The Reachability Problem for Acyclic Join-Free Petri nets is NP-complete

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Abstract. The reachability problem for Petri nets is the task to decide, for a given Petri net N and a marking m, whether m can be reached from the initial configuration of N by firing a valid sequence of transitions. In this paper, we show that the reachability problem for acyclic join-free Petri nets is NP-complete.

1 Introduction

The concept of Petri nets goes back to Carl Adam Petri [5, 24] and has been further developed in the following years, particularly by the contributions of Anatol Holt [15] and the group around him. Today, Petri nets are a well-established language for modeling concurrent processes and distributed systems. Different dialects of Petri nets and their extensions such as, for example, 1-safe Petri nets, colored Petri nets or timed Petri nets, have applications in several areas such as, for example, performance evaluation [27]; communication protocols [3, 26]; modeling and analysis of distributed software systems [19, 25]; synthesis of speed independent asynchronous cicuits [13]; modeling and verification in hardware design [2, 7].

Petri net *analysis* deals, for example, with the task to deduce behavioral properties of the model like reachability, liveness or deadlock, and is subject of a dedicated annual contest [1].

In this paper, we deal with a special instance of a central algorithmic problem of Petri net analysis: The *reachability problem* for Petri nets is the task to decide, for a given Petri net N and a marking m, whether m is reachable from the initial configuration of N by firing a sequence of transitions of N.

In theoretical computer science, this problem has been investigated for many years from both the computability and the complexity point of view: while the decidability status of the problem has been open for a long time, it was finally shown in [21] that a decision algorithm exists, and several works aimed at improved and less complex decision methods [18, 20]. Unfortunately, the reachability problem for Petri nets is intrinsically hard to solve: while its was already proven to be EXPSPACE-hard in [6], a recent work showed that the problem needs even a tower of exponentials of time and space [8].

However, from the complexity point of view, better results can be obtained for structural restricted Petri nets: for example, the complexity of the reachability problem boils down to PSPACE-complete for 1-safe Petri nets; the problem is NP-complete for acyclic Petri nets [4], for acyclic 1-safe Petri nets [28], and for conflict-free Petri nets [16] as well. On the other hand, the reachability problem is solvable in polynomial-time for S-systems and marked graphs [14, 5], and 1-safe conflict-free Petri nets [4].

Free-choice Petri nets define one of the most important sub-classes of Petri nets. The reachability problem generally remains EXPSPACE-hard for these nets, but becomes PSPACE-complete for 1-safe free-choice Petri nets, and it is NP-complete for live 1-safe free-choice Petri nets [12]. Moreover, the problem is polynomial for reversible free-choice Petri nets [10], and for cyclic extended free-choice systems [11], and for sound extended free-choice workflow nets [30].

Join-free Petri nets build a particular yet useful subclass of Petri nets and allow every transition to have at most one input place [9, 17, 29]. To the best of our knowledge, the complexity status of the reachability problem has not yet been characterized for this class. In this paper, we partially close this gap by showing that the reachability problem for *acyclic* join-free Petri nets is NP-complete: On the one hand, the problem inherits NP-membership from the more general class of acyclic Petri nets [28]. On the other hand, we show by a reduction of a particular SAT-problem, namely CUBIC MONOTONE 1 IN 3 3SAT, that the reachability problem for acyclic join-free Petri nets is NP-hard, even if the the net is additionally free-choice, that is, its arc weights are restricted to zero and one.

The paper is structured as follows: The next Section 2 provides basic definitions and supports them with some examples. After that, Section 3 provides the announced hardness result. Finally, Section 4 briefly closes the paper.

2 Preliminaires

In this section, we introduce relevant basic notions around Petri nets and show some examples.

Definition 1 (Petri Nets). *A* Petri net*, also Petri net,* $N = (P, T, f, m_0)$ *consists of finite and disjoint sets of* places P *and* transitions T*, a (total)* flow $f : ((P \times T) \cup (T \times P)) \to \mathbb{N}$ *and an* initial marking $m_0 : P \to \mathbb{N}$*.*

Definition 2 (Preset, postset). *The* nodes *of a Petri net* $N = (P, T, f, m_0)$ *are* $P \cup T$*. The* preset *of a node* x *is defined by* $\bullet x = \{y \mid f(y, x) > 0\}$*, the* postset *of* x *is defined by* $x\bullet = \{y \mid f(x, y) > 0\}$*. Notice that* $\bullet x \cap x\bullet$ *is not necessarily empty. For a transition* t*, the* pre-places *are all places in* $\bullet t$*; the* post-places *are all places in* $t\bullet$*.* Pre-transitions *and* post-transitions *for a place are defined analogously.*

Subclasses of Petri nets. Within the class of Petri nets, we can determine various subclasses, which differ in their flow relation structure: A Petri net $N = (P, T, f, m_0)$ is called *acyclic* if its underlying directed graph $G = (V, E)$ with vertices $V = P \cup T$ and edges $E = \{(x, y) \mid x, y \in V, f(x, y) > 0\}$ has no directed circuits. Moreover, N is *join-free*, if every transition t has at most one pre-place, that is, $| \bullet t| \leq 1$. If the co-domain of the flow f is restricted to $\{0, 1\}$, then N is called *plain*. Furthermore, N is *free-choice*, if it is plain and, additionally, if it holds for two arbitrary distinct transitions t and $t' \in T$: $\bullet t \cap \bullet t' \neq \emptyset \Rightarrow \bullet t = \bullet t'$. This means that if two transitions share one pre-place, then they share all pre-places. Notice that a plain join-free Petri net is free-choice by triviality, since in this case there is no transition with several pre-places.

Figure 1 shows the net N that is obviously acyclic and join-free and free-choice as well.

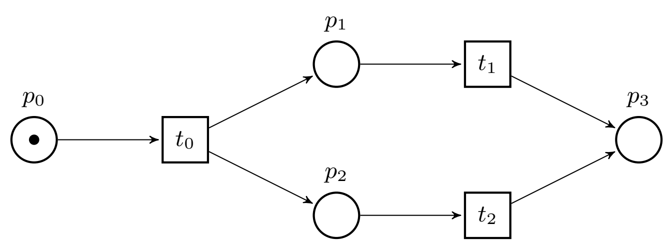

Fig. 1. The acyclic, join-free Petri net N, which is also free-choice.

The behavior of a Petri net is defined by the transition rule.

Definition 3 (Transition Rule). *Let $N = (P, T, f, m_0)$ be a Petri net. Transition t is* enabled *in marking m if $\forall p : m(p) \geq f(p, t)$.* Firing *a Transition t leads from marking m to marking m' if t is enabled in m and $\forall p : m'(p) = m(p) - f(p, t) + f(t, p)$, denoted as $m \xrightarrow{t} m'$.*

Definition 4 (Firing Sequence, Reachability Set). *Let $N = (P, T, f, m_0)$ be a Petri net and m, m' some (not necessarily distinct) markings of N. We say a* sequence (of transitions) *$\sigma = \omega \in T^*$ can* fire *at m and its firing leads to m', denoted by $m \xrightarrow{\sigma} m'$ if and only if either $\sigma = \varepsilon$ and $m = m'$ or $\sigma = \omega a$, where $\omega \in T^*$ and $a \in T$, and there is a marking m'' of N such that $m \xrightarrow{\omega} m''$ and $m'' \xrightarrow{a} m'$. We call $RS(N) = \{m \mid \exists \sigma \in T^* : m_0 \xrightarrow{\sigma} m\}$ the* reachability set *(of N), which contains all of N's reachable markings.*

Definition 5 (Set of transitions of a sequence). *Let $N = (P, T, f, m_0)$ be a Petri net and let $\sigma \in T^*$. We define the* set S_σ of transitions of σ *inductively as follows: If $\sigma = \varepsilon$, then $S_\sigma = \emptyset$ and, otherwise, if $\sigma = \omega a$ with $\omega \in T^*$ and $a \in T$, then $S_\sigma = \{a\} \cup S_\omega$.*

Using the transition rule, a Petri net induces a labeled transition system, called the *reachability graph*.

Definition 6 (Labeled Transition System, Reachability Graph). *A (deterministic)* transition system $TS = [S, s_0, R, A]$ *is a directed labeled graph with the set of nodes* S *(called* states*), an* initial state $s_0 \in S$*, and a* transition relation $R \subseteq S \times A \times S$ *with some set of* actions A*. The* reachability graph *of a Petri net* N *is a transition system, where the set of states is* $RS(N)$*,* m_0 *serves as the initial state and* $(m, t, m') \in R$ *iff* $m \xrightarrow{t} m'$*.*

In this paper, we consider the following instance of the reachability problem:

REACHABILITY FOR ACYCLIC JOIN-FREE PETRI NETS	
Input:	A tuple (N, m) where $N = (P, T, f, m_0)$ is an acyclic and join-free Petri net and m is a marking of N.
Question:	Does there exist a firing sequence $\sigma \in T^*$ such that $m_0 \xrightarrow{\sigma} m$?

An example shall illustrate this problem.

Example 1. Let N be the given net in Figure 1 with the initial marking $m_0 = (1, 0, 0, 0)$ whereas, with a little abuse of definition, the tuple is an abbreviation of $m_0(p_{H_0}) = 1, m_0(p_{H_1}) = m_0(p_{H_2}) = m_0(p_{H_3}) = 0$. In m_0, only t_0 is activated. Firing t_0 leads to marking $m_1 = (0, 1, 1, 0)$. The marking $m_2 = (0, 0, 0, 2)$ is reachable, because the firing sequence $t_0t_1t_2$ leads us from m_0 to m_2. On the contrary, marking $m_3 = (1, 0, 0, 1)$ is not reachable, as for producing tokens on p_{H_3}, we need to consume the token from p_{H_0} first to activate the transition t_1 and t_2.

3 Hardness Result

The following theorem provides the main result of this paper:

Theorem 1. REACHABILITY FOR ACYCLIC JOIN-FREE PETRI NETS *is NP-complete.*

The remainder of this paper is dedicated to the proof of Theorem 1: First of all, by a result of [23], if $N = (P, T, f, m_0)$ is an acyclic Petri net and m a marking, then m is reachable from m_0 if and only if the well-known state equation $m = m_0 + C \cdot x$ has a non-negative integer solution x. In other words, the reachability problem is reducible to the problem LINEAR INTEGER PROGRAMMING, which is well-known to be NP-complete. Hence, the reachability problem for acyclic join-free Petri nets belongs to NP.

Consequently, in order to complete the proof of Theorem 1, it remains to show that REACHABILITY FOR ACYCLIC JOIN-FREE PETRI NETS is NP-hard. The proof of the NP-hardness is based on a polynomial-time reduction of the following particular SAT-problem, which is known to be NP-complete from [22]:

CUBIC MONOTONE 1 IN 3 3SAT (CM 1 IN 3 3SAT)	
Input:	A pair (V, F) consisting of a set V of boolean variables and a set $F = \{C_0, \dots, C_{n-1}\}$ consisting of 3-variable-clauses, such that $C_i = \{X_{i_0}, X_{i_1}, X_{i_2}\} \subseteq V$ and $i_0 < i_1 < i_2$ for all $i \in \{0, \dots, n-1\}$. Every variable $X \in V$ appears in exactly three different clauses.
Question:	Does there exist a one-in-three model for (V, F), i.e. a set $S \subseteq V$ such that $\lvert S \cap C_i \rvert = 1$ for all $i \in \{0, \dots, n-1\}$?

Example 2. The instance (V, F) of CM 1 IN 3 3SAT with set of variables $V = \{X_0, X_1, \dots, X_5\}$ and set of 3-variable-clauses $F = \{C_0, C_1, \dots, C_5\}$, where

- $C_0 = \{X_0, X_1, X_2\}$
- $C_1 = \{X_0, X_1, X_3\}$
- $C_2 = \{X_0, X_1, X_5\}$
- $C_3 = \{X_2, X_3, X_4\}$
- $C_4 = \{X_2, X_4, X_5\}$
- $C_5 = \{X_3, X_4, X_5\}$

allows a positive decision, since $S = \{X_0, X_4\}$ is a one-in-three model for (V, F).

Notice that the number of variables and the number of clauses are equal for any arbitrary instance of CM 1 IN 3 3SAT. This concludes from the problem definition of CM 1 IN 3 3SAT, as every variable occurs exactly in three different clauses and each clause consists of three different variables.

In the remainder of this paper, unless explicitly stated otherwise, let (V, F) be an arbitrary but fixed input of CM 1 IN 3 3SAT, where $V = \{X_0, X_1, \dots, X_{n-1}\}$ and $F = \{C_0, C_1, \dots, C_{n-1}\}$ such that $C_i = \{X_{i_0}, X_{i_1}, X_{i_2}\}$ and $i_0 < i_1 < i_2$ for all $i \in \{0, \dots, n-1\}$.

Our reduction uses the following simple yet crucial fact:

Fact 1. *If $S \subseteq V$ is a one-in-three model for F, then $|S| = \frac{n}{3}$.*

Proof. Let S be a one-in-three model for F. Since every variable of V occurs in exactly three distinct clauses and, moreover, $|S \cap C_i| = 1$ for all $i \in \{0, \dots, n-1\}$, we have that $|F| = 3|S|$. This implies $|S| = \frac{n}{3}$. □

The reduction. In order to prove the hardness part of Theorem 1, we translate the instance (V, F) into an input (N, m) of REACHABILITY FOR ACYCLIC JOIN-FREE PETRI NETS such that there is a one-in-three model for (V, F) if and only if m is reachable from the initial marking of N.

For the Petri net N, we introduce the following components:

- for every $i \in \{0, \dots, n-1\}$, the place p_{C_i} that represents the clause C_i and is initially marked by one token: $m_0(p_{C_i}) = 1$;
- for every $i \in \{0, \dots, n-1\}$, the initially empty place p_{X_i}: $m(p_{X_i}) = 0$;
- for every $i \in \{0, \dots, n-1\}$, three transitions $t^0_{X_i}, t^1_{X_i}$ and $t^2_{X_i}$ that represent the three occurrences of the variable X_i in the clauses of F: if $C_{\ell_0}, C_{\ell_1}, C_{\ell_2}$ with $\ell_0 < \ell_1 < \ell_2 \in \{0, \dots, n-1\}$ are exactly the clauses that contain X_i, then, for all $j \in \{0, 1, 2\}$, it is $p_{C_{\ell_j}}$ the unique pre-place of the transition $t^j_{X_i}$,

where $f(p_{C_{\ell_j}}, t^j_{X_i}) = 1$; moreover, for all $j \in \{0,1,2\}$, the place p_{X_i} is the only post-place of $t^j_{X_i}$, where $f(t^j_{X_i}, p_{X_i}) = 1$;

– for every $i \in \{0, \dots, n-1\}$, three other helper-transitions $t^0_{H_i}, t^1_{H_i}, t^2_{H_i}$ and two helper-places p_{H_i} and q_{H_i}, such that
  - $f(t^0_{H_i}, p_{H_i}) = 1$ and $f(t^0_{H_i}, p_{X_i}) = 1$,
  - $f(p_{H_i}, t^1_{H_i}) = 1$ and $f(t^1_{H_i}, q_{H_i}) = 1$ and $f(t^1_{H_i}, p_{X_i}) = 1$,
  - $f(q_{H_i}, t^2_{H_i}) = 1$ and $f(t^2_{H_i}, p_{X_i}) = 1$,
  - $m(p_{H_i}) = m(q_{H_i}) = 0$.

The reduction yields the Petri net $N = (P, T, f, m_0)$ with places and transitions as follows:

$$P = \{p_{C_0}, p_{C_1}, \dots, p_{C_{n-1}}\} \cup \{p_{X_0}, p_{X_1}, \dots, p_{X_{n-1}}\} \cup \{p_{H_0}, q_{H_0}, \dots, p_{H_{n-1}}, q_{H_{n-1}}\},$$
$$T = \{t^0_{X_0}, t^1_{X_0}, t^2_{X_0}, t^0_{X_1}, t^1_{X_1}, t^2_{X_1}, \dots, t^0_{X_{n-1}}, t^1_{X_{n-1}}, t^2_{X_{n-1}}\}$$
$$\cup \{t^0_{H_0}, t^1_{H_0}, t^2_{H_0}, t^0_{H_1}, t^1_{H_1}, t^2_{H_1}, \dots, t^0_{H_{n-1}}, t^1_{H_{n-1}}, t^2_{H_{n-1}}\}.$$

Notice that the resulting net is plain and thus also free-choice. Let the marking m, whose reachability is shown to be equivalent to the existence of a one-in-three model for (V, F), be defined by $m(p_{X_0}) = \dots = m(p_{X_{n-1}}) = 3$ and $m(p) = 0$ for all $p \in P \setminus \{p_{X_0}, \dots, p_{X_{n-1}}\}$.

The following example will make the reduction technique more clear:

Example 3. For the instance (V, F) of CM 1 IN 3 3SAT presented in Example 2, we build the Petri net N, illustrated in Figure 2.

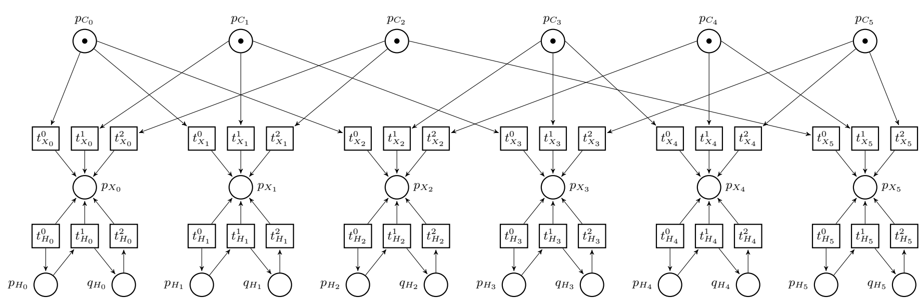

Fig. 2. The Petri net N according to the reduction for the CM 1 IN 3 3SAT-instance of Example 2.

In the following, we argue that the reduction actually satisfies the functionality introduced. The following lemma proves that if (V, F) allows a positive decision, then so does (N, m):

Lemma 1. *If there is a one-in-three model for F, then m is a reachable marking of N.*

Proof. Recall that any one-in-three model for F has exactly $\frac{n}{3}$ elements by Fact 1. Let $i_0, \dots, i_{\frac{n}{3}-1} \in \{0, \dots, n-1\}$ be $\frac{n}{3}$ pairwise distinct indices, such that $S = \{X_{i_0}, \dots, X_{i_{\frac{n}{3}-1}}\}$ is a one-in-three model for F, that is, for all $i \in \{0, \dots, n-1\}$, it holds $S \cap C_i = \{X_{i_\ell}\}$ for some $\ell \in \{0, \dots, \frac{n}{3}-1\}$. Moreover, let $j_0, \dots, j_{\frac{2n}{3}-1} \in \{0, \dots, n-1\}$ be the $\frac{2n}{3}$ pairwise distinct indices such that $\{X_{j_0}, \dots, X_{j_{\frac{2n}{3}-1}}\} = V \setminus S$. We now show, that the marking m is reachable from the marking m_0 of N by a firing sequence, which can be derived from S. Initially, only the clause places $p_{C_0}, \dots, p_{C_{n-1}}$ are marked with one token each. Therefore, for every $\ell \in \{0, \dots, \frac{n}{3}-1\}$, the transitions $t^0_{X_{i_\ell}}, t^1_{X_{i_\ell}}, t^2_{X_{i_\ell}}$, which represent the occurrences of the variable X_{i_ℓ} in the clauses of F, are all activated. Since S is a one-in-three model, for all $j, \ell \in \{0, \dots, \frac{n}{3}-1\}$ and all $h, k \in \{0, 1, 2\}$, the following is true: if $(j, h) \neq (\ell, k)$, then $\bullet t_{X^h_{i_j}} \cap \bullet t_{X^k_{i_\ell}} = \emptyset$. Hence, the following sequence can fire at m_0 (and results in the marking m_1, defined below):

$$\sigma_1 = \underbrace{t^0_{X_{i_0}} t^1_{X_{i_0}} t^2_{X_{i_0}}}_{m_1(p_{X_{i_0}})=3} \underbrace{t^0_{X_{i_1}} t^1_{X_{i_1}} t^2_{X_{i_1}}}_{m_1(p_{X_{i_1}})=3} \dots \underbrace{t^0_{X_{i_{\frac{n}{3}-1}}} t^1_{X_{i_{\frac{n}{3}-1}}} t^2_{X_{i_{\frac{n}{3}-1}}}}_{m_1(p_{X_{i_{\frac{n}{3}-1}}})=3}$$

For all $\ell \in \{0, \dots, \frac{n}{3}-1\}$ and all $j \in \{0, 1, 2\}$, firing $t^j_{X_{i_\ell}}$ consumes the only token from its unique pre-place and produces exactly one token on its unique post-place $p_{X_{i_\ell}}$. Consequently, firing the transition sequence σ_1 leads from m_0 to the marking m_1, such that $m_1(p_{X_{i_\ell}}) = 3$ for all $\ell \in \{0, \dots, \frac{n}{3}-1\}$ and $m_1(p) = 0$ for all other places $p \in P \setminus \{p_{X_{i_0}}, \dots, p_{X_{i_{\frac{n}{3}-1}}}\}$.

In order to obtain m, we extend σ_1 by the sequence

$$\sigma_2 = \underbrace{t^0_{H_{j_0}}\ t^1_{H_{j_0}}\ t^2_{H_{j_0}}}_{m(p_{X_{j_0}})=3} \underbrace{t^0_{H_{j_1}}\ t^1_{H_{j_1}}\ t^2_{H_{j_1}}}_{m(p_{X_{j_1}})=3} \dots \underbrace{t^0_{H_{j_{\frac{2n}{3}-1}}}\ t^1_{H_{j_{\frac{2n}{3}-1}}}\ t^2_{H_{j_{\frac{2n}{3}-1}}}}_{m(p_{X_{j_{\frac{2n}{3}-1}}})=3}$$

It is easy to see, that σ_2 can be fired at m_1, since $t^0_{H_{j_\ell}}$ does not have a pre-place for all $\ell \in \{0, \dots, \frac{2n}{3}-1\}$. Moreover, for every $\ell \in \{0, \dots, \frac{2n}{3}-1\}$, the firing of the subsequence $t^0_{H_{j_\ell}}\ t^1_{H_{j_\ell}}\ t^2_{H_{j_\ell}}$ does nothing else than to put three tokens on $p_{X_{j_\ell}}$: $t^0_{H_{j_\ell}}$ puts a token on $p_{H_{j_\ell}}$ and a token on $p_{X_{j_\ell}}$; after that $t^1_{H_{j_\ell}}$ consumes the token from $p_{H_{j_\ell}}$ and puts a token on $q_{H_{j_\ell}}$ and another one on $p_{X_{j_\ell}}$; finally, $t^2_{H_{j_\ell}}$ consumes the token from $q_{H_{j_\ell}}$ and puts a third token on $p_{X_{j_\ell}}$. In particular, $m_0 \xrightarrow{\sigma_1\sigma_2} m$, which proves the lemma. □

Conversely, by the following lemma, if (N, m) is a yes-instance, then (V, F) is a yes-instance as well:

Lemma 2. *Let $\sigma \in T^*$ be a firing sequence of N that leads to the marking m and let $i \in \{0, \dots, n-1\}$ be arbitrary but fixed.*

1. *If $t^0_{H_i} \in S_\sigma$, then $t^j_{X_i} \notin S_\sigma$ for all $j \in \{0, 1, 2\}$.*

2. *If there is* $j \in \{0,1,2\}$ *such that* $t^j_{X_i} \in S_\sigma$*, then* $t^0_{X_i}, t^1_{X_i}, t^2_{X_i} \in S_\sigma$*.*
3. *The set* $S = \{X_i \mid t^0_{X_i}, t^1_{X_i}, t^2_{X_i} \in S_\sigma\}$ *defines a one-in-three model for* F*.*

Proof. (1): The transition $t^0_{H_i}$ puts a token on p_{H_i} and a token on p_{X_i}. Since $m(p_{H_i}) = 0$ and $p_{H_i}\bullet = \{t^1_{H_i}\}$, we have that $t^1_{H_i} \in S_\sigma$. Similarly, transition $t^1_{H_i}$ puts a token on q_{H_i} and on p_{X_i} and as well. Since $m(q_{H_i}) = 0$ and $q_{H_i}\bullet = \{t^2_{H_i}\}$, the transition $t^2_{H_i}$ occurs in σ. Finally, since the firing of $t^2_{H_i}$ puts a token on p_{X_i}, and $p_{X_i}\bullet = \emptyset$, and $m(p_{X_i}) = 3$, we have that $\bullet p_{X_i} \cap S_\sigma = \{t^0_{H_i}, t^1_{H_i}, t^2_{H_i}\}$. This implies the claim.

(2): Since $t^j_{X_i} \in S_\sigma$, by (1), we have that $t^0_{H_i} \notin S_\sigma$, which certainly implies $t^1_{H_i} \notin \sigma$ and $t^2_{H_i} \notin \sigma$. Hence, by $m(p_{X_i}) = 3$, we have that every of the transitions $t^0_{X_i}, t^1_{X_i}$ and $t^2_{X_i}$ has put a token on p_{X_i}, which implies the claim.

(3): Recall that the i-th clause is given by $C_i = \{X_{i_0}, X_{i_1}, X_{i_2}\}$. Since $m(p_{C_i}) = 0$, there are $j \in \{0,1,2\}$ and $k \in \{0,1,2\}$, such that $t^k_{X_{i_j}} \in S_\sigma$. By (2), this implies $t^0_{X_{i_j}}, t^1_{X_{i_j}}, t^2_{X_{i_j}} \in S_\sigma$ and thus $X_{i_j} \in C_i \cap S$. In particular, we have that $S \cap C_i \neq \emptyset$.

If $|C_i \cap S| \geq 2$, then there are distinct $j, \ell \in \{0,1,2\}$ such that $X_{i_j}, X_{i_\ell} \in C \cap S$. By definition of S, this implies $t^0_{X_{i_j}}, t^1_{X_{i_j}}, t^2_{X_{i_j}} \in S_\sigma$ and $t^0_{X_{i_\ell}}, t^1_{X_{i_\ell}}, t^2_{X_{i_\ell}} \in S_\sigma$ and, by the construction of N, this implies $|p_{C_i} \bullet \cap S_\sigma| \geq 2$ as well. This is a contradiction, since $m_0(p_{C_i}) = 1$ and $\bullet p_{C_i} = \emptyset$ and $f(p_{C_i}, t) = 1$ for all $t \in p_{C_i}\bullet$. Consequently, $|C_i \cap S| = 1$.

Finally, by the arbitrariness of i, we have that $|C_i \cap S| = 1$ for all $i \in \{0, \ldots, n-1\}$, which proves the claim. □

Since the reduction is obviously polynomial and Reachability for Acyclic Join-free Petri Nets belongs to NP, by Lemma 1 and Lemma 2, we have proven Theorem 1.

4 Conclusion

In this paper, we show that the well-known reachability problem of Petri nets is NP-complete for the class of acyclic join-free Petri nets, which are also free-choice. The hardness-proof bases on the reduction of a particular SAT-problem. The membership in NP heavily bases on the fact that the nets addressed are acyclic. Hence, it remains future work to determine the complexity of the reachability problem for join-free Petri nets that may contain cycles.

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References

1. Amparore, E.G., Berthomieu, B., Ciardo, G., Dal-Zilio, S., Gallà, F., Hillah, L., Hulin-Hubard, F., Jensen, P.G., Jezequel, L., Kordon, F., Botlan, D.L., Liebke, T., Meijer, J., Miner, A.S., Paviot-Adet, E., Srba, J., Thierry-Mieg, Y., van Dijk, T., Wolf, K.: Presentation of the 9th edition of the model checking contest. In: TACAS (3). Lecture Notes in Computer Science, vol. 11429, pp. 50–68. Springer (2019)
2. Azéma, P., Valette, R., Diaz, M.: Petri nets as a common tool for design verification and hardware simulation. In: Humcke, D.J., Galey, J.M., Szygenda, S.A., Pistilli, P.O., Dooner, N.P., Brinsfield, J.G., Olila, J.S. (eds.) Proceedings of the 13th Design Automation Conference, DAC '76, San Francisco, California, USA, June 28-30, 1976. pp. 109–116. ACM (1976). https://doi.org/10.1145/800146.804803, https://doi.org/10.1145/800146.804803
3. Berthelot, G., Terrat, R.: Petri nets theory for the correctness of protocols. IEEE Trans. Commun. **30**(12), 2497–2505 (1982). https://doi.org/10.1109/TCOM.1982.1095452, https://doi.org/10.1109/TCOM.1982.1095452
4. Cheng, A., Esparza, J., Palsberg, J.: Complexity results for 1-safe nets. In: Shyamasundar, R.K. (ed.) Foundations of Software Technology and Theoretical Computer Science, 13th Conference, Bombay, India, December 15-17, 1993, Proceedings. Lecture Notes in Computer Science, vol. 761, pp. 326–337. Springer (1993). https://doi.org/10.1007/3-540-57529-4_66, https://doi.org/10.1007/3-540-57529-4_66
5. Commoner, F., Holt, A.W., Even, S., Pnueli, A.: Marked directed graphs. J. Comput. Syst. Sci. **5**(5), 511–523 (1971). https://doi.org/10.1016/S0022-0000(71)80013-2, https://doi.org/10.1016/S0022-0000(71)80013-2
6. of Computer Science, Y.U.D., Lipton, R.: The reachability problem requires exponential space. Research report (Yale University. Department of Computer Science), Department of Computer Science, Yale University (1976), https://books.google.de/books?id=7iSbGwAACAAJ
7. Cortadella, J., Kishinevsky, M., Kondratyev, A., Lavagno, L., Yakovlev, A.: Hardware and petri nets: Application to asynchronous circuit design. In: Nielsen, M., Simpson, D. (eds.) Application and Theory of Petri Nets 2000, 21st International Conference, ICATPN 2000, Aarhus, Denmark, June 26-30, 2000, Proceeding. Lecture Notes in Computer Science, vol. 1825, pp. 1–15. Springer (2000). https://doi.org/10.1007/3-540-44988-4_1, https://doi.org/10.1007/3-540-44988-4_1
8. Czerwinski, W., Lasota, S., Lazic, R., Leroux, J., Mazowiecki, F.: The reachability problem for petri nets is not elementary. In: Charikar, M., Cohen, E. (eds.) Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019. pp. 24–33. ACM (2019). https://doi.org/10.1145/3313276.3316369, https://doi.org/10.1145/3313276.3316369
9. Delosme, J., Hujsa, T., Kordon, A.M.: Polynomial sufficient conditions of well-behavedness for weighted join-free and choice-free systems. In: Carmona, J., Lazarescu, M.T., Pietkiewicz-Koutny, M. (eds.) 13th International Conference on Application of Concurrency to System Design, ACSD 2013, Barcelona, Spain, 8-10 July, 2013. pp. 90–99. IEEE Computer Society (2013). https://doi.org/10.1109/ACSD.2013.12, https://doi.org/10.1109/ACSD.2013.12

10. Desel, J., Esparza, J.: Reachability in reversible free choice systems. In: Choffrut, C., Jantzen, M. (eds.) STACS 91, 8th Annual Symposium on Theoretical Aspects of Computer Science, Hamburg, Germany, February 14-16, 1991, Proceedings. Lecture Notes in Computer Science, vol. 480, pp. 384–397. Springer (1991). https://doi.org/10.1007/BFb0020814, https://doi.org/10.1007/BFb0020814
11. Desel, J., Esparza, J.: Reachability in cyclic extended free-choice systems. Theor. Comput. Sci. **114**(1), 93–118 (1993)
12. Esparza, J.: Reachability in live and safe free-choice petri nets is np-complete. Theor. Comput. Sci. **198**(1-2), 211–224 (1998). https://doi.org/10.1016/S0304-3975(97)00235-1, https://doi.org/10.1016/S0304-3975(97)00235-1
13. Gallo, O., Necas, T., Lehocki, F.: A tool for the synthesis of asynchronous speed-independent circuits. In: Donatelli, S., Kleijn, J., Machado, R.J., Fernandes, J.M. (eds.) Proceedings of the Workshops of the 31st International Conference on Application and Theory of Petri Nets and Other Models of Concurrency (PETRI NETS 2010) and of the 10th International Conference on Application of Concurrency to System Design (ACSD 2010), Braga, Portugal, June, 2010. CEUR Workshop Proceedings, vol. 827, pp. 207–211. CEUR-WS.org (2010), http://ceur-ws.org/Vol-827/16_OndrejGallo_article.pdf
14. Genrich, H.J., Lautenbach, K.: Synchronisationsgraphen. Acta Informatica **2**, 143–161 (1973)
15. Holt, A.W.: Coordination technology and petri nets. In: Applications and Theory in Petri Nets. Lecture Notes in Computer Science, vol. 222, pp. 278–296. Springer (1985)
16. Howell, R.R., Rosier, L.E.: Completeness results for conflict-free vector replacement systems. J. Comput. Syst. Sci. **37**(3), 349–366 (1988). https://doi.org/10.1016/0022-0000(88)90013-X, https://doi.org/10.1016/0022-0000(88)90013-X
17. Hujsa, T., Delosme, J., Kordon, A.M.: On the reversibility of live equal-conflict petri nets. In: Devillers, R.R., Valmari, A. (eds.) Application and Theory of Petri Nets and Concurrency - 36th International Conference, PETRI NETS 2015, Brussels, Belgium, June 21-26, 2015, Proceedings. Lecture Notes in Computer Science, vol. 9115, pp. 234–253. Springer (2015). https://doi.org/10.1007/978-3-319-19488-2_12, https://doi.org/10.1007/978-3-319-19488-2_12
18. Kosaraju, S.R.: Decidability of reachability in vector addition systems (preliminary version). In: Lewis, H.R., Simons, B.B., Burkhard, W.A., Landweber, L.H. (eds.) Proceedings of the 14th Annual ACM Symposium on Theory of Computing, May 5-7, 1982, San Francisco, California, USA. pp. 267–281. ACM (1982). https://doi.org/10.1145/800070.802201, https://doi.org/10.1145/800070.802201
19. Krämer, B.J.: Stepwise construction of non-sequential software systems using a net-based specification language. In: Rozenberg, G., Genrich, H.J., Roucairol, G. (eds.) Advances in Petri Nets 1984, European Workshop on Applications and Theory in Petri Nets, covers the last two years which include the workshop 1983 in Toulouse and the workshop 1984 in Aarhus, selected papers. Lecture Notes in Computer Science, vol. 188, pp. 307–330. Springer (1984). https://doi.org/10.1007/3-540-15204-0_18, https://doi.org/10.1007/3-540-15204-0_18
20. Lambert, J.: A structure to decide reachability in petri nets. Theor. Comput. Sci. **99**(1), 79–104 (1992). https://doi.org/10.1016/0304-3975(92)90173-D, https://doi.org/10.1016/0304-3975(92)90173-D
21. Mayr, E.W.: An algorithm for the general petri net reachability problem. In: Proceedings of the 13th Annual ACM Symposium on Theory of Comput-

ing, May 11-13, 1981, Milwaukee, Wisconsin, USA. pp. 238–246. ACM (1981). https://doi.org/10.1145/800076.802477, https://doi.org/10.1145/800076.802477
22. Moore, C., Robson, J.M.: Hard tiling problems with simple tiles. Discrete & Computational Geometry **26**(4), 573–590 (2001). https://doi.org/10.1007/s00454-001-0047-6
23. Murata, T.: Petri nets: Properties, analysis and applications. Proceedings of the IEEE **77**(4), 541–580 (1989)
24. Petri, C.A.: Concepts of net theory. In: MFCS. pp. 137–146. Mathematical Institute of the Slovak Academy of Sciences (1973)
25. Richta, T., Janousek, V., Kocí, R.: Dynamic software architecture for distributed embedded control systems. In: Moldt, D., Rölke, H., Störrle, H. (eds.) Proceedings of the International Workshop on Petri Nets and Software Engineering (PNSE'15), including the International Workshop on Petri Nets for Adaptive Discrete Event Control Systems (ADECS 2015) A satellite event of the conferences: 36th International Conference on Application and Theory of Petri Nets and Concurrency Petri Nets 2015 and 15th International Conference on Application of Concurrency to System Design ACSD 2015, Brussels, Belgium, June 22-23, 2015. CEUR Workshop Proceedings, vol. 1372, pp. 133–150. CEUR-WS.org (2015), http://ceur-ws.org/Vol-1372/paper8.pdf
26. Rodríguez, A., Kristensen, L.M., Rutle, A.: On modelling and validation of the MQTT iot protocol for M2M communication. In: Moldt, D., Kindler, E., Rölke, H. (eds.) Proceedings of the International Workshop on Petri Nets and Software Engineering (PNSE'18), co-located with the39th International Conference on Application and Theory of Petri Nets and Concurrency Petri Nets 2018 and the 18th International Conference on Application of Concurrency to System Design ACSD 2018, Bratislava, Slovakia, June 24-29, 2018. CEUR Workshop Proceedings, vol. 2138, pp. 99–118. CEUR-WS.org (2018), http://ceur-ws.org/Vol-2138/paper5.pdf
27. Sifakis, J.: Use of petri nets for performance evaluation. In: Beilner, H., Gelenbe, E. (eds.) Measuring, Modelling and Evaluating Computer Systems, Proceedings of the Third International Symposium, Bonn - Bad Godesberg, Germany, October 3-5, 1977. pp. 75–93. North-Holland (1977)
28. Stewart, I.A.: Reachability in some classes of acyclic petri nets. Fundam. Informaticae **23**(1), 91–100 (1995). https://doi.org/10.3233/FI-1995-2314, https://doi.org/10.3233/FI-1995-2314
29. Teruel, E., Colom, J.M., Suárez, M.S.: Choice-free petri nets: a model for deterministic concurrent systems with bulk services and arrivals. IEEE Trans. Syst. Man Cybern. Part A **27**(1), 73–83 (1997). https://doi.org/10.1109/3468.553226, https://doi.org/10.1109/3468.553226
30. Yamaguchi, S.: Polynomial time verification of reachability in sound extended free-choice workflow nets. IEICE Trans. Fundam. Electron. Commun. Comput. Sci. **97-A**(2), 468–475 (2014)