PROPERTIES OF THE LAMBERT FUNCTION W(z)

The first order, non-linear, ODE-

$$\frac{dW(z)}{dz} = \frac{\exp[-W(z)]}{[1+W(z)]} \quad \textit{subject to } W(0) = 0$$

can be solved by the simple integration-

$$\int_0^z dz = \int_0^{W(z)} [1+W(z)]\exp[1+W(z)]dW(z)$$

to yield the implicit solution-

$$z = W(z)\exp[W(z)]$$

where W(z) is the Lambert function.

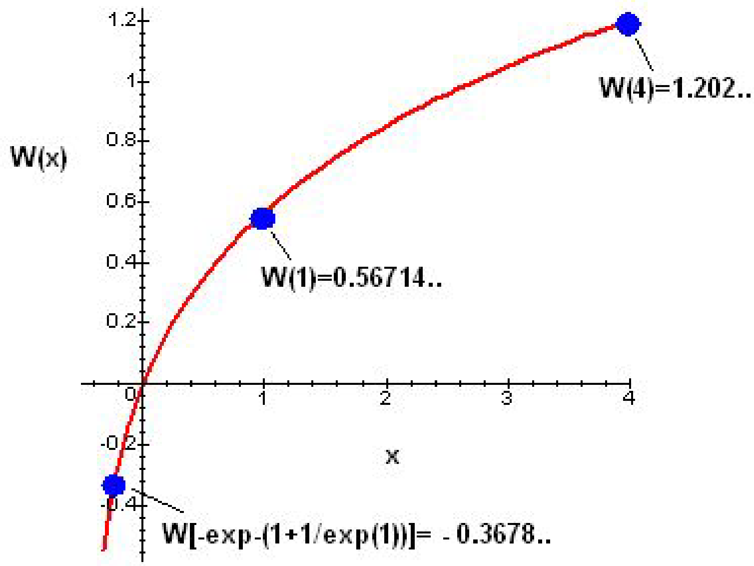
One can expand this function in a Taylor series-

$$W(z) = W(0) + \frac{dW(0)}{dz} z + \frac{d^2W(0)}{dz^2}\frac{z^2}{2!} + ...$$

to obtain-

$$W(z) = z - z^2 + \frac{3z^3}{2} - \frac{8z^4}{3} + O(z^5) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n$$

A plot of W(z) for z=x in the range -0.3678<x<4 follows-



There are several direct applications of the Lambert function. One of the better known of these is in finding the limit of an iteration connected with the tetration N^(N^(N^(N . Tetration of a number N can be represented by the iteration-

$$a[n+1] = N^{a[n]} = \exp(a[n]\ln N) \ with \ a[0] = N$$

For this iteration to converge one must have that-

$$a[\infty]\exp(a[\infty]\ln(\frac{1}{N})) = 1$$

which is equivalent to-

$$a[\infty] = \frac{W(\ln\frac{1}{N})}{\ln(\frac{1}{N})}$$

Thus one has that tetration for N=a[0]=i yields a[∞]=0.43828293..+i 0.36059247..

Another place where the Lambert function is encountered is in the solution of the difference equation-

$$\frac{dx(t)}{dt} = c\ x(t-1)$$

We try x(t)=exp(b t) to yield-

$$b = c\ \exp(-b)\ \textit{or equivalent}\ \ b = W(c)$$

so that the equation yields the solution-

$$x(t) = \exp[\,W\,(c)\ t\,]$$

Also one can find certain values of z for which W(z) assumes simple closed forms. Start with the function-

$$F(z) = \frac{W(\ln(z))}{\ln(z)} = \exp(-W(\ln(z))$$

We find this function has the exact values F[1/sqrt(2)]=2, F[1]=1, and F[4]=0.5. From these results one can infer, for example, that-

$$W(2\ln 2) = \ln(2)\ \ \textit{and}\ \ W[\ln(\frac{1}{\sqrt{2}})] = -\ln(2)$$

This last result in turn suggests one try W[ln(a)]=ln(b). This leads to-

$$W[\ln(a)]\exp W[\ln(a)] = \ln(b)\exp(\ln(b)) = \ln(a)$$

from which follows that $a=b^b$ so that we have the identities-

$$W(b\ln b) = \ln(b)\ \ \textit{and}\ \ W[c\exp(c)] = c$$

where c= ln(b). From these last identities follow the equalities-

$$W[\exp(1)] = 1\,,\; W[\exp(-1)] = -1\,,\; W(\frac{-\pi}{2}) = i\frac{\pi}{2}$$

Also by setting z=i, we obtain the identity-

$$\pi = -2i\{W(i) + \ln[W(i)]\}$$