Journal of Computational and Applied Mathematics 236 (2012) 2754–2762

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A method for solving differential equations of fractional order

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ARTICLE INFO

Article history:
Received 3 May 2010
Received in revised form 3 January 2012

MSC:
23A33
26A33
34A99

Keywords:
Fractional derivatives and integrals
Fractional differential equations
Existence of solutions

ABSTRACT

In this paper, we consider Caputo type fractional differential equations of order $0 < \alpha < 1$ with initial condition $x(0) = x_0$. We introduce a technique to find the exact solutions of fractional differential equations by using the solutions of integer order differential equations. Generalization of the technique to finite systems is also given. Finally, we give some examples to illustrate the applications of our results.

1. Introduction

In recent years, fractional order differential equations have become an important tool in mathematical modeling [1]. A geometric interpretation of fractional integral and derivative is given in [2]. Although there are very many possible generalizations of $\frac{d^n}{dt^n} f(t)$, the most commonly used definitions are Riemann–Liouville and Caputo fractional derivatives. The former concept is historically the first and the theory about this concept has been established very well by now, but there are some difficulties with applying it to real-life problems [3]. In order to overcome these difficulties, the latter concept, Caputo type derivative is defined. This new concept is closely related to the Riemann–Liouville derivative. In this study, we consider nonlinear Caputo type fractional differential equations of order $0 < \alpha < 1$, which are used in modeling physical and biological facts [4–7]. Several numerical solution techniques for this kind of equations were studied in earlier works [6,8–12]. In most of these techniques, either the solutions of integer order differential equation versions of the given fractional differential equations or the series expansions in the neighborhood of the initial conditions are used.

In this paper, we use a transformation in the equivalent fractional Volterra integral equation of given fractional differential equation (FDE) and obtain its exact solution in terms of the solution of an integer order differential equation.

Some examples are also given to show that this technique works properly. For the equations that could not be solved analytically, comparison is made using the numerical solutions given in [12,13]. Finally, an application of this technique is given to solve a fractional order epidemic model, numerically.

2. Preliminaries

Definition 1. The Riemann–Liouville type fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau.$$

Here and elsewhere Γ denotes the Gamma function.

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doi:10.1016/j.cam.2012.01.005

Definition 2. The Riemann–Liouville type fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is defined by

$$\mathbf{D}^{\alpha} f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

Definition 3. The Caputo type fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau)\, d\tau$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

Some of the main properties of the Riemann–Liouville fractional integral and derivative are given below (see [12]).

(i) If $f \in C[0, \infty)$, then the Riemann–Liouville fractional order integral has the following important property:

$$I^{\alpha} \left(I^{\beta} f(t) \right) = I^{\alpha+\beta} f(t)$$

where $\alpha > 0$ and $\beta > 0$.

(ii) For $\alpha > 0, t > 0$,

$$\mathbf{D}^{\alpha} \left(I^{\alpha} f(t) \right) = f(t)$$

i.e. the Riemann–Liouville fractional derivative is the left inverse of the Riemann–Liouville fractional integral of the same order.

(iii) If the fractional derivative of a function of order α is integrable then,

$$I^{\alpha} \left(\mathbf{D}^{\alpha} f(t) \right) = f(t) - \sum_{j=1}^{n} \left[\mathbf{D}^{\alpha-j} f(t) \right]_{t=0} \frac{t^{\alpha-j}}{\Gamma(\alpha-j+1)}$$

where $n = [\alpha] + 1$ and if $m < 0$, $\mathbf{D}^m f(t)$ is defined as

$$\mathbf{D}^m f(t) = I^{-m} f(t).$$

A particular case of (iii) can be given by

$$I^{\alpha} \left(\mathbf{D}^{\alpha} f(t) \right) = f(t)$$

for $0 < \alpha \leq 1$.

(iv) Riemann–Liouville fractional derivative of a constant c is given by

$$\mathbf{D}^{\alpha}(c) = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}.$$

One of the most important advantages of using a Caputo type fractional derivative is that the Caputo derivative of a constant is zero, which means this kind of derivative can be used to model the rate of change.

3. Main results

Consider the initial value problem (IVP) with Caputo type FDE given by

$$\begin{aligned} D^{\alpha} x(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $f \in C([0, T] \times R, R)$, $0 < \alpha < 1$.

Since f is assumed to be a continuous function, every solution of the IVP given by (1) is also a solution of the following Volterra fractional integral equation.

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau, \quad t \in [0, T]. \tag{2}$$

Moreover, every solution of (2) is a solution of the IVP (1).

We note that IVP (1) is equivalent to the IVP

$$\begin{aligned} \mathbf{D}^{\alpha}(x(t) - x_0) &= f(t, x(t)) \\ x(0) &= x_0. \end{aligned}$$

The following existence theorem is given for (1) in [14].

Theorem 4. *Assume that* $f \in C[R_0, R]$ *where* $R_0 = [(t, x) : 0 \le t \le a$ *and* $|x - x_0| \le b]$ *and let* $|f(t, x)| \le M$ *on* R_0. *Then there exists at least one solution for the IVP* (1) *on* $0 \le t \le \gamma$ *where* $\gamma = \min\left(a, \left[\frac{b}{M}\Gamma(\alpha+1)\right]^{\frac{1}{\alpha}}\right), 0 < \alpha < 1$.

Theorem 5. *Consider the IVP given by* (1). *Let*

$$g(v, x_*(v)) = f\left(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}, x\left(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}\right)\right)$$

and assume that the conditions of Theorem 4 hold. Then, a solution of (1), $x(t)$, *is given by*

$$x(t) = x_*(t^\alpha/\Gamma(\alpha+1))$$

where $x_*(v)$ *is a solution of the integer order differential equation*

$$\frac{d(x_*(v))}{dv} = g(v, x_*(v)) \tag{3}$$

with the initial condition

$$x_*(0) = x_0. \tag{4}$$

Proof. The existence of the solution of (1) follows from Theorem 4. If $x(t)$ is a solution of (1) then, it is also a solution of (2). Let $\tau = t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}$. So, Volterra fractional integral equation (2) can be written as

$$\begin{aligned} x(t) &= x_0 + \int_0^{t^\alpha/\Gamma(\alpha+1)} f\left(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}, x\left(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}\right)\right) dv \\ &= x_0 + \int_0^{t^\alpha/\Gamma(\alpha+1)} g(v, x_*(v))\, dv. \end{aligned} \tag{5}$$

On the other hand, consider the IVP given by (3)–(4). Every solution of (3)–(4) is also a solution of the Volterra integral equation given below and vice versa.

$$x_*(v) = x_0 + \int_0^v g(s, x_*(s))\, ds, \quad 0 \le v \le a^\alpha/\Gamma(\alpha+1). \tag{6}$$

Since $0 \le t^\alpha/\Gamma(\alpha+1) \le a^\alpha/\Gamma(\alpha+1)$, the right-hand side of equation (5) is equal to $x_*(t^\alpha/\Gamma(\alpha+1))$. □

The theorems given below are simple generalizations of Theorems 4 and 5, respectively.

Theorem 6. *Let* $\|\cdot\|$ *denote any convenient norm on* R^n. *Assume that* $f \in C[R_1, R^n]$, *where* $R_1 = \left[(t, X) : 0 \le t \le a \text{ and } \|X - X_0\| \le b\right], f = (f_1, f_2, \ldots, f_n)^T, X = (x_1, x_2, \ldots, x_n)^T$ *and let* $\|f(t, X)\| \le M$, *on* R_1. *Then, there exists at least one solution for the system of FDE's given by*

$$D^\alpha X(t) = f(t, X(t)) \tag{7}$$

with initial conditions

$$X(0) = X_0 \tag{8}$$

on $0 \le t \le \beta$ *where* $\beta = \min\left(a, \left[\frac{b}{M}\Gamma(\alpha+1)\right]^{\frac{1}{\alpha}}\right), 0 < \alpha < 1$.

Theorem 7. *Consider the IVP given by* (7)–(8) *of order* α, $0 < \alpha < 1$. *Let*

$$g(v, X_*(v)) = f\left(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}, X\left(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}\right)\right)$$

and assume that the conditions of Theorem 6 hold. Then, a solution of (1), $X(t)$, *can be given by*

$$X(t) = X_*(t^\alpha/\Gamma(\alpha+1))$$

where $X_*(v)$ *is a solution of the system of integer order differential equations*

$$\frac{d(X_*(v))}{dv} = g(v, X_*(v))$$

with the initial conditions

$$X_*(0) = X_0.$$

Remark 8. Although the Caputo derivative is more commonly used in applied problems, in [5,15] the models are given with Riemann–Liouville type derivative. Theorem 5 also holds if

$$\mathbf{D}^{\alpha}(x(t)-x_0)=f(t,x(t))$$
$$x(0)=x_0.$$

Riemann–Liouville type IVP is considered. But, generally the IVPs are given in the form

$$\mathbf{D}^{\alpha}x(t)=f(t,x(t))$$
$$x(0)=x_0.$$

To apply the given solution technique to these kind of problems, one should set

$$h(t,x(t))=f(t,x(t))-\frac{x_0t^{-\alpha}}{\Gamma(1-\alpha)}$$

and solve the problem

$$D^{\alpha}x(t)=h(t,x(t))$$
$$x(0)=x_0.$$

Most of the fractional differential equations of order α, $0<\alpha<1$, are given in the following form

$$D^{\alpha}(x(t))=f(t,x(t)). \tag{9}$$

In order to use Theorem 5 to solve (9) with the initial condition $x(0)=x_0$, set $h(t,x(t))=f(t,x(t))-\frac{x_0t^{-\alpha}}{\Gamma(1-\alpha)}$ and solve

$$D^{\alpha}(x(t)-x_0)=h(t,x(t)).$$

4. Examples

We now give four examples to illustrate our results. The first three examples are chosen such that the exact solutions can be evaluated analytically to show that the technique given in this paper works properly. The last example is a fractional order epidemic model. For this example the technique is used to evaluate the numerical solution of the system.

Example 9. Consider the fractional order IVP given by

$$\begin{aligned}&D^{\frac{1}{2}}x(t)=t,\\&x(0)=x_0.\end{aligned} \tag{10}$$

For this example,

$$g(v)=2\sqrt{t}\Gamma\left(\frac{3}{2}\right)v-v^2\Gamma^2\left(\frac{3}{2}\right).$$

The solution of the corresponding integer order IVP given in Theorem 5 is

$$x_1(v)=\sqrt{t}\Gamma\left(\frac{3}{2}\right)v^2-\frac{\Gamma^2\left(\frac{3}{2}\right)v^3}{3}+x_0.$$

So, the solution of the given fractional order IVP is

$$x(t)=x_1\left(\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right)=\frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}+x_0. \tag{11}$$

Indeed, it can be shown that (11) is a solution of (10), by using the fractional derivative.

For the following examples, the analytical solutions of the fractional order IVP's could not be evaluated in the recent works. So, the comparison is made with subject to given approximations to the solutions in [12].

Example 10. Consider the linear fractional differential equation given by

$$D^{\frac{1}{2}}x(t)=t+x(t) \tag{12}$$

with initial condition

$$x(0)=x_0.$$

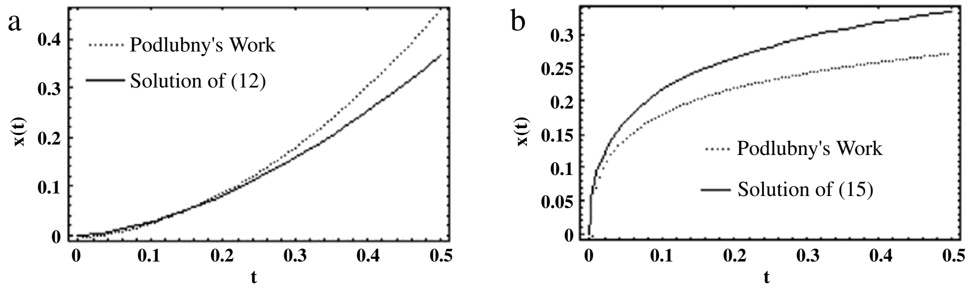

Fig. 1. (a) Solution of IVP given by (12) with initial conditions $x(0) = 0$. (b) Solution of IVP given by (15) with initial conditions $x(0) = 0$ and $u_0 = 1$.

The corresponding differential equation of this fractional IVP is

$$\frac{dx_1(v)}{dv} = f_1(v) = x_1(v) + 2\sqrt{t}\Gamma\left(\frac{3}{2}\right)v - v^2\Gamma^2\left(\frac{3}{2}\right)$$
$$x(0) = x_0.$$

The solution of this integer order linear IVP is

$$x_1(v) = -2\sqrt{t}\Gamma\left(\frac{3}{2}\right)(v+1) + \Gamma^2\left(\frac{3}{2}\right)\left(v^2+2v+2\right) + e^v\left(x_0 + 2\sqrt{t}\Gamma\left(\frac{3}{2}\right) - 2\Gamma^2\left(\frac{3}{2}\right)\right),$$

consequently, the solution of the given fractional order IVP is

$$x(t) = x_1\left(\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right) = -t + \frac{\pi}{2} + e^{2\sqrt{t}/\sqrt{\pi}}\left(x_0 + \sqrt{t\pi} - \frac{\pi}{2}\right). \tag{13}$$

In [12], the numerical solution for an IVP for one of the simplest fractional differential equations that can be used in applied problems [16],

$$D^{\alpha}x(t) + Ax(t) = f(t), \quad (t > 0) \tag{14}$$
$$x^{(k)} = 0, \quad (k = 0, 1, \dots, n-1)$$

is given by

$$x_k = 0, \quad (k = 1, 2, \dots, n-1)$$
$$x_m = -Ah^{\alpha}x_{m-1} - \sum_{j=1}^{m} w_j^{(\alpha)}x_{m-j} + h^{\alpha}f_m, \quad (m = n, n+1, \dots),$$

where

$$t_m = mh, \qquad x_m = x(t_m), \qquad f_m = f(t_m), \quad (m = 0, 1, 2, \dots);$$
$$w_j^{(\alpha)} = (-1)^j\binom{\alpha}{j}, \quad (j = 0, 1, 2, \dots).$$

(12) is a particular case of (14) with $A = -1$ and $f(t) = t$. The result of the computation for $h = 0.005$ and the solution (13) are shown in Fig. 1(a).

Example 11. An example of nonlinear fractional differential equations which is used to solve an initial-boundary value problem describing the process of cooling of a semi-infinite body by radiation is given by

$$D^{\frac{1}{2}}(x(t)) - \alpha(u_0 - x(t))^4 = 0 \tag{15}$$

with initial condition $x(0) = 0$ in [12]. Using Theorem 5, the solution of this problem can be found as

$$x(t) = u_0 - \left(\frac{u_0^3\sqrt{\pi}}{6\sqrt{t} + \sqrt{\pi}}\right)^{1/3}. \tag{16}$$

For $\alpha = 1$, $u_0 = 1$, a numerical solution of (15), using the technique given in [12], for $h = 0.05$ and the solution (16) are shown in Fig. 1(b).

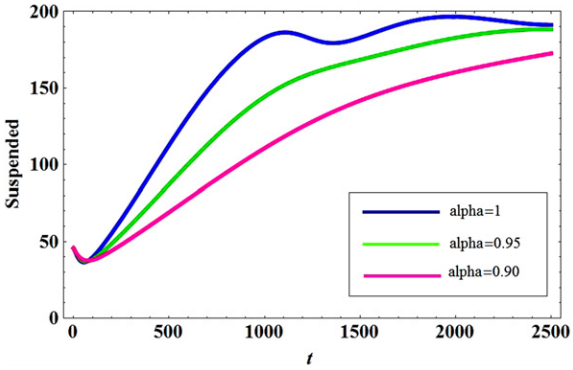

Fig. 2. Numerical solution (S) of (17), (20) with parameter values (19) using the technique given in [13] for different α values.

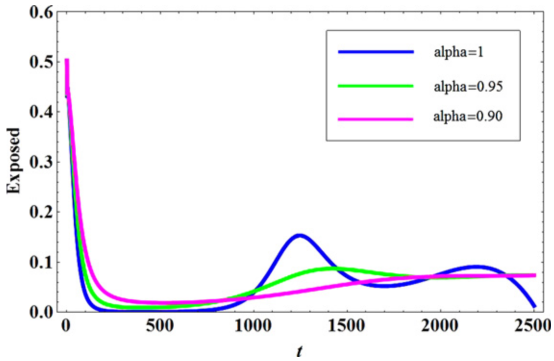

Fig. 3. Numerical solution (E) of (17), (20) with parameter values (19) using the technique given in [13] for different α values.

Examples 9–11 include fractional differential equations that can be solved exactly by the technique given in Theorem 5. In fact, the reason of this is that the corresponding integer order differential equations are exactly solvable equations.

Integer order differential equations have been used in mathematical modeling for long time, but in the recent studies, FDEs are being used as new and strong tools to model real-life phenomena. In order to solve integer order differential equations numerically, various advanced techniques have been constructed for years. However, for FDEs, the numerical techniques are not as strong as them. One of the effective numerical methods, so far, to solve FDEs, is a generalized Adams–Bashford–Moulton algorithm [13].

In the following model problem studied in [17,18], we get the numerical solution of the system by applying the 4th-order Runge–Kutta method to the corresponding integer order system, which is at least ten times faster and effective than the method given in [13].

Example 12. We now consider an epidemic model given by

$$\begin{aligned}
D^{\alpha}S &= bN - pbE - qbI - r\frac{SI}{N} - d(N)S \\
D^{\alpha}E &= pbE + qbI + r\frac{SI}{N} - \beta E - d(N)E \\
D^{\alpha}I &= \beta E - \theta I - \gamma I - d(N)I \\
D^{\alpha}R &= \gamma I - d(N)R
\end{aligned} \tag{17}$$

$$S(0) = S_0, \qquad E(0) = E_0, \qquad I(0) = I_0, \qquad R(0) = R_0 \tag{18}$$

where $0 < \alpha \leq 1$, $N = S + E + I + R$, $(S, E, I, R) \in R_+^4$. Here, $\beta, \gamma > 0$, $\theta \geq 0$ are real constants and d is a continuous and non decreasing function on R^+. A detailed analysis of this model is given in [17]. Numerical solution of this model using the technique given in [13] is also evaluated in [17] (Figs. 2–5) with the parameter values

$$\begin{aligned}
&b = 0.001555, \qquad p = 0.8, \qquad q = 0.95, \qquad r = 0.05, \qquad \beta = 0.05, \\
&\theta = 0.002, \qquad \gamma = 0.003, \qquad d(N) = 0.00001 + 0.000007\,N
\end{aligned} \tag{19}$$

and the initial conditions

$$S(0) = 140, \qquad E(0) = 0.01, \qquad I(0) = 0.02, \qquad N(0) = 141. \tag{20}$$

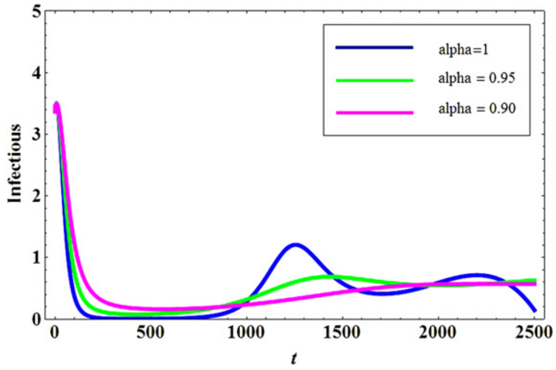

Fig. 4. Numerical solution (I) of (17), (20) with parameter values (19) using the technique given in [13] for different α values.

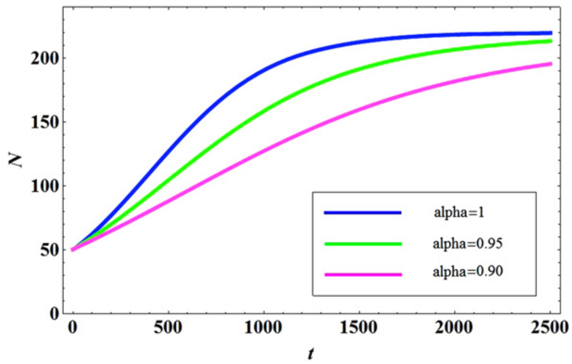

Fig. 5. Numerical solution (N) of (17), (20) with parameter values (19) using the technique given in [13] for different α values.

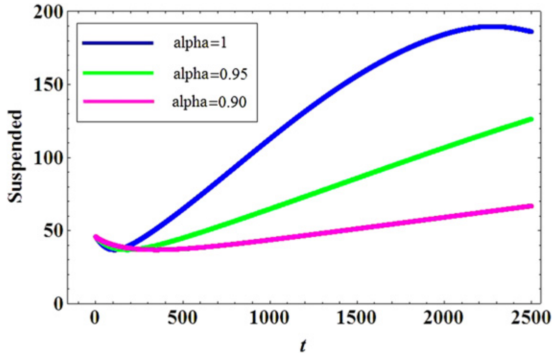

Fig. 6. Numerical solution (S) of (17), (20) with parameter values (19) using the technique given by Theorem 5 for different α values.

The corresponding integer order system given in Theorem 5 is

$$\frac{dS^*}{d\nu} = bN^* - pbE^* - qbI^* - r\frac{S^*I^*}{N^*} - d(N^*)S^*$$
$$\frac{dE^*}{dt} = pbE^* + qbI^* + r\frac{S^*I^*}{N^*} - \beta E^* - d(N^*)E^*$$
$$\frac{dI^*}{dt} = \beta E^* - \theta I^* - \gamma I^* - d(N^*)I^*$$
$$\frac{dR^*}{dt} = \gamma I^* - d(N^*)R^*.$$

If the solution of this integer order system is in $(S^*(\nu), E^*(\nu), I^*(\nu), R^*(\nu))$, the solution of the IVP (17)–(18) is $(S^*(t^\alpha/\Gamma(\alpha+1)), E^*(t^\alpha/\Gamma(\alpha+1)), I^*(t^\alpha/\Gamma(\alpha+1)), R^*(t^\alpha/\Gamma(\alpha+1)))$. For the numerical solution of the IVP (17)–(20) for the parameter values (19) is evaluated using the technique of Theorem 6, see Figs. 6–9 [18].

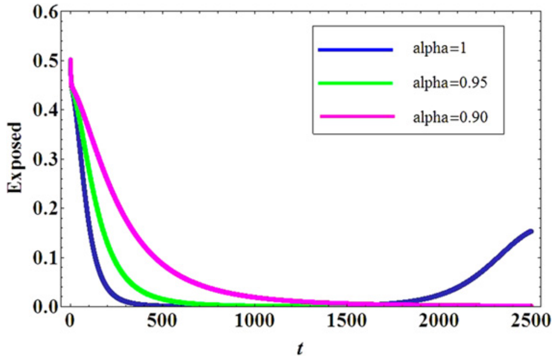

Fig. 7. Numerical solution (E) of (17), (20) with parameter values (19) using the technique given by Theorem 5 for different α values.

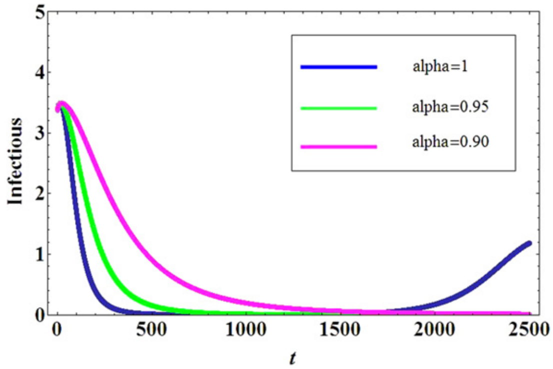

Fig. 8. Numerical solution (I) of (17), (20) with parameter values (19) using the technique given by Theorem 5 for different α values.

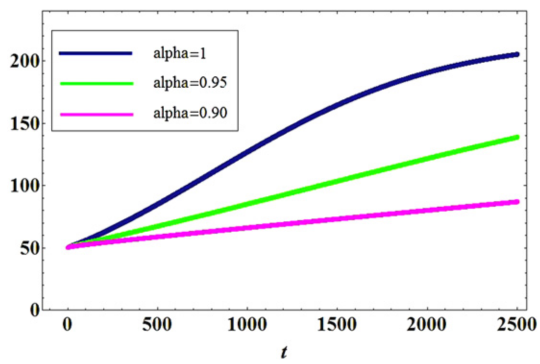

Fig. 9. Numerical solution (N) of (17), (20) with parameter values (19) using the technique given by Theorem 5 for different α values.

5. Conclusions

In this paper, a technique to solve nonlinear Caputo fractional differential equations of order $0 < \alpha < 1$ with initial condition $x(0) = x_0$ is studied. We defined an integer order differential equation using the given FDE and studied the relationship between their solutions. A generalization of the method to finite systems is also given. The main advantage of the technique is being able to write the exact solutions of FDE's in terms of solutions of integer order differential equations. This is not only important in the theory of FDEs but also very valuable in applications of FDEs. Although there are different methods to find numerical solutions of applied problems including FDEs, the methods for integer order equations are stronger in accuracy and rate of convergence. By the technique given in this paper one can use the numerical methods for integer order differential equations for the numerical solutions of FDEs. Some examples are given and the exact solutions found by this technique are compared with the numerical solutions given in [12]. We also give a mathematical model and solve this model numerically using this technique.

Acknowledgments

The authors are very grateful to the referees for their valuable suggestions, which helped to improve the paper significantly.

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