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GENERALIZED HYPERGEOMETRIC FUNCTIONS

BY

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PREFACE

This book should really be attributed to Bailey and Slater. It was Professor Bailey's intention to write a comprehensive work on hypergeometric functions, with my assistance. This present work is based in part on notes for a series of lectures which he gave in 1947–50 at Bedford College, London University. The rest of the book contains the results of my own researches into the general theory. It also covers the great advances made in the subject since 1936 when W. N. Bailey's Cambridge Tract 'Generalized Hypergeometric Series' was first published.

The theory of generalized hypergeometric functions is fundamental in the field of Mathematical Physics, since the general functions studied here contain as special cases all the commonly used functions of analysis. The present work should prove of use and interest to all Mathematical Analysts and Theoretical Physicists. The generalized Gauss function is also used increasingly in Mathematical Statistics, and the basic analogues of the Gauss functions have many interesting applications in the field of Number Theory.

I should like to thank Dr Theo Chaundy for a very careful reading of the manuscript, and several helpful comments.

L. J. S.

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