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Numerical solution of fractional differential equations using the generalized block pulse operational matrix

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ABSTRACT

The Riemann–Liouville fractional integral for repeated fractional integration is expanded in block pulse functions to yield the block pulse operational matrices for the fractional order integration. Also, the generalized block pulse operational matrices of differentiation are derived. Based on the above results we propose a way to solve the fractional differential equations. The method is computationally attractive and applications are demonstrated through illustrative examples.

1. Introduction

Fractional calculus has become the focus of interest for many researchers in different disciplines of science and technology [1–5]. A great deal of research has shown the advantageous use of the fractional calculus in the modeling and control of many dynamical systems [1–8]. However, the fractional order systems are represented in the frequency domain by irrational transfer functions, which correspond to linear time varying differential equations in the time domain. One of the main difficulties is how to solve the fractional differential equations, so some techniques were proposed to solve them. The most commonly used ones are Adomian decomposition method (ADM) [9], Variational Iteration Method (VIM) [10], Fractional Differential Transform Method (FDTM) [11], Operational Matrix Method [12], Homotopy Analysis Method [13,14], Fractional Difference Method (FDM) [15] and Power Series Method [16]. Also there are some classical solution techniques, e.g. Laplace Transform Method [17].

It is somewhat surprising that among different solution techniques few papers reported application of the orthogonal function method for the fractional order differential equations [18–21]. However, through the analysis of the orthogonal function method, we hold that it should be applicable to solve the fractional order systems.

For an ordinary dynamical system, the orthogonal function method is based on converting the underlying differential equations into integral equations through integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix of integration to eliminate the integral operations. Typical examples are the Walsh functions [22,23], block pulse functions [24–27], Legendre polynomials [28–30], Chebyshev polynomials [31], Laguerre polynomials [32,33], and Fourier series [34,35].

In this paper, our purpose is to extend the orthogonal function method to solve the fractional linear differential equations. Similar to the process of using the orthogonal function method to solve the ordinary dynamical systems, for the fractional systems, we need to convert the underlying fractional differential equations into integral equations through the fractional integration, expand various signals involved in the equation by block pulse functions and using the operational matrix for the fractional integration to eliminate the fractional integral operations. So, there are some questions to be answered:

(i) How to derive block pulse operational matrices of the fractional integration and differentiation.
(ii) How to solve the fractional order linear systems via block pulse operational matrices of the fractional integration.

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2. Brief review of block pulse functions (BPFs) and the related operational matrices [36]

A set of BPFs $\Phi_m(t)$ containing m component functions in the semi-open interval $[0, T)$ is given by

$$\Phi_m(t) \triangleq [\varphi_0(t)\ \varphi_1(t)\ \cdots\ \varphi_i(t)\ \cdots\ \varphi_{m-1}(t)]^T, \tag{1}$$

where T denotes transpose.

The ith component $\varphi_i(t)$ of the BPF vector $\Phi_m(t)$ is defined as

$$\varphi_i(t) = \begin{cases} 1 & iT/m \le t < (i+1)T/m \\ 0 & \text{otherwise} \end{cases}$$

where $i = 0, 1, 2, \ldots, (m-1)$.

A square integrable time function $f(t)$ of Lebesgue measure may be expanded into an m-term BPF series in $t \in [0, T)$ as

$$f(t) = [c_0\ c_1\ \cdots\ c_i\ \cdots\ c_{(m-1)}]\Phi_m(t) \triangleq C^T\Phi_m(t). \tag{2}$$

The constant coefficients c_i's in Eq. (2) are given by

$$c_i = (1/h)\int_{ih}^{(i+1)h} f(t)\mathrm{d}t, \tag{3}$$

where $h = T/m$ is the duration of each component BPF along time scale.

In the m-term BPF domain, an operational matrix for integration P_m has been given by Deb et al. [37] as the following upper triangular matrix:

$$P_m \triangleq h\begin{bmatrix} 1/2 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1/2 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1/2 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1/2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1/2 \end{bmatrix}. \tag{4}$$

The matrix P_m performs as an integrator in the BPF domain and it is pivotal in any BPF domain analysis. Thus, approximate integration of a function $f(t)$, using Eqs. (2) and (4), is

$$\int_0^T f(t)\mathrm{d}t \approx C^T P_m \Phi_m(t). \tag{5}$$

3. Block pulse operational matrices for fractional calculus

We now derive the operational matrix for the fractional calculus. Several definitions of a fractional calculus have been proposed [2,3,5]. we formulate the problem in terms of the Riemann–Liouville fractional integration, which is defined as

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \quad 0 \le t < T, \tag{6}$$

where α is the order of the integration, $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1} * f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$. Now if $f(t)$ is expanded in block pulse functions, as shown in Eq. (2), the Riemann–Liouville fractional integral becomes

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t) \approx C^T \frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * \Phi_m(t)\}. \tag{7}$$

Thus if $t^{\alpha-1} * f(t)$ can be integrated, then expanded in block pulse functions, the Riemann–Liouville fractional integral is solved via the block pulse functions.

According to the linear nature of Laplace transform, in order to compute the convolution product $C^T\frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * \Phi_m(t)\}$, we only need to compute $\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * \varphi_i(t)$.

For $\varphi_i(t)$, applying definition of the convolution, we have

$$t^{\alpha-1} * \varphi_i(t) = \int_0^t \tau^{\alpha-1} \cdot \varphi_i(t-\tau)\mathrm{d}\tau. \tag{8}$$

Since $\varphi_i(t) = 1$ in $[iT/m, (i+1)T/m)$, then we have

$$t^{\alpha-1} * \varphi_0(t) = \begin{cases} \int_0^t \tau^{\alpha-1} \mathrm{d}\tau = \dfrac{t^\alpha}{\alpha}, & 0 \le t < T/m \\ \int_{t-T/m}^t \tau^{\alpha-1} \mathrm{d}\tau = \dfrac{t^\alpha - (t-T/m)^\alpha}{\alpha}, & T/m \le t < T \end{cases} \tag{9}$$

$$t^{\alpha-1} * \varphi_1(t) = \begin{cases} 0, & 0 \le t < T/m \\ \dfrac{(t-T/m)^\alpha}{\alpha}, & T/m \le t < 2T/m \\ \dfrac{(t-T/m)^\alpha - (t-2T/m)^\alpha}{\alpha}, & 2T/m \le t < T \end{cases} \tag{10}$$

$$t^{\alpha-1} * \varphi_2(t) = \begin{cases} 0, & 0 \le t < 2T/m \\ \dfrac{(t-T/m)^\alpha}{\alpha}, & 2T/m \le t < 3T/m \\ \dfrac{(t-2T/m)^\alpha - (t-3T/m)^\alpha}{\alpha}, & 3T/m \le t < T \end{cases} \tag{11}$$

$$\vdots$$

$$t^{\alpha-1} * \varphi_{m-1}(t) = \begin{cases} 0, & 0 \le t < (m-1)T/m \\ \dfrac{[t-(m-1)T/m]^\alpha}{\alpha}, & (m-1)T/m \le t < T. \end{cases} \tag{12}$$

Set

$$(I^\alpha \Phi_m)(t) \approx F_\alpha \Phi_m(t). \tag{13}$$

Next, we derive Block pulse operational matrix for fractional integration. For $(I^\alpha \varphi_0)(t)$ we have

$$(I^\alpha \varphi_0)(t) = f_{11}\varphi_0(t) + f_{12}\varphi_1(t) + \cdots + f_{1m}\varphi_{m-1}(t) \tag{14}$$

where

$$f_{11} = \frac{m}{b} \int_0^{\frac{b}{m}} \frac{t^\alpha}{\Gamma(\alpha+1)} \mathrm{d}t = \left(\frac{b}{m}\right)^\alpha \cdot \frac{1}{\Gamma(\alpha+2)} \tag{15}$$

$$f_{1i} = \frac{m}{T} \int_{(i-1)T/m}^{iT/m} \frac{t^\alpha - (t-T/m)^\alpha}{\Gamma(\alpha+1)} \mathrm{d}t = \left(\frac{b}{m}\right)^\alpha \cdot \frac{1}{\Gamma(\alpha+2)} [i^{\alpha+1} - 2\cdot(i-1)^{\alpha+1} + (i-2)^{\alpha+1}],$$
$$i = 2, \ldots, m. \tag{16}$$

For another $(I^\alpha \varphi_i)(t), i = 1, \ldots, m-1$, applying shifting property, we can directly obtain the coefficients expanded by BPFs.

Finally, we can get

$$F_\alpha = h^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

where

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}, \quad (k = 1, 2, \ldots, m-1). \tag{18}$$

F_α is called the block pulse operational matrix of fractional integration. When $\alpha = 1$, matrix F_α is equal to matrix P_m, so matrix F_α is a generalization of the block pulse operational matrix for integration P_m.

Let D_α is the block pulse operational matrix for the fractional differentiation, According to the property of fractional calculus, $D_\alpha F_\alpha = I$, we can easily get matrix D_α by inverting the F_α matrix.

According to matrix theory, we know that the inverse matrix of an upper triangular matrix is also upper triangular matrix. So, we can write

$$\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} d_0 & d_1 & d_2 & \cdots & d_{m-1} \\ 0 & d_0 & d_1 & \cdots & d_{m-2} \\ 0 & 0 & d_0 & \cdots & d_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d_0 \end{bmatrix}. \tag{19}$$

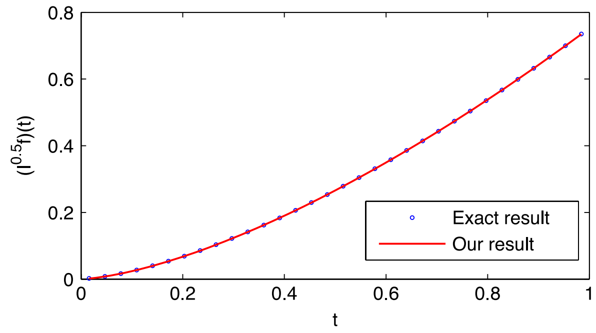

Fig. 1. 0.5-order integration of the function $f(t) = t$.

That is

$$\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d_0 & d_1 & d_2 & \cdots & d_{m-1} \\ 0 & d_0 & d_1 & \cdots & d_{m-2} \\ 0 & 0 & d_0 & \cdots & d_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d_0 \end{bmatrix} = I_m, \tag{20}$$

where I_m is the identity matrix of order $m \times m$. After solving the matrix equation of Eq. (20), we get

$$d_0 = 1, \qquad d_1 = -\xi_1 d_0, \ldots, d_{m-1} = -\sum_{k=1}^{m-1} \xi_k d_{m-k-1}. \tag{21}$$

For example, let $\alpha = 0.5$, $m = 8$, $T = 1$, the operational matrices $F_{0.5}$ and $D_{0.5}$ are computed below:

$$F_{0.5} = \begin{bmatrix} 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 & 0.0894 & 0.0816 & 0.0755 \\ 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 & 0.0894 & 0.0816 \\ 0 & 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 & 0.0894 \\ 0 & 0 & 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 & 0.1001 \\ 0 & 0 & 0 & 0 & 0.2660 & 0.2203 & 0.1434 & 0.1160 \\ 0 & 0 & 0 & 0 & 0 & 0.2660 & 0.2203 & 0.1434 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2660 & 0.2203 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2660 \end{bmatrix}$$

$$D_{0.5} = \begin{bmatrix} 3.7599 & -3.1148 & 0.5527 & -0.4178 & -0.0091 & -0.0998 & -0.0442 & -0.0460 \\ 0 & 3.7599 & -3.1148 & 0.5527 & -0.4178 & -0.0091 & -0.0998 & -0.0442 \\ 0 & 0 & 3.7599 & -3.1148 & 0.5527 & -0.4178 & -0.0091 & -0.0998 \\ 0 & 0 & 0 & 3.7599 & -3.1148 & 0.5527 & -0.4178 & -0.0091 \\ 0 & 0 & 0 & 0 & 3.7599 & -3.1148 & 0.5527 & -0.4178 \\ 0 & 0 & 0 & 0 & 0 & 3.7599 & -3.1148 & 0.5527 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7599 & -3.1148 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.7599 \end{bmatrix}.$$

The fractional integration and differentiation of the function t was selected to verify the correctness of matrices F_α and D_α. That is because the fractional integration and differentiation of the function $f(t) = t$ is easily obtained as following $(I^\alpha f)(t) = \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1}$ and $(D^\alpha f)(t) = \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}$, respectively, which is easily used to compare the results obtained by the proposed method. When $\alpha = 0.5$, $T = 1$, $m = 32$, the comparison results for the fractional integration and differentiation are shown in Figs. 1 and 2, respectively.

4. Solution of the fractional differential equations by the generalized block pulse operational matrix

4.1. Linear multi-order fractional differential equation

In this section, we are concerned with providing a numerical solution to multi-order fractional linear system of the form

$$D^\alpha y(t) = \sum_{j=1}^{n} a_j(t) D^{\beta_j} y(t) + a_0(t) y(t) + f(t) \tag{22}$$

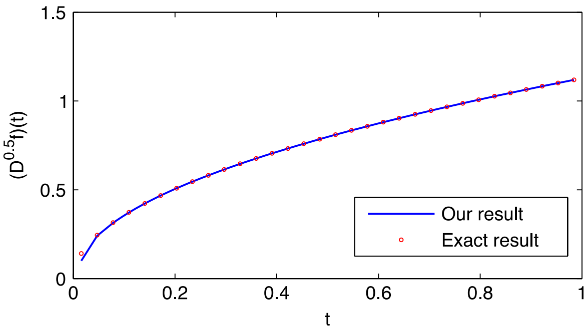

Fig. 2. 0.5-order differentiation of the function $f(t) = t$.

subject to the initial conditions

$$y^{(k)}(0) = b_k, \quad k = 0, 1, \ldots, \lceil \alpha \rceil - 1, \tag{23}$$

where $\alpha > \beta_1 > \beta_2 > \cdots > \beta_n$, D^α denotes the Caputo fractional derivative of order α, $a_j(t)$ is known function for $j = 0, 1, \ldots, n$, and $f(t)$ is input signal, $y(t)$ is output response.

The general procedure of numerical solution of fractional differential equations consists of two steps.

First, initial conditions are used to reduce a given initial-value problem to a problem with zero initial conditions. At this stage, instead of a given equation a modified equation, incorporating the initial values, is obtained.

Then, the generalized block pulse operational matrix of fractional integration is used to transform the fractional differential equation into an algebraic equation.

The solution of the initial-value problem Eq. (22) can be written in the form

$$y(t) = y_*(t) + z(t) \tag{24}$$

where $y_*(t)$ is some known function, satisfying the conditions $y^{(k)}(0) = b_k, k = 0, 1, \ldots, \lceil \alpha \rceil - 1$, and $z(t)$ is a new unknown function.

Substituting Eq. (24) into the Eq. (22) and the initial conditions Eq. (23), we obtain for the function $z(t)$ an initial-value problem with zero initial conditions.

$$D^\alpha z(t) = \sum_{j=1}^{n} a_j(t) D^{\beta_j} z(t) + a_0(t) z(t) + g(t) \tag{25}$$

subject to the initial conditions

$$z^{(k)}(0) = 0, \quad k = 0, 1, \ldots, \lceil \alpha \rceil - 1. \tag{26}$$

The input signal $g(t)$ and output response $D^\alpha z(t)$ may be expanded by the BPFs as

$$g(t) \approx G^T \Phi_m(t), \tag{27}$$

$$D^\alpha z(t) \approx C^T \Phi_m(t) \tag{28}$$

where $G = [g_0, g_1, \ldots, g_m]^T$ is a known but $C = [c_0, c_1, \ldots, c_m]^T$ is an unknown $m \times 1$ column vector.

Similarly, $a_j(t)$ for $j = 0, 1, \ldots, n$, may also be expanded by the BPFs as

$$a_j(t) \approx A_j^T \Phi_m(t) \tag{29}$$

where A_j is a known $m \times 1$ column vector.

Using Eq. (28) together with property of fractional calculus, we have

$$D^{\beta_j} z(t) = I^{\alpha-\beta_j}[D^\alpha z(t)] = I^{\alpha-\beta_j}[C^T \Phi_m(t)] = C^T F_{\alpha-\beta_j} \Phi_m(t). \tag{30}$$

Substituting Eqs. (27)–(30) into (25), we have

$$C^T \Phi_m(t) = \sum_{j=1}^{n} A_j^T \Phi_m(t) [\Phi_m(t)]^T [F_{\alpha-\beta_j}]^T C + A_0^T \Phi_m(t) [\Phi_m(t)]^T [F_\alpha]^T C + G^T \Phi_m(t). \tag{31}$$

According to the properties of BPFs, we can get

$$\Phi_m(t)[\Phi_m(t)]^T = \begin{bmatrix} \varphi_1(t) & & & O \\ & \varphi_2(t) & & \\ & & \ddots & \\ O & & & \varphi_m(t) \end{bmatrix}. \tag{32}$$

Set $[F_{\alpha-\beta_j}]^T C = X_j = [x_{j1}, x_{j2}, \dots, x_{jm}]^T$, then we have

$$\Phi_m(t)[\Phi_m(t)]^T[F_{\alpha-\beta_j}]^T C = \begin{bmatrix} x_{j1} & & & O \\ & x_{j2} & & \\ & & \ddots & \\ O & & & x_{jm} \end{bmatrix} \Phi_m(t) = diag(X_j)\Phi_m(t). \tag{33}$$

Using Eq. (33) we can rewrite Eq. (31) as

$$C^T\Phi_m(t) = \sum_{j=1}^{n} A_j^T diag(X_j)\Phi_m(t) + A_0^T(diag[F_\alpha]^T C)\Phi_m(t) + G^T\Phi_m(t) \tag{34}$$

or

$$C = \sum_{j=1}^{n} diag(X_j)A_j + (diag[F_\alpha]^T C)A_0 + G. \tag{35}$$

Solving the system of algebraic equations, we can obtain the coefficients C^T. Then, we can get the output response

$$z(t) = C^T F_\alpha \Phi_m(t). \tag{36}$$

4.2. Nonlinear multi-order fractional differential equation

Consider the nonlinear multi-order fractional differential equation

$$D^\alpha y(t) = \sum_{j=1}^{n} a_j D^{\beta_j} y(t) + a_0[y(t)]^m + g(t) \tag{37}$$

subject to the initial conditions

$$y^{(k)}(0) = c_k, \quad k = 0, 1, \dots, \lceil\alpha\rceil - 1 \tag{38}$$

where $\alpha > \beta_1 > \beta_2 > \cdots > \beta_n$, D^α denotes the Caputo fractional derivative, a_j is constant for $j = 0, 1, \dots, n$.

Compared with the linear case, a major distinction lies in the computation of $[y(t)]^i$, when we use the generalized Block pulse operational matrix to solve the nonlinear multi-order fractional differential equation.

Set

$$D^\alpha y(t) \approx C^T\Phi_m(t) \tag{39}$$

then we have

$$y(t) \approx C^T F_\alpha \Phi_m(t). \tag{40}$$

Set $C^T F_\alpha = [x_1,\ x_2,\ \dots,\ x_m]$, then we have

$$[y(t)]^i = [x_1^i, x_2^i, \dots, x_m^i]\Phi_m(t). \tag{41}$$

The derivation is similar to the linear case, we have

$$C^T\Phi_m(t) = \sum_{j=1}^{n} a_j C^T F_{\alpha-\beta_j}\Phi_m(t) + a_0[x_1^i, x_2^i, \dots, x_m^i]\Phi_m(t) + G^T\Phi_m(t). \tag{42}$$

This is a nonlinear system of algebraic equation, we use the "fsolve" in Matlab to solve it.

5. Numerical examples

Example 1. Consider the fractional equation [11]

$$D^2 y(t) + D^\alpha y(t) + y(t) = 8, \quad t > 0,\ 0 < \alpha < 2. \tag{43}$$

Subject to

$$y(0) = 1, \qquad y'(0) = 0.$$

This problem was solved in [11], for $\alpha = 0.5$ and $\alpha = 1.5$. Our results compared to Ref. [11] are given in Table 1, where $h = 0.001$. According to the exact solution given in Ref. [11], ours results are in high agreement with results obtained using the FDTM and better than those obtained using the ADM. We show the approximate solution in Fig. 3 for $h = 0.1$.

Table 1
Our results compared to Ref. [11].

t	$\alpha = 0.5$			$\alpha = 1.5$		
	Ours	FDTM [11]	ADM [11]	Ours	FDTM [11]	ADM [11]
0.1	0.039754	0.039750	0.039874	0.033510	0.033507	0.036478
0.2	0.157043	0.157036	0.158512	0.125226	0.125221	0.140640
0.3	0.347373	0.347370	0.353625	0.267611	0.267609	0.307485
0.4	0.604699	0.604695	0.622083	0.455439	0.455435	0.533284
0.5	0.921768	0.921768	0.960047	0.684336	0.684335	0.814757
0.6	1.290458	1.290457	1.363093	0.950395	0.950393	1.148840
0.7	1.702007	1.702008	1.826257	1.249959	1.249959	1.532571
0.8	2.147286	2.147287	2.344224	1.579558	1.579557	1.963033
0.9	2.616998	2.617001	2.911278	1.935832	1.935832	2.437331
1.0	3.101902	3.101906	3.521462	2.315526	2.315526	2.952567

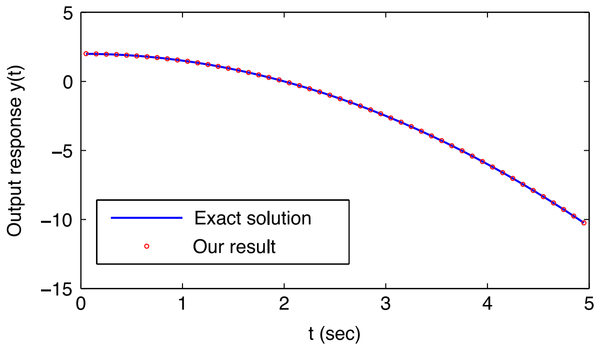

Fig. 3. Numerical and exact solution of Example 2 for $h = 0.1$.

Table 2
Absolute error in $y(t)$ for different values of h.

t	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.00625$
0.5	0.0024	0.6084E−3	0.1524E−3	0.3815E−4	0.9549E−5
1.5	0.0020	0.5058E−3	0.1267E−3	0.3173E−4	0.7945E−5
2.5	0.0018	0.4403E−3	0.1102E−3	0.2759E−4	0.6902E−5
3.5	0.0016	0.4124E−3	0.1032E−3	0.2581E−4	0.6457E−5
4.5	0.0016	0.3995E−3	0.0999E−3	0.2499E−4	0.6249E−5

Example 2. Consider the fractional variable coefficient linear differential equation [38]

$$aD^2y(t) + b(t)D^{\alpha_2}y(t) + c(t)Dy(t) + e(t)D^{\alpha_1}y(t) + k(t)y(t) = f(t), \quad t \in [0, T] \tag{44}$$

where $0 < \alpha_1 \le 1$, $1 < \alpha_2 \le 2$, and

$$f(t) = a - \frac{b(t)}{\Gamma(3-\alpha_2)}t^{2-\alpha_2} - c(t)t - \frac{e(t)}{\Gamma(3-\alpha_1)}t^{2-\alpha_1} + k(t)\left(2 - \frac{1}{2}t^2\right),$$

subject to

$$y(0) = 2, \qquad y'(0) = 0.$$

It is easily verified that the exact solution of this problem is

$$y(t) = 2 - \frac{1}{2}t^2.$$

For $a = 1, b(t) = t^{1/2}, c(t) = t^{1/3}, e(t) = t^{1/4}, k(t) = t^{1/5}, \alpha_1 = 0.5, \alpha_2 = 1.5$ we present numerical values of the solution to Eq. (44) by our method with $h = 0.01$ and exact solution in Fig. 3, which is in perfect agreement with the exact solutions, the absolute errors of $y(t)$ at given points for different values of h are shown in Table 2.

Example 3. Consider the nonlinear fractional differential equation, which has been studied in [9,11].

$$D^\alpha y(t) = y^2(t) + 1, \quad t \in (0, 1),\ m - 1 < \alpha \le m, \tag{45}$$

Table 3
Our results compared to Refs. [9,11].

t	$\alpha = 0.5$			$\alpha = 1.5$		
	Ours	FDTM [11]	ADM [11]	Ours	FDTM [11]	ADM [11]
0.1	0.023800	0.023790	0.023790	0.000954	0.000952	0.000951
0.2	0.067335	0.067330	0.067330	0.005385	0.005383	0.005382
0.3	0.123900	0.123896	0.123896	0.014836	0.014833	0.014833
0.4	0.191368	0.191362	0.191362	0.030455	0.030450	0.030449
0.5	0.268862	0.268856	0.268856	0.053203	0.053197	0.053196
0.6	0.356244	0.356238	0.356238	0.083931	0.083925	0.083924
0.7	0.453956	0.453950	0.453950	0.123418	0.123412	0.123412
0.8	0.563014	0.563007	0.563007	0.172397	0.172391	0.172391
0.9	0.685067	0.685056	0.685056	0.231582	0.231574	0.231574
1.0	0.822525	0.822509	0.822511	0.301686	0.301676	0.301676

subject to

$$y^k(0) = 0, \quad k = 0, 1, \dots, m-1.$$

It has been studied by using FDTM [11] and ADM [9]. Our results compared to Refs. [9,11] are given in Table 3, where $h = 0.01$. Our results are in good agreement with the results in Refs. [9,11].

6. Conclusion

In this paper, we derive a numerical method for the fractional differential equations based on the operational matrix F_α for the fractional integration and differentiation. A general procedure of forming this matrix F_α is summarized. Several examples are given to demonstrate the powerfulness of the proposed method. The matrix F_α can also be used to solve problems such as fractional system identification, fractional order optimal control.

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