On Representing Concepts in High-Dimensional Linear Spaces

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Abstract. Producing a mathematical model of concepts is a very important issue in artificial intelligence, because if such a model were found this, besides being a very interesting result in its own right, would also contribute to the emergence of what we could call the 'mathematics of thought.' One of the most interesting attempts made in this direction is P. Gärdenfors' theory of conceptual spaces, a theory which is mostly presented by its author in an informal way. The main aim of the present article is contributing to Gärdenfors' theory of conceptual spaces by discussing some of the advantages which derive from the possibility of representing concepts in high-dimensional linear spaces.

1 Introduction

Producing a mathematical model of concepts is a very important issue in artificial intelligence, because if such a model were found this, besides being a very interesting result in its own right, would also contribute to the emergence of what we could call the 'mathematics of thought.' One of the most interesting attempts made in this direction is P. Gärdenfors' theory of conceptual spaces, a theory which is mostly presented by its author in an informal way.

The main aim of the present article is contributing to Gärdenfors' theory of conceptual spaces by discussing some of the advantages which derive from the possibility of representing concepts in high-dimensional linear spaces.

In what follows, section 2 is dedicated to providing the main features of and motivations behind Gärdenfors's theory of conceptual spaces.

Section 3 discusses the application of high-dimensional linear spaces to the representation of concepts without engaging in a preliminary discussion of the several attempts made to formalize conceptual spaces present in the literature.[4] (The lack of the above mentioned survey is due to requirements concerning article length).

[4] See on this [Chella et al., 1997], [Chella et al., 1998], [Raubal, 2004], [Raubal, 2009], [Rickard, 2006], [Rickard et al., 2007], [Augello et al., 2013].

In section 4 we examine some of the advantages and limitations of our approach to conceptual spaces; and, eventually, section 5 brings the paper to a close with providing a very short resume of what we think we have achieved in this article.

2 Gärdenfors conceptual spaces

Gärdenfors in [Gärdenfors, 2004] describes a cognitive architecture for modelling representations. In this architecture an intermediate level, called 'geometrical conceptual space', is introduced between a linguistic-symbolic level and an associationistic sub-symbolic level to produce a mathematical representation of concepts and their use.[5]

According to Gärdenfors, conceptual spaces represent concepts exploiting geometrical structure rather than symbols or connections between neurons. This geometrical representation is based on the existence/construction of a space endowed with a number of what Gärdenfors calls 'quality dimensions,' quality dimensions whose main function is that of representing different qualities of objects such as brightness, temperature, height, width, depth. For him, many important conceptual spaces are metric spaces, i.e. they are sets of points on which a distance function is defined.

Within conceptual spaces, objects are represented as points and concepts as regions. These regions may have various shapes, although to some concepts — those which refer to natural kinds or natural properties[6] — correspond regions which are characterized by convexity.[7] According to Gärdenfors, this latter type of region is strictly related to the notion of prototype, i.e., to those entities that may be regarded as the archetypal representatives of a given category of objects (the centroids of the convex regions).[8]

For him, judgments of similarity play a crucial rôle in cognitive processes, and he conjectures that the smaller is the distance between the representations of two given objects (in a conceptual metric space) the more similar to each other the objects represented are.

Gärdenfors's motivation for the introduction of conceptual spaces is that, even if the symbolic approach to modeling representations is very rich and expressive, it has some

[5] On Gärdenfors tripartite cognitive architecture see: [Augello et al., 2013], [Augello et al., 2014], [Augello et al., 2015].

[6] [Gärdenfors, 2004], Chapter 3, §3.5, p. 71:

> CRITERION P A *natural property* is a convex region of a domain in a conceptual space.

[7] [Gärdenfors, 2004], Chapter 3, §3.4, p. 69:

> DEFINITION 3.3 A subset C of a conceptual space S is said to be *convex* if, for all points x and y in C, all points between x and y are also in C.

[8] [Gärdenfors, 2004], Chapter 3, §3.9, p. 88:

> [A]ssuming that a Euclidean metric is defined on the subspace that is subject to categorization, a set of prototypes will by this method [Voronoi tesselation] generate a unique partitioning of the subspace into convex regions.

intrinsic limitations represented, for example, by the 'symbol grounding problem,'[9] and by the well known A.I. 'frame problem'.[10] On the other hand, the associationist approach suffers from its low-level nature, which makes it unsuited for modeling complex representations.

According to Gärdenfors, conceptual spaces strike a happy medium between the two above mentioned systems used for modeling representations, a happy medium given by the consideration that:

1. conceptual spaces, in contrast with associationist approaches, are suited for modeling complex representations;
2. within conceptual spaces problems affecting the symbolic approach, like the symbol grounding problem, can be solved.

Let us now consider a simple example of a conceptual space. Assume the existence of what we are going to call 'ground space,' that is, the space in which a cognitive agent $\mathcal{A}$ acts according to certain rules or by using given tools; that such a space is none other than $\mathbb{R}^2$ (see Figure 1); and that $\mathcal{A}$ can operate in $\mathbb{R}^2$ using pencil, straightedge, and compasses.

Now, using compasses, $\mathcal{A}$ can draw circles of any centre $p \in \mathbb{R}^2$, and any finite radius $r \in \mathbb{R}$, for $0 < r$; and can also measure the length of the radius of the circles he draws. Since 'x has a radius of the same length as y' is an equivalence relation on the collection C of all circles in $\mathbb{R}^2$, it follows that C is partitioned by it into mutually disjoint equivalence classes in such a way that any two circles in C having radii of the same length belong to the same equivalence class.

Note that the equivalence classes of congruent circles in $\mathbb{R}^2$ can be represented as points of the one-dimensional conceptual space $[C]$ with coordinates $x \in (0, \infty) \subseteq \mathbb{R}$. This is a conceptual space which, for obvious reasons, we are going to call **circle**. In **circle**, any $x \in (0, \infty)$ expresses the length of the radius of the circles belonging to the equivalence class with coordinate x. (We use x as the index of the equivalence class of which x is the coordinate.)

If we define the distance in **circle** between any two equivalence classes as the absolute value of the difference between the lengths of the radii of any two of their representatives then **circle** is a metric space, and, clearly, the smaller is the distance between two

[9] The following quotation from [Harnad, 1990] is to be found in [Gärdenfors, 2004], Chapter 2, §2.2.2, p. 38:

> How can the semantic interpretation of a formal symbol system be made *intrinsic* to the system, rather than just parasitic on the meaning in our heads? How can the meanings of the meaningless symbol tokens, manipulated solely on the basis of their (arbitrary) shapes, be grouped in anything but other meaningless symbols?

[10] [Gärdenfors, 2004], Chapter 2, §2.2.2, p. 37:

> The frame problem can be defined as the problem of specifying on the symbolic level what changes and what stays constant in the particular domain when a particular action is performed.

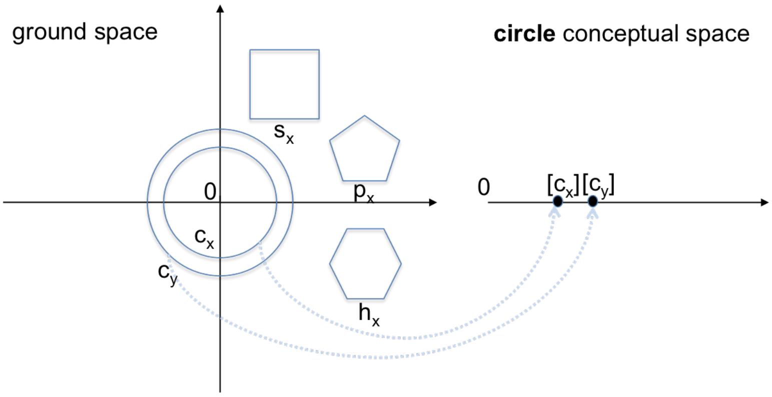

Fig. 1. From Ground to Conceptual Space

points $[c_x]$ and $[c_y]$ of **circle**, the more similar to each other any $s \in [c_x]$ and $l \in [c_y]$ will be. Other interesting mathematical examples of conceptual spaces are **rectangle**, and **direction**.[11]

What we intend to do in the remaining part of this paper is contributing to Gärdenfors's theory of conceptual spaces by:

[11] The conceptual space **rectangle** is discussed by Gärdenfors in [Gärdenfors, 2004], Chapter 3, §3.10, pp. 93–94. With regard to the conceptual space **direction**, if you define the parallelism relation on $\mathbb{R}^2$ in such a way that any straight line l in $\mathbb{R}^2$ is parallel to itself, then the parallelism relation becomes an equivalence relation on the set of straight lines S in $\mathbb{R}^2$. From this we have that this equivalence relation partitions S into mutually disjoint equivalence classes.

Now, if $[S]$ is the set of the above mentioned equivalence classes, let τ be the function

$$\tau : [S] \to [0, \pi)$$

that associates each $[s] \in [S]$ to the angle made by each $s \in [s]$ with the x-axis of $\mathbb{R}^2$. (Note that τ is injective and surjective.) If $\tau([s]) = \theta$ and $\tau([l]) = \lambda$, and we put $d([s], [l]) = d(\tau([s]), \tau([l]))$, where $d(\tau([s]), \tau([l]))$ is the usual metric on $[0, \pi)$, we have:

$$d([s], [l]) = d(\tau([s]), \tau([l])) \quad (1)$$
$$= d(\theta, \lambda) \quad (2)$$
$$= |\theta - \lambda|. \quad (3)$$

Therefore, $[S]$, with the above metric, is the 1-dimensional conceptual metric space **direction**.

Consider that in **direction** each $[s] \in [S]$ is a direction, and that in it we can distinguish a region $\mathcal{R}_1$, the points of which are equivalence classes of straight lines with an angle θ with the x-axis such that $\theta \in (0, \frac{\pi}{2})$ (which have a positive derivative); from the region $\mathcal{R}_2$, the points of which are equivalence classes of straight lines with an angle θ with the x-axis such that $\theta \in (\frac{\pi}{2}, \pi)$ (which have a negative derivative). Clearly, $\mathcal{R}_1$ and $\mathcal{R}_2$ are convex sets.

Lastly, for any $[s], [l] \in$ **direction**, if $d([s], [l]) \to 0$ then also $|\theta - \lambda| \to 0$ and, consequently, any $s \in [s]$ and $l \in [l]$ will be more similar to each other in the sense that they will tend to become parallel to each other.

(a) providing plausible similarity measures between patterns[12] using a particular example of conceptual spaces — high-dimensional linear spaces — and the so-called 'kernels' (see next section);
(b) individuating a general mathematical strategy for solving, within high-dimensional linear spaces, the problem concerning the possibility of assigning perceptual input either to the region of a conceptual space representing a given concept or to its complement when exemplars of patterns falling under the concept (positive exemplars) and of patterns not falling under the concept (negative exemplars) are provided.

Both these questions are of vital importance for the theory of conceptual spaces, because:

(i) much of what Gärdenfors intends to do (with his theory) hangs on the existence of an effective measure of similarity between patterns;
(ii) the meaningfulness and usefulness of a concept in general and, in particular, of a concept given in terms of 'positive' and 'negative' exemplars, rest on the possibility of recognizing whether perceptual input falls under it or not.

3 Kernels, and high-dimensional linear spaces

Imagine you are in a farm and, while you are walking about in its grounds, you come across a group of horses. These horses are either ponies or normal height horses. You are not an expert on horses and, although you recognize some of them as ponies and others as normal height horses, you would like to have a criterion which, exploiting the few ponies and normal height horses you recognize, might enable you to assign to each horse in the group either the label 'pony' or the label 'normal height horse.'

Given the present state of your knowledge about horses, your wish can be granted only if you are able to establish, for each horse in the group, that either this is more similar to the ponies you recognize than to the normal height horses you recognize or *vice versa*. Of course, this ability presupposes the possibility of defining a relevant similarity measure on the group of horses.

We can give a mathematical representation of this problem in the following way. Call $\mathfrak{D}$, *domain*, the group of horses, and let P and Q be the two distinguished subsets of $\mathfrak{D}$ whose elements are, respectively, the positive (ponies you recognize) and the negative (normal height horses you recognize) exemplars.

The problem of finding a (relevant) similarity measure on the group of horses now becomes the problem of finding a function k on $\mathfrak{D} \times \mathfrak{D}$ into the reals, $k : \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}$, such that, given any ordered pair of horses $(a, b) \in \mathfrak{D} \times \mathfrak{D}$, associates to it the real number expressing how similar the first and the second element of the pair are to one other (with regard to the relevant feature). The function k above is known in the literature as *kernel*.

It is important to say that similarity measures (kernels) vary according to the type of elements we find in the domain $\mathfrak{D}$, e.g. horses, cats, dogs, pigs, lawyers, politicians,

[12] Here the concept of pattern is more general than that of object. For the concepts of pattern and object see: [Resnik, 1981], [Dennett, 1991], [Oliveri, 1997], [Oliveri, 1998], [Shapiro, 2000], [Resnik, 2001], [Oliveri, 2007], [Oliveri, 2012], [Bombieri, 2013].

etc., and that the choice of the right/relevant similarity measure for the elements of a certain domain is not, in general, a trivial matter.

However, to see a particularly simple example of kernels at work as similarity measures, let us assume that there exists a function σ which embeds $\mathfrak{D}$ (our group of horses) into a real inner product linear space W, which we are going to call *feature space*;[13] and that we can define the appropriate kernel on $\mathfrak{D} \times \mathfrak{D}$ using the canonical dot product in W[14] in the following way:
for any $a, b \in \mathfrak{D}$,

$$k(a,b) = (\sigma(a) \cdot \sigma(b)) \quad (4)$$
$$= (w_a \cdot w_b). \quad (5)$$

Indeed, if, for $w_a, w_b \in W$, we consider the vectors $\overline{w}_a = \frac{w_a}{\|w_a\|}$ and $\overline{w}_b = \frac{w_b}{\|w_b\|}$, we have that $\|\overline{w}_a\| = \|\overline{w}_b\| = 1$, and that $(\overline{w}_a \cdot \overline{w}_b) = \cos\theta$, where θ is the angle between $\overline{w}_a$ and $\overline{w}_b$. And to see how the dot product in W defines a similarity measure k on $\mathfrak{D}$, it is sufficient to observe that when $\theta \to 0$ the distance between the two vectors $\overline{w}_a$ and $\overline{w}_b$, $d(\overline{w}_a, \overline{w}_b) = \|\overline{w}_a - \overline{w}_b\|$, tends to 0 as well and, therefore, as a consequence of the embedding σ, a and b become more and more similar to one another.

Having seen how kernels can offer a satisfactory similarity measure on $\mathfrak{D}$ (point **(a)** of section 2), let us now illustrate an example of a simple algorithm for solving, within high-dimensional linear spaces, the problem concerning the possibility of assigning perceptual input either to the region of a conceptual space representing a given concept or to its complement when positive and negative exemplars are provided (point **(b)** §2).

Assume that $\mathfrak{D}$, k, W, P, Q and σ are as above, and that $w_{x_1}, \ldots, w_{x_n} \in \sigma(P)$ are the vectors in W representing the positive exemplars (the ponies you recognize), whereas $w_{y_1}, \ldots, w_{y_m} \in \sigma(Q)$ are the vectors in W representing the negative exemplars (the normal height horses you recognize).

Now, given an arbitrary horse $h \in \mathfrak{D}$, and its vectorial representation w_h in W, determine the mean vector w_x of $w_{x_1}, \ldots, w_{x_n}$, and the mean vector w_y of $w_{y_1}, \ldots, w_{y_m}$. Having done so, calculate the distance of w_h from w_x and from w_y; h is a pony if and only if $\|w_h - w_x\| < \|w_h - w_y\|$; and, of course, h is a normal height horse if and only if $\|w_h - w_y\| < \|w_h - w_x\|$.

As a matter of fact, it is possible to show that the hyperplane, in the feature space W, separating the vectorial representations of ponies from the vectorial representations of normal height horses depends only on a proper subset S_v of the set of vectors $\sigma(P) \bigcup \sigma(Q)$ whose elements are the vectorial representations of the positive and the negative exemplars. The elements of S_v are called 'support vectors,' and the idea here is that the separating hyperplane is determined only by those vectors representing the positive and negative exemplars which are closest to it.

A last important consideration about kernels is that, if we are dealing with classifications which are separable by hyperplanes (linearly separable), and operate the right

[13] Here the fact that, as a consequence of the embedding σ, $\mathfrak{D}$ is isomorphic to $\sigma(\mathfrak{D})$ is of crucial importance.

[14] If W is an n-dimensional vector space and $v, w \in W$, where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, then $(v \cdot w) = \Sigma_{i=1}^{n} v_i w_i$.

choice of high-dimensional feature space W, it is possible to show that there exists an optimal separating hyperplane in W, that is, a hyperplane which is 'distinguished by the maximum margin of separation between any [vector representing a] training point [a positive or negative exemplar] and the hyperplane' (§1.4, p. 10); and that '[b]y the use of a kernel function ... it is possible to compute the separating hyperplane without explicitly carrying out the [embedding] map into the feature space' (§1.4, p.13).

To see the relevance of kernels to the possibility of giving a solution to problem **(b)** of §2, consider that, if the vectorial representations of the ponies and of the normal height horses belonging to our group of horses are linearly separable within a given high-dimensional real linear space W, then W well deserves to be called 'conceptual space.' For, since it is possible to find in W an optimal hyperplane which separates the region of vectors representing ponies from that representing normal height horses, it follows that:

(i) it is also possible to draw (in W) a sharp distinction between the concepts 'x is a pony belonging to our group of horses,' and 'x is a normal height horse belonging to our group of horses;'
(ii) the region of vectors representing ponies and that representing normal height horses (and the hyperplane) are clearly convex regions of W.

Secondly, the existence of a kernel function which gives us the possibility of computing the separating hyperplane in W provides us with an effective procedure for solving the problem concerning the possibility of assigning perceptual input either to the region of a conceptual space representing a given concept or to its complement when exemplars of patterns falling under the concept (positive exemplars) and of patterns not falling under the concept (negative exemplars) are provided (problem **(b)**, §2).

4 Kernels and conceptual spaces

In attempting to assess the relevance of the application of kernel methods to conceptual spaces, besides what we said at the end of section 2 with regard to the fact that kernels provide: (α) effective similarity measures between patterns, and (β) algorithms for deciding whether or not perceptual input falls under a concept given in terms of positive and negative exemplars, we need to consider the following points.

First, there is a strong connection between the mathematical character of our approach — consisting of an application of ideas and techniques belonging to linear algebra and geometry — and Gärdenfors' declared main aim in developing his theory of conceptual spaces: describing the geometry of thought.

Secondly, in contrast with some of Gärdenfors' original ideas, pattern-recognition algorithms based on support vectors do not appeal to the rather controversial notion of prototype. We find the notion of prototype to be controversial, because since the prototypical horse, the prototypical bird, etc. are just fictitious entities produced by the imagination of artists, the decision algorithms based on prototypes,[15] are bound to be deeply flawed.

[15] An example of decision algorithm based on prototypes is the following: let h and p be, respectively, the representations, within conceptual space C, of the normal height horses, and of the

On the other hand, this is not the case with decision algorithms based on support vectors, because since the support vectors are some of the positive and negative exemplars produced/exhibited to generate a concept, and/or train a device in recognizing certain patterns, they can hardly be thought to be fictitious.

Thirdly, our attempt to model conceptual spaces is particularly relevant to that stage in the activity of a cognitive agent known as learning concepts from exemplars. Learning concepts from exemplars is very important for the theory of conceptual spaces, because it provides, among other things, a convincing account of a possible way in which concepts and, therefore, conceptual spaces come about.

Fourthly, if we identify the input domain $\mathfrak{D}$ of §3 with the set of measures taken by the sensors of a given artificial agent, or with the perceptual input of a human being, such an input is highly non-linear in the sense that if we increase (or decrease) the stimulation beyond certain threshold values, the perception of the agent does not increase (or decrease) accordingly. An important advantage of the use of kernels consists in the linearization of the input, a linearization which makes possible the use of vector spaces.

Lastly, there seem to be cases in which our approach to modeling conceptual spaces is not applicable. One of these is represented by those conceptual spaces which are not metric spaces, and another is that family of conceptual spaces in which concepts are not generated from positive and negative exemplars.

5 Conclusions

In writing this paper we intended to contribute to Gärdenfors' theory of conceptual spaces (§2) by showing how it is possible to use linear algebra to give a mathematical representation of conceptual spaces.

In particular, by exploiting kernel functions and high-dimensional linear spaces, we answered two problems which are central to the theory of conceptual spaces:

1. Is it possible to provide similarity measures for input patterns?
2. Are there procedures of decision for input patterns falling under concepts given/learned by means of positive and negative exemplars? (§3)

We also discussed some of the merits and defects of our approach to formalizing conceptual spaces (§4). The questions we touched upon were:

(i) the harmony existing between our way of formalizing conceptual spaces, and Gärdenfors' main aim in introducing his theory of conceptual spaces, which is that of providing a description of the geometry of thought;
(ii) the superiority of our pattern-recognition algorithms, which exploit support vectors, over those based on prototypes;
(iii) the relevance of our approach to conceptual spaces to the important problem of learning concepts from exemplars;

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ponies prototypes. If the representation of horse x in C is closer to h than it is to p, the horse represented by x is a normal height horse, etc. Gärdenfors discusses this type of algorithm in [Gärdenfors, 2004], Chapter 3, §3.9, p. 87.

(iv) the importance of using kernels for applying linear algebra to the problem of providing a fruitful mathematical representation of perceptual input;
(v) some of the limitations present in our way of dealing mathematically with conceptual spaces.

A word of warning concerning point **(iv)** above, and the general framework within which this paper should be read. Our approach to conceptual spaces is meant to provide 'a' possible way of formalizing them, alongside many others which have been offered. This is a way of formalizing conceptual spaces which proves to be particularly effective in circumstances like those we have discussed.

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