CHAPTER 1

Quantum Field Theory Basics

Introduction

This chapter is devoted to basic aspects of quantum field theory, ranging from the foundations to perturbation theory and renormalization, and is limited to the canonical formalism (functional methods are treated in Chapter 2) and to the traditional workflow (Lagrangian $\longrightarrow$ Feynman rules $\longrightarrow$ time-ordered products of fields $\longrightarrow$ scattering amplitudes) for the calculation of scattering amplitudes (the spinor-helicity formalism and on-shell recursion are considered in Chapter 4). The problems of this chapter deal with questions in scalar field theory and quantum electrodynamics, while non-Abelian gauge theories are discussed in Chapter 3.

Non-interacting Field Theory

A non-interacting field theory may be defined by a quadratic Lagrangian. In the simplest case of a scalar field theory, it reads

$$\mathcal{L} \equiv \int d^3\boldsymbol{x} \left\{ \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2} m^2 \phi^2 \right\}. \tag{1.1}$$

Such a Lagrangian defines a dynamical system with infinitely many degrees of freedom, corresponding to the values taken by $\phi(x)$ at every point $\boldsymbol{x}$ of space. The momentum canonically conjugate to $\phi(x)$ is given by

$$\Pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi(x))} = \partial^0 \phi(x), \tag{1.2}$$

which leads to the Hamiltonian

$$\mathcal{H} \equiv \int d^3\boldsymbol{x}\; \Pi(x)\partial^0\phi(x) - \mathcal{L} = \int d^3\boldsymbol{x} \left\{ \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \tfrac{1}{2} m^2 \phi^2 \right\}. \tag{1.3}$$

From the Hamiltonian or Lagrangian, one obtains the equation of motion of the field, which in the present example reads

$$\big(\square_x + m^2\big)\phi(x) = 0, \tag{1.4}$$

known as the *Klein–Gordon equation*. Generic real solutions of this linear equation are superpositions of plane waves:

$$\phi(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2E_{\boldsymbol{k}}} \Big\{\alpha^*_{\boldsymbol{k}}\, e^{-ik\cdot x} + \alpha_{\boldsymbol{k}}\, e^{ik\cdot x}\Big\}, \tag{1.5}$$

where $E_{\boldsymbol{k}} \equiv \sqrt{\boldsymbol{p}^2 + m^2}$ is the dispersion relation associated with the wave equation (1.4), and $\alpha_{\boldsymbol{k}}$ is a function of momentum that depends on the boundary conditions imposed on the solution.

Canonical quantization consists in promoting the coefficients $\alpha_{\boldsymbol{k}}, \alpha^*_{\boldsymbol{k}}$ into annihilation and creation operators $a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}$ that obey the following commutation relations:

$$\big[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\big] \equiv (2\pi)^3 2E_{\boldsymbol{p}}\, \delta(\boldsymbol{p} - \boldsymbol{q}). \tag{1.6}$$

The normalization in Eqs. (1.5) and (1.6) is chosen so that $\big[\mathcal{H}, a^\dagger_{\boldsymbol{p}}\big] = E_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}}$ and $\big[\mathcal{H}, a_{\boldsymbol{p}}\big] = -E_{\boldsymbol{p}} a_{\boldsymbol{p}}$, which means that $a^\dagger_{\boldsymbol{p}}$ increases the energy of the system by $E_{\boldsymbol{p}}$ while $a_{\boldsymbol{p}}$ decreases it by the same amount. As a consequence, this setup describes a collection of non-interacting particles. The commutation relation (1.6) implies the following equal-time commutation relation between the field operator and its conjugate momentum:

$$\big[\phi(x), \Pi(y)\big] \underset{x^0=y^0}{=} i\,\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{1.7}$$

which one may view as the quantum version of the classical Poisson bracket between a coordinate and its conjugate momentum.

Interacting Field Theory and Interaction Representation

Interactions are introduced via terms of degree higher than two in the Lagrangian:

$$\mathcal{L} \equiv \int d^3\boldsymbol{x}\, \Big\{ \underbrace{\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2}_{\mathcal{L}_0,\ \text{non-interacting theory}} \underbrace{-V(\phi)}_{\text{interactions}} \Big\}. \tag{1.8}$$

(In order to have a causal theory, the potential $V(\phi)$ must be a local function of the field $\phi(x)$; see Problem 4.) In the presence of interactions, the Klein–Gordon equation of motion becomes

$$\big(\square_x + m^2\big)\phi(x) + V'(\phi(x)) = 0. \tag{1.9}$$

Since the degree of $V(\phi)$ is higher than two, this equation is non-linear, which induces a mixing between the Fourier modes of the field and prevents writing its solutions as superpositions of plane waves.

By assuming that the interactions are turned off at large times, $x^0 \to \pm\infty$, we may define free fields ϕ_{in} and ϕ_{out} that coincide with the interacting field ϕ of the Heisenberg representation, respectively when $x^0 \to -\infty$ and $x^0 \to +\infty$. For instance, ϕ and ϕ_{in} are related by

$$\phi(x) = U(-\infty, x^0)\,\phi_{in}(x)\,U(x^0, -\infty),$$
$$U(t_2, t_1) \equiv T\,\exp\left(-i\int_{t_1}^{t_2} dx^0 d^3\boldsymbol{x}\; V(\phi_{in}(x))\right). \tag{1.10}$$

In this representation, the time dependence of the field $\phi(x)$ is split into a trivial one that comes from the free field ϕ_{in} and the time evolution operator U that depends on the interactions. Since they are free fields obeying Eq. (1.4), ϕ_{in} and ϕ_{out} can be written as superpositions of plane waves, with coefficients $a_{\boldsymbol{p},in}$, $a^\dagger_{\boldsymbol{p},in}$ and $a_{\boldsymbol{p},out}$, $a^\dagger_{\boldsymbol{p},out}$, respectively. These two sets of creation and annihilation operators define two towers of *Fock states*, i.e., states with a definite particle content at $x^0 = -\infty$ and $x^0 = +\infty$, respectively.

Lehmann–Symanzik–Zimmermann Reduction Formulas

Experimentally measurable quantities, such as cross-sections, may be related to correlation functions of the field operator as follows. An intermediate step involves the transition amplitudes between *in* and *out* states,

$$\big\langle \boldsymbol{q}_1 \cdots \boldsymbol{q}_{n\,out} \big| \boldsymbol{p}_1 \cdots \boldsymbol{p}_{m\,in} \big\rangle \equiv (2\pi)^4 \delta\Big(\sum_i p_i - \sum_j q_j\Big)\; \mathcal{T}(\boldsymbol{q}_{1\cdots n} | \boldsymbol{p}_{1\cdots m}), \tag{1.11}$$

in terms of which a cross-section in the center of momentum frame is given by

$$\sigma_{12\to 1\cdots n}\Big|_{\substack{\text{center of}\\ \text{momentum}}} = \frac{1}{4\sqrt{s}\,|\boldsymbol{p}_1|} \int d\Gamma_n(\boldsymbol{p}_{1,2})\, \Big|\mathcal{T}(\boldsymbol{q}_{1,\cdots,n}|\boldsymbol{p}_{1,2})\Big|^2, \tag{1.12}$$
$$\text{with}\;\; d\Gamma_n(\boldsymbol{p}_{1,2}) \equiv \prod_j \frac{d^3\boldsymbol{q}_j}{(2\pi)^3 2E_{\boldsymbol{q}_j}}\; (2\pi)^4\delta\big(p_1 + p_2 - \sum_j q_j\big), \qquad s \equiv (p_1+p_2)^2.$$

In turn, the transition amplitudes from *in* to *out* states are expressed in terms of expectation values of time-ordered products of field operators by the Lehmann–Symanzik–Zimmermann (LSZ) reduction formulas:

$$\big\langle \boldsymbol{q}_1 \cdots \boldsymbol{q}_{n\;out} \big| \boldsymbol{p}_1 \cdots \boldsymbol{p}_{m\;in} \big\rangle = \frac{i^{m+n}}{Z^{\frac{m+n}{2}}} \int \prod_{i=1}^{m} d^4x_i\; e^{-ip_i\cdot x_i}\; (\square_{x_i} + m^2)$$
$$\times \int \prod_{j=1}^{n} d^4y_j\; e^{iq_j\cdot y_j}\; (\square_{y_j} + m^2)\; \big\langle 0_{out} \big| T\,\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n) \big| 0_{in} \big\rangle, \tag{1.13}$$

where Z is the wavefunction renormalization factor.

Generating Functional, Feynman Propagator

The calculation of expectation values of time-ordered products of field operators is usually organized by encapsulating them in a generating functional

$$\big\langle 0_{out} \big| T\,\phi(x_1)\cdots\phi(x_n) \big| 0_{in} \big\rangle = \frac{\delta^n Z[j]}{i\delta j(x_1)\cdots i\delta j(x_n)}\bigg|_{j\equiv 0}, \tag{1.14}$$

$$\text{with}\quad Z[j] \equiv \big\langle 0_{\text{out}} \big| \mathrm{T} \, \exp i \int d^4x \; j(x)\phi(x) \big| 0_{\text{in}} \big\rangle \tag{1.15}$$

$$= \exp\left(-i \int d^4x \; V\left(\frac{\delta}{i\delta j(x)}\right)\right) \underbrace{\big\langle 0_{\text{in}} \big| \mathrm{T} \, \exp i \int d^4x \; j(x)\phi_{\text{in}}(x) \big| 0_{\text{in}} \big\rangle}_{Z_0[j],\quad \text{non-interacting theory}} . \tag{1.16}$$

The last factor, the generating functional of the non-interacting theory, is a Gaussian in the auxiliary source j:

$$Z_0[j] = \exp\left(-\frac{1}{2} \int d^4x d^4y \; j(x) j(y) G^0_{F}(x, y)\right), \tag{1.17}$$

where $G^0_F(x, y)$ is the free *Feynman propagator*, which can be expressed in various equivalent ways:

$$G^0_F(x, y) = \big\langle 0_{\text{in}} \big| \mathrm{T} \, \phi_{\text{in}}(x)\phi_{\text{in}}(y) \big| 0_{\text{in}} \big\rangle \tag{1.18}$$

$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} \Big(\theta(x^0 - y^0)\, e^{-ip\cdot(x-y)} + \theta(y^0 - x^0)\, e^{+ip\cdot(x-y)} \Big), \tag{1.19}$$

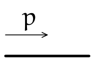
$$G^0_F(p) = \frac{i}{p^2 - m^2 + i0^+}. \tag{1.20}$$

Feynman Rules of Scalar Field Theory

The effect of interactions can be calculated order-by-order by expanding the first exponential in Eq. (1.16). The successive terms of this expansion are obtained from a diagrammatic expansion, where each diagram is converted into a formula by means of *Feynman rules*. Below we list these rules in momentum space, for a scalar field theory:

1. Draw all the graphs with as many external lines as field operators in the correlation function, and a number of vertices equal to the desired order. The vertices allowed in these graphs must have valences equal to the degrees of the terms in $V(\phi)$. Graphs with multiple connected components need not be considered in the calculation of scattering amplitudes.

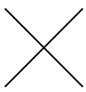
2. A 4-momentum k is assigned to each internal line of the graph, and the associated Feynman rule is a free propagator $G^0_F(k)$:

   $$= \frac{i}{p^2 - m^2 + i0^+}.$$

   No propagator should be assigned to the external lines of a graph when calculating a scattering amplitude (because of the factors $\square + m^2$ in the reduction formulas).

3. For an interaction $\frac{\lambda}{n!}\phi^n$, each vertex of valence n brings a factor $-i\lambda(2\pi)^4\delta\big(\sum_i k_i\big)$, where the k_i are the momenta incoming into this vertex:

   $$= -i\lambda.$$

3. All the internal momenta that are not constrained by the delta functions at the vertices should be integrated over with a measure $d^4k/(2\pi)^4$. In a connected graph with $n_{_I}$ internal lines and n_V vertices, there are $n_{_L} = n_{_I} - n_V + 1$ of them, which is also the number of *loops* in the graph.

4. Each graph must be weighted by a *symmetry factor*, defined as the inverse of the order of the discrete symmetry group of the graph (assuming interaction terms properly symmetrized, as in $V(\phi) = \phi^n/n!$).

Dimensional Regularization

The momentum integrals that correspond to loops in Feynman diagrams may be divergent at large momentum. Convergence may be assessed from the *superficial degree of divergence* of a graph, $\omega(\mathcal{G}) \equiv 4n_{_L} - 2n_{_I}$ for a graph with $n_{_L}$ loops and $n_{_I}$ internal lines in a scalar field theory with quartic coupling in four spacetime dimensions: the graph $\mathcal{G}$ is convergent if $\omega(\mathcal{G}) < 0$ and the superficial degree of divergence of all its subgraphs is negative as well. In order to safely manipulate possibly divergent loop integrals, the first step is to introduce a *regularization*, i.e., a modification of the Feynman rules such that all loop integrals become well defined. Many regularization methods are possible: Pauli–Villars subtraction, lattice discretization, momentum cutoff, dimensional regularization.

Dimensional regularization, based on the observation that loop integrals calculated in an arbitrary number D of dimensions have an analytical continuation which is well defined at all D's except a discrete set of values, is particularly adapted to analytical calculations. With this regularization scheme, some common (Euclidean) loop integrals are given by

$$\begin{aligned}
&\int \frac{d^D k_{_E}}{(2\pi)^D}\,\frac{1}{(k_{_E}^2+\Delta)^n} = \frac{\Delta^{\frac{D}{2}-n}}{(4\pi)^{\frac{D}{2}}}\,\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)},\\
&\int \frac{d^D k_{_E}}{(2\pi)^D}\,\frac{k_{_E}^\mu k_{_E}^\nu}{(k_{_E}^2+\Delta)^n} = \frac{g^{\mu\nu}}{2}\,\frac{\Delta^{\frac{D}{2}+1-n}}{(4\pi)^{\frac{D}{2}}}\,\frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n)},\\
&\int \frac{d^D k_{_E}}{(2\pi)^D}\,\frac{k_{_E}^\mu k_{_E}^\nu k_{_E}^\rho k_{_E}^\sigma}{(k_{_E}^2+\Delta)^n} = \frac{g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}}{4}\,\frac{\Delta^{\frac{D}{2}+2-n}}{(4\pi)^{\frac{D}{2}}}\,\frac{\Gamma(n-2-\frac{D}{2})}{\Gamma(n)},\\
&\int \frac{d^D k_{_E}}{(2\pi)^D}\,\frac{k_{_E}^{\mu_1}\cdots k_{_E}^{\mu_{2n+1}}}{(k_{_E}^2+\Delta)^n} = 0.
\end{aligned} \tag{1.21}$$

The first of these equations is obtained by integration in D-dimensional spherical coordinates, and the subsequent equations follow from Lorentz invariance.

Renormalization

The list of correlation functions that exhibit ultraviolet divergences can be obtained from the superficial degree of divergence $\omega(\mathcal{G})$ (except in situations where a symmetry produces cancellations that cannot be anticipated by power counting). For a scalar field theory with a quartic coupling, one has $\omega(\mathcal{G}) = 4 - n_{_E} + (D-4)n_{_L}$ in D spacetime dimensions, where

n_E is the number of external points and n_L the number of loops. The *Weinberg convergence theorem* states that a graph is ultraviolet convergent if and only if the superficial degree of divergence of the graph, and of any of its subgraphs, is negative.

In $D = 4$ spacetime dimensions, $\omega(\mathcal{G})$ is negative for all correlation functions with $n_E > 4$ points, implying that only a finite number of correlation functions have intrinsic divergences. Moreover, these divergent correlation functions are the expectation values of the operators already present in the Lagrangian, $(\partial_\mu\phi)^2, \phi^2, \phi^4$. The divergences that appear in these functions can be subtracted order-by-order via a redefinition of their coefficients in the Lagrangian, i.e., Z (this one is usually not explicit in the bare Lagrangian because it is set to 1), m^2 and λ, respectively. Such a quantum field theory is called *renormalizable*.

In $D > 4$ dimensions, $\omega(\mathcal{G})$ increases with the number of loops at fixed n_E. This implies that any correlation function exhibits intrinsic ultraviolet divergences beyond a certain loop order. Removing these divergences would require that one adds arbitrarily many new terms in the Lagrangian, reducing considerably the predictive power of such a theory (but it may nevertheless be of some use in an effective sense, at low loop orders). It is called *non-renormalizable*.

When $D < 4$, the superficial degree of divergence of any correlation function eventually becomes negative after a certain loop order. These theories have a finite number of ultraviolet divergent Feynman graphs, whose calculation is sufficient to determined the renormalized Lagrangian once and for all. These theories are called *super-renormalizable*.

For general interactions in arbitrary dimensions, the above criteria can be expressed in terms of the mass dimension of the prefactor that accompanies the operator in the Lagrangian. The corresponding operator is renormalizable if the mass dimension of its coupling constant is zero, non-renormalizable if this dimension is negative, super-renormalizable if it is positive.

Renormalization Group

In a renormalized quantum field theory, one may still freely choose the *renormalization scale* μ at which the conditions that define the parameters of the renormalized Lagrangian (masses, couplings, etc.) are imposed. Physical results should not depend on this scale. The dependence of various renormalized quantities with respect to μ is controlled by the *Callan–Symanzik equations*, also known as *renormalization group* equations. For the renormalized n-point correlation function G_n, this equation reads

$$\Big(\underbrace{\mu\partial_\mu + \beta\partial_\lambda + \gamma_m\, m\partial_m}_{\equiv \mathcal{D}_\mu} + n\gamma\Big)\, G_n = 0, \tag{1.22}$$

$$\text{with}\quad \gamma \equiv \frac{1}{2}\frac{\partial \ln(Z)}{\partial \ln(\mu)}, \quad \beta \equiv \frac{\partial \lambda}{\partial \ln(\mu)}, \quad \gamma_m \equiv \frac{\partial \ln(m)}{\partial \ln(\mu)} \tag{1.23}$$

(γ is called an *anomalous dimension*, and β is the β *function*). Physical quantities are invariant under the action of $\mathcal{D}_\mu$, i.e., under the simultaneous change of the scale μ and of the parameters Z, λ, m as prescribed by the above differential equations (the solutions $\lambda(\mu)$ and $m(\mu)$ are called the running coupling and running mass, respectively). The curves $(Z(\mu), \lambda(\mu), m(\mu))$ in the parameter space of the renormalized theory, along which physical quantities are invariant, define a vector field called the *renormalization flow*.

From the Callan–Symanzik equation satisfied by the propagator, $(\mathcal{D}_\mu + 2\gamma)G_2 = 0$, one obtains the corresponding flow equations for the pole mass m_p (defined from the value of p^2

at the pole of the propagator) and for the residue Z at the pole:

$$\mathcal{D}_\mu \, m_p = 0, \quad (\mathcal{D}_\mu + 2\gamma) \, Z = 0. \tag{1.24}$$

Thus, a n-point scattering amplitude $\mathcal{A}_n \sim Z^{-n/2} G_n$ also satisfies $\mathcal{D}_\mu \mathcal{A}_n = 0$. Amputated correlation functions $\Gamma_n \equiv (G_2)^{-n} G_n$ obey $(\mathcal{D}_\mu - n\gamma)\Gamma_n = 0$.

Spin-1/2 Fields

The representation of the Lorentz algebra of lowest even dimension is defined by the generators $M^{\mu\nu}_{1/2} \equiv \frac{i}{4}\big[\gamma^\mu, \gamma^\nu\big]$, where the γ^μ are the *Dirac* 4×4 *matrices*, which obey $\big\{\gamma^\mu, \gamma^\nu\big\} = 2\, g^{\mu\nu}$. Under a Lorentz transformation $\Lambda \equiv \exp\big(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\big)$, a *Dirac spinor* is a four-component field that transforms as

$$\psi(x) \quad \rightarrow \quad \exp\big(-\tfrac{i}{2}\omega_{\mu\nu} M^{\mu\nu}_{1/2}\big)\, \psi(\Lambda^{-1} x). \tag{1.25}$$

In the absence of interactions, such a field obeys the – Lorentz invariant – *Dirac equation*,

$$\big(i\gamma^\mu \partial_\mu - m\big)\, \psi = 0, \tag{1.26}$$

which can be obtained as the equation of motion that results from the following Lagrangian:

$$\mathcal{L} = \overline{\psi}\, \big(i\gamma^\mu \partial_\mu - m\big)\, \psi, \quad \text{with } \; \overline{\psi} \equiv \psi^\dagger \gamma^0. \tag{1.27}$$

The canonical quantization of a free spinor (i.e., a solution of the Dirac equation (1.26)) consists in replacing its Fourier coefficients by creation and annihilation operators:

$$\psi(x) \equiv \sum_{s=\pm} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \Big\{ d^\dagger_{s\mathbf{p}} v_s(\mathbf{p}) e^{+ip\cdot x} + b_{s\mathbf{p}} u_s(\mathbf{p}) e^{-ip\cdot x} \Big\}. \tag{1.28}$$

Since ψ is not Hermitian, the two operators in this decomposition need not be mutual conjugates (except in the special case of Majorana fermions). The spinors u_s, v_s are a basis of free spinors in momentum space defined by

$$\big(\gamma^\mu p_\mu - m\big) u_s(\mathbf{p}) = 0, \quad \big(\gamma^\mu p_\mu + m\big) v_s(\mathbf{p}) = 0, \tag{1.29}$$

$$u^\dagger_r(\mathbf{p}) u_s(\mathbf{p}) = 2E_{\mathbf{p}} \delta_{rs}, \quad v^\dagger_r(\mathbf{p}) v_s(\mathbf{p}) = 2E_{\mathbf{p}} \delta_{rs}. \tag{1.30}$$

For the Hamiltonian of this system to have a well-defined ground state, these creation and annihilation operators must obey anti-commutation relations. The non-zero ones read

$$\{d_{s\mathbf{p}}, d^\dagger_{s'\mathbf{p}'}\} = \{b_{s\mathbf{p}}, b^\dagger_{s'\mathbf{p}'}\} = (2\pi)^3 2E_{\mathbf{p}} \delta_{ss'} \delta(\mathbf{p} - \mathbf{p}'), \tag{1.31}$$

or, equivalently,

$$\{\psi_\alpha(x), \psi^\dagger_\beta(y)\} \underset{x^0 = y^0}{=} \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{y}). \tag{1.32}$$

The Dirac Lagrangian has a $U(1)$ symmetry, $\psi \to e^{-i\alpha}\psi$, which by Noether's theorem leads to a conserved current $J^\mu \equiv \overline{\psi}\gamma^\mu\psi$ and conserved charge

$$Q \equiv \int d^3\mathbf{x}\; J^0 = \sum_{s=\pm}\int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\,\big(b^\dagger_{s\mathbf{p}} b_{s\mathbf{p}} - d^\dagger_{s\mathbf{p}} d_{s\mathbf{p}}\big). \tag{1.33}$$

Spin-1 Fields

A *vector field* $A^\mu(x)$ is a four-component field that transforms as $A^\mu(x) \to \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}x)$ under a Lorentz transformation Λ. The simplest such (massless) field is the electromagnetic field, whose Lagrangian reads

$$\mathcal{L} = -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu}, \quad \text{with } F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1.34}$$

The corresponding equation of motion, $\partial_\mu F^{\mu\nu} = 0$, has several remarkable properties:

- *Gauge invariance*: for any function θ, $A^\mu - \partial^\mu\theta$ is a solution if A^μ is a solution.
- The field A^0 is not dynamical, but given by a constraint from the spatial components A^i.
- Only the transverse (i.e., transverse to the momentum k^i in Fourier space) components of A^i are constrained by the equation of motion.

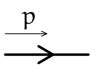
The unphysical redundancy due to gauge invariance is removed by imposing a *gauge condition* – e.g., $\partial_\mu A^\mu = 0$ (Lorenz gauge), $\partial_i A^i = 0$ (Coulomb gauge), $A^0 = 0$ (temporal gauge) – leaving only two independent dynamical solutions per Fourier mode. The quantization of the vector field A^μ amounts to replacing the coefficients in its Fourier decomposition by creation and annihilation operators:

$$A^\mu(x) \equiv \sum_{\lambda=1,2}\int \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|}\,\Big\{a^\dagger_{\lambda\mathbf{p}}\,\epsilon^{\mu *}_\lambda(\mathbf{p})\, e^{ip\cdot x} + a_{\lambda\mathbf{p}}\,\epsilon^\mu_\lambda(\mathbf{p})\, e^{-ip\cdot x}\Big\}, \tag{1.35}$$

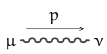
with the canonical commutation relation $\big[a_{\lambda\mathbf{p}}, a^\dagger_{\lambda'\mathbf{p}'}\big] = (2\pi)^3 2|\mathbf{p}|\delta_{\lambda\lambda'}\delta(\mathbf{p}-\mathbf{p}')$ and where the objects $\epsilon^\mu_\lambda(\mathbf{p})$ are *polarization vectors* that encode the Lorentz indices of a vector field of polarization λ and momentum $\mathbf{p}$. The polarization vectors may depend on the choice of gauge condition, but always satisfy $p_\mu \epsilon^\mu_\lambda(\mathbf{p}) = 0$.

Quantum Electrodynamics

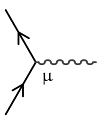
The conserved charge of the Dirac fermions can be interpreted as an electrical charge. Interactions between these fermions and photons are introduced by *minimal coupling*, i.e., by requesting that the modified Dirac Lagrangian is invariant under spacetime-dependent $U(1)$ transformations, $\psi(x) \to e^{-ie\theta(x)}\psi(x)$. This is achieved by replacing the ordinary derivative by a *covariant derivative*, $D_\mu \equiv \partial_\mu - ieA_\mu$. Perturbation theory in QED has the following Feynman rules:



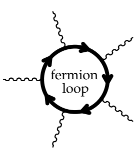
$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i0^+}$$



$$= \frac{i\,C^{\mu\nu}(p)}{p^2 + i0^+}$$



$$= -i\,e\,\gamma^\mu$$



= (minus sign)

The numerator $C^{\mu\nu}$ in the photon propagator depends on the gauge fixing (for instance, $C^{\mu\nu} = -g^{\mu\nu}$ in Feynman gauge).

Ward–Takahashi Identity

A crucial property of QED amplitudes with external photons is the *Ward–Takahashi identity*, namely

$$p_\mu \, \Gamma^{\mu\cdots}(p, \ldots) = 0, \tag{1.36}$$

where $\Gamma^{\mu\cdots}(p, \ldots)$ is an amplitude amputated of its external propagators, containing a photon of momentum p with Lorentz index μ. The dots represent the other external lines, either photons or charged particles. The conditions of validity of this identity, which follows from the conservation of the electrical current, are the following:

- All the external lines corresponding to charged particles must be on-shell, and contracted in the appropriate spinors if they are fermions.
- The gauge fixing condition must be linear in the gauge potential, in order not to have three- and four-photon vertices.

The Ward–Takahashi identity plays a crucial role in ensuring that QED scattering amplitudes are gauge invariant, and that they fulfill the requirements of unitarity despite the presence of non-physical photon polarization in certain gauges.

Unitarity, the Optical Theorem and Cutkosky's Cutting Rules

The time evolution operator from $x^0 = -\infty$ to $x^0 = +\infty$ (also called the S-matrix) is unitary, $SS^\dagger = 1$. Writing it as $S \equiv 1 + iT$ to separate the interactions, this implies the *optical theorem*:

$$\mathrm{Im}\,\big\langle \alpha_{\mathrm{in}}\big|T\big|\alpha_{\mathrm{in}}\big\rangle = \frac{1}{2}\sum_{\text{states }\beta} \big|\big\langle \alpha_{\mathrm{in}}\big|T\big|\beta_{\mathrm{in}}\big\rangle\big|^2.$$

This relation implies that the total probability of scattering from the state α to any state β (with at least one interaction) equals twice the imaginary part of the forward scattering amplitude $\alpha \to \alpha$. In perturbation theory, the imaginary part of a transition amplitude Γ can be obtained by means of *Cutkosky's cutting rules*:

$$\mathrm{Im}\,\Gamma = \frac{1}{2}\sum_{\text{cuts }\gamma} \big[\Gamma\big]_\gamma,$$

where a cut is a fictitious line that divides the graph into two subgraphs, with at least one external leg on each side of the cut. A cut graph $\big[\Gamma\big]_\gamma$ is calculated with the following rules:

- Left of the cut: use the propagator $G^0_{++}(p) = \frac{i}{p^2-m^2+i0^+}$ and the vertex $-i\lambda$,
- Right of the cut: use the propagator $G^0_{--}(p) = \frac{-i}{p^2-m^2-i0^+}$ and the vertex $+i\lambda$,
- The propagators traversing the cut should be $G^0_{+-}(p) = 2\pi\,\theta(-p^0)\delta(p^2)$.

About the Problems of this Chapter

- **Problem 1** establishes a crucial relationship between the field operators ϕ (Heisenberg representation) and $\phi_{\rm in}$ (interaction representation), namely that the former obeys the interacting equation of motion if the latter obeys the free Klein–Gordon equation.
- In **Problem 2**, we derive an explicit form of the elements of the *little group* for massless particles. This is then used in **Problem 9** in order to show that, in a theory with massless spin-1 bosons, the Lorentz invariance of scattering amplitudes implies a property that may be viewed as a weak form of the Ward–Takahashi identity. This observation, due to Weinberg, is extended to gravity in **Problem 10**.
- **Problem 3** establishes some formal relationships between various expressions for the time evolution operator and the S-matrix. Then, **Problem 4** shows that the expression for the S-matrix as the time-ordered exponential of a local interaction term is to a large extent a consequence of causality.
- In **Problem 5**, we derive a set of conditions, known as the *Landau equations*, for a given loop integral to have infrared or collinear singularities. An explicit multi-loop integration is studied in **Problem 6**, which provides another point of view on these conditions.
- **Problem 7** establishes *Weinberg's convergence theorem* in the simple case of scalar field theory, a crucial result in the discussion of renormalization since it clarifies the role of the superficial degree of divergence in assessing whether a particular diagram is ultraviolet divergent.
- The electron anomalous magnetic moment is calculated at one loop in **Problem 8**. This is a classic QED calculation of great historical importance, which has now been pushed to five loops and provides one of the most precise agreements between theory and experiment in all of physics.
- **Problem 11** derives the *Lee–Nauenberg theorem*, an important result about soft and collinear singularities which states that such divergences are removed by summing transition probabilities over degenerate states, thereby providing a link between the finiteness of a quantity and its practical measurability.
- In **Problem 12**, we discuss the external classical field approximation, thanks to which a heavy charged object may be replaced by its classical Coulomb field.
- **Problems 13** and **14** are devoted to a derivation of the *Low–Burnett–Kroll theorem*, a result that states that the emission probability of a soft photon is proportional to the probability of the underlying hard process, at the first two orders in the energy of the emitted photon.
- *Coherent states* are introduced in **Problem 15** and their main properties established. They will be discussed further in **Problems 20, 21** and **22**.
- **Problems 16** and **17** study the running coupling in a scalar field theory with two fields, and in a QCD-like theory at two-loop order.