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Hybrid Functions Approach for the Nonlinear Volterra-Fredholm Integral Equations

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Abstract

An approximation method based on hybrid Legendre and Block-Pulse functions used for the solution of nonlinear Volterra-Fredholm integral equations (NV-FIEs). These hybrid functions operational matrices are presented and are utilized to reduce a nonlinear Volterra-Fredholm integral equation to a system of nonlinear algebraic equations. In addition, convergence analysis and numerical examples that illustrate the pertinent features of the method are presented.

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Keywords: Hybrid functions, Nonlinear Volterra-Fredholm Integral equations, Operational Matrix, Product matrix, Coefficient matrix.

1. Introduction

Integral equation has been one of the principal tools in various areas of applied mathematics, physics and engineering. Integral equation is encountered in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communication theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, etc. Many of these integral equations are nonlinear, see([1]-[5]).

In this paper we dealing with nonlinear Volterra-Fredholm integral equations as follows:

$$u(x) = f(x) + \lambda_1 \int_0^x k_1(x,s)\psi_1(s,u(s))ds + \lambda_2 \int_0^1 k_2(x,s)\psi_2(s,u(s))ds, \quad (1)$$

where the parameters λ_1, λ_2 and functions $f(x)$, $\psi_1(s,u(s))$, $\psi_2(s,u(s))$, $k_1(x,s)$ and $k_2(x,s)$ are known and in $L^2[0,1)$ and $u(x)$ is an unknown function. In this work we suppose $\psi_1(s,u(s)) = u(s)^\alpha$ and $\psi_2(s,u(s)) = u(s)^\beta$ where α, β are positive integers.

We use the Hybrid Legendre and Block-Pulse functions as basis for reducing these NV-FIEs to a system of nonlinear algebraic equations. We present Hybrid Legendre and Block-Pulse useful properties such as operational matrix of integration, product matrix, integration of the cross product and coefficient matrix and use them for transform our NV-FIE. As showed in our examples our method in analogy to existed methods works better. This paper is organized as follows: In section 2 we introduce hybrid functions and its properties . In Section 3 we apply these set of Hybrid functions for approximating the solution of NV-FIEs. Convergence analysis is given in section 4.

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In section 5 the proposed method is tested with some examples and the results have been compared by some existed methods results. And finally section 6 give the conclusion.

2. Definition and some properties of hybrid functions of Block-Pulse and Legendre

2.1. Hybrid functions of Block-Pulse and Legendre

Consider the Legendre polynomials $L_m(x)$ on the interval $[-1,1]$:

$$L_0(x) = 1, L_1(x) = x, (m+1)L_{m+1}(x) = (2m+1)xL_m(x) - mL_{m-1}(x), m = 1,2,3,\dots$$

set $\{L_m(x): m = 0,1,\dots\}$ in Hilbert space $L^2[-1,1]$ is a complete orthogonal set. A set of Block-Pulse functions $b_i(x), i = 1,2,\dots,n$ and the orthogonal set of hybrid functions, $j = 0,1,\dots,m-1$ that produces by Legendre polynomials and Block-Pulse functions on $[0,1)$ are defined as follows respectively:

$$b_i(x) = \begin{cases} 1, & \frac{i-1}{n} \le x < \frac{i}{n} \\ 0, & \text{otherwise} \end{cases}, \quad h_{ij}(x) = \begin{cases} L_j(2nx - 2i + 1), & \frac{i-1}{n} \le x < \frac{i}{n} \\ 0, & \text{otherwise} \end{cases}.$$

2.2. function approximation

Any function $u(x) \in L^2[0,1)$ can be expanded as $u(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} h_{ij}(x)$, where the hybrid coefficients are given by

$$c_{ij} = \frac{(u(x), h_{ij}(x))}{(h_{ij}(x), h_{ij}(x))}, i = 1,2,\dots,\infty, j = 0,1,\dots,\infty,$$

so that $(\cdot,\cdot)$ denotes the inner product. If $u(x)$ is piecewise constant or may be approximated as piecewise constant, then the sum may be terminated after nm terms, that is, $u(x) \cong \sum_{i=1}^{n} \sum_{j=0}^{m-1} c_{ij} h_{ij}(x) = C^T \mathbf{h}(x),$

where

$$C = [c_{10}, \dots, c_{1,m-1}, c_{20}, \dots, c_{2,m-1}, \dots, c_{n0}, \dots, c_{n,m-1}]^T, \tag{2}$$

$$\mathbf{h}(x) = [h_{10}(x), \dots, h_{1m-1}(x), h_{20}(x), \dots, h_{2m-1}(x), \dots, h_{nm-1}(x)]^T. \tag{3}$$

We can also approximate the function $k(x,s) \in L^2([0,1) \times [0,1))$ as follows:

$$k(x,s) \cong \mathbf{h}^T(x) K \mathbf{h}(s),$$

so that

$$K_{ij} = \frac{\left(\mathbf{h}_{(i)}(x), \left(k(x,s), \mathbf{h}_{(j)}(s)\right)\right)}{\left(\mathbf{h}_{(i)}(x), \mathbf{h}_{(i)}(x)\right)\left(\mathbf{h}_{(j)}(s), \mathbf{h}_{(j)}(s)\right)}, \quad i,j = 1,2,\dots,nm$$

.

2.3. Operational matrix of integrstion

The integration of the vector $\mathbf{h}(x)$ defined in Eq.(3) is given by

$$\int_0^x \mathbf{h}(x')dx' \cong P\mathbf{h}(x), \tag{4}$$

where P is the $nm \times nm$ operational matrix for integration and is given in [6] in details.

2.4. The integration of the cross product

The integration of the cross product of two hybrid function vectors can be obtained as

$$D = \int_0^1 \mathbf{h}(x)\mathbf{h}^T(x)dx = \begin{bmatrix} L & O & \cdots & O \\ O & L & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & L \end{bmatrix}, \quad (5)$$

where matrix L is a $m \times m$ diagonal matrix that can be seen in [6].

2.5. Product operational matrix

It is always necessary to evaluate the product of $\mathbf{h}(x)$ and $\mathbf{h}^T(x)$, that be called the product matrix of hybrid functions. Let

$$\mathbf{H}(x) = \mathbf{h}(x)\mathbf{h}^T(x), \quad (6)$$

where $\mathbf{H}(x)$ is $nm \times nm$ matrix. By multiplying the matrix $\mathbf{H}(x)$ in vector C that defined in Eq.(2) we obtain

$$\mathbf{H}(x)C = \tilde{C}\mathbf{h}(x), \quad (7)$$

where $\tilde{C}$ is $nm \times nm$ matrix and called the coefficient matrix. Basic multiplication properties of arbitrary two hybrid function $h_{ij}(x)$ and $h_{kl}(x)$ are described in [6].

3. Outline of the method for NV-FIEs via Hybrid functions

Consider the nonlinear Volterra-Fredholm integral equation (1).
We put

$$u(x) \cong U^T\mathbf{h}(x), \quad (8)$$

where U is an unknown nm -vector and $\mathbf{h}(x)$ is given by Eq.(3). Likewise, $k_1(x,s)$, $k_2(x,s)$ and $f(x)$ are expanded into the hybrid functions as follows

$$k_1(x,s) \cong \mathbf{h}^T(x)K_1\mathbf{h}(s), \qquad k_2(x,s) \cong \mathbf{h}^T(x)K_2\mathbf{h}(s). \quad (9)$$

$$f(x) \cong F^T\mathbf{h}(x), \quad (10)$$

where K_1, K_2 are known $nm \times nm$ -matrices and F is a known nm -vector.
After substituting the approximate equations (8), (9), (10) in (1) we get

$$U^T\mathbf{h}(x) \cong F^T\mathbf{h}(x) + \lambda_1\mathbf{h}^T(x)K_1\int_0^x \mathbf{h}(s)\psi_1\left(s, U^T\mathbf{h}(s)\right)ds + \lambda_2\mathbf{h}^T(x)K_2\int_0^1 \mathbf{h}(s)\psi_2\left(s, U^T\mathbf{h}(s)\right)ds. \quad (11)$$

Functions $\psi_1\left(s, U^T h(s)\right) = \left(U^T h(s)\right)^\alpha$ and $\psi_2\left(s, U^T h(s)\right) = \left(U^T h(s)\right)^\beta$ are known which can be expanded into the hybrid functions as

$$\left(u(s)\right)^\alpha \cong U_\alpha^T\mathbf{h}(s), \qquad \left(u(s)\right)^\beta \cong U_\beta^T\mathbf{h}(s). \quad (12)$$

In the next subsection, we consider computing U_α and U_β in terms of U , which U_α, U_β are mn -vectors whose elements are nonlinear combination of the elements of the vector U . Substitute Eq.(12) in Eq.(11) produces

$$U^T\mathbf{h}(x) \cong F^T\mathbf{h}(x) + \lambda_1\mathbf{h}^T(x)K_1\int_0^x \mathbf{h}(s)\mathbf{h}^T(s)U_\alpha ds + \lambda_2\mathbf{h}^T(x)K_2\int_0^1 \mathbf{h}(s)\mathbf{h}^T(s)U_\beta ds. \quad (13)$$

Note that by use of Eq.(4) and Eq.(7) we have $\int_0^x \mathbf{h}(s)\mathbf{h}^T(s)U_\alpha ds = \int_0^x \widetilde{U_\alpha}\mathbf{h}(s)ds = \widetilde{U_\alpha}P\mathbf{h}(x)$, by this relation and Eq.(5) we get

$$U^T\mathbf{h}(x) \cong F^T\mathbf{h}(x) + \lambda_1\mathbf{h}^T(x)K_1\widetilde{U_\alpha}P\mathbf{h}(x) + \lambda_2\mathbf{h}^T(x)(K_2DU_\beta). \quad (14)$$

In order to find U we collocate Eq.(14) in nm nodal points of Newton-Cotes as,

$$x_p = \frac{2p-1}{2nm}, \qquad p = 1,2,\dots,nm. \quad (15)$$

then we have following system of nonlinear equations

$$U^T\mathbf{h}(x_p) \cong F^T\mathbf{h}(x_p) + \lambda_1\mathbf{h}^T(x_p)K_1\widetilde{U_\alpha}P\mathbf{h}(x_p) + \lambda_2\mathbf{h}^T(x_p)(K_2DU_\beta), \qquad p = 1,2,\dots,nm. \quad (16)$$

This nonlinear system of equations can be solved by Newton's method. We used the Mathematica software to solve this nonlinear system. After solving above nonlinear system we can achieve U, then we will have our unknown $u(x)$ as $U^T\mathbf{h}(x)$, that is the approximate solution of NV-FIE (1).

3.1. Evaluating U_α and U_β

For numerical implementation of the method explained in section 3, we need to evaluate U_α and U_β, so that the elements of each one are nonlinear combination of the elements of the vector U. From Eqs.(7) and (8), We have

$$\begin{aligned}(u(x))^2 &\cong (U^T\mathbf{h}(x))(U^T\mathbf{h}(x)) = U^T\mathbf{h}(x)\mathbf{h}^T(x)U \\ &= U^T\tilde{U}\mathbf{h}(x) = U_2\mathbf{h}(x),\end{aligned} \quad (17)$$

where the vector $U_2 = U^T\tilde{U}$ is a mn -row vector, then for $(u(s))^3$ we get

$$\begin{aligned}(u(x))^3 &\cong (U^T\mathbf{h}(x))(U_2\mathbf{h}(x)) = U^T\mathbf{h}(x)\mathbf{h}^T(x)U_2^T \\ &= U^T\widetilde{U_2^T}\mathbf{h}(x) = U_3\mathbf{h}(x).\end{aligned} \quad (18)$$

Therefore with this method we can approximate $(u(s))^\alpha$ and $(u(s))^\beta$ for arbitrary α and β. Suppose that this method holds for $\alpha-1$ where $(u(x))^{\alpha-1} = U_{\alpha-1}\mathbf{h}(x)$, we shall obtain it for α as follows

$$\begin{aligned}(u(x))^\alpha &= u(x)u(x)^{\alpha-1} \cong (U^T\mathbf{h}(x))(U_{\alpha-1}\mathbf{h}(x)) \\ &= U^T\mathbf{h}(x)\mathbf{h}^T(x)U_{\alpha-1}^T \\ &= U^T\widetilde{U_{\alpha-1}^T}\mathbf{h}(x) = U_\alpha\mathbf{h}(x).\end{aligned} \quad (19)$$

we have similar relation for β. So, the components of U_α and U_β can be computed in terms of components of unknown vector U.

4. Convergence analysis

We assume the following conditions on k_1, k_2 and ψ_1, ψ_2 for Eq.(1).

1. $M_1 \equiv \sup_{0\le x,s\le 1}|k_1(x,s)| < \infty$, $M_2 \equiv \sup_{0\le x,s\le 1}|k_2(x,s)| < \infty$;
2. $\psi_1(s,x), \psi_2(s,x)$ are continuous in $s \in [0,1]$ and Lipschitz continuous in $x \in R$, i.e., there exists constants $C_1, C_2 > 0$ for which

$$|\psi_1(s,x_1) - \psi_1(s,x_2)| \le C_1|x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in R,$$

$$|\psi_2(s,x_1) - \psi_2(s,x_2)| \le C_2|x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in R,$$

Theorem 4.1: *The solution of Nonlinear Volterra-Fredholm Integral Equation by using hybrid functions converges if* .

Proof. For NV-FIE by assumption $\int_0^x |k_1(x,t)|dt \le \int_0^1 |k_1(x,t)|dt$, for ; We see that there exists a constant $\gamma = |\lambda_1| M_1 C_1 + |\lambda_2| M_2 C_2 > 0$ such that

$$\| u_{nm}(x) - u(x) \| = \max_{x\in[0,1]} |u_{nm}(x) - u(x)|$$

$$\le \max_{x\in[0,1]} |\lambda_1| \int_0^x |k_1(x,s)| |\psi_1(s,u_{nm}(s)) - \psi_1(s,u(s))| ds$$

$$+ \max_{x\in[0,1]} |\lambda_2| \int_0^1 |k_2(x,s)| |\psi_2(s,u_{nm}(s)) - \psi_2(s,u(s))| ds$$

$$\le (|\lambda_1| M_1 C_1 + |\lambda_2| M_2 C_2) \max_{x\in[0,1]} |u_{nm}(x) - u(x)| \le \gamma \max_{x\in[0,1]} |u_{nm}(x) - u(x)| .$$

$$0 < \gamma <$$

We get $(1-\gamma) \| u_{nm}(x) - u(x) \| \le 0$ and choose , when $n \to \infty$, it implies $\| u_{nm}(x) - u(x) \| \to 0$.

5. Numerical results

In this section, some examples are provided to show the efficiency of the proposed method. In our examples we get the results by $m = 8$ and $n = 2,4,8,16$. In all of them we compared our answers with some existed methods, and its explicit that our method works better than them because in lower n we could get better results.

5.1. Example 1

Consider the nonlinear Volterra-Fredholm equation given by

$$u(x) = -\frac{1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4} + \int_0^x (x-s)u^2(s)ds + \int_0^1 (x+s)u(s)ds,$$

with the the exact solution $u(x) = x^2 - 2$ [7]. The comparison among the hybrid solutions beside the exact solutions are shown in Table 1.

Table 1. Approximate and exact solutions for Example 1.

x	Solution with $n = 2$	Solution with $n = 4$	Solution with $n = 8$	Solution with $n = 16$	Method in [7] with $k = 16$	Exact
0.0	-2.004858	-2.001333	-2.000340	-2.000085	-1.995	-2
0.1	-1.996105	-1.991671	-1.990420	-1.990104	-1.989	-1.99
0.2	-1.967892	-1.962063	-1.960511	-1.960125	-1.965	-1.96
0.3	-1.919818	-1.912286	-1.910599	-1.910152	-1.912	-1.91
0.4	-1.851187	-1.842838	-1.840683	-1.840174	-1.841	-1.84
0.5	-1.761040	-1.753034	-1.750775	-1.750195	-1.752	-1.75

0.6	-1.650618	-1.643166	-1.640844	-1.640208	-1.643	-1.64
0.7	-1.521064	-1.513388	-1.510834	-1.510204	-1.498	-1.51
0.8	-1.371005	-1.362753	-1.360742	-1.360192	-1.359	-1.36
0.9	-1.198995	-1.192383	-1.190576	-1.190149	-1.185	-1.19
1.0	-1.003603	-1.001267	-1.000339	-1.000086	-0.994	-1

6. Conclusion

This work present a numerical approach for solving NV-FIEs based on hybrid legendre and Block-Pulse functions. By some useful properties of these hybrid functions such as, operational matrix, The integration of the cross product matrix, product matrix and coefficient matrix together with collocation method, a NV-FIE can be transformed to a system of algebraic equations. A considerable advantage of the method is that the problem has been reduced to solving a set of algebraic equations and they are represented in matrix form. Illustrative examples are given to demonstrate the validity and applicability of proposed method and our answers compared with the answers of some existed method.

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