets on several items in Ada's book, the total balance is still given by (10.1).

In the Łukasiewicz infinite-valued calculus, letting $\mathrm{Form}(X_1, \ldots, X_k)$ denote the set of MV-terms (called here "formulas") in the variables $X_1, \ldots, X_k$, a "possible world" is rigorously defines as a *valuation*, i.e., a function

$$V \colon \mathrm{Form}(X_1, \ldots, X_k) \to [0,1]$$

such that $V(\neg\phi) = 1 - V(\phi)$, and $V(\phi \oplus \psi) = \min(1, V(\phi) + V(\psi))$. Two formulas $\phi, \psi \in \mathrm{Form}(X_1, \ldots, X_k)$ are *equivalent* if $V(\phi) = V(\psi)$ for all valuations V. The equivalence class of ϕ is denoted f_ϕ. The set of equivalence classes of formulas over k variables, equipped with the operations $\neg f_\phi = f_{\neg\phi}$ and $f_\phi \oplus f_\psi = f_{\phi\oplus\psi}$, forms an MV-algebra denoted $\mathcal{L}_k$. [1]

The following result [37] solves Paris' problem:

Theorem 10.0.3 *Let* $\psi_1, \ldots, \psi_n \in \mathrm{Form}(X_1, \ldots, X_k)$ *and* $\beta_1, \ldots, \beta_n \in [0,1]$. *Then the following are equivalent:*

(i) The set $\{\langle \psi_i, \beta_i \rangle \mid i = 1, \ldots, n\}$ *satisfies the condition*

$$\text{for no } \sigma_1, \ldots, \sigma_n \in \mathbb{R}, \quad 0 > \sum_{i=1}^{n} \sigma_i(\beta_i - V(\psi_i)) \quad \text{for every valuation } V.$$

(ii) $\beta_i = s(f_{\psi_i})$ *for some* state s *of* $\mathcal{L}_k$ *i.e., a map* $s \colon \mathcal{L}_k \to [0,1]$ *satisfying the conditions of* normality: $s(1) = 1$, *and* additivity: $s(f \oplus g) = s(f) + s(g)$ *whenever* $f \odot g = 0$.

[1] It is not hard to see that $\mathcal{L}_k$ is the free MV-algebra over k generators.

One can similarly deal with infinite sets of formulas, and with the case when the formulas ψ_i are subject to logical constraints, such as "ψ_1 implies ψ_2", or "ψ_1 is incompatible with ψ_2". When the β_i are rational numbers and the ψ_i are subject to a finite number of logical constraints, there is an algorithm to decide whether or not Ada's book is Dutch.

Chapter 11

Γ, with applications

In every ℓ-group G with order-unit u one automatically has such desirable properties as the existence of maximal ℓ-ideals and of unit-preserving ℓ-homomorphisms into the reals. Further, (G, u) is representable as an ℓ-group of (possibly nonstandard) real-valued functions; when the intersection of its maximal ℓ-ideals is zero, (G, u) can be identified with an ℓ-group of continuous real-valued functions over a compact Hausdorff space, with the constant function 1 in place of u. In general, ℓ-groups do not have these properties—but they form an equational class; by contrast, the archimedean property of the order-unit is not even definable in first-order logic. MV-algebras are doubly blessed: they are defined by a small number of simple equations, and they also enjoy all dividends offered by order-units: existence of maximal ideals and of homomorphisms into $[0, 1]$, functional representation, and much more. This wealth of structure in MV-algebras is an effect of their being essentially the same as ℓ-groups with order-unit.

11.1 MV-algebras and ℓ-groups with order-unit

The categorical equivalence $\mathbf{\Gamma}$. As we have seen in Proposition 4.1.4, $\mathbf{\Gamma}$ is a functor from unital ℓ-groups into MV-algebras. Conversely, for any MV-algebra A, let $\mathbf{\Lambda}(A)$ be the Chang ℓ-group of A with the one-term good sequence $1 = (1)$ as a distinguished positive element. By the Subdirect Representation Theorem, together with our analysis of Chang ℓ-group, $\mathbf{\Lambda}(A)$ is a unital ℓ-group. Given MV-algebras A and B and a homomorphism $\theta\colon A \to B$, let

$$\mathbf{\Lambda}(\theta)\colon \mathbf{\Lambda}(A) \to \mathbf{\Lambda}(B)$$

be the canonical extension of the map sending any good sequence $(x_1, x_2, \ldots) \in M_A$ into the good sequence $(\theta(x_1), \theta(x_2), \ldots) \in M_B$. In this way we obtain a functor $\mathbf{\Lambda}$ from MV-algebras into unital ℓ-groups.

The following strengthening of Theorem 6.0.15 was originally proved in [31]. A simpler proof can be found in [10, Corollary 7.1.8]). [1]

[1] The result has been generalized to noncommutative unital lattice-ordered groups by

Theorem 11.1.1 *The $\mathbf{\Gamma}$ functor is a categorical equivalence between unital ℓ-groups and MV-algebras. The lattice-order of A agrees with that of $\mathbf{\Lambda}(A)$.*

This theorem has many important consequences, both for MV-algebras and for ℓ-groups, with or without order-unit. For instance, using the amalgamation property for ℓ-groups [43], we easily get

Corollary 11.1.2 *The variety of MV-algebras has the amalgamation property.*

11.2 Maximal spectral theory

For any MV-algebra A we let $\mathcal{M}(A)$ denote its maximal ideal space equipped with the *spectral* topology: a basis of closed sets for $\mathcal{M}(A)$ is given by the *zerosets* $Z_a = \{\mathfrak{m} \in \mathcal{M}(A) \mid a \in \mathfrak{m}\}$, letting a range over all elements of A. Equivalently, a basis of open sets is given by all sets of the form $support(a) = \{\mathfrak{m} \in \mathcal{M}(A) \mid a \notin \mathfrak{m}\}$. This definition is similar in spirit to the usual definition of "hull-kernel" topogy for ℓ-groups, or for rings. As a straightforward consequence of the definition one obtains

Lemma 11.2.1 *$\mathcal{M}(A)$ is a nonempty compact Hausdorff space*

Proof. Essentially, [10, 3.4.3]. ■

For any compact Hausdorff space X we denote by $Cont(X)$ the MV-algebra of all continuous $[0,1]$-valued functions on X with the pointwise operations of $[0,1]$, $\neg x = 1 - x$ and $x \oplus y = \min(1, x + y)$.

Lemma 11.2.2 *Suppose X is a compact Hausdorff space and D is a* separating *subalgebra of $Cont(X)$, in the sense that for any two distinct $x, y \in X$ there is $g \in D$ such that $g(x) = 0$ and $g(y) > 0$. Then the map $\iota\colon x \in X \mapsto \{f \in D \mid f(x) = 0\}$ is a homeomorphism of X onto $\mathcal{M}(D)$.*

Proof. Essentially, [10, 3.4.4]. ■

In the particular case of free MV-algebras, arguing as in [10, 3.4.6-3.4.9] we have

Lemma 11.2.3 *For any cardinal κ we have:*

1. *$Free_\kappa$ is a separating subalgebra of $Cont([0,1]^\kappa)$.*
2. *Each finitely generated ideal of $Free_\kappa$ is an intersection of maximal ideals.*

An MV-algebra A is *simple* if its only ideal is $\{0\}$. The preparatory lemmas above yield the following characterization:

Theorem 11.2.4 [10, Section 3.5] *Up to isomorphism we have:*

Dvurečenskij and, further, by Tsinakis and Galatos.

(i) *Every finite simple MV-algebra coincides with some finite chain* $\mathbf{L}_n$.

(ii) *Finite MV-algebras are the same as finite products of finite chains.*

(iii) *Simple MV-algebras are the same as subalgebras of* $[0, 1]$.

Part (ii) in the theorem above yields

An MV-algebra A is said to be *semisimple* if for each nonzero $x \in A$ there is a homomorphism $\eta: A \to [0, 1]$ with $\eta(x) \neq 0$. Equivalently, $\bigcap \mathcal{M}(A) = \{0\}$.

Lemma 11.2.5 (*i*) *For any MV-algebra A and ideal $\mathfrak{m} \in \mathcal{M}(A)$, there is an isomorphism $\overline{\mathfrak{m}}$ of the quotient $A/\mathfrak{m}$ onto a subalgebra of* $[0, 1]$.
(*ii*) *If in addition, A is semisimple the map $a \in A \mapsto f_a \in [0, 1]^{\mathcal{M}(A)}$ defined by $f_a(\mathfrak{m}) = \overline{\mathfrak{m}}\,(a/\mathfrak{m})$, is an isomorphism of A onto a separating subalgebra A^* of $Cont(\mathcal{M}(A))$.*

Proof. Respectively from [10, 1.2.10, 3.5.1] and [10, 3.6]. ■

In conclusion we have

Theorem 11.2.6 [6], [10, Corollary 3.6.8] *The following conditions are equivalent for any MV-algebra A:*

(i) *A is semisimple.*

(ii) *Up to isomorphism, A is an MV-algebra of $[0, 1]$-valued functions over some set X.*

(iii) *Up to isomorphism, A is a separating MV-algebra of continuous $[0, 1]$-valued functions over some compact Hausdorff space X.*

Owing to the archimedean property of real numbers, the MV-algebra $[0, 1]$ is semisimple. Therefore, any MV-algebra of $[0, 1]$-valued functions over some set X, with the pointwise MV-operations of $[0, 1]$, is semisimple. The lemma above states that there are no other examples of semisimple MV-algebras: Intuitively, boolean algebras stand to $\{0, 1\}$-valued functions as semisimple MV-algebras stand to $[0, 1]$-valued functions.

11.3 Γ and the spectral topology

The topology of $\mathcal{M}(A)$ of Lemma 11.2.1 turns out to be the MV-algebraic counterpart of the spectral topology of the maximal ideal space $\mathcal{M}(G)$ of the unital ℓ-group (G, u) corresponding to A via the $\mathbf{\Gamma}$ functor, [2], [10, 7.2.3].

Using [10, 7.2.6], Lemma 11.2.5 and Theorem 11.2.6 can be strengthened as follows:

Theorem 11.3.1 *For any MV-algebra A and ideal $\mathfrak{m} \in \mathcal{M}(A)$, there is a* unique *isomorphism $\overline{\mathfrak{m}}$ of the quotient $A/\mathfrak{m}$ onto a subalgebra of $[0, 1]$. When A is semisimple, the map $a \in A \mapsto f_a \in [0, 1]^{\mathcal{M}(A)}$ defined by $f_a(\mathfrak{m}) = \overline{\mathfrak{m}}\,(a/\mathfrak{m})$, is an isomorphism of A onto a separating subalgebra A^* of $Cont(\mathcal{M}(A))$.*

States and maximal ideals. Following [21] and [18], for any ℓ-group G with order-unit u, let $S(G,u)$ denote the convex set of *states* of (G,u), i.e., the unit-preserving order-preserving homomorphisms of (G,u) into $(\mathbb{R},1)$. Let $\partial_{\rm e}S(G,u)$ denote the set of extremal states of (G,u). One immediately sees that $\partial_{\rm e}S(G,u)$ is a closed subspace of the product space $\prod_{a\in G}[-n_a,n_a]$, whence $\partial_{\rm e}S(G,u)$ is a compact Hausdorff space. As proved in [20, 3.2], $\partial_{\rm e}S(G,u)$ coincides with the set of all ℓ-homomorphisms $\chi\colon G\to\mathbb{R}$ such that $\chi(u)=1$. Further, the map

$$(11.1)\quad \chi\mapsto\ker(\chi)$$

induces a one-one correspondence between $\partial_{\rm e}S(G,u)$ and the maximal ℓ-ideal space $\mathcal{M}(G)$. In particular, $\partial_{\rm e}S(G,u)$ is nonempty. The inverse map sends each $\mathfrak{m}\in\mathcal{M}(G)$ to the homomorphism $\chi\colon G\to\mathbb{R}$ given by the quotient map

$$(11.2)\quad \chi(g)=g/\mathfrak{m}\in\mathbb{R},\quad (g\in G).$$

Here we are tacitly using the ℓ-group-theoretical counterpart of Theorem 11.3.1, namely Hölder theorem [2]: the latter states that that any ℓ-group without non-zero ℓ-ideals is isomorphic to $\mathbb{R}$, and the isomorphism is unique, if it is required to preserve units.

Let now $S(A)$ (resp., $\partial_{\rm e}S(A)$) denote the convex set of states (resp., extremal states) of an MV-algebra A. With a little more effort one can show

Theorem 11.3.2 *Let (G,u) be a unital $\sigma\ell$-group and $A=\mathbf{\Gamma}(G,u)$. Then the map $\chi\mapsto\ker(\chi)$ is a homeomorphism of $\partial_{\rm e}S(G,u)$ onto $\mathcal{M}(G)$. The map $\chi\mapsto\ker\chi\cap[0,u]$ is a homeomorphism of $\partial_{\rm e}S(G,u)$ onto the maximal ideal space $\mathcal{M}(A)$.*

If-then-else Consider the following generalized definition by cases:

$$(11.3)\begin{cases} & \text{if } h_1 \text{ holds} & \text{then} & e_1 \text{ follows,}\\ \text{else} & \text{if } h_2 \text{ holds} & \text{then} & e_2 \text{ follows,}\\ \dots & \dots & \dots & \dots\\ \text{else} & \text{if } h_n \text{ holds} & \text{then} & e_n \text{ follows.}\end{cases}$$

In many concrete cases, the hypotheses h_i do not form a boolean partition, but they are still thought of as forming an irredundant and exhaustive set of incompatible propositions in some logic L: working within L one may wish to establish some kind of logical interrelation between "causes" $\{h_1,...,h_n\}$ and "effects" $\{e_1,...,e_n\}$, generalizing what is done in the boolean case. It turns out that this program is feasible when L is the infinite-valued calculus of Łukasiewicz. Indeed, in every MV-algebra A one has a satisfactory generalization of the notion of boolean partition: by Theorem 11.1.1 A can be realized as the unit interval $A=[0,u]=\mathbf{\Gamma}(G,u)$ of *a unique* abelian lattice-ordered group G with a distinguished order-unit u. Thus we can say that a set of elements $h_1,\dots,h_n\in A$ forms a (nonboolean) partition iff $h_1+\dots+h_n=u$, and the set $h_1,\dots,h_n$ is linearly independent (in G). For more information on MV-algebraic partitions see [34].

From MV-algebras to ℓ-groups via $\mathbf{\Gamma}$. Since this tutorial is about MV-algebras, we shall mention only two applications of $\mathbf{\Gamma}$-functor theory to ℓ-groups. These are given by exporting to ℓ-groups the notion of Schauder basis associated to a unimodular triangulation of the k-cube $[0,1]^k$.

We refer to [10, 3.2, 9.1, 9.2] for the elementary notions and facts about polyhedra used in the following lines. All polyhedra considered here are contained in the k-cube $[0,1]^k$, and all vertices of all polyhedra have rational coordinates. Suppose $[0,1]^k$ is triangulated by a simplicial complex Σ: in other words, any two simplexes of Σ intersect in a common face, and the point-set union of the simplexes in Σ coincides with $[0,1]^k$. For short, we say that Σ is a *triangulation* of $[0,1]^k$.

For $\mathbf{z}$ a vertex of (some simplex in) Σ, let $d_{\mathbf{z}}$ be the least common denominator of the coordinates of $\mathbf{z}$. Then the *hat at* $\mathbf{z}$ (*over* Σ) is the uniquely determined continuous piecewise linear function $h_{\mathbf{z}}\colon [0,1]^k \to [0,1]$ which attains the value $1/d_{\mathbf{z}}$ at $\mathbf{z}$, vanishes at all remaining vertices of Σ, and is linear on each simplex of Σ. We say that Σ is *unimodular* if each hat $h_{\mathbf{z}}$ happens to be a McNaughton function.[2] In this case $h_{\mathbf{z}}$ is said to be a *Schauder hat.* The *Schauder basis* H_Σ *over* Σ is the set of Schauder hats $\{h_{\mathbf{z}} \mid \mathbf{z} \text{ is a vertex of } \Sigma\}$.

Schauder bases in free MV-algebras, as well as their homogeneous linear counterparts in ℓ-groups, are the key tool for the proof of the following two results:

Theorem 11.3.3 [33] *Every free ℓ-group G is* ultrasimplicial, *in the sense that for any finite set of elements $p_1, \ldots, p_k \in G^+$ there is a finite set $B \subseteq G^+$ of independent elements such that every p_i lies in the monoid generated by B.*

This result was extended by Marra to all ℓ-groups [28], thus solving a long-standing problem by Handelman.

Theorem 11.3.4 [27] *An ℓ-group is finitely generated and projective iff it is presentable by a single word in the language of lattices.*

The celebrated Baker-Beyon theory [17] only yields that G is finitely generated and projective iff it is presented by an ℓ-group word.

For more information on unimodular triangulations and their associated bases of Schauder hats, see [27, 29, 35, 36, 39].

[2] This definition is equivalent to the usual one [10, 9.1.1].

Chapter 12

σ-complete MV-algebras

As we have seen, every MV-algebra A carries a definable lattice structure, which turns out to coincide with the restriction of the lattice structure of its corresponding unital ℓ-group. A is said to be *σ-complete* if so is its underlying lattice; σ-complete MV-algebras have a central role in the generalization of boolean algebraic probability theory (see [47] and references therein). The $\mathbf{\Gamma}$-equivalents of σ-complete MV-algebras are known as Dedekind σ-complete ℓ-groups with order-unit (for short, *unital $\sigma\ell$-groups*). As shown in [21, Section II, 13-14], and [20], unital $\sigma\ell$-groups naturally arise[1] from an interesting class of $\aleph_0$-continuous regular rings, and finite Rickart algebras. Unital $\sigma\ell$-groups are interesting objects per se, and as such they are attracting increasing attention, [44], [8].

Since every σ-complete MV-algebra A is semisimple [10, 6.6.2], the map $a \mapsto f_a$ of Lemma 11.2.5 is an isomorphism of A onto the subalgebra $A^* \subseteq Cont(\mathcal{M}(A))$. For each maximal ideal $\mathfrak{m}$ be a of A we can safely identify A with A^*, and $A/\mathfrak{m}$ with $\overline{\mathfrak{m}}(A/\mathfrak{m}) \subseteq [0,1]$. Then for each $f \in A$ the quotient map $f \mapsto f/\mathfrak{m}$ amounts to evaluating f at $\mathfrak{m}$, in symbols, $f/\mathfrak{m} = f(\mathfrak{m})$. Accordingly, the basic open sets of $\mathcal{M}(A)$ can be realized as the sets of the form

$$support(f) = \{\mathfrak{m} \in \mathcal{M}(A) \mid f(\mathfrak{m}) > 0, \text{ where } f \in A\}.$$

Following [10, 1.5.2, 1.5.4] let $B(A)$ denote the subalgebra of A given by the *boolean* elements of A, those $b \in A$ such that $b \oplus b = b$. In the particular case when A is σ-complete, a trivial adaptation of the proof of [10, 6.6.5(i)] shows that $B(A)$ is a σ-complete boolean algebra; further, for any sequence $b_i \in B(A)$ the supremum of the b_i in $B(A)$ coincides with their supremum in A.

The mutual relations between A, $B(A)$ and $Cont(\mathcal{M}(A))$ are summarized in the following proposition, which follows from various results proved in [18] for Dedekind σ-complete unital ℓ-groups:

Proposition 12.0.5 *For any σ-complete MV-algebra A we have*

(i) The map $\xi\colon \mathfrak{m} \mapsto \mathfrak{m} \cap B(A)$ is a homeomorphism of $\mathcal{M}(A)$ onto the Stone space $\mathcal{M}(B(A))$;

[1] via Grothendieck functor $\mathbf{K_0}$

(ii) $\mathcal{M}(A)$ is a basically disconnected *(compact Hausdorff) space, in the sense that the closure of every open* F_σ*-set in $\mathcal{M}(A)$ is open.*

(iii) Identifying A with the subalgebra $A^* \subseteq Cont(\mathcal{M}(A))$ of Lemma 11.2.5 and supposing $f \in A$ to be the supremum in A of a sequence of elements a_i of A, it follows that the supremum of the a_i in $Cont(\mathcal{M}(A))$ exists and equals f.

Combining the $\mathbf{\Gamma}$ functor with the celebrated Goodearl-Handelman-Lawrence functional representation theorem [21, Proposition 1.4.7, Theorem 1.9.4], [18, 9.12-9.15] we have the following result:

Theorem 12.0.6 *Let A be a σ-complete MV-algebra. Let*

$$A' = \{f \in Cont(\mathcal{M}(A)) \mid f(\mathfrak{m}) \in \overline{\mathfrak{m}}(A/\mathfrak{m}) \quad \forall \mathfrak{m} \in \mathcal{M}(A)\},$$

where $\overline{\mathfrak{m}}$ is the canonical isomorphism of Lemma 11.2.5(i). Then the map $a \mapsto f_a$ of Lemma 11.2.5(ii) is an isomorphism of A onto A'.

12.1 Bounded finite rank

Let A be a σ-complete MV-algebra. A maximal ideal $\mathfrak{m}$ of A is said to have *finite rank* if for some integer $n \geq 1$ the quotient $A/\mathfrak{m}$ is (isomorphic to) the finite Łukasiewicz chain $\mathbb{Z}\frac{1}{n} \cap [0,1]$. When this is the case we write $\mathrm{rank}(\mathfrak{m}) = n$. Otherwise we say that $\mathfrak{m}$ has infinite rank, and we write $\mathrm{rank}(\mathfrak{m}) = \infty$. It is well known that if $\mathfrak{m}$ has infinite rank then $A/\mathfrak{m}$ is uniquely isomorphic to $[0,1]$, via the map $\bar{\mathfrak{m}}$ of Lemma 11.2.5(i).

The *numerical spectrum* of a *σ-complete* MV-algebra A is the set of integers $r \geq 1$ such that there is $\mathfrak{m} \in \mathcal{M}(A)$ of rank r. A is said to have *bounded finite rank* if there is an upper bound to the cardinality of its *finite* maximal quotients. Note that having bounded finite rank does not exclude the possibility that some (possibly every) maximal ideal of A has infinite rank. Similarly, a unital $\sigma\ell$-group (G,u) is said to have *bounded finite rank* if there is an upper bound on the integers n such that $G/\mathfrak{m} \cong \mathbb{Z}\frac{1}{n}$, letting $\mathfrak{m}$ range over $\mathcal{M}(G)$. Unital $\sigma\ell$-groups with bounded finite rank strictly include $\sigma\ell$-groups (G,u) with order-unit of finite index. By [19, 4.4] (also see the main result of [4]) any such (G,u) can be written as $(K_0(R),[R])$ for some regular, biregular ring with bounded index of nilpotency, satisfying suitable continuity properties.

The category $\mathbb{W}$. Following [8] we let $\mathbb{DED}$ denote the category whose objects are Dedekind σ-complete ℓ-groups with a distinguished order-unit, and whose morphisms are the unit preserving ℓ-group homomorphisms that also preserve all denumerable infima and suprema. We further denote by $\mathbf{\Gamma}(\mathbb{DED})$ the corresponding category of MV-algebras. It is not hard to see that objects in $\mathbf{\Gamma}(\mathbb{DED})$ are precisely the σ-complete MV-algebras, and morphisms are those homomorphisms that preserve all denumerable infima and suprema. The full subcategory of $\mathbb{DED}$ whose objects are the $\sigma\ell$-groups of bounded finite rank will be denoted by $\mathbb{BFR}$. By $\mathbf{\Gamma}(\mathbb{BFR})$ we shall denote the category of σ-complete MV-algebras with bounded finite rank.

In the rest of this section we shall define a duality between $\mathbf{\Gamma}(\mathbb{BFR})$ and a category $\mathbb{W}$ whose objects are given by the following definition (morphisms will be defined in 12.1.3 below):

Definition 12.1.1 *Objects* of $\mathbb{W}$ are triples $\langle X, S, \varphi\rangle$ such that X is a basically disconnected compact Hausdorff space, S is a (possibly empty) set of natural numbers and φ is a one-one map from S into the set of subsets of X satisfying the following three conditions for all $m, n \in S$:

(A1) $\varphi(n)$ is a non-empty special closed subset of X;

(A2) $\varphi(n) \supsetneqq \bigcup\{\varphi(j) : j \in S,\ \ j < n,\ \ j \text{ divides } n\}$;

(A3) $\varphi(m) \cap \varphi(n) = \bigcup\{\varphi(j) : j \in S,\ \ j \text{ is a common divisor of } m \text{ and } n\}$.

Notation: For each object $\langle X, S, \varphi\rangle$ in $\mathbb{W}$, and $n \in S$, we let

$$\varphi(n)' = \varphi(n) \setminus \bigcup\{\varphi(i) : n > i \in S \ \text{ and } \ i \text{ divides } n\}.$$

Lemma 12.1.2 *Let $\langle X, S, \varphi\rangle$ be an object in $\mathbb{W}$. For each $x \in \bigcup_{n\in S}\varphi(n)$ there is $n_x \in S$ with $x \in \varphi(n_x)$ having the additional property that for every $m \in S$, $x \in \varphi(m)$ if and only if n_x divides m. In other words, n_x is the minimum $n \in S$ such that $x \in \varphi(n)$.*

Let $\langle X, S, \varphi\rangle$ be an object in $\mathbb{W}$ and let $x \in X$. In the light of Lemma 12.1.2, we define the *virtual rank* $\mathrm{vrank}(x)$ of x as the minimum $n \in S$ such that $x \in \varphi(n)$ in case $x \in \bigcup_{n\in S}\varphi(n)$, and we set $\mathrm{vrank}(x) = \infty$ otherwise.

It follows that, for each $n \in S$, $\varphi(n)' = \{x \in X : \mathrm{vrank}(x) = n\}$. Hence (A2) asserts that there is at least one $x \in X$ such that $\mathrm{vrank}(x) = n$.

Given basically disconnected spaces X, Y, we say that a function $f : X \to Y$ is *σ-continuous* if it is continuous, and for each sequence $(U_n : n \in \mathbb{N})$ of clopen subsets of Y, we have the identity

$$\text{(12.1)}\quad \mathrm{int}\left(\bigcap_{n\in\mathbb{N}} f^{-1}(U_n)\right) = f^{-1}\left(\mathrm{int}\left(\bigcap_{n\in\mathbb{N}} U_n\right)\right).$$

Note that $f : X \to Y$ is σ-continuous if and only if it induces a σ-homomorphism from the dual σ-boolean algebra of Y into the dual σ-boolean algebra of X (cf.[49, §22]).

We can now complete the definition of the category $\mathbb{W}$:

Definition 12.1.3 Whenever $\langle X, S, \varphi\rangle$ and $\langle Y, T, \psi\rangle$ are objects in $\mathbb{W}$, by a *morphism* $\langle X, S, \varphi\rangle \to \langle Y, T, \psi\rangle$ we understand a σ-continuous function

$$f : X \to Y$$

such that for every $x \in X$, if $\mathrm{vrank}\,(x) < \infty$, then $\mathrm{vrank}\,(f(x))$ divides $\mathrm{vrank}\,(x)$.

Theorem 12.1.4 [8] *let $\mathbb{W}_{\mathrm{fin}}$ be the full subcategory of $\mathbb{W}$ whose objects are all triples $\langle X, S, \varphi\rangle$ with S finite. There is an equivalence between $\mathbb{W}_{\mathrm{fin}}$ and the opposite of $\mathbf{\Gamma}(\mathbb{BFR})$.*

For σ-complete MV-algebras with bounded finite rank one has an alternative functional representation, besides the one given in Theorem 12.0.6, as follows: For every topological space X, a function $f\colon X \to [0,1]$ is called *rectangular* if there is a clopen $C \subseteq X$ such that f is equal to a constant over C and zero over the complementary set $X \setminus C$.

A function $f\colon \mathcal{M}(A) \to [0,1]$ is said to be A-*admissible* if for each $\mathfrak{m} \in \mathcal{M}(A)$ with $\mathrm{rank}(\mathfrak{m}) = r < \infty$, $f(\mathfrak{m})$ is an integer multiple of $1/r$. Identifying A with $A^* \subseteq Cont(\mathcal{M}(A))$ we immediately see that all elements of A are A-admissible.

Theorem 12.1.5 *Suppose A is a σ-complete MV-algebra with bounded finite rank. Canonically identify A with the algebra $A^* \subseteq Cont(\mathcal{M}(A))$ of Lemma 11.2.5. A function $f\colon \mathcal{M}(A) \to [0,1]$ belongs to A if and only if f is continuous and A-admissible. Specifically, every function $f \in A$ is the supremum in A of a countable sequence of rational-valued, rectangular, A-admissible functions; f also coincides with the supremum of these functions in $Cont(\mathcal{M}(A))$.*

Problem. Extend the last two theorems to larger classes of σ-complete MV-algebras.

Chapter 13

Miscellanea

In this chapter various kinds of results are collected, in order to show the flexibility of MV-algebras and their deep connections with other mathematical areas.

13.1 The word problem for MV-algebras

Consider the following problem:

INSTANCE: An MV-term $\tau = \tau(x_1, \ldots, x_n)$.

QUESTION: Does the identity $\tau = 0$ identically hold in all MV-algebras ? (Equivalently, does the identity hold in the free MV-algebra $Free_n$ over n free generators ? Equivalently, does it hold in $[0, 1]$?)

We shall give a short proof that the problem is co-NP-complete, i.e., its complementary problem is NP-complete.[1] Letting the variable X_i represent the ith projection function π_i, and proceeding as in Section 3, τ will represent in $Free_n$ a McNaughton function $\tau^{Free_n} = f_\tau(x_1, \ldots, x_n)$. Let $\mathrm{occ}(\tau)$ denote the number of occurrences of variable symbols in τ. For all points $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ with $\mathbf{x} \neq \mathbf{y}$, the *one-sided derivative* of f_τ at $\mathbf{x}$ along direction $\mathbf{d} = \mathbf{y} - \mathbf{x}$ is defined by

$$f'_\tau(\mathbf{x}; \mathbf{d}) = \lim_{\epsilon \downarrow 0} \frac{f_\tau(\mathbf{x} + \epsilon \mathbf{d}) - f_\tau(\mathbf{x})}{\epsilon}.$$

We let $||\mathbf{d}||$ denote the euclidean norm of $\mathbf{d} \in \mathbb{R}^n$. Arguing by induction on $\mathrm{occ}(\tau)$ we immediately get

(13.1) $\quad |f'_\tau(\mathbf{x}; \mathbf{d})| \quad \leq \quad ||\mathbf{d}|| \cdot \mathrm{occ}(\tau).$

Let $p(x_1, \ldots, x_n) = c + m_1 x + \cdots + m_n x_n$ be a linear polynomial with integer coefficients $c, m_1, \ldots, m_n$. Suppose f_τ coincides with p over an n-dimensional simplex $T \subseteq [0, 1]^n$. Then by (13.1), $\max(|m_1|, \ldots, |m_n|) \leq \mathrm{occ}(\tau)$. Assume f_τ does not identically vanish over $[0, 1]^n$. Then f_τ attains its maximum at

[1] Readers of this section should have some familiarity with computational complexity theory

some point $\mathbf{z} \in [0,1]^n$ where $n+1$ linear pieces of f_τ have the same value (with a trivial modification in case $\mathbf{z}$ lies on a face of $[0,1]^n$). Elementary linear algebra, together with Hadamard's determinant inequality yields a point $\mathbf{z}^* = (a_1/b, \ldots, a_n/b) \in [0,1]^n$ with $a_i, b \in \mathbb{Z}$ and $0 \le a_i \le b$ $(i = 1, \ldots, n)$, such that $f_\tau(\mathbf{z}^*) > 0$ and $0 < b < 2^{(4\,\mathrm{occ}(\tau)^2)}$.

Theorem 13.1.1 [32] *The tautology problem for the infinite-valued calculus of Łukasiewicz (i.e., the problem of deciding if an MV-term is identically equal to* 1*) is co-NP-complete.*

Proof. We shall deal with the dual problem of deciding in $\tau = 0$.

Claim 1. The problem is in co-NP.

Indeed, after guessing a rational point $\mathbf{z}^* = (a_1/b, \ldots, a_n/b) \in [0,1]^n$ such that $f_\tau(\mathbf{z}^*) > 0$ and $0 < b < 2^{(4\,\mathrm{occ}(\tau)^2)}$, we quickly check that $f_\tau(\mathbf{z}^*) > 0$ as follows: we write each coordinate a_i/b as a pair of binary integers; denoting by $[[a_i]]$ and $[[b]]$ the number of bits of a_i and b, we have $[[a_i]] \le [[b]] \le 4\,\mathrm{occ}(\tau)^2$ for all $i = 1, \ldots, n$. Once $\mathbf{z}^*$ is written down as a sequence of pairs of binary numbers, its length $[[\mathbf{z}^*]]$ will satisfy the inequalities

$$[[\mathbf{z}^*]] \le 1 + n(2 + [[b]] + \max_i [[a_i]]) \le 1 + n(2 + 8\,\mathrm{occ}(\tau)^2) \le 11\,\mathrm{occ}(\tau)^3.$$

Since the operations of negation and truncated addition do not increase denominators, for some fixed polynomial q (independent of τ) the value $f_\tau(\mathbf{z}^*)$ is computable a deterministic Turing machine within a number of steps $\le q(\mathrm{occ}(\tau))$.

Having thus settled our first claim, in order to prove co-NP-hardness, for all integers $i \ge 1$ and $t \ge 2$ we define the MV-terms $\psi_{i,t}$, and $\rho_{n,t}$ by $\psi_{i,t} = (X_i \vee \neg X_i) \odot \ldots \odot (X_i \vee \neg X_i)$ (t times), and $\rho_{n,t} = \psi_{1,t} \odot \ldots \odot \psi_{n,t}$. We shall write $f_{n,t}$ to denote the McNaughton function corresponding to $f_{\rho_{n,t}}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{2^n}$ be the vertices of the cube $[0,1]^n$. Let $\mathcal{E}_{j1}, \ldots, \mathcal{E}_{jn}$ be the edges of $[0,1]^n$ adjacent to $\mathbf{v}_j$. For each $i = 1, \ldots, n$ and $t \ge 2$ let $\mathbf{y}_{ji}$ be the point lying on edge $\mathcal{E}_{ji}$ at a distance $1/t$ from $\mathbf{v}_j$. Let $\mathcal{T}_j$ be the n-simplex with vertices $\mathbf{v}_j, \mathbf{y}_{j1}, \ldots, \mathbf{y}_{jn}$. Then a tedious but straightforward verification shows that

(i) $f_{n,t}(\mathbf{v}_j) = 1$;

(ii) $f_{n,t}(\mathbf{y}_{ji}) = 0$;

(iii) $f_{n,t}$ is linear over each simplex $\mathcal{T}_j$;

(iv) $f_{n,t}$ vanishes over $[0,1]^n \setminus \bigcup_{j=1}^{2^n} \mathcal{T}_j$.

Claim 2. Let $\phi = \phi(X_1, \ldots, X_n)$ be an MV-term, and suppose $t = \mathrm{occ}(\phi)$ with $t \ge 2$. Then ϕ (with $\oplus$ read as boolean disjunction) is a tautology in the boolean calculus iff $\neg\rho_{n,t} \oplus \phi$ is a tautology in the infinite-valued calculus (iff $f_{n,t} \le f_\phi$).

One direction is trivial. For the converse, assume ϕ to be a tautology in the boolean calculus. By (iv) above, the inequality $f_{n,t} \le f_\phi$. holds over the set $[0,1]^n \setminus \bigcup_j \mathcal{T}_j$. By way of contradiction, assume $f_{n,t}(\mathbf{x}) > f_\phi(\mathbf{x})$ for some $j = 1, \ldots, 2^n$ and $\mathbf{x} \in \mathcal{T}_j$. By continuity we can assume $\mathbf{x}$ to be in the interior of $\mathcal{T}_j$, whence in particular, $\mathbf{x} \ne \mathbf{v}_j$. Let $\mathbf{w}$ be the unit vector in the direction from

$\mathbf{x}$ to $\mathbf{v}_j$. By (iii), $f'_{n,t}(\mathbf{y};\mathbf{w}) \geq t$ for each point $\mathbf{y} \neq \mathbf{v}_j$ lying in the interval $[\mathbf{x}, \mathbf{v}_j]$. On the other hand, by (13.1), $f'_\phi(\mathbf{y};\mathbf{w}) \leq t$, whence $f'_{n,t}(\mathbf{y};\mathbf{w}) \geq f'_\phi(\mathbf{y};\mathbf{w})$. By our assumption about ϕ, $f_{n,t}(\mathbf{v}_j) = f_\phi(\mathbf{v}_j) = 1$. Since $f_{n,t}$ is linear on the interval $[\mathbf{v}_j, \mathbf{x}]$ and f_ϕ is (continuous and) piecewise-linear on $[\mathbf{v}_j, \mathbf{x}]$, we conclude that $f_{n,t} \leq f_\phi$ over $[\mathbf{x}, \mathbf{v}_j]$, a contradiction.

We have exhibited a polytime reduction of the boolean tautology problem to the tautology problem for the infinite-valued Łukasiewicz calculus. This completes the proof. ∎

13.2 Finite-valued MV-algebras.

Chang Completeness Theorem states that the MV-algebra $[0,1]$ generates the variety (= equational class) of all MV-algebras. One can similarly study the variety $\mathcal{MV}_n$ generated by the MV-algebra $\mathbf{L}_n$, $n = 2, 3, \ldots$. A complete axiomatization of $\mathcal{MV}_n$ was given by Grigolia as follows:

Theorem 13.2.1 [23], [10, 8.5] *An MV-algebra A is a member of $\mathcal{MV}_n$ iff it satisfies $(n-1)x = nx$, together with the equations $px^{p-1} = nx^p$, for every integer $p = 2, 3, \ldots, n-2$ not dividing $n-1$. In particular, $\mathcal{MV}_2$ coincides with the variety of boolean algebras. Also, an MV-algebra is in $\mathcal{MV}_3$ iff it satisfies $x \oplus x \oplus x = x \oplus x$.*

Corollary 13.2.2 *Fix $n = 2, 3, \ldots$. Given MV-terms σ and τ, the following conditions are equivalent for the equation $\sigma = \tau$:*

(i) The equation holds in the MV-algebra $\mathbf{L}_n$.

(ii) The equation follows from the equations in Theorem 13.2.1, via substitutions of equals for equals.

13.3 The most general MV-algebra.

Di Nola's representation theorem, yields a functional representation for the most general MV-algebra. The proof requires familiarity with $\mathbf{\Gamma}$-functor theory [10, Section 7] and model-theory, with particular reference to elementary ultraproduct embeddings, and to the elementary theory of totally ordered divisible abelian groups:

Theorem 13.3.1 [10, 9.5.1] *Up to isomorphism, every MV-algebra A is an algebra of $[0,1]^*$-valued functions over some set, where $[0,1]^*$ is an ultrapower of $[0,1]$ only depending on the cardinality of A.*

Proof. By Theorem 2.1.2, A is embeddable into the MV-algebra $\prod\{A/I \mid I \in \mathcal{P}(A)\}$. For each prime ideal I of A there is a (uniquely determined) totally ordered abelian group G_I with order-unit u_I satisfying $\mathbf{\Gamma}(G_I, u_I) \cong A/I$. We canonically embed G_I into a totally ordered *divisible* abelian group K_I with the same strong unit u_I. It follows that A/I is embeddable into the MV-algebra $D_I = \mathbf{\Gamma}(K_I, u_I)$. Since any totally ordered divisible abelian group is elementarily equivalent to the additive group $\mathbb{R}$ of real numbers with natural order, it follows that the MV-algebras D_I and $[0,1]$ are elementarily equivalent. By Frayne theorem, D_I is elementarily embeddable in an ultrapower $[0,1]^{*_I}$ of $[0,1]$. The joint embedding property of first-order logic now yields an ultrapower $[0,1]^*$ (only depending on the cardinality of A), such that every MV-algebra $[0,1]^{*_I}$ is elementarily embeddable into $[0,1]^*$. Thus every quotient MV-algebra A/I is embeddable into $[0,1]^*$, whence the desired conclusion immediately follows. ∎

13.4 MV-algebras and AF C^*-algebras.

An *approximately finite-dimensional C^*-algebra* (for short, *AF C^*-algebra*) [3] is the norm closure of the union of a sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ of finite-dimensional C^*-algebras, where each $\mathcal{F}_i$ is a *-subalgebra of $\mathcal{F}_{i+1}$. Among others, AF C^*-algebras are used for a rigorous description of infinite spin systems. Elliott's celebrated classification of AF C^*-algebras goes back to his 1976 paper [15]. We shall give a succinct presentation of the relations between AF C^*-algebras and countable MV-algebras, as follows:

Given an AF C^*-algebra $\mathcal{A}$, two projections[2] $p, q \in \mathcal{A}$ are *equivalent* iff there exists an element $v \in \mathcal{A}$ such that $vv^* = p$ and $v^*v = q$. We denote by $[p]$ the equivalence class of p, and by $L(\mathcal{A})$ the set of equivalence classes of projections of $\mathcal{A}$. The *Murray-von Neumann order* over $L(\mathcal{A})$ is defined by: $[p] \leq [q]$ iff p is equivalent to a projection r such that $rq = r$.

Elliott partial addition is the partial operation $+$ on $L(\mathcal{A})$ obtained by adding two projections whenever they are orthogonal. This operation is associative, commutative, monotone, and has the following *residuation* property: For every projection $p \in \mathcal{A}$, among all classes $[q]$ such that $[p] + [q] = [1_{\mathcal{A}}]$ there is a smallest one, denoted $\neg[p]$, namely the class $[1_{\mathcal{A}} - p]$. Here $1_{\mathcal{A}}$ denotes the unit element of $\mathcal{A}$. As a corollary of the main results of [31], in [38] one finds a proof of the following

Theorem 13.4.1 *For every AF C^*-algebra $\mathcal{A}$ we have:*

(i) There is at most one extension of Elliott partial addition to an associative, commutative, monotone operation $\oplus$ over $L(\mathcal{A})$, satisfying the residuation property. Such extension $\oplus$ exists iff $L(\mathcal{A})$ is a lattice.

(ii) Letting $\mathcal{K}(\mathcal{A}) = (L(\mathcal{A}), [0], \neg, \oplus)$ the map $\mathcal{A} \mapsto \mathcal{K}(\mathcal{A})$ is a one-one correspondence between (all isomorphism classes of) AF C^*-algebras whose Murray-von Neumann order over $L(\mathcal{A})$ is a lattice, and countable MV-algebras.

(iii) In particular, the map $\mathcal{A} \mapsto \mathcal{K}(\mathcal{A})$ determines a one-one correspondence between commutative AF C^*-algebras and countable Boolean algebras, between finite-dimensional C^*-algebras and finite MV-algebras, between Glimm's UHF algebras [14] and rational subalgebras of $[0,1]$.

[2] a *projection* p is a self-adjoint idempotent element $p = p^* = p^2$

Subsequent work by Effros, Handelman, Shen, Goodearl and others showed that Elliott's partial monoid can be replaced by a certain class of countable partially ordered abelian groups, called dimension groups, and that the classifying functor is (a suitable order-theoretic enrichment of) Grothendieck's functor $\mathbf{K}_0$. Since dimension groups are a generalization of unital ℓ-groups, the one-one correspondence of Theorem 13.4.1 turns out to be induced by the composite functor $\mathbf{\Gamma} \circ \mathbf{K}_0$.

Since countable MV-algebras are the algebras of Łukasiewicz infinite-valued calculus over countably many variables [10, 4.6.9], the combinatorial and algorithmic machinery of Łukasiewicz logic can be transferred to AF C^*-algebras via the inverse of $\mathbf{\Gamma} \circ \mathbf{K}_0$, [9, 35]. Remarkably enough, countable free MV-algebras are transformed by the inverse of $\mathbf{\Gamma} \circ \mathbf{K}_0$ into AF C^*-algebras with universal AF C^*-algebraic properties, [31, Corollary 8.8]. Last, but not least, Marra's ultrasimplicial theorem [28] yields a functor $\mathbf{U}$ from countable unital ℓ-groups to AF C^*-algebras $\mathcal{A}$ whose Murray-von Neumann order of projections is a lattice, in such a way that $\mathbf{U}(\mathbf{K}_0(\mathcal{A})) \cong \mathcal{A}$.

Further Reading

In the last 20 years the literature devoted to MV-algebras has been rapidly increasing.[3] Here we shall only quote a few selected examples of surveys and monographs. In the book [10], which is entirely devoted to Łukasiewicz logic and MV-algebras, one can find self-contained proofs of all fundamental theorems about MV-algebras and Łukasiewicz logic. Hájek's monograph [25] devotes ample space to MV-algebras, and so does Gottwald's book [22]. In the second edition of the Handbook of Philosophical Logic, Urquhart's classical chapter has been updated and expanded [52]. Further, one finds there a new chapter, by Hähnle [24], on the complexity of many-valued proof-theory. As shown in the monograph [13] and in the pioneering textbook [46], MV-algebras and their states also yield an important specimen of "quantum structures". The second volume of the Handbook of Measure Theory [40] includes a chapter entirely devoted MV-algebraic probability theory [47] and other chapters where MV-algebras have an important role for the non-boolean approach to algebraic measure theory á la Carathéodory. The survey [16] gives a detailed account of the universal algebraic properties of the equational class of MV-algebras. The survey [29] is especially devoted to the relationship between MV-algebras and lattice-ordered groups. One can also find in the literature several surveys and original papers devoted to the Rényi-Ulam game-theoretic interpretation of infinite-valued logic, and its applications to error-correcting codes, fault-tolerant search, algorithmic learning and logic programming [42, 7, 26].

[3] The literature has expanded so rapidly that in the year 2000 the AMS Classification Index introduced the special item 06D35 for MV-algebras.

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