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Harmonic Functions Starlike in the Unit Disk

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Complex-valued harmonic functions that are univalent and sense-preserving in the unit disk Δ can be written in the form $f = h + \bar{g}$, where h and g are analytic in Δ. We give univalence criteria and sufficient coefficient conditions for normalized harmonic functions that are starlike of order α, $0 \le \alpha < 1$. These coefficient conditions are also shown to be necessary when h has negative and g has positive coefficients. These lead to extreme points and distortion bounds. © 1999 Academic Press

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1. INTRODUCTION

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathscr{C}$ if both u and v are real harmonic in $\mathscr{C}$. In any simply connected domain $\mathscr{D} \subset \mathscr{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in $\mathscr{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathscr{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathscr{D}$. See Clunie and Sheil-Small [2].

Denote by $\mathscr{H}$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{z\colon |z| < 1\}$ for which

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$h(0) = f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathscr{H}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

Note that $\mathscr{H}$ reduces to the class of normalized analytic univalent functions if the co-analytic part of its members is zero. In 1984 Clunie and Sheil-Small [2] investigated the class $\mathscr{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathscr{H}$ and its subclasses. For more references see Duren [3]. In this note, we look at two subclasses of $\mathscr{H}$ and provide univalence criteria, coefficient conditions, extreme points, and distortion bounds for functions in these classes.

For $0 \leq \alpha < 1$ we let $\mathscr{S}_{\mathscr{H}}(\alpha)$ denote the subclass of $\mathscr{H}$ consisting of harmonic starlike functions of order α. A function f of the form (1) is harmonic starlike of order α, $0 \leq \alpha < 1$, for $|z| = r < 1$ (e.g., see Sheil-Small [4, p. 244]) if

$$\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right) \geq \alpha, \qquad |z| = r < 1. \tag{2}$$

We further denote by $\mathscr{T}_{\mathscr{H}}(\alpha)$ the subclass of $\mathscr{S}_{\mathscr{H}}(\alpha)$ such that the functions h and g in $f = h + \bar{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n. \tag{3}$$

2. MAIN RESULTS

It was shown by Sheil-Small [4, Theorem 7] that $|a_n| \leq (n+1) \cdot (2n+1)/6$ and $|b_n| \leq (n-1)(2n-1)/6$ if $f = h + \bar{g} \in \mathscr{S}^o_{\mathscr{H}}(0)$. The subclass of $\mathscr{S}_{\mathscr{H}}(\alpha)$ where $\alpha = b_1 = 0$ is denoted by $\mathscr{S}^o_{\mathscr{H}}(0)$. These bounds are sharp and thus give necessary coefficient conditions for the class $\mathscr{S}^o_{\mathscr{H}}(0)$. Avci and Zlotkiewicz [1] proved that the coefficient condition $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1$ is sufficient for functions $f = h + \bar{g}$ to be in $\mathscr{S}^o_{\mathscr{H}}(0)$. Silverman [6] proved that this coefficient condition is also necessary if $b_1 = 0$ and if a_n and b_n in (1) are negative. We note that both results obtained in [1, 6] are subject to the restriction that $b_1 = 0$. The argument presented in this paper provides sufficient coefficient conditions for functions $f = h + \bar{g}$ of the form (1) to be in $\mathscr{S}_{\mathscr{H}}(\alpha)$ where $0 \leq \alpha < 1$ and b_1 is not necessarily zero. It is shown that these conditions are also necessary when $f \in \mathscr{T}_{\mathscr{H}}(\alpha)$.

THEOREM 1. *Let $f = h + \bar{g}$ be given by* (1). *Furthermore, let*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \le 2, \tag{4}$$

where $a_1 = 1$ and $0 \le \alpha < 1$. Then f is harmonic univalent in Δ, and $f \in \mathcal{S}_{\mathcal{H}}(\alpha)$.

Proof. First we note that f is locally univalent and sense-preserving in Δ. This is because

$$\begin{aligned} |h'(z)| &\ge 1 - \sum_{n=2}^{\infty} n|a_n|r^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \ge 1 - \sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n| \\ &\ge \sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}|b_n| \ge \sum_{n=1}^{\infty} n|b_n| > \sum_{n=1}^{\infty} n|b_n|r^{n-1} \ge |g'(z)|. \end{aligned}$$

To show that f is univalent in Δ we notice that if $g(z) \equiv 0$, then $f(z)$ is analytic and the univalence of f follows from its starlikeness (e.g., see [5]). If $g(z) \not\equiv 0$, then we show that $f(z_1) \ne f(z_2)$ when $z_1 \ne z_2$.

Suppose $z_1, z_2 \in \Delta$ so that $z_1 \ne z_2$. Since Δ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in \Delta$, where $0 \le t \le 1$. Then we can write

$$f(z_2) - f(z_1) = \int_0^1 \left[(z_2 - z_1)h'(z(t)) + \overline{(z_2 - z_1)g'(z(t))}\right] dt.$$

Dividing the above equation by $z_2 - z_1 \ne 0$ and taking the real parts we obtain

$$\begin{aligned} \operatorname{Re}\frac{f(z_2) - f(z_1)}{z_2 - z_1} &= \int_0^1 \operatorname{Re}\left[h'(z(t)) + \frac{\overline{z_2 - z_1}}{z_2 - z_1}\overline{g'(z(t))}\right] dt \\ &> \int_0^1 \left[\operatorname{Re} h'(z(t)) - |g'(z(t))|\right] dt. \end{aligned} \tag{5}$$

On the other hand

$$\begin{aligned} \operatorname{Re} h'(z) - |g'(z)| &\ge \operatorname{Re} h'(z) - \sum_{n=1}^{\infty} n|b_n| \\ &\ge 1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n| \\ &\ge 1 - \sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n| - \sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}|b_n| \\ &\ge 0, \quad \text{by (4)}. \end{aligned}$$

This in conjunction with the inequality (5) leads to the univalence of f.

Now we show that $f \in \mathcal{S}_{\mathcal{H}}(\alpha)$. According to the condition (2) we only need to show that if (4) holds then

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \operatorname{Im}\left(\frac{\partial}{\partial\theta}\log f(re^{i\theta})\right) = \operatorname{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(s) + \overline{g(z)}}\right) \geq \alpha,$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $0 \leq r < 1$, and $0 \leq \alpha < 1$.

Using the fact that $\operatorname{Re} w \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0, \tag{6}$$

where $B(z) = h(z) + \overline{g(z)}$ and $A(z) = zh'(z) - \overline{zg'(z)}$.

Substituting for $B(z)$ and $A(z)$ in (6) yields

$$\begin{aligned}
&|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \\
&\quad = \left|(1-\alpha)h(z) + zh'(z) + \overline{(1-\alpha)g(z) - zg'(z)}\right| \\
&\qquad - \left|(1+\alpha)h(z) - zh'(z) + \overline{(1+\alpha)g(z) + zg'(z)}\right| \\
&\quad = \left|(2-\alpha)z + \sum_{n=2}^{\infty}(n+1-\alpha)a_n z^n - \overline{\sum_{n=1}^{\infty}(n-1+\alpha)b_n z^n}\right| \\
&\qquad - \left|-\alpha z + \sum_{n=2}^{\infty}(n-1-\alpha)a_n z^n - \overline{\sum_{n=1}^{\infty}(n+1+\alpha)b_n z^n}\right| \\
&\quad \geq (2-\alpha)|z| - \sum_{n=2}^{\infty}(n+1-\alpha)|a_n|\,|z|^n - \sum_{n=1}^{\infty}(n-1+\alpha)|b_n|\,|z|^n \\
&\qquad - \alpha|z| - \sum_{n=2}^{\infty}(n-1-\alpha)|a_n|\,|z|^n - \sum_{n=2}^{\infty}(n+1+\alpha)|b_n|\,|z|^n \\
&\quad = 2(1-\alpha)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n|\,|z|^{n-1} - \sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}|b_n|\,|z|^{n-1}\right\} \\
&\quad \geq 2(1-\alpha)|z|\left\{1 - \left(\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}|b_n|\right)\right\} \geq 0, \quad \text{by } (4).
\end{aligned}$$

The starlike harmonic mappings

$$f(z) = z + \sum_{n=2}^{\infty}\frac{1-\alpha}{n-\alpha}x_n z^n + \sum_{n=1}^{\infty}\frac{1-\alpha}{n+\alpha}\bar{y}_n \bar{z}^n, \tag{7}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=2}^{\infty}|y_n| = 1$, show that the coefficient bound given by (4) is sharp.

The functions of the form (7) are in $\mathscr{S}_{\mathscr{H}}(\alpha)$ because

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) = 1 + \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 2.$$

The restriction placed in Theorem 1 on the moduli of the coefficients of $f = h + \bar{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic starlike and univalent. Our next theorem establishes that such coefficient bounds cannot be improved.

THEOREM 2. *Let $f = h + \bar{g}$ be given by* (3). *Then $f \in \mathscr{T}_{\mathscr{H}}(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \le 2, \tag{8}$$

where $a_1 = 1$ and $0 \le \alpha < 1$.

Proof. The *if* part follows from Theorem 1 upon noting that if the analytic and co-analytic parts of $f = h + \bar{g} \in \mathscr{S}_{\mathscr{H}}(\alpha)$ are of the form (3) then $f \in \mathscr{T}_{\mathscr{H}}(\alpha)$.

For the *only if* part, we show that $f \notin \mathscr{T}_{\mathscr{H}}(\alpha)$ if the condition (8) does not hold.

Note that a necessary and sufficient condition for $f = h + \bar{g}$ given by (3) to be starlike of order α, $0 \le \alpha < 1$, is that $\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) - \alpha \ge 0$, $0 \le \alpha < 1$. This is equivalent to

$$\begin{aligned}
&\operatorname{Re}\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - \alpha \\
&\quad = \operatorname{Re}\frac{(1-\alpha)z - \sum_{n=2}^{\infty}(n-\alpha)|a_n|z^n - \sum_{n=1}^{\infty}(n+\alpha)|b_n|\bar{z}^n}{z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\bar{z}^n} \\
&\quad \ge 0.
\end{aligned}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$ we must have

$$\frac{(1-\alpha) - \sum_{n=2}^{\infty}(n-\alpha)|a_n|r^{n-1} - \sum_{n=1}^{\infty}(n+\alpha)|b_n|r^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n|r^{n-1} + \sum_{n=1}^{\infty}|b_n|r^{n-1}} \ge 0. \tag{9}$$

If the condition (8) does not hold then the numerator in (9) is negative for r sufficiently close to 1. Thus there exists a $z_o = r_o$ in (0, 1) for which the quotient in (9) is negative. This contradicts the required condition for $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$ and so the proof is complete.

Next we determine the extreme points of the closed convex hulls of $\mathcal{T}_{\mathcal{H}}(\alpha)$, denoted by clco $\mathcal{T}_{\mathcal{H}}(\alpha)$.

THEOREM 3. *$f \in$ clco $\mathcal{T}_{\mathcal{H}}(\alpha)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n), \tag{10}$$

where $h_1(z) = z$, $h_n(z) = z - \frac{1-\alpha}{n-\alpha} z^n$ $(n = 2, 3, \cdots)$, $g_n(z) = z + \frac{1-\alpha}{n+\alpha}\bar{z}^n$ $(n = 1, 2, 3, \cdots)$, $\sum_{n=1}^{\infty}(X_n + Y_n) = 1$, $X_n \geq 0$, and $Y_n \geq 0$. In particular, the extreme points of $\mathcal{T}_{\mathcal{H}}(\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For functions f of the form (10) we have

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) = \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\alpha}{n-\alpha} X_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{n+\alpha} Y_n \bar{z}^n.$$

Then

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}\left(\frac{1-\alpha}{n-\alpha} X_n\right) + \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha}\left(\frac{1-\alpha}{n+\alpha} Y_n\right) = \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1$$

and so $f \in$ clco $\mathcal{T}_{\mathcal{H}}(\alpha)$.

Conversely, suppose that $f \in$ clco $\mathcal{T}_{\mathcal{H}}(\alpha)$. Set $X_n = \frac{n-\alpha}{1-\alpha}|a_n|$ $(n = 2, 3, \cdots)$ and $Y_n = \frac{n+\alpha}{1-\alpha}|b_n|$ $(n = 1, 2, 3, \cdots)$. Then note that by Theorem 2, $0 \leq X_n \leq 1$ $(n = 2, 3, \cdots)$ and $0 \leq Y_n \leq 1$ $(n = 2, 2, 3, \cdots)$. We define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$ and note that, by Theorem 2, $X_1 \geq 0$. Consequently, we obtain $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$ as required.

Using Theorem 2 it is easily seen that $\mathcal{T}_{\mathcal{H}}(\alpha)$ is convex and closed, so clco $\mathcal{T}_{\mathcal{H}}(\alpha) = \mathcal{T}_{\mathcal{H}}(\alpha)$. Then the statement of Theorem 3 is really for $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$.

Finally we give the distortion bounds for functions in $\mathcal{T}_{\mathcal{H}}(\alpha)$, which yield a covering result for $\mathcal{T}_{\mathcal{H}}(\alpha)$.

THEOREM 4. *If $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$ then*

$$|f(z)| \le (1+|b_1|)r + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)r^2, \qquad |z| = r < 1,$$

and

$$|f(z)| \ge (1-|b_1|)r - \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)r^2, \qquad |z| = r < 1.$$

Proof. Let $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$. Taking the absolute value of f we obtain

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \\
&\le (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^2 \\
&= (1+|b_1|)r + \frac{1-\alpha}{2-\alpha}\sum_{n=2}^{\infty}\left(\frac{2-\alpha}{1-\alpha}|a_n| + \frac{2-\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\le (1+|b_1|)r + \frac{1-\alpha}{2-\alpha}\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\le (1+|b_1|)r + \frac{1-\alpha}{2-\alpha}\left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right)r^2, \quad \text{by (8)}, \\
&= (1+|b_1|)r + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)r^2,
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\ge (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \\
&\ge (1-|b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^2 \\
&= (1-|b_1|)r - \frac{1-\alpha}{2-\alpha}\sum_{n=2}^{\infty}\left(\frac{2-\alpha}{1-\alpha}|a_n| + \frac{2-\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\ge (1-|b_1|)r - \frac{1-\alpha}{2-\alpha}\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\ge (1-|b_1|)r + \frac{1-\alpha}{2-\alpha}\left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right)r^2 \quad \text{by (8)}, \\
&= (1-|b_1|)r - \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)r^2.
\end{aligned}$$

The bounds given in Theorem 4 for the functions $f = h + \bar{g}$ of the form (3) also hold for functions of the form (1) if the coefficient condition (4) is satisfied. The functions

$$f(z) = z + |b_1|\bar{z} + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)\bar{z}^2$$

and

$$f(z) = (1 - |b_1|)z - \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)z^2$$

for $|b_1| \leq (1-\alpha)/(1+\alpha)$ show that the bounds given in Theorem 4 are sharp.

The following covering result follows from the left hand inequality in Theorem 4.

COROLLARY. *If* $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$ *then*

$$\left\{w : |w| < \frac{1}{2-\alpha}(1 + (2\alpha - 1)|b_1|\right\} \subset f(\Delta).$$

Remark. For $\alpha = b_1 = 0$ the covering result in the above corollary coincides with that given in [2, Theorem 5.9] for harmonic convex functions.

A function $f \in \mathcal{H}$ is harmonic convex of order α, $0 \leq \alpha < 1$ for $|z| = r < 1$ (see [4] p. 244) if $\frac{\partial}{\partial\theta}(\arg(\frac{\partial}{\partial\theta}f(re^{i\theta})) \geq \alpha$, $|z| = r < 1$.

The corresponding definition for harmonic convex functions of order α leads to analogous coefficient bounds and extreme points.

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