# Towards Implementations for Advanced Equivalence Checking in Answer-Set Programming

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**Abstract.** In recent work, a general framework for specifying program correspondences under the answer-set semantics has been defined. The framework allows to define different notions of equivalence, including the well-known notions of *strong* and *uniform equivalence*, as well as refined equivalence notions based on the *projection* of answer sets, where not all parts of an answer set are of relevance (like, e.g., removal of auxiliary letters). In the general case, deciding the correspondence of two programs lies on the fourth level of the polynomial hierarchy and therefore this task can (presumably) not be efficiently reduced to answer-set programming. In this paper, we describe an approach to compute program correspondences in this general framework by means of *quantified Boolean formulas* (QBFs). We provide linear-time constructible reductions from program correspondence problems to the evaluation problem of QBFs. We can thus use extant solvers for QBFs as back-end inference engines for solving program correspondence problems. We also describe how our translations provide a method to construct *counterexamples* in case a program correspondence does not hold.

## 1 Introduction

*Answer-set programming* (ASP) is widely recognised as a fruitful paradigm for declarative knowledge representation and reasoning. It is based on the idea that problems are encoded in terms of theories of some suitable language, associated with a declarative semantics, such that the solutions of the given problems are determined by the models of the corresponding theories. Among the different instances of the ASP paradigm, the class of nonmonotonic logic programs under the answer-set semantics [14], with which we are concerned with in this paper, represents the canonical and, due to the availability of efficient answer-set solvers, like DLV [18], Smodels [26], and ASSAT [22], arguably most widely used ASP approach.

An important issue for the further development of ASP is to provide methods and tools for *engineering ASP solutions*. This includes techniques for the *simplification* and (offline) *optimisation* of programs, tools for supporting the user with *debugging* or *verification* features, and methods for *modular programming*. Crucial for all these issues are mechanisms for determining the *equivalence of* (*parts of*) *logic programs*.

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In previous work [13], a general framework for specifying correspondences between logic programs under the answer-set semantics has been introduced. In this framework, the correspondence of two programs is determined in terms of a class $\mathcal{C}$ of *context programs* and a comparison relation $\rho$: Correspondence between two programs $P$ and $Q$ holds iff the answer sets of $P \cup R$ and $Q \cup R$ satisfy $\rho$, for any program $R \in \mathcal{C}$. The framework includes as special cases the well-known notions of *strong equivalence* [20], *uniform equivalence* [10], and relativised notions thereof [28], as well as the practicably important case of program comparison under *projected* answer sets. In the latter setting, not a whole answer set of a program $P$ is of interest, but only its intersection on a subset of all letters; this includes, in particular, removal of auxiliary letters in computation.

For the case of propositional disjunctive logic programs, correspondence checking in the above framework under projected answer sets is surprisingly hard, viz. $\Pi_4^P$-complete in general [13], i.e., lying on the fourth level of the polynomial hierarchy. Hence, this task can (presumably) not be efficiently reduced to propositional answer-set programming. Such an approach (used, e.g., by Oikarinen and Janhunen [23] for ordinary equivalence) reduces equivalence checking to problems like program consistency such that equivalence holds iff the resultant program possesses no answer set. Taking the results of Eiter *et al.* [9] into account, a compact reduction as such cannot even be obtained by using non-ground programs as long as we restrict the arities of predicates to a fixed constant. This indicates that advanced equivalence tests in answer-set programming cannot be straightforwardly solved using ASP-systems themselves.

In this paper, we describe an approach to compute program correspondences in the framework of Eiter *et al.* [13] by means of efficient reductions to *quantified propositional logic*. The latter is an extension of classical propositional logic characterised by the condition that its sentences, usually referred to as *quantified Boolean formulas* (QBFs), are permitted to contain quantifications over atomic formulas. More specifically, our reductions enjoy the following properties:

1. a solution correspondence under projected answer sets between two given logic programs holds iff the associated QBF is valid in quantified propositional logic,
2. the reduction is constructible in *linear time and space*, and
3. determining the validity of the resultant QBFs under the translations is not computationally harder than checking the original correspondence problem.

Besides the reduction of correspondence problems, we also describe how our translations provide a method to construct *counterexamples* in case a program correspondence does not hold.

The rationale to consider a reduction approach to QBFs is twofold: On the one hand, complexity results about quantified propositional logic imply that decision problems from the polynomial hierarchy can be efficiently represented in terms of QBFs, and, on the other hand, practicably efficient solvers for quantified propositional logic have been presented in recent years (like, e.g., the solvers QuBE [15], semprop [19], or others—see [17, 16]). Hence, solvers for QBFs can be used as back-end inference engines to compute the correspondence problems under consideration. We note that a similar reduction approach to QBFs has been successfully applied in diverse fields like nonmonotonic reasoning [6, 5, 12], paraconsistent reasoning [3, 1, 2], planning [25], and automated deduction [7].

## 2 Preliminaries

We deal with propositional disjunctive logic programs, which are finite sets of rules of form

$$a_1 \vee \cdots \vee a_l \leftarrow a_{l+1}, \ldots, a_m, not\, a_{m+1}, \ldots, not\, a_n, \tag{1}$$

$n \geq m \geq l \geq 0$, where all $a_i$ are propositional atoms from a universe $\mathcal{U}$ and $not$ denotes default negation. If all atoms occurring in a program $P$ are from a given set $A \subseteq \mathcal{U}$ of atoms, we say that $P$ is a program *over* $A$. The set of all programs over $A$ is denoted by $\mathcal{P}_A$.

Following Gelfond and Lifschitz [14], an interpretation $I$, i.e., a set of atoms, is an *answer set* of a program $P$ iff it is a minimal model of the *reduct* $P^I$, resulting from $P$ by (i) deleting all rules containing default negated atoms $not\, a$ such that $a \in I$ and (ii) deleting all default negated atoms in the remaining rules. The set of all answer sets of a program $P$ is denoted by $\mathcal{AS}(P)$. The relation $I \models P$ between an interpretation $I$ and a program $P$ is defined as usual.

Under the answer-set semantics, two programs $P$ and $Q$ are regarded as (ordinarily) equivalent iff $\mathcal{AS}(P) = \mathcal{AS}(Q)$. The more restrictive form of *strong equivalence* [20] has recently been generalised as follows [28]: Let $P, Q$ be programs over $\mathcal{U}$, and let $A \subseteq \mathcal{U}$. Then, $P$ and $Q$ are *strongly equivalent relative to* $A$ iff, for any $R \in \mathcal{P}_A$ $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$. If $A = \mathcal{U}$, strong equivalence relative to $A$ reduces to strong equivalence; if $A = \emptyset$, we obtain ordinary equivalence.

We use the following notation: For an interpretation $I$ and a set $\mathcal{S}$ of interpretations (resp., pairs of interpretations), we write $\mathcal{S}|_I = \{Y \cap I \mid Y \in \mathcal{S}\}$ (resp., $\mathcal{S}|_I = \{(X \cap I, Y \cap I) \mid (X, Y) \in \mathcal{S}\}$). If $\mathcal{S} = \{s\}$, we usually write $s|_I$ instead of $\mathcal{S}|_I$.

For any $A \subseteq \mathcal{U}$, a pair $(X, Y)$ of interpretations, where $Y \subseteq \mathcal{U}$, is an *$A$-SE-interpretation* (over $\mathcal{U}$) iff either $X = Y$ or $X \subset Y|_A$. $(X, Y)$ is an *$A$-SE-model* of a program $P$ iff (i) $Y \models P$, (ii) for all $Y' \subset Y$ with $Y'|_A = Y|_A$, $Y' \not\models P^Y$, and (iii) $X \subset Y$ implies the existence of an $X' \subseteq Y$ with $X'|_A = X$ such that $X' \models P^Y$ holds. A pair $(X, Y)$ is *total* iff $X = Y$, and *non-total* otherwise. The set of all $A$-SE-models of $P$ is denoted by $SE^A(P)$.

For $A = \mathcal{U}$, the notion of an $A$-SE-interpretation (resp., $A$-SE-model) coincides with the notion of an *SE-interpretation* (resp., *SE-model*) as defined by Turner [27], and we write $SE(P)$ instead of $SE^{\mathcal{U}}(P)$. Thus, $(X, Y) \in SE(P)$ iff $X \subseteq Y$, $Y \models P$, and $X \models P^Y$.

**Proposition 1 ([28]).** *Two programs $P$ and $Q$ are strongly equivalent relative to $A$ iff $SE^A(P) = SE^A(Q)$.*

*Example 1.* Consider the following two programs, $P$ and $Q$:

$$P = P_0 \cup \{c \vee d \leftarrow a;\ c \vee d \leftarrow b\},$$
$$Q = P_0 \cup \{c \vee d \leftarrow a, b;\ d \leftarrow b, not\, c;\ c \leftarrow a, not\, d\},$$

for $P_0 = \{a \leftarrow c;\ b \leftarrow c;\ a \leftarrow d;\ b \leftarrow d;\ \leftarrow not\, c, not\, d\}$.

They have the following SE-models:[1]

$$SE(P) = \{(\emptyset, abc), (\emptyset, abd), (\emptyset, abcd), (abcd, abcd), (abc, abcd), (abd, abcd), (abc, abc), (abd, abd)\},$$
$$SE(Q) = SE(P) \cup \{(b, abc), (a, abd), (b, abcd), (a, abcd)\}.$$

Hence, $P$ and $Q$ are not strongly equivalent. On the other hand, $\mathcal{AS}(P) = \mathcal{AS}(Q) = \emptyset$, i.e., $P$ and $Q$ are (ordinarily) equivalent. Moreover, $P$ and $Q$ are strongly equivalent relative to $A$ iff $A \cap \{a, b\} = \emptyset$. For $A = \{a, b\}$, we get

$$SE^A(P) = \{(\emptyset, abc),\ (\emptyset, abd),\ (abc, abc),\ (abd, abd)\},$$
$$SE^A(Q) = SE^A(P) \cup \{(b, abc),\ (a, abd)\}.$$

Hence, $P$ and $Q$ are not strongly equivalent relative to $A = \{a, b\}$. For instance, adding a fact $a \leftarrow$ yields $\mathcal{AS}(P \cup \{a \leftarrow\}) = \{abc, abd\}$, while $\mathcal{AS}(Q \cup \{a \leftarrow\}) = \{abc\}$. □

A set $\mathcal{S}$ of SE-interpretations is *complete* iff, for each $(X, Y) \in \mathcal{S}$, also $(Y, Y) \in \mathcal{S}$ as well as $(X, Z) \in \mathcal{S}$, for any $Z$ such that $Y \subseteq Z$ and $(Z, Z) \in \mathcal{S}$. It can be shown that the set $SE(P)$ of SE-models of any program $P$ is always complete. Conversely, any complete set $\mathcal{S}$ of SE-interpretations can be represented by some program $P$. As a general result, taking also a restricted alphabet $A$ into account, the following result holds:

**Proposition 2.** *Let $\mathcal{S}$ be a complete set of SE-interpretations, and let $A$ be a set of atoms. Then, there exists a program $P_{\mathcal{S},A} \in \mathcal{P}_A$ such that $SE(P_{\mathcal{S},A})|_A = \mathcal{S}|_A$.*

One possibility to obtain $P_{\mathcal{S},A}$ from $\mathcal{S}$ is as follows:

1. for each $Y \subseteq A$ with $(Y, Y) \notin \mathcal{S}|_A$, add rules $\bot \leftarrow Y, not\ (A \setminus Y)$, and
2. for each $X \subset Y$ with $(X, Y) \notin \mathcal{S}|_A$ and $(Y, Y) \in \mathcal{S}|_A$, add rules $\bigvee_{p \in (Y \setminus X)} p \leftarrow X, not\ (A \setminus Y)$.

## 3 Correspondence Checking

In order to deal with differing notions of program equivalence in a uniform manner, taking in particular strong equivalence and its relativised version, as well as equivalence notions based on the projection of answer sets into account, Eiter *et al.* [13] introduced a general framework for specifying differing notions of equivalence. In this framework, one parameterises, on the one hand, the set $R$ of rules to be added to the programs $P$ and $Q$, and, on the other hand, the relation that has to hold between the collection of answer sets of $P \cup R$ and $Q \cup R$.

**Definition 1.** *A* correspondence frame, *or simply* frame, *$\mathcal{F}$, is a triple $(\mathcal{U}, \mathcal{C}, \rho)$, where* (*i*) *$\mathcal{U}$ is a set of atoms, called the* universe of $\mathcal{F}$, (*ii*) *$\mathcal{C} \subseteq \mathcal{P}_\mathcal{U}$, called the* context of $\mathcal{F}$, *and* (*iii*) *$\rho \subseteq 2^{2^\mathcal{U}} \times 2^{2^\mathcal{U}}$.*

*For any program $P, Q \in \mathcal{P}_\mathcal{U}$, $P$ and $Q$ are* $\mathcal{F}$-corresponding, *in symbols $P \simeq_\mathcal{F} Q$, iff, for all $R \in \mathcal{C}$, $(\mathcal{AS}(P \cup R), \mathcal{AS}(Q \cup R)) \in \rho$.*

[1] We write $abc$ instead of $\{a, b, c\}$, $a$ instead of $\{a\}$, etc.

It is quite obvious that the equivalence notions discussed above are special cases of $\mathcal{F}$-correspondence. Indeed, for any universe $\mathcal{U}$ and any $A \subseteq \mathcal{U}$, strong equivalence relative to $A$ coincides with $(\mathcal{U}, \mathcal{P}_A, =)$-correspondence, and ordinary equivalence coincides with $(\mathcal{U}, \{\emptyset\}, =)$-correspondence.

Following Eiter *et al.* [13], we are mainly concerned with correspondence frames of form $(\mathcal{U}, \mathcal{P}_A, \subseteq_B)$ and $(\mathcal{U}, \mathcal{P}_A, =_B)$, where $A, B \subseteq \mathcal{U}$ are sets of atoms, and $\subseteq_B$ and $=_B$ are projections of the standard subset and set-equality relation, respectively, defined as follows: for any set $\mathcal{S}, \mathcal{S}'$ of interpretations, $\mathcal{S} \subseteq_B \mathcal{S}'$ iff $\mathcal{S}|_B \subseteq \mathcal{S}'|_B$, and $\mathcal{S} =_B \mathcal{S}'$ iff $\mathcal{S}|_B = \mathcal{S}'|_B$.

A *correspondence problem*, $\Pi$, (*over* $\mathcal{U}$) is a quadruple $(P, Q, \mathcal{C}, \rho)$, where $P, Q \in \mathcal{P}_\mathcal{U}$ and $(\mathcal{U}, \mathcal{C}, \rho)$ is a frame. We say that $\Pi$ *holds* iff $P \simeq_{(\mathcal{U},\mathcal{C},\rho)} Q$ holds. For a correspondence problem $\Pi = (P, Q, \mathcal{C}, \rho)$ over $\mathcal{U}$, we usually leave $\mathcal{U}$ implicit, assuming that it consists of all atoms occurring in $P$, $Q$, and $\mathcal{C}$. We call $\Pi$ an *equivalence problem* if $\rho$ is given by $=_B$, and an *inclusion problem* if $\rho$ is given by $\subseteq_B$, for some $B \subseteq \mathcal{U}$. Note that $(P, Q, \mathcal{C}, =_B)$ holds iff $(P, Q, \mathcal{C}, \subseteq_B)$ and $(Q, P, \mathcal{C}, \subseteq_B)$ jointly hold.

For inclusion problems, we define the concept of a *counterexample*, which is easily extended to equivalence problems.

**Definition 2.** *A pair $(Y, R)$, where $Y$ is an interpretation and $R \in \mathcal{C}$, is called a* counterexample *for $(P, Q, \mathcal{C}, \subseteq_B)$ iff $Y \in \mathcal{AS}(P \cup R)$ and, for each $Z$ with $Z =_B Y$, $Z \notin \mathcal{AS}(Q \cup R)$.*

*Example 2.* We have already seen that for $P$, $Q$ from Example 1, $(P, Q, \mathcal{P}_A, \subseteq_\mathcal{U})$ does not hold for $A = \{a, b\}$ and $\mathcal{U} = \{a, b, c, d\}$. What happens if we restrict the comparison of answer sets from $\mathcal{U}$ to $A$, i.e., does $(P, Q, \mathcal{P}_A, \subseteq_A)$ hold? Note that, e.g., $\mathcal{AS}(P \cup \{a \leftarrow\})|_A = \mathcal{AS}(Q \cup \{a \leftarrow\})|_A = \{ab\}$. Hence, the counterexample $(abc, \{a \leftarrow\})$ from Example 1 is no longer a counterexample for $(P, Q, \mathcal{P}_A, \subseteq_A)$. As we shall see below, there still exist counterexamples for this problem, but these are more involving. □

As shown by Eiter *et al*. [13], inclusion problems with projection may possess only counterexamples which are exponential in the size of the compared programs. Hence, instead of guessing concrete programs and checking whether they are counterexamples for a given inclusion problem, Eiter *et al.* provide a semantical structure, called *spoiler*, which operates on the compared programs alone, together with the notion of a *partial spoiler*.

**Definition 3.** *Let $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ be an inclusion problem, $Y$ an interpretation, and $\mathcal{S} \subseteq SE^A(Q) \cap \{(X, Z) \mid Z =_{A \cup B} Y\}$ a complete set of $A$-SE-interpretations. The pair $(Y, \mathcal{S})$ is a* spoiler for $\Pi$ *iff*

1. *$(Y, Y) \in SE^A(P)$,*
2. *each $(Z, Z) \in SE^A(Q)$ such that $Z =_{A \cup B} Y$ is also in $\mathcal{S}$,*
3. *for each $(Z, Z) \in \mathcal{S}$, some non-total $(X, Z) \in \mathcal{S} \cap SE^A(Q)$ exists, and*
4. *for each non-total $(X, Z) \in \mathcal{S}$, $(X, Y) \notin SE^A(P)$.*

*For a spoiler $(Y, \mathcal{S})$, the interpretation $Y$ is referred to as a* partial spoiler *for* $\Pi$.

Intuitively, in a spoiler $(Y, \mathcal{S})$, the interpretation $Y$ is an answer set of $P \cup R$ but not of $Q \cup R$, where $R$ is some program which is semantically given by $\mathcal{S}$.

We collect and rephrase the main results from [13].

**Proposition 3.** *Let $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ be an inclusion problem. Then, $\Pi$ holds iff there exists no spoiler $(Y, \mathcal{S})$ for $\Pi$.*

As an immediate consequence, we obtain that a correspondence problem $\Pi$ holds iff there exists no partial spoiler $Y$ for $\Pi$. Moreover, we are able to connect spoilers to counterexamples using the generic programs $P_{\mathcal{S},A}$, as introduced in Section 2.

**Proposition 4.** *If $(Y, \mathcal{S})$ is a spoiler for an inclusion problem $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$, then $(Y, P_{\mathcal{S},A})$ is a counterexample for $\Pi$.*

*Example 3.* For $P_1$ and $P_2$ from Example 1 and $A = \{a, b\}$, the pairs $(Y_1, \mathcal{S})$ and $(Y_2, \mathcal{S})$ are the only spoilers for $(P_1, P_2, \mathcal{P}_A, \subseteq_A)$, where $Y_1 = \{abc\}$ and $Y_2 = \{abd\}$ are the partial spoilers for $(P_1, P_2, \mathcal{P}_A, \subseteq_A)$, and $\mathcal{S} = \{(a, abd), (b, abc), (abc, abc), (abd, abd)\}$. Invoking our program construction, we obtain $P_{\mathcal{S},A} = \{\bot \leftarrow a, not\, b;\ \bot \leftarrow b, not\, a;\ \bot \leftarrow not\, a, not\, b;\ a \vee b \leftarrow\}$. One can verify that both $Y_1$ and $Y_2$ are contained in $\mathcal{AS}(P_1 \cup P_{\mathcal{S},A})$, while no interpretation $Z$ with $Z =_A Y_1$ is an answer set of $P_2 \cup P_{\mathcal{S},A}$. □

Finally, we recall the computational complexity of checking whether an equivalence or inclusion problem holds. As shown by Eiter *et al.* [13], deciding $(P, Q, \mathcal{P}_A, =_B)$ is of a significantly higher complexity compared to more restricted notions of equivalence, like strong equivalence (which is coNP-complete) or ordinary equivalence and relativised strong equivalence (which are both $\Pi_2^P$-complete).

**Proposition 5 ([13]).** *Given programs $P$, $Q$, sets of atoms $A$, $B$, and $\rho \in \{\subseteq_B, =_B\}$, deciding whether a correspondence problem $(P, Q, \mathcal{P}_A, \rho)$ holds is $\Pi_4^P$-complete.*

## 4 Reductions

In this section, we provide two approaches to map inclusion problems $(P, Q, \mathcal{P}_A, \subseteq_B)$ into quantified Boolean formulas. By combining the reductions for $(P, Q, \mathcal{P}_A, \subseteq_B)$ and $(Q, P, \mathcal{P}_A, \subseteq_B)$, we straightforwardly obtain a method to check whether an equivalence problem $(P, Q, \mathcal{P}_A, =_B)$ holds. We start with a brief recapitulation of the basic facts about the quantified version of propositional logic.

### 4.1 Quantified Propositional Logic

Quantified propositional logic is an extension of classical propositional logic in which formulas are permitted to contain quantifications over propositional variables. More formally, formulas of quantified propositional logic are built from atomic formulas using the primitive sentential connectives $\neg$ and $\wedge$, the logical constant $\top$, and unary operators of form $\forall p$ (where $p$ is some atom), called *universal quantifiers*. The operators $\vee$, $\rightarrow$, $\leftrightarrow$, as well as the symbol $\bot$, are defined from the primitive ones, $\neg$, $\wedge$, and $\top$, as usual.

Furthermore, similar to first-order logic, $\exists p$ is defined as the operator $\neg\forall p\neg$, referred to as an *existential quantifier*. Formulas of this language are also called *quantified Boolean formulas* (QBFs) and we denote them by Greek upper-case letters.

An occurrence of an atom $p$ is *free* in a QBF $\Phi$ if it does not occur in the scope of a quantifier $\mathsf{Q}p$, $\mathsf{Q} \in \{\exists, \forall\}$. In what follows, we tacitly assume that every subformula $\mathsf{Q}p\,\Phi$ of a QBF contains a free occurrence of $p$ in $\Phi$, and for two different subformulas $\mathsf{Q}p\,\Phi$, $\mathsf{Q}q\,\Psi$ of a QBF we require $p \neq q$. Moreover, given a finite set $P$ of atoms, $\mathsf{Q}P\,\Psi$ stands for any QBF $\mathsf{Q}p_1\mathsf{Q}p_2 \ldots \mathsf{Q}p_n\Psi$ such that the variables $p_1, \ldots, p_n$ are pairwise distinct and $P = \{p_1, \ldots, p_n\}$.

Towards the definition of the semantics of QBFs, we introduce the following concept. For an atom $p$ (resp., a set $P$ of atoms) and a set $I$ of atoms, $\Phi[p/I]$ (resp., $\Phi[P/I]$) denotes the QBF resulting from $\Phi$ by replacing each free occurrence of $p$ (resp., any $p \in P$) in $\Phi$ by $\top$ if $p \in I$ and by $\bot$ otherwise.

For an interpretation $I$ and a QBF $\Phi$, the relation $I \models \Phi$ is inductively defined as follows:

1. $I \models \top$,
2. $I \models p$ iff $p \in I$,
3. $I \models \neg\Phi$ iff $I \not\models \Phi$,
4. $I \models \Phi_1 \wedge \Phi_2$ iff $I \models \Phi_1$ and $I \models \Phi_2$, and
5. $I \models \forall p\ \Phi$ iff $I \models \Phi[p/\{p\}]$ and $I \models \Phi[p/\emptyset]$.

The truth conditions for $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\exists p$, for any $p$, follow from the above in the usual way.

A QBF $\Phi$ is *true under* $I$ iff $I \models \Phi$, otherwise $\Phi$ is *false under* $I$. A QBF is *valid* iff it is true under any interpretation. Note that a *closed* QBF, i.e., a QBF without free variable occurrences, is either true under any $I$ or false under any $I$.

A QBF $\Phi$ is said to be in *prenex normal form* iff it is closed and of the form

$$\mathsf{Q}_n P_n \ldots \mathsf{Q}_1 P_1\, \phi, \tag{2}$$

$n \geq 0$, where $\phi$ is a propositional formula, $\mathsf{Q}_i \in \{\exists, \forall\}$ such that $\mathsf{Q}_i \neq \mathsf{Q}_{i+1}$ for $1 \leq i \leq n-1$, and $(P_1, \ldots, P_n)$ is a partition of the propositional variables occurring in $\phi$, and $P_i \neq \emptyset$, for each $1 \leq i \leq n$. We call a QBF of the form (2) an $(n, \mathsf{Q}_n)$-*QBF*.

Without going into details, we mention that any closed QBF $\Phi$ is easily transformed into an equivalent QBF in prenex normal form such that each quantifier from the original QBF corresponds to a quantifier in the prenex normal form. Call such a QBF the *prenex normal form of* $\Phi$. However, similar as in first-order logic, depending on the structure of the quantifier occurrences in the formula-tree, there are different ways how to obtain an equivalent prenex QBF (cf. [8] for more details on this issue).

The following property is essential:

**Proposition 6.** *For every $k \geq 0$, deciding the truth of a given $(k, \exists)$-QBF (resp., $(k, \forall)$-QBF) is $\Sigma_k^P$-complete (resp., $\Pi_k^P$-complete).*

Hence, any decision problem $\mathcal{D}$ in $\Sigma_k^P$ (resp., $\Pi_k^P$) can be mapped in polynomial time to a $(k, \exists)$-QBF (resp., $(k, \forall)$-QBF) $\Phi$ such that $\mathcal{D}$ holds iff $\Phi$ is valid. In particular, Proposition 5 implies therefore that any correspondence problem $(P, Q, \mathcal{P}_A, \rho)$, for $\rho \in \{\subseteq_B, =_B\}$, can be reduced to a $(4, \forall)$-QBF. In what follows, we construct two such mappings which are moreover constructible in *linear space and time*.

### 4.2 Encodings

For our encodings, we use the following building blocks. We assume indexed sets $V$ of atoms, and we use (pairwise) disjoint copies $V_i = \{v_i \mid v \in V\}$, for any $i$. In fact, we use subscripts as a general renaming schema for interpretations, formulas, and rules. For instance, formula $\phi_i$ is the result of replacing each occurrence of an atom $p$ in $\phi$ by $p_i$, for any $i$.

The following abbreviations allow for comparing different subsets of $V$:

1. $(V_i \leq V_j) := \bigwedge_{v \in V}(v_i \rightarrow v_j)$,
2. $(V_i < V_j) := (V_i \leq V_j) \wedge \neg(V_j \leq V_i)$, and
3. $(V_i = V_j) := (V_i \leq V_j) \wedge (V_j \leq V_i)$,

with the latter being equivalent to $\bigwedge_{v \in V}(v_i \leftrightarrow v_j)$.

**Proposition 7.** *Let $I$ be an interpretation, $A, X, Y \subseteq V$ such that, for some $i, j$, $I|_{V_i} = X_i$ and $I|_{V_j} = Y_j$. Then,*

1. *$X|_A \subseteq Y|_A$ iff $I \models (A_i \leq A_j)$,*
2. *$X|_A \subset Y|_A$ iff $I \models (A_i < A_j)$, and*
3. *$X|_A = Y|_A$ iff $I \models (A_i = A_j)$.*

For a rule $r$ of form (1), we define $H(r) = a_1 \vee \cdots \vee a_l$, $B^+(r) = a_{l+1} \wedge \cdots \wedge a_m$, and $B^-(r) = \neg a_{m+1} \wedge \cdots \wedge \neg a_n$. Furthermore, for a program $P$, we define $P_{i,j} = \bigwedge_{r \in P}\big((B^+(r_i) \wedge B^-(r_j)) \rightarrow H(r_i)\big)$.

**Proposition 8.** *Let $P$ be a program over atoms $V$, $I$ an interpretation, and $X, Y \subseteq V$ such that, for some $i$, $j$, $I|_{V_i} = X_i$ and $I|_{V_j} = Y_j$. Then, $X \models P^Y$ iff $I \models P_{i,j}$.*

Intuitively, this allows to refer to the reduct of $P$ (in case that $i \neq j$) and to the classical formula associated to $P$ (in case that $i = j$) simultaneously. The latter is seen by the fact that for any program $P$ and any interpretation $Y$, $Y \models P$ iff $Y \models P^Y$.

The central characterisation towards our encodings is as follows. It is obtained by replacing the concept of an $A$-SE-model in Definition 3 by the test over program reducts, following the definition of $A$-SE-models.

**Proposition 9.** *An interpretation $Y$ is a partial spoiler for $(P, Q, \mathcal{P}_A, \subseteq_B)$ iff*

(*a*) *$Y \models P$,*
(*b*) *for each $Y' \subset Y$ with $Y' =_A Y$, $Y' \not\models P^Y$, and*
(*c*) *for each $Z =_{A \cup B} Y$, $Z \models Q$ implies the existence of a $X \subset Z$ such that $X \models Q^Z$ and, if $X \subset Z|_A = Y|_A$, then, for each $X' \subseteq Y$ with $X' =_A X$, $X' \not\models P^Y$.*

**Definition 4.** *Let $P, Q$ be programs over $V$ and let $A, B \subseteq V$. Furthermore, consider $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$. Then,*

$$\begin{aligned}
\boldsymbol{S}_\Pi(V_1) &:= P_{1,1} \wedge \boldsymbol{S}^1(P, A) \wedge \forall V_3\big(\boldsymbol{S}^2(Q, A, B) \rightarrow \boldsymbol{S}^3(P, Q, A)\big), \text{ where} \\
\boldsymbol{S}^1(P, A) &:= \forall V_2\big((A_2 = A_1) \wedge (V_2 < V_1) \rightarrow \neg P_{2,1}\big), \\
\boldsymbol{S}^2(Q, A, B) &:= \big((A \cup B)_3 = (A \cup B)_1\big) \wedge Q_{3,3}, \text{ and} \\
\boldsymbol{S}^3(P, Q, A) &:= \exists V_4\big((V_4 < V_3) \wedge Q_{4,3} \wedge \big((A_4 < A_1) \rightarrow \\
&\qquad \forall V_5((A_5 = A_4) \wedge (V_5 \leq V_1) \rightarrow \neg P_{5,1})\big)\big).
\end{aligned}$$

**Lemma 1.** *Let $P$ and $Q$ be programs over $V$, and let $A, B, Y \subseteq V$. Then, $Y$ is a partial spoiler for $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ iff $Y_1 \models \boldsymbol{S}_\Pi(V_1)$.*

We do not give a formal proof here, but just provide the following explanations. The subformula $P_{1,1} \wedge \boldsymbol{S}^1(P, A)$ of $\boldsymbol{S}_\Pi(V_1)$ takes care of Conditions (a) and (b) from Proposition 9; we use atoms $V_1$ to refer to $Y$, and atoms $V_2$ to refer to the $Y'$ therein. Note that $(A_2 = A_1) \wedge (V_2 < V_1)$ thus guarantees that we take only those $Y'$ for testing $Y' \models P^Y$ into account, where $Y' \subset Y$ and $Y' =_A Y$. The next subformula, $\boldsymbol{S}^2(Q, A, B)$, "returns" all $Z$ (via assignments to $V_3$) such that $Z =_{A \cup B} Y$ and $Z \models Q$. Finally, for each such $Z$, $\boldsymbol{S}^3(P, Q, A)$ has to be true. By $(V_4 < V_3)$ we let the assignments to $V_4$ (which refer to the $X$ in Item (c) of Proposition 9) be a proper subset of those to $V_3$, i.e., we require $X \subset Z$. Then we test whether $X \models Q^Z$ via $Q_{4,3}$, as follows from Proposition 8, and in the case $X|_A \subset Y|_A$ (checked via $A_4 < A_3$), the remaining formula encodes the test whether for all $X'$ (assignments to $V_5$) with $X' =_A X$ and $X' \subseteq Y$, $X' \not\models P^Y$, i.e., $P_{5,1}$ is false under the current assignment to $V_1$ and $V_5$.

In what follows, we give a more compact encoding, which in particular reduces the number of universal quantifications. The idea is to save on the fixed assignments as, e.g., in $\boldsymbol{S}^2(Q, A, B)$ where we have $(A \cup B)_3 = (A \cup B)_1$. That is, in $\boldsymbol{S}^2(Q, A, B)$, we implicitly ignore all assignments to $V_3$ where atoms from $A$ or $B$ have different truth values as those in $V_1$. Therefore, it makes sense to consider only atoms from $V_3 \setminus (A_3 \cup B_3)$ and use $A_1 \cup B_1$ instead of $A_3 \cup B_3$ in $Q_{3,3}$.

This calls for a more subtle renaming schema for programs, however. Let $\mathcal{V}$ be a set of indexed atoms, and let $r$ be a rule. Then, $r^{\mathcal{V}}_{i,k}$ results from $r$ by replacing each atom $x$ in $r$ by $x_i$, providing $x_i \in \mathcal{V}$, and by $x_k$ otherwise. For a program $P$, we define

$$P^{\mathcal{V}}_{i,j,k} := \bigwedge_{r \in P} \big((B^+(r^{\mathcal{V}}_{i,k}) \wedge B^-(r^{\mathcal{V}}_{j,k})) \rightarrow H(r^{\mathcal{V}}_{i,k})\big).$$

Moreover, for any $i \geq 0$, any set $V$ of atoms, and any set $C$, $V^C_i := (V \setminus C)_i$.

**Definition 5.** *Let $P, Q$ be programs over $V$, let $A, B \subseteq V$, and $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$. Furthermore, let $\mathcal{V} = V_1 \cup V_2^A \cup V_3^{A \cup B} \cup V_4 \cup V_5^A$. Then,*

$$\begin{aligned}
\boldsymbol{T}_\Pi(V_1) &:= P_{1,1} \wedge \boldsymbol{T}^1(P, A, \mathcal{V}) \wedge \forall V_3^{A \cup B}\big(Q^{\mathcal{V}}_{3,1,1} \rightarrow \boldsymbol{T}^3(P, Q, A, \mathcal{V})\big), \text{ where} \\
\boldsymbol{T}^1(P, A, \mathcal{V}) &:= \forall V_2^A\big((V_2^A < V_1^A) \rightarrow \neg P^{\mathcal{V}}_{2,1,1}\big) \text{ and} \\
\boldsymbol{T}^3(P, Q, A, \mathcal{V}) &:= \exists V_4\big(\big(V_4 < ((A \cup B)_1 \cup V_3^{A \cup B})\big) \wedge Q^{\mathcal{V}}_{4,3,1} \wedge \big((A_4 < A_1) \rightarrow \\
&\qquad \forall V_5^A((V_5^A \leq V_1^A) \rightarrow \neg P^{\mathcal{V}}_{5,1,4})\big)\big).
\end{aligned}$$

Note that the subformula $V_4 < ((A \cup B)_1 \cup V_3^{A \cup B})$ in $\boldsymbol{T}^3(P, Q, A, \mathcal{V})$ denotes

$$\big(\big((A \cup B)_4 \leq (A \cup B)_1\big) \wedge (V_4 \leq V_1)\big) \wedge \neg\big(\big((A \cup B)_1 \leq (A \cup B)_4\big) \wedge (V_1 \leq V_4)\big).$$

Also note that, compared to the first encoding $\boldsymbol{S}_\Pi(V_1)$, we do not have a pendant to subformula $\boldsymbol{S}^2$ here, which reduces simply to $Q^{\mathcal{V}}_{3,1,1}$ due to the new renaming schema.

**Lemma 2.** *Let $P, Q$ be programs over $V$, and let $A, B, Y \subseteq V$. Then, $Y$ is a partial spoiler for $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ iff $Y_1 \models \boldsymbol{T}_\Pi(V_1)$.*

For illustration, consider the two programs $P = \{a \vee b \leftarrow c\}$ and $Q = \{a \leftarrow c, not\, b\}$, $A = \{a\}$, and $B = \{b\}$. The encodings for the problem $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ are as follows:

$$\begin{aligned}
\boldsymbol{S}_\Pi(V_1) &= (c_1 \rightarrow a_1 \vee b_1) \wedge \boldsymbol{S}^1(P, A) \wedge \\
&\quad \forall a_3 b_3 c_3 (\boldsymbol{S}^2(Q, A, B) \rightarrow \boldsymbol{S}^3(P, Q, A)), \\
\boldsymbol{S}^1(P, A) &= \forall a_2 b_2 c_2 \big((a_2 \leftrightarrow a_1) \wedge (\{b_2, c_2\} < \{b_1, c_1\}) \rightarrow \neg(c_2 \rightarrow a_2 \vee b_2)\big), \\
\boldsymbol{S}^2(Q, A, B) &= (a_3 \leftrightarrow a_1) \wedge (b_3 \leftrightarrow b_1) \wedge (c_3 \wedge \neg b_3 \rightarrow a_3), \\
\boldsymbol{S}^3(P, Q, A) &= \exists a_4 b_4 c_4 \big((\{a_4, b_4, c_4\} < \{a_3, b_3, c_3\}) \wedge (c_4 \wedge \neg b_3 \rightarrow a_4) \wedge \\
&\qquad (\{a_4\} < \{a_1\}) \rightarrow \forall a_5 b_5 c_5 ((a_5 \leftrightarrow a_4) \wedge \\
&\qquad (\{a_5, b_5, c_5\} \leq \{a_1, b_1, c_1\}) \rightarrow \neg(c_5 \rightarrow a_5 \vee b_5)))\big); \\
\boldsymbol{T}_\Pi(V_1) &= (c_1 \rightarrow a_1 \vee b_1) \wedge \boldsymbol{T}^1(P, A, \mathcal{V}) \wedge \\
&\quad \forall c_3 \big((c_3 \wedge \neg b_1 \rightarrow a_1) \rightarrow \boldsymbol{T}^3(P, Q, A, \mathcal{V})\big), \\
\boldsymbol{T}^1(P, A, \mathcal{V}) &= \forall b_2 c_2 ((\{b_2, c_2\} < \{b_1, c_1\}) \rightarrow \neg(c_2 \rightarrow a_1 \vee b_2), \\
\boldsymbol{T}^3(P, Q, A, \mathcal{V}) &= \exists a_4 b_4 c_4 \big((\{a_4, b_4, c_4\} < \{a_1, b_1, c_3\}) \wedge (c_4 \wedge \neg b_1 \rightarrow a_4) \wedge \\
&\qquad (\{a_4\} < \{a_1\}) \rightarrow \forall b_5 c_5 ((\{b_5, c_5\} \leq \{b_1, c_1\}) \rightarrow \\
&\qquad \neg(c_5 \rightarrow a_4 \vee b_5)))\big).
\end{aligned}$$

As mentioned before, the optimised encoding $\boldsymbol{T}_\Pi(\cdot)$ saves "fixed assignments", like $(a_2 \leftrightarrow a_1)$, which occur in $\boldsymbol{S}_\Pi(\cdot)$, by employing the advanced renaming schema in such a way that, instead of atom $a_2$, atom $a_1$ is used in the encoding. One effect of this refinement is the decrease of universally quantified atoms.

**Theorem 1.** *For any inclusion problem $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$, the following statements are equivalent:* (*i*) *$\Pi$ holds;* (*ii*) *$\neg\exists V_1 \boldsymbol{S}_\Pi(V_1)$ is valid; and* (*iii*) *$\neg\exists V_1 \boldsymbol{T}_\Pi(V_1)$ is valid.*

**Corollary 1.** *Let $\Pi = (P, Q, \mathcal{P}_A, =_B)$ be an equivalence problem. Then, for $\Pi' = (P, Q, \mathcal{P}_A, \subseteq_B)$ and $\Pi'' = (Q, P, \mathcal{P}_A, \subseteq_B)$, the following statements are equivalent:* (*i*) *$\Pi$ holds;* (*ii*) *$\neg\exists V_1 \boldsymbol{S}_{\Pi'}(V_1) \wedge \neg\exists V_1 \boldsymbol{S}_{\Pi''}(V_1)$ is valid; and* (*iii*) *$\neg\exists V_1 \boldsymbol{T}_{\Pi'}(V_1) \wedge \neg\exists V_1 \boldsymbol{T}_{\Pi''}(V_1)$ is valid.*

### 4.3 Applicability and Adequacy of the Encodings

In order to employ off-the-shelves QBF-solvers for deciding answer-set correspondence, we have to transform above encodings into prenex normal form. The propositional part of these prenex QBFs additionally has to be reduced to CNF, which can be accomplished by usual techniques. We thus focus here just on possible prenex normal forms of our encodings.

Recall that there are several ways to transform a QBF into prenex normal form. For our encodings, the situation is as follows. Take, e.g., the existential closure of $\boldsymbol{S}_\Pi(V_1)$, given by $\exists V_1 \boldsymbol{S}_\Pi(V_1)$: for this closed QBF, different prenex forms can be obtained, e.g.,

$$\exists V_1 \forall (V_2 \cup V_3) \exists V_4 \forall V_5\, \phi \quad \text{or} \quad \exists V_1 \forall V_3 \exists V_4 \forall (V_5 \cup V_2)\, \phi,$$

where $\phi$ represents the so-called *propositional skeleton* of the QBF $\boldsymbol{S}_\Pi(V_1)$ (cf. [8]), which, roughly speaking, results from $\boldsymbol{S}_\Pi(V_1)$ by deleting all quantifiers. For later purposes, we use in the following the second variant, and define $\boldsymbol{S}^{\mathsf{p}}_\Pi := \exists V_1 \forall V_3 \exists V_4 \forall (V_5 \cup V_2)\, \phi$. Likewise, we use $\boldsymbol{T}^{\mathsf{p}}_\Pi := \exists V_1 \forall V_3^{A \cup B} \exists V_4 \forall (V_5^A \cup V_2^A)\, \psi$ as a prenex form for $\exists V_1 \boldsymbol{T}_\Pi(V_1)$, where $\psi$ is the propositional skeleton of $\boldsymbol{T}_\Pi(V_1)$.

**Theorem 2.** *For any inclusion problem* $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$*, the following statements are equivalent:* (*i*) $\Pi$ *holds;* (*ii*) $\neg \boldsymbol{S}^{\mathsf{p}}_\Pi$ *is valid;* (*iii*) $\neg \boldsymbol{T}^{\mathsf{p}}_\Pi$ *is valid.*

These prenex forms also give evidence that our encodings are *adequate* in a certain theoretical sense: Following [3], given decision problems $\mathcal{D} \subseteq \mathcal{L}$ and $\mathcal{D}' \subseteq \mathcal{L}'$ in languages $\mathcal{L}$ and $\mathcal{L}'$, respectively, we call an encoding $f : \mathcal{L} \to \mathcal{L}'$ *adequate* iff, for each $s \in \mathcal{L}$, (i) $s \in \mathcal{D}$ iff $f(s) \in \mathcal{D}'$, (ii) $f(s)$ is constructible in polynomial time from $s$, and (iii) deciding whether $f(s) \in \mathcal{D}'$ is not computationally harder than deciding whether $s \in \mathcal{D}$.

From Proposition 5, we get that the complementary problem of inclusion checking, i.e., checking whether, for given $P$, $Q$, $A$, $B$, the problem $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ does not hold, is $\Sigma^P_4$-complete. Note that, for any such $\Pi$, (i) $\boldsymbol{S}^{\mathsf{p}}_\Pi$ is valid iff $\Pi$ does not hold (by Theorem 2), (ii) $\boldsymbol{S}^{\mathsf{p}}_\Pi$ is always computable in polynomial time (indeed, in linear time) in the size of $\Pi$ (as is easily verified from the definitions), and (iii) $\boldsymbol{S}^{\mathsf{p}}_\Pi$ is a $(4, \exists)$-QBF. From Proposition 6, we know that determining the truth of $\boldsymbol{S}^{\mathsf{p}}_\Pi$ is thus in the same complexity class (viz. $\Sigma^P_4$) as the encoded problem. All these properties hold for $\boldsymbol{T}^{\mathsf{p}}_\Pi$ as well. Hence, both of our encodings are adequate.

## 5 Obtaining Counterexamples

In this section, we provide a theoretical basis how to use our encodings to obtain counterexamples for an inclusion problem $(P, Q, \mathcal{P}_A, \subseteq_B)$. To this end, we use the concept of *policies* for prenex QBFs, along the lines of Coste-Marquis *et al*. [4].

**Definition 6.** *The set* $P(k, \mathsf{Q}, X_k, \ldots, X_1)$ *of* policies *for a* $(k, \mathsf{Q})$*-QBF of the form* $\mathsf{Q}_k X_k \ldots \mathsf{Q}_1 X_1 \phi$ *is inductively defined as follows:*

1. $P(0, \mathsf{Q}) = \{\lambda\}$,
2. $P(k, \exists, X_k, \ldots, X_1) = \{(I, \pi) \mid I \subseteq X_k,\ \pi \in P(k-1, \forall, X_{k-1}, \ldots, X_1)\}$*, and*
3. $P(k, \forall, X_k, \ldots, X_1) = \{\pi \mid \pi : 2^{X_k} \to P(k-1, \exists, X_{k-1}, \ldots, X_1)\}$,

*where* $\lambda$ *represents the empty policy.*

Note that policies for $(k, \exists)$-QBFs are pairs $(I, \pi)$, where $I$ is an interpretation over atoms from the outermost group of quantifiers and $\pi$ is a policy itself, whereas policies for $(k, \forall)$-QBFs are functions assigning to each interpretation over atoms of the outermost group of quantifiers a policy.

**Definition 7.** *A* $(k, \mathsf{Q})$*-QBF* $\Phi = \mathsf{Q}_k X_k \ldots \mathsf{Q}_1 X_1 \phi$ *is* satisfied *by a policy* $\pi$ (*for* $\Phi$) *iff one the following conditions applies* (*inductively*)*:*

1. $k = 0$, $\pi = \lambda$*, and* $\phi$ *is true,*

2. $k > 0$, $\mathsf{Q} = \exists$, $\pi = (I, \pi')$, *and* $\forall X_{k-1} \ldots \mathsf{Q}_1 X_1 \phi[X_k/I]$ *is satisfied by* $\pi'$,
3. $k > 0$, $\mathsf{Q} = \forall$, *and for any* $I \subseteq X_k$, $\exists X_{k-1} \ldots \mathsf{Q}_1 X_1 \phi[X_k/I]$ *is satisfied by* $\pi(I)$.

*Denote by* $SP(\Phi)$ *the set of satisfying policies for a prenex QBF* $\Phi$.

**Proposition 10.** *A prenex QBF* $\Phi$ *is valid iff* $SP(\Phi) \neq \emptyset$.

For illustration, consider $\phi = (p \rightarrow q) \wedge (q \rightarrow p)$ and the following QBFs:[2]

$$\Phi_1 = \exists pq\, \phi, \quad \Phi_2 = \forall pq\, \phi, \quad \Phi_3 = \exists p \forall q\, \phi, \quad \text{and} \quad \Phi_4 = \forall p \exists q\, \phi.$$

The set of policies for $\Phi_1$ is given by $\{(I, \lambda) \mid I \subseteq \{p, q\}\}$, i.e., the satisfying policies for $\Phi_1$ are in a one-to-one correspondence to the models of $\phi$, and are given by $(\emptyset, \lambda)$ and $(\{p, q\}, \lambda)$. For $\Phi_2$, the only policy is the function $\pi$ assigning to each $I \subseteq \{p, q\}$ the empty policy $\lambda$. Note that $\pi$ is not satisfying $\Phi_2$ since, for instance, with $I = \{p\}$, we get $\pi(I) = \lambda$, but $\phi[\{p, q\}/I] = (\top \rightarrow \bot) \wedge (\bot \rightarrow \top)$ is not true. For $\Phi_3$, we get as policies $\pi_1 = (\{p\}; \pi')$ and $\pi_2 = (\emptyset; \pi')$, where $\pi'$ is defined as $\pi'(\{q\}) = \pi'(\emptyset) = \lambda$. It can be shown that neither $\pi_1$ nor $\pi_2$ satisfy $\Phi_3$, by similar arguments as for the case of $\Phi_2$. Finally, $\Phi_4$ yields four policies, given as follows:

$$\begin{array}{llll} \pi(p) = (q, \lambda), & \pi(\emptyset) = (q, \lambda); & \pi'(p) = (q, \lambda), & \pi'(\emptyset) = (\emptyset, \lambda); \\ \pi''(p) = (\emptyset, \lambda), & \pi''(\emptyset) = (q, \lambda); & \pi'''(p) = (\emptyset, \lambda), & \pi'''(\emptyset) = (\emptyset, \lambda). \end{array}$$

One can verify that $\pi'$ is the only satisfying policy for $\Phi_4$.

We now use the concept of policies to obtain the counterexamples from the satisfying policies of our encodings. Note that, in the definition below, we make use of our renaming schema as used in the encodings; e.g., $Z_3 = \{z_3 \mid z \in Z\}$.

**Definition 8.** *Let* $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ *be an inclusion problem,* $\boldsymbol{S}^{\mathsf{p}}_{\Pi}$ *and* $\boldsymbol{T}^{\mathsf{p}}_{\Pi}$ *as in Subsection 4.3, and* $\Omega \in \{\boldsymbol{S}, \boldsymbol{T}\}$. *Then,*

$$\sigma(\Omega, \Pi) := \{(Y, \Sigma_{\Omega, Y, \pi}) \mid (Y_1, \pi) \in SP(\Omega^{\mathsf{p}}_{\Pi})\},$$

*where*

$$\begin{array}{rl} \Sigma_{\boldsymbol{S}, Y, \pi} := \{(X, Z), (Z, Z) \mid & Z =_{A \cup B} Y, (Z, Z) \in SE^A(Q), \\ & \pi(Z_3) = (X_4, \pi'), \text{ for some } \pi'\} \quad \text{and} \\ \Sigma_{\boldsymbol{T}, Y, \pi} := \{(X, Y \dot{+} Z), (Y \dot{+} Z, Y \dot{+} Z) \mid & (Y \dot{+} Z, Y \dot{+} Z) \in SE^A(Q), \\ & \pi(Z_3) = (X_4, \pi'), \text{ for some } \pi'\}, \end{array}$$

*and* $Y \dot{+} Z$ *stands for* $Y|_{A \cup B} \cup Z$.

These two projections, $\sigma(\boldsymbol{S}, \cdot)$ and $\sigma(\boldsymbol{T}, \cdot)$, on the satisfying policies for our two encodings are actually identical. Hence, our final two results in this section apply to both encodings.

**Theorem 3.** *Let* $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ *be an inclusion problem and* $\Omega \in \{\boldsymbol{S}, \boldsymbol{T}\}$. *Then, each* $(Y, \Sigma) \in \sigma(\Omega, \Pi)$ *is a spoiler for* $\Pi$.

[2] In what follows, we sometimes omit brackets "{" and "}" for ease of notation.

In view of the construction of Proposition 2, we can thus construct counterexamples directly from the satisfying policies of our encodings.

**Corollary 2.** *Let* $\Pi = (P, Q, \mathcal{P}_A, \subseteq_B)$ *be an inclusion problem and* $\Omega \in \{\boldsymbol{S}, \boldsymbol{T}\}$. *Then, each* $(Y, \Sigma) \in \sigma(\Omega, \Pi)$ *induces a counterexample* $(Y, P_{\Sigma,A})$ *for* $\Pi$.

From Proposition 10 and Theorem 2, in turn, we obtain that in case no satisfying policy for our encodings exists, the considered inclusion problem holds, and therefore does not possess any counterexample.

## 6 Special Cases

Finally, we analyse our encodings in the light of special instantiations of correspondence problems and give pointers to related work.

In what follows, for every equivalence problem $\Pi = (P, Q, \mathcal{P}_A, =_B)$, let $\Pi' = (P, Q, \mathcal{P}_A, \subseteq_B)$ and $\Pi'' = (Q, P, \mathcal{P}_A, \subseteq_B)$ be the associated inclusion problems (see also Corollary 1).

In case of *strong equivalence* [20], i.e., for problems of form $\mathit{\Pi} = (P, Q, \mathcal{P}_A, =_A)$ with $A = \mathcal{U}$, the encodings $\boldsymbol{T}_{\Pi'}(V_1)$ and $\boldsymbol{T}_{\Pi''}(V_1)$, as defined in Definition 5, can be drastically simplified since $V_2^A = V_3^A = V_5^A = \emptyset$. In particular, $\boldsymbol{T}_{\Pi'}(V_1)$ is equivalent to

$$P_{1,1} \wedge \big(Q_{1,1} \rightarrow \exists V_4\big((V_4 < V_1) \wedge Q_{4,1} \wedge \neg P_{4,1}\big)\big).$$

Note that the composed encoding for deciding strong equivalence, i.e., the closed QBF $\neg\exists V_1 \boldsymbol{T}_{\Pi'}(V_1) \wedge \neg\exists V_1 \boldsymbol{T}_{\Pi''}(V_1)$, amounts to a propositional unsatisfiability test, witnessing the coNP-completeness complexity for checking strong equivalence [24]. One can show that the reductions due to Pearce *et al*. [24] and Lin [21] for testing strong equivalence in terms of propositional logic are simple variants thereof.

For strong equivalence *relative* to a set $A$ of atoms [28], i.e., for $\mathit{\Pi}$ being of form $(P, Q, \mathcal{P}_A, =_B)$ with $B = \mathcal{U}$ but with arbitrary $A$, our encodings $\boldsymbol{T}_{\Pi'}(V_1)$ and $\boldsymbol{T}_{\Pi''}(V_1)$ can still be simplified since $V_3^{A \cup B} = \emptyset$. Indeed, $\boldsymbol{T}^{\mathsf{p}}_{\Pi'}$ and $\boldsymbol{T}^{\mathsf{p}}_{\Pi''}$ are then $(2, \exists)$-QBFs, reflecting the complexity of strong equivalence relative to $A$, which is on the second level of the polynomial hierarchy [28].

Next, we address the case of *bounded* relativised strong equivalence, as investigated by Eiter *et al*. [11]. This notion applies to problems of form $\Pi = (P, Q, \mathcal{P}_A, =)$, where the cardinality of $(\mathcal{U} \setminus A)$, i.e., the number of atoms missing in $A$, is bounded by a constant. Hereby, the sets $V_2^A$ and $V_5^A$, which build the only universal quantifiers in the encoding $\boldsymbol{T}_{\mathit{\Pi}'}(V_1)$ for relativised strong equivalence, are sets of a fixed size. Hence, we can eliminate these quantifiers according to the semantics and still get an adequate encoding for this particular notion of equivalence. Consequently, bounded relativised strong equivalence can be checked with a polynomial unsatisfiability test, once again reflecting the coNP-complexity of this problem [11].

Finally, we address the case of ordinary equivalence, i.e., considering problems of form $\Pi = (P, Q, \mathcal{P}_A, =)$ with $A = \emptyset$, which is well known to be $\mathit{\Pi}_2^P$-complete [10]. Here, the encoding $\boldsymbol{S}_{\Pi'}(V_1)$ from Definition 4 can be simplified as follows:

$$P_{1,1} \wedge \forall V_2\big((V_2 < V_1) \rightarrow \neg P_{2,1}\big) \wedge \big(Q_{1,1} \rightarrow \exists V_4((V_4 < V_1) \wedge Q_{4,1})\big).$$

One can observe that this encoding is related to encodings for computing stable models via QBFs, as discussed by Egly *et al*. [6] and Pearce *et al*. [24]. Indeed, taking the two main conjuncts from $\boldsymbol{S}_{\Pi'}(V_1)$, $\Phi = P_{1,1} \wedge \forall V_2\big((V_2 < V_1) \rightarrow \neg P_{2,1}\big)$ and $\Psi = Q_{1,1} \rightarrow \exists V_4\big((V_4 < V_1) \wedge Q_{4,1})\big)$, we get, for any assignment $Y_1 \subseteq V_1$, $Y_1 \models \Phi$ iff $Y$ is an answer set of $P$, and $Y_1 \models \Psi$ iff $Y$ is not an answer set of $Q$. Note that once more the encodings reflect the inherent complexity of the reduced equivalence checking task, viz. the $\Pi_2^P$-completeness for ordinary equivalence in this case.

## 7 Conclusion

In this paper, we discussed a novel decision procedure for advanced program comparison in answer-set programming (ASP) via encodings into quantified propositional logic. This approach was motivated by the high computational complexity we have to face for this task, making a direct realisation via ASP hard to accomplish. Furthermore, we showed how to obtain counterexamples from policies, which satisfy these encodings, and discussed special instances of the considered correspondence problems. Since currently practicably efficient solvers for quantified propositional logic are available, they can be used as back-end inference engines to compute the correspondence problems under consideration using the proposed encodings. Moreover, since these correspondence problems are one of the few natural problems lying above the second level of the polynomial hierarchy, yet still part of the polynomial hierarchy, we believe that our encodings also provide valuable benchmarks for evaluating QBF-solvers, for which there is actually a lack of structured problems with more than one quantifier alternation (see [17, 16]).

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