

WEAK CONVERGENCE AND UNIFORM NORMALIZATION IN INFINITARY REWRITING

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ABSTRACT. We study infinitary term rewriting systems containing finitely many rules. For these, we show that if a weakly convergent reduction is not strongly convergent, it contains a term that reduces to itself in one step (but the step itself need not be part of the reduction). Using this result, we prove the starkly surprising result that for any orthogonal system with finitely many rules, the system is weakly normalizing under weak convergence iff it is strongly normalizing under weak convergence iff it is weakly normalizing under strong convergence iff it is strongly normalizing under strong convergence.

As further corollaries, we derive a number of new results for weakly convergent rewriting: Systems with finitely many rules enjoy unique normal forms, and acyclic orthogonal systems are confluent. Our results suggest that it may be possible to recover some of the positive results for strongly convergent rewriting in the setting of weak convergence, if systems with finitely many rules are considered. Finally, we give a number of counterexamples showing failure of most of the results when infinite sets of rules are allowed.

1. Introduction

In term rewriting, weak normalization is the property that every term has a normal form (“there exists at least one reduction to normal form”), whereas strong normalization, also called termination, is the property that every reduction from every term is finite (“all reductions will eventually lead to a normal form”). For some subclasses of term rewriting systems (TRSs), it is known that the property of *uniform normalization* holds: A system is weakly normalizing iff it is strongly normalizing. This property holds, for example, for the class of *orthogonal, non-erasing* systems, that is, every variable occurring in the left-hand side of a rule must also occur on the right-hand side of that rule [18].

In the elegant paper [19], Klop and de Vrijer argue that when lifting the concepts of weak and strong normalization to infinitary rewriting, only weak normalization should be lifted in the obvious way: From every term, there is a, possibly infinite, reduction to normal form. But strong normalization should be treated differently. In the infinitary setting, strong normalization should instead be the property that every well-behaved reduction is *convergent*: Every possible infinite reduction satisfying some very basic integrity constraints

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should have a well-defined limit to which it converges, regardless of whether that limit is a normal form.

Klop and de Vrijer prove the following remarkable result:

Theorem 1.1 (Klop, de Vrijer [19]). *For any orthogonal iTRS R : All strongly continuous reductions can be extended to strongly convergent reductions (“ R is strongly normalizing”) iff every term of R reduces to a normal form by a strongly convergent reduction (“ R is weakly normalizing”).*

Thus, the theorem states that strong and weak normalization *coincide* in the setting of strongly convergent reductions.

Strong convergence means that not only do reductions converge in the complete metric space of (potentially infinite) trees [1, 3] (called *weak* convergence), but the number of rewrite steps occurring at each finite depth is finite along *any* reduction. While weak convergence was the first notion of infinitary rewriting studied [4], it has by now been established that weak convergence in general does not have the desirable properties normally true for syntactically well-behaved (that is, *orthogonal*) rewriting systems [25] while strong convergence does [11, 9].

As a contribution towards showing positive results for weak convergence, we will show that the equivalence in Theorem 1.1 may be extended to hold in the setting of weakly convergent rewriting with the proviso that we only consider systems with a finite number of rules. This should be contrasted with the counterexamples given in [25] that relied crucially on systems with an *infinite* number of rules.

The key to our results is a characterization of the difference between weakly and strongly convergent rewriting: If there exists a weakly convergent reduction $s \rightarrow t_1 \rightarrow t_2 \rightarrow \dots \rightarrow t$ that is not also strongly convergent, there will be some—possibly infinite—term t' occurring in $s \rightarrow t_1 \rightarrow t_2 \rightarrow \dots \rightarrow t$ such that $t' \rightarrow t'$. That is, the rewrite system *admits a cycle of length 1*.

The observation that the existence of cycles of lengths 1 is *the* crucial difference between strong and weak convergence is the focal point of the paper and will furnish a small host of derivative results. The core results of the paper are Theorems 3.16 and 4.1; these contain the main (and somewhat surprising) equivalence results, in particular the latter contains the result that strong and weak normalization in weak and strong convergence *all* coincide for systems with a finite number of rules.

Another interesting new result derived from the above concerns confluence in weakly convergent rewriting: If an orthogonal system with a finite number of rules is acyclic, it will also be confluent. We believe this to be one of the first general and non-trivial confluence results in weakly convergent rewriting.

A remark on concessions to readability

The author has made a concession to readability in this paper, at the expense of further generality: All of the results are proven for (first-order) infinitary term rewriting systems (iTRSs). Bar the explicit counterexamples we give, none of our proofs rely on the first-order nature of these systems. The results of this paper *also* hold true for both infinitary lambda calculus [10] and for infinitary combinatory reduction systems (iCRSs) [14, 15], as the reasoning employed in our proofs essentially only uses the machinery of metric spaces and the abstract notion of depth in the distinction between weak and strong convergence.

The author believes that an abstract account of all our results can be given in the topological setting of Kahrs' work on meta-theory for infinitary rewriting [6], but we feel that a very concrete "syntactical" account such as we give may reach a wider audience.

1.1. Related work

Uniform normalization has been studied in various forms of finitary rewriting, including first-order term rewriting [23, 17], lambda calculus [26, 22] and higher-order rewriting [16].

The original study on weakly convergent rewriting [4] provided a number of results related to normalization. Many of these results concern *top-termination*, a concept equivalent to strong normalization in the setting of strongly convergent reductions, and most of the results thus also hold for the latter, but have been subsumed by later work in that setting. In addition, the definition of "normal form" in [4] concerns infinite terms s with the property that the only possible reduct of s is s . This is in contrast to this and most other papers where infinite normal forms are terms that allow no rewrite steps starting from them.

Klop and de Vrijer were the first to extend the study of uniform normalization to the infinitary setting [19]; they prove Theorem 1.1 and several other interesting results, but do not consider weak convergence.

Ketema extends Theorem 1.1 and several other results of [19] to higher-order infinitary rewriting in the setting of strong convergence [13].

Lucas proves several results related to normalization and confluence in weakly convergent rewriting [21]; all those results concern constructor systems.

Zantema considers strong normalization in a general setting allowing infinite left-hand sides in rules [27]. He characterizes the property of every reduction being strongly convergent using a novel variation of monotone algebras and shows applicability of a simple generalization of termination-by-matrix-analysis from the finitary setting to the setting of strongly convergent infinitary rewriting. It has later been pointed out that Zantema's results hold for reductions of length $\leq \omega$, and that continuous algebras are needed for reductions of greater ordinal length [5]. Weak convergence is not considered in these papers.

Kahrs has taken the first steps in establishing a working theory of general infinitary rewriting that in particular tackles the inherent difficulties of weakly convergent rewriting [6, 7]. He considers modularity of strong normalization in [7], but not the equivalence of strong and weak convergence.

Rodenburg defines infinite reduction sequences over terms with symbols having infinite arities, but his terms are built by induction over such function symbols, hence have only finite depth [24]. It is not hard to see that every infinitary term rewriting system in the sense of the present paper (finite arities, but a liberal notion of convergence) can be simulated by one of Rodenburg's systems if the latter is allowed to have an infinite number of function symbols and rules. Unfortunately, this is the case even if the original system has only a finite number of symbols and rules, the most pertinent restriction in the present paper. Furthermore, even though Rodenburg's reductions correspond to weakly convergent reductions, his main results concern termination and a generalization of Newman's lemma for systems with a complexity measure satisfying certain properties (so-called *weakly* and *strongly descending* systems). These properties do not seem to be satisfied by a sufficiently large class of systems, and we hence feel that a technical exposition involving translation to his systems would not be fruitful in our setting.

Finally, the author of the present paper shows that most of the usual methods for procuring nice-to-have properties of orthogonal systems do not hold in the setting of weak convergence [25]. In particular, confluence of (even non-collapsing) orthogonal systems does not hold when an infinite number of rules is allowed.

2. Preliminaries on infinitary term rewriting

Throughout the paper, we presuppose a working knowledge of ordinals [20]; the least infinite ordinal is denoted by ω , the least uncountable ordinal by Ω . The general theory for infinitary rewriting is laid out for weak convergence in [4] and for strong convergence in [9]. We give the briefest of definitions below to keep the paper self-contained.

Definition 2.1. Assume a denumerably infinite set V of variables and a first-order signature Σ of function symbols. The set of (finite) *terms* over Σ with variable set V , denoted $Ter(\Sigma, V)$, is defined inductively as follows: (i) every $x \in V$ is a term, (ii) if $f \in \Sigma$ is an n -ary function symbol and s_1, \dots, s_n are terms, then $f(s_1, \dots, s_n)$ are terms. A *position* is any finite sequence of positive integers. Given a term s , the set of *positions of s* is the least set of positions such that (i) the empty string ϵ is a position, and (ii) if $s = f(t_1, \dots, t_i, \dots, t_m)$ and p is a position of term t_i , then $i \cdot p$ is a position of s and we say that t_i is *the subterm of s at position $i \cdot p$* , writing $s|_{i \cdot p} = t_i$. The *length* of a position is denoted $|p|$. The *depth* of a finite term s is the length of the longest position in s .

Definition 2.2. The *term metric* is the metric on $Ter(\Sigma, V)$ defined by $d(s, t) = 0$ if $s = t$ and if $s \neq t$ by $d(s, t) = 2^{-k}$ where k is the length of the shortest position at which s and t differ. The set of *finite and infinite terms*, denoted $Ter^\infty(\Sigma, V)$ is the metric completion of $Ter(\Sigma, V)$ with respect to the metric d —that is, the set obtained by augmenting $Ter(\Sigma, V)$ by the set of all limits of Cauchy sequences of elements of $Ter(\Sigma, V)$. An *infinitary term rewriting system* (iTRS) (over $Ter^\infty(\Sigma, V)$) is a set R of pairs (l, r) , written $l \rightarrow r$ where (i) $l \in Ter(\Sigma, R) \setminus V$, (ii) $r \in Ter^\infty(\Sigma, V)$, and (iii) all variables in r occur in l . The pairs $l \rightarrow r$ are called *rules*. A rule is *collapsing* if $r \in V$.

Thus, if $\Sigma = \{f, g\}$ where f and g a binary and unary symbols respectively, then $s = f(g(x), s) = f(g(x), f(g(x), f(g(x), f(g(x), \dots)))$ is an infinite term (it is the limit of the Cauchy sequence $f(g(x), y), f(g(x), f(g(x), y)), f(g(x), f(g(x), f(g(x), y))), \dots$

Observe that left-hand sides of rules are finite, but right-hand sides may be infinite. This is a standard technical convenience that also serves to retain decidability of applicability of a given rewrite rule at a given position in a term¹.

Definition 2.3. A *substitution* is a map $\theta : V \rightarrow Ter^\infty(\Sigma, V)$ (where θ is usually specified only on a finite subset of V). Any substitution can be extended to a map $\theta : Ter^\infty(\Sigma, V) \rightarrow Ter^\infty(\Sigma, V)$ by setting $\theta(f(s_1, \dots, s_n)) = f(\theta(s_1), \dots, \theta(s_n))$ for all n -ary function symbols $f \in \Sigma$. A *one-hole context* is a term over $Ter^\infty(\Sigma, V \cup \square)$ with exactly one occurrence of \square . Let t be any term, and σ be the substitution $\{\square \mapsto t\}$; we write $s = C[t]$ if $s = \sigma(C[\square])$. A *rewrite step* of rule $l \rightarrow r$ is a pair $C[\theta(l)] \rightarrow C[\theta(r)]$ where θ is a substitution. The rewrite step *occurs at position p* (and *at depth $|p|$*) if p is the position of \square in $C[\square]$.

Thus, if $R = \{f(x) \rightarrow g(x, a)\}$, then $f(f(a)) \rightarrow f(g(a, a))$ is a rewrite step at position 1 (and at depth 1).

¹Note that left-linearity is also required for decidability for iTRSs. Both left-linearity and fully-extendedness are required for decidability in iCRSs.

Definition 2.4. An iTRS R is said to be left-linear if, for all of its rules $l \rightarrow r$, every variable x occurs at most once in l . R is said to be *orthogonal* if it is left-linear, and for all pairs of rules $(l_1 \rightarrow r_1, l_2 \rightarrow r_2)$, the following holds: If there is a context $C[\]$ with the hole at position p and substitutions σ, θ such that $\sigma(l_1) = C[\theta(l_2)]$, then either (i) a variable in l_1 is at a prefix position of p , or (ii) $p = \epsilon$ and $l_1 \rightarrow r_1 = l_2 \rightarrow r_2$.

Example 2.5. The iTRSs $R_1 = \{f(x, x) \rightarrow x\}$ and $R_2 = \{f(g(x), a) \rightarrow a, g(a) \rightarrow a\}$ are not orthogonal. The iTRS $R_3 = \{f(g(x), a) \rightarrow a, f(a, x) \rightarrow g(x)\}$ is orthogonal.

When $C[\]$ is a one-hole context, we usually write C^ω for the infinite term $C[C[C[\dots]]]$.

Definition 2.6. Let α be an ordinal. A transfinite reduction with domain $\alpha > 0$ is a sequence of (terms, positions, rules) $(s_\beta, p_\beta, (l \rightarrow r)_\beta)_{\beta < \alpha}$ such that, for each $\beta + 1 < \alpha$ we have $s_\beta \rightarrow s_{\beta+1}$ by contraction of a redex of rule $l \rightarrow r$ at position p_β . The reduction is *open* if α is a limit ordinal and *closed* if α is a successor ordinal. The *length* of an open reduction is α , and the length of a closed reduction $\alpha - 1$.

A transfinite reduction is *weakly* (also known as *Cauchy*) continuous if for every limit ordinal $\gamma < \alpha$, it holds that s_β converges to s_γ in the metric d as β approaches γ from below. The reduction is *weakly convergent* if it is weakly continuous and closed. We write $s \rightarrow_w t$ if there is a weakly convergent reduction $(s_\beta)_{\beta < \alpha' + 1}$ with $s_0 = s$ and $t = s_{\alpha'}$.

For every rewrite step $s_\beta \rightarrow s_{\beta+1}$, let d_β denote the depth of the contracted redex. The reduction is strongly continuous if it is weakly continuous and if, for every limit ordinal $\gamma < \alpha$, the depth d_β tends to infinity as β approaches γ from below. The reduction is strongly convergent if it is strongly continuous and closed. We write $s \rightarrow_s t$ if there is a strongly convergent reduction $(s_\beta)_{\beta < \alpha' + 1}$ with $s = s_0$ and $t = s_{\alpha'}$.

The requirement that the length, α , of a convergent reduction be a successor ordinal is to ensure that the limit term of a continuous reduction is included. Most of the literature on infinitary rewriting has focused on strong convergence as it is the more pliable of the two notions of convergence under technical manipulation, and as the demand that rewrite steps eventually occur deeper and deeper corresponds to computational intuition about manipulation of potentially infinite data structures in finite time.

Definition 2.7. An *extension* of a reduction $S : s_0 \rightarrow s_1 \rightarrow \dots$ of length α is a reduction $T : t_0 \rightarrow t_1 \rightarrow \dots$ of length $\zeta \geq \alpha$ such that for all $0 \leq \beta + 1 < \alpha$, $s_\beta = t_\beta$ and the steps $s_\beta \rightarrow s_{\beta+1}$ and $t_\beta \rightarrow t_{\beta+1}$ are contractions of redexes at identical positions and of identical rules (informally: S is a prefix of T).

Example 2.8. Consider the orthogonal iTRS $\{a(x) \rightarrow a(b(x))\}$. The reduction

$$a(x) \rightarrow a(b(x)) \rightarrow a(b(b(x))) \rightarrow \dots a(b^\omega)$$

is weakly convergent, but not strongly convergent. Observe that there are no strongly convergent reductions from $a(x)$ to $a(b^\omega)$.

The reduction

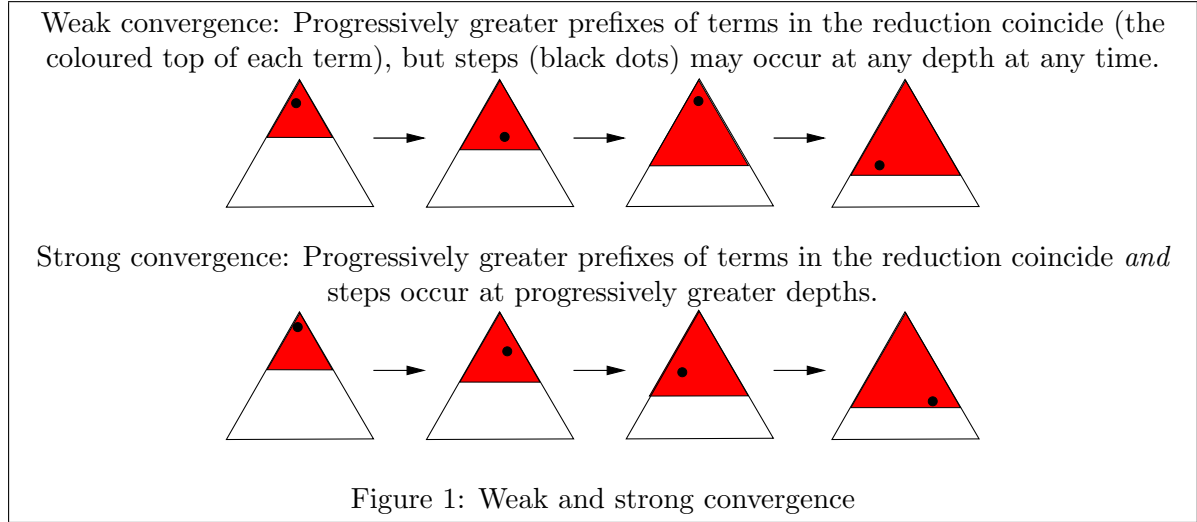
$$a^\omega \rightarrow a(b(a^\omega)) \rightarrow a(b(a(b(a^\omega)))) \rightarrow \dots a(b(a(b(a(b(\dots))))))$$

is strongly convergent, hence also weakly convergent.

Strongly convergent reductions are of countable length:

Proposition 2.9. *Every strongly convergent reduction is of countable length.*

Proof. See for example [11, Lemma 3.5]. ■



2.1. Preliminaries on weak and strong normalization

The concept of normal form is lifted to infinitary rewriting in the obvious way:

Definition 2.10. A *normal form* is a term t such that no rewrite step starts from t .

A term s is *normalizing under weak convergence*, denoted $WN_{\mathbb{W}}^{\infty}(s)$, if there is a normal form t such that $s \rightarrow_{\mathbb{W}} t$. The term s is *normalizing under strong convergence*, denoted $WN^{\infty}(s)$ if there is a normal form t such that $s \rightarrow_{\mathbb{S}} t$.

The iTRS R is *normalizing under weak convergence*, denoted $WN_{\mathbb{W}}^{\infty}(R)$, if every s satisfies $WN_{\mathbb{W}}^{\infty}(s)$, and R is *normalizing under strong convergence*, denoted $WN^{\infty}(R)$, if every term s satisfies $WN^{\infty}(s)$.

It is by now well-established [8, 19, 27, 13] that the proper way to sensibly extend the notion of *strong* normalization (“termination”) to infinitary rewriting is to require that every strongly continuous reduction can be extended to a strongly convergent one (informally: “If the reduction has well-defined prefixes, then it is convergent”). We follow Ketema [13] in the wording of the definition below, extending it to strong normalization under *weak* convergence in the obvious way.

In the setting of *weak* convergence, we might define strong normalization analogously, but this would be a poor choice of nomenclature—it could be that no *normalization* occurs. For example, using the rule $a \rightarrow a$, the weakly continuous reduction $a \rightarrow a \rightarrow a \rightarrow \dots$ of length ω can be extended to a weakly convergent reduction by simply adding a at the end; but a has no normal form. So, certainly, the ability to extend any weakly continuous reduction to a weakly convergent one does *not* imply that a has a normal form. We therefore prefer to use the term *extendable* for this property of weakly convergent reductions.

One could argue that there is need for a generalization of the concept of termination for weakly convergent rewriting; by analogue with termination in finitary rewriting, we have chosen what we believe to be the most obvious generalization: To extend the definition of $EXT_{\mathbb{W}}^{\infty}(R)$ with the demand that there is a “maximal” length of reductions. As we shall later see the existence of such an ordinal is *equivalent* to the assumption that every weakly convergent reduction is also strongly convergent (as always in this paper, when the system has a finite number of rules).

(Finitary) term rewriting	Infinitary rewriting (strong convergence)	Infinitary rewriting (weak convergence)
(Weak) normalization (WN): Every term has a finite reduction to normal form	WN^∞ : Every term has a strongly convergent reduction to normal form	$WN_{\mathbb{W}}^\infty$: Every term has a weakly convergent reduction to normal form
Termination (SN): Every reduction is finite	SN^∞ : Every strongly continuous reduction can be extended to a strongly convergent reduction	$SN_{\mathbb{W}}^\infty$: Every weakly continuous reduction can be extended to a weakly convergent reduction <i>and</i> there is an ordinal upper bound on the length of weakly convergent reductions

Table 1: Corresponding notions for normalization in ordinary and infinitary rewriting

Definition 2.11. Term s is said to be *extendable under weak convergence*, denoted $EXT_{\mathbb{W}}^\infty(s)$ if, for every ordinal α , any open, weakly continuous reduction of length α starting from s can be extended to a weakly convergent reduction. A term s is *strongly normalizing under strong convergence*, denoted $SN^\infty(s)$ if, for every ordinal α , any open, strongly continuous reduction of length α can be extended to a strongly convergent reduction. The iTRS R is said to be extendable under weak convergence, denoted $EXT_{\mathbb{W}}^\infty(R)$, if all of its terms s satisfy $EXT_{\mathbb{W}}^\infty(s)$, and strongly normalizing under strong convergence, denoted $SN^\infty(R)$, if all of its terms s satisfy $SN^\infty(s)$. Finally, R is said to be *strongly normalizing under weak convergence*, denoted $SN_{\mathbb{W}}^\infty(R)$ if $EXT_{\mathbb{W}}^\infty(R)$ and there exists an ordinal α such that every weakly convergent reduction has length $< \alpha$.

Our definition of $SN_{\mathbb{W}}^\infty$ is similar to Rodenburg's [24]: Namely that each term t has an upper bound on the length of convergent reductions starting from t (note that Rodenburg's notion of term and convergence differs from the one in modern infinitary rewriting). For iTRSs, the two notions are equivalent: If $SN_{\mathbb{W}}^\infty(R)$, there is trivially an upper bound on the length of convergent reductions starting from any term. Conversely, let α_t be an upper bound on the length of convergent reductions starting from term t ; as the set of all infinite terms can be indexed by the least uncountable ordinal, $\gamma = \sup_t \alpha_t$ is an ordinal and is an upper bound on the length of all convergent reductions, whence $SN_{\mathbb{W}}^\infty(R)$.

Example 2.12 (Ketema [13]). The iTRS $R_1 = \{a \rightarrow a\}$ satisfies $EXT_{\mathbb{W}}^\infty(R_1)$ as the only possible reductions are those starting $a \rightarrow a \rightarrow \dots$, all of which are weakly convergent.

The iTRS $R_2 = \{f(x) \rightarrow g(f(x))\}$ satisfies both $EXT_{\mathbb{W}}^\infty(R_2)$ and $SN^\infty(R_2)$ as (i) exactly one new redex is created by contraction of a redex and, (ii) for each depth m , every term has only a finite number of redexes at depth at most m and the depth of the new redex created by a contraction is exactly one greater than that of the creating redex, (iii) the rewrite rule strictly increases the depth of its argument x .

The iTRS $R_3 = \{a \rightarrow b, b \rightarrow a\}$ satisfies neither $EXT_{\mathbb{W}}^\infty(R_3)$, nor $SN^\infty(R_3)$, as $a \rightarrow b \rightarrow a \rightarrow b \rightarrow \dots$, and the only convergent reductions starting from a (or b) are finite.

We end this section by noting that in systems with a finite number of rules, normalization of individual terms does not depend on the notion of convergence used:

Proposition 2.13. *Let R be an orthogonal iTRS consisting of a finite number of rules and s be a term. If $s \rightarrow_w t$ where t is a normal form, then $s \rightarrow_s t$.*

Proof. See e.g. [11, Thm. 9.1]. ■

3. Weak, but not strong, convergence entails existence of a cycle of length 1

This section is devoted to proving that every weakly convergent reduction $s \rightarrow_{\mathbb{W}} t$ that is not strongly convergent can be written as $s \rightarrow_{\mathbb{W}} t' \rightarrow_{\mathbb{W}} t$ where t' is a term that reduces to itself in one step: $t' \rightarrow t'$; i.e., there is a *cycle of length 1*.

Before outlining the proof idea, we introduce the concept of a *cofinal map*².

Definition 3.1. If $g : \beta \rightarrow \alpha$, then g is *cofinal* if, for all $\gamma < \alpha$, there exists $\zeta < \beta$ such that $\gamma < g(\zeta)$.

Definition 3.2. Let $S : s \rightarrow_{\mathbb{W}} s_{\alpha}$ be a weakly convergent reduction of length α , let $\alpha' \leq \alpha$, and let $g : \beta \rightarrow \alpha'$ be strictly monotonic. The *g -pick* of S is a sequence $(s_{g(\gamma)})_{\gamma \in \beta}$ where each $s_{g(\gamma)}$ occurs in S . We say that the g -pick $(s_{g(\gamma)})_{\gamma \in \beta}$ is *induced* by g . The g -pick is said to be *cofinal* if g is cofinal. We shall occasionally suppress the g and speak merely of a *pick*.

A pick is not a reduction, nor does it necessarily induce a reduction in any meaningful sense. Picks are simply a way of “picking out” terms from the reduction S . That the function g is cofinal means that the terms in the pick occur “densely” before the term $s_{\alpha'}$. For example, if $\alpha = \omega^2$ and $\alpha' = \omega$, the pick induced by $g : \omega \rightarrow \omega$ where $g(\gamma) = \gamma \cdot 2 + 1$ is cofinal.

The idea of the proof of the splitting $s \rightarrow_{\mathbb{W}} t' \rightarrow_{\mathbb{W}} t$ is quite simple: There is a position p such that only a finite number of steps occur above p in $s \rightarrow_{\mathbb{W}} t$ and such that an infinite number of steps occur at p ; due to the fact that only a finite number of rules are present, one rule, $l \rightarrow r$, must be used an infinite number of times at p . By taking the least limit ordinal α' such that an infinite number of such steps occur before α' , we employ the fact that the prefix, $s \rightarrow_{\mathbb{W}} t'$, of $s \rightarrow_{\mathbb{W}} t$ of length α' is weakly continuous to show that the sequence of subterms at position p converges at α' , and that the subterm $t'|_p$ satisfies $t'|_p = \theta(l) = \theta(r)$ for some substitution θ . The result then follows immediately, as $t' = t'[\theta(l)]_p \rightarrow t'[\theta(r)]_p = t'[\theta(l)]_p = t'$.

Proposition 3.3. Let R be an *iTRS* consisting of a finite number of rules, and let $S : s \rightarrow_{\mathbb{W}} t$ be a weakly convergent reduction of length α that is not strongly convergent. Then there is a rule $l \rightarrow r \in R$, a limit ordinal $\alpha' \leq \alpha$, an ordinal $\gamma < \alpha'$, a position p and an infinite pick $(s_{g(\zeta)})_{\zeta < \beta}$ induced by a cofinal $g : \beta \rightarrow \alpha'$ such that:

- For all $\gamma < \delta < \alpha'$, no step $s_{\delta} \rightarrow s_{\delta+1}$ occurs at a position $q < p$ in S .
- For every $\zeta < \beta$, the rewrite step $s_{g(\zeta)} \rightarrow s_{g(\zeta)+1}$ occurs at position p and employs rule $l \rightarrow r$.

Proof. As any finite reduction is strongly convergent, we have $\alpha \geq \omega$. By standard results (see for example [9, Ex. 12.3.6]), there is a position of minimal depth m in S such that the number of redex contractions at depth m is infinite, as S would otherwise be strongly convergent. Observe that as m is minimal among such depths, there is an ordinal $\gamma < \alpha$ such that for any $\gamma \leq \delta < \alpha$, the rewrite step $s_{\delta} \rightarrow s_{\delta+1}$ does not occur at a position of length $< m$.

Let $\alpha' \leq \alpha$ be the least limit ordinal such that an infinite number of contractions at depth m occur in the prefix of $s \rightarrow_{\mathbb{W}} t$ of length α' . By weak convergence of S , this prefix converges to some term t' .

²Nomenclature taken from [7]. Cofinality is a standard notion in set theory, see e.g. [20, Ch. 1]; a related concept from [25] is that of an α' -frequent property.

Claim: There is an infinite, cofinal pick induced by a $g : \beta \rightarrow \alpha'$ such that the redex employed in the step $s_{g(\zeta)} \rightarrow s_{g(\zeta)+1}$ occurs at depth m for every $\zeta < \beta$.

The claim follows by contradiction: If the claim did not hold, there would be a limit ordinal $\alpha'' < \alpha'$ with an infinite number of contractions at a depth $\leq m$, contradicting the above observations.

As t' has only a finite number of positions at depth m , there must be a position p with $|p| = m$ and an infinite cofinal pick induced by $g' : \beta' \rightarrow \alpha'$ in which every rewrite step occurs at p .

As R consists of a finite number of rules and the pick induced by g' is infinite, the pigeon-hole principle yields that there is a rule $l \rightarrow r$ and an infinite pick induced by $g'' : \beta'' \rightarrow \alpha'$ with $\beta'' \leq \beta'$ in which every rewrite step occurs at p and is a contraction of a redex of rule $l \rightarrow r$. As α' was the minimal limit ordinal with an infinite number of steps at depth $m = |p|$, the pick induced by g must be cofinal, as desired. ■

Example 3.4. To illustrate the proof of Proposition 3.3, we give a short example.

Let R be the (non-orthogonal) iTRS consisting of the rules $\{f(x) \rightarrow f(g(x)), g(x) \rightarrow g(f(x))\}$, and consider the following reduction S where we have underlined the root symbol of the contracted redex in each step:

$$\begin{aligned} \underline{g}(x) &\rightarrow g(\underline{f}(x)) \\ &\rightarrow g(\underline{f}(g(x))) \\ &\rightarrow g(\underline{f}(g(f(x)))) \\ &\rightarrow g(\underline{f}(g(g(f(x))))) \\ &\rightarrow g(\underline{f}(g(g(g(f(x))))) \\ &\rightarrow \dots \\ \dots &g(f(g^\omega)) \end{aligned}$$

where we assume that ω steps are performed. The above reduction is weakly convergent, but not strongly so. Writing the reduction as $g(x) = s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \dots s_\omega = g(f(g^\omega))$, the step $s_0 \rightarrow s_1$ occurs at position ϵ ; for $k \geq 1$, the step $s_{2k} \rightarrow s_{2k+1}$ occurs at position 1, and the step $s_{2k+1} \rightarrow s_{2k+2}$ occurs at position $1 \cdot 1^k$. Define $g : \omega \rightarrow \omega$ by $g(k) = 2k$. Then $(s_{2k})_{k \in \omega}$ is a cofinal pick induced by g where each step occurs at position $p = 1$, and each step is of the rule $f(x) \rightarrow f(g(x))$; note that there are only a finite number of steps occurring above p in S .

We shall also need the standard concept of a unifier:

Definition 3.5. Let s and t be terms. A *unifier* of s and t is a substitution θ such that $\theta(s) = \theta(t)$.

We refer to [2] for details on unification of infinite terms.

Proposition 3.6. If $l \rightarrow r$ is a rule and θ is a unifier of l and r , then $\theta(l) \rightarrow \theta(r)$

Proof. By definition of the rewrite relation, we have $\theta(l) \rightarrow \theta(r)$. As θ is a unifier of l and r , we have $\theta(r) = \theta(l)$, and the result follows. ■

Lemma 3.7. Let R be an iTRS with a finite number of rules. With notation as in Proposition 3.3, let $s = s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \dots s_\xi \rightarrow s_{\xi+1} \rightarrow \dots s_{\alpha'} = t'$ be the closed prefix of $s \rightarrow_w t$ of length α' . Then there is a rule $l \rightarrow r \in R$ and a (countable!) sequence $(\sigma_\xi)_{\xi < \omega}$ of substitutions such that:

- For every x occurring in l , the sequence $(\sigma_\xi(x))_{\xi < \omega}$ converges in the tree metric to some term t_x .
- The substitution σ_ω defined by $\sigma_\omega(x) = t_x$ is a unifier of l and r .
- $t'|_p = \sigma_\omega(l) = \sigma_\omega(r)$.

Proof. As $s \rightarrow_W t$ is weakly convergent, it converges at α' to t' . As the g -pick $(s_{g(\gamma)})_{\gamma \in \beta}$ is cofinal and $g : \beta \rightarrow \alpha'$, the sequence $(s_{g(\gamma)})_{\gamma < \beta}$ converges to t' . For each $\gamma < \beta$, we have $s_{g(\gamma)}|_p = \sigma_\gamma(l)$ for some substitution σ_γ .

By weak convergence, there is thus for each natural number m , a substitution σ_m such that $d(\sigma_m(l), \sigma_m(r)) < 2^{-m}$ and $d(\sigma_m(l), t'|_p) < 2^{-m}$.

Let p_i be the position of any occurrence of variable x_i in l . By the above observations, $k \geq m$ implies that $d(\sigma_m(l)|_{p_i}, \sigma_k(l)|_{p_i}) < 2^{-m+p_i}$. But $\sigma_k(l)|_{p_i} = \sigma_k(x_i)$, whence the sequence $(\sigma_k(x_i))_{k < \omega}$ converges in the tree metric to some term t_i . To see that σ_ω is a unifier of l and r , consider $d(\sigma_\omega(l), \sigma_\omega(r))$. For each natural number m we then have:

$$\begin{aligned} d(\sigma_\omega(l), \sigma_\omega(r)) &\leq d(\sigma_\omega(l), \sigma_{m+2}(l)) + d(\sigma_{m+2}(l), \sigma_\omega(r)) \\ &\leq 2^{-(m+2)} + d(\sigma_{m+2}(l), \sigma_{m+2}(r)) + d(\sigma_{m+2}(r), \sigma_\omega(r)) \\ &\leq 2^{-(m+2)} + 2^{-(m+2)} + 2^{-(m+2)} \\ &< 2^{-m} \end{aligned}$$

As m was arbitrary, we obtain $\sigma_\omega(l) = \sigma_\omega(r)$, and as the sequence $(\sigma_k(l))_{k < \omega}$ converges to $t'|_p$ in the tree metric, we have $t'|_p = \sigma_\omega(l) = \sigma_\omega(r)$. ■

We have come to the main ancillary result of the paper:

Theorem 3.8. *Let R consist of a finite number of rules. If there exists a weakly convergent reduction $s \rightarrow_w t$ that is not strongly convergent, then $s \rightarrow_w t$ may be written as $s \rightarrow_w t' \rightarrow_w t$ where t' is a term with $t' \rightarrow t'$.*

Proof. Let notation be as in Proposition 3.3 and write $s \rightarrow_w t$ as $s \rightarrow_W t' \rightarrow_W t$ where by Lemma 3.7 we have $t'|_p = \sigma_\omega(l) = \sigma_\omega(r)$ for some substitution σ_ω . By Proposition 3.6, we then have $t'|_p \rightarrow t'|_p$ and thus $t' \rightarrow t'$. ■

Theorem 3.8 fails in the presence of an infinite number of rules:

Example 3.9. Let R be the orthogonal iTRS with infinite rule set $\{g^n(c) \rightarrow g^{n+1}(c) : n \geq 1\}$. Then there is a weakly convergent reduction $g(c) \rightarrow g(g(c)) \rightarrow \dots g^\omega$ where the redex is contracted at the root in each step, whence the reduction is not strongly convergent. Observe that no term on the form $g^n(c)$ is cyclic (in fact, R is acyclic), and g^ω is a normal form; hence, the assumption of finiteness of the set of rules in Theorem 3.8 cannot be omitted.

Remark 3.10. An anonymous referee has kindly directed the author's attention to [12] where Lemma 4.3.2 states (using the terminology of the present paper) that for an orthogonal iTRS, there does *not* exist a rule $l \rightarrow r$ where l unifies with r iff all weakly convergent reductions are strongly convergent iff all weakly convergent reductions are top-terminating. Whereas Lemma 4.3.2 is formulated for arbitrary iTRSs, the authors of [12] almost certainly meant for their iTRSs to have a finite number of rules: Example 3.9 exhibits an orthogonal iTRS with an infinite number of rules such that for no rule $l \rightarrow r$ does l unify with r ; this iTRS contains a weakly convergent reduction that is not strongly convergent.

We now give a number of corollaries showing the usefulness of Theorem 3.8.

Corollary 3.11. *Let R consist of a finite number of rules. If R does not admit a cycle of length 1, then every weakly convergent reduction is strongly convergent.*

Corollary 3.12. *Let R consist of a finite number of rules. Then there is a weakly convergent reduction of uncountable length iff R admits a cycle of length 1.*

Proof. If R admits a cycle of length 1, there are weakly convergent reductions of any ordinal length. If R does not admit a cycle of length 1, then every weakly convergent reduction is strongly convergent, hence of countable length by Proposition 2.9. ■

Recall that left-linear systems enjoy the *compression property* in strongly convergent rewriting: If $s \rightarrow_S t$, then there is a strongly convergent reduction from s to t of length $\leq \omega$. As a curiosity, we mention the following result showing that failure of compression to length at most ω entails existence of a cycle of length 1:

Corollary 3.13. *Let R be a left-linear i TRS with a finite number of rules. If there is a reduction $s \rightarrow_W t$ such that there is no reduction from s to t of length at most ω , then $s \rightarrow_W t' \rightarrow_W t$ where $t' \rightarrow t'$.*

Proof. $s \rightarrow_W t$ cannot be strongly convergent (as $s \rightarrow_S t$ would entail a strongly convergent reduction of length at most ω from s to t by compression). The result then follows from Theorem 3.8. ■

We can also reason about the possible lengths of weakly convergent reductions depending on whether they admit cycles of length 1:

Corollary 3.14. *Let R be an i TRS having a finite number of rules. Then for any ordinal $\alpha \geq \Omega$, there exists a weakly convergent reduction of length α iff R admits a cycle of length 1.*

Proof. If there is a weakly convergent reduction of uncountable length, it cannot be strongly convergent by Proposition 2.9, and Theorem 3.8 proves existence of a cycle of length 1. Conversely, if R admits a cycle of length 1, there are weakly convergent reductions of any ordinal length. ■

Corollary 3.15. *Let R be an i TRS having a finite number of rules. Then R admits a cycle of length 1 iff there is no ordinal α such that all weakly convergent reductions are of length $< \alpha$.*

Proof. If R admits a cycle of length 1, there are weakly convergent reductions of any ordinal length. If R does not admit a cycle of length 1, then by Theorem 3.8, every weakly convergent reduction is strongly convergent, whence Proposition 2.9 yields that every weakly convergent reduction has length $< \Omega$, concluding the proof. ■

We summarize the results of this section in the following theorem:

Theorem 3.16. *Let R be an i TRS consisting of a finite number of rules. The following are equivalent:*

- (1) *Every weakly convergent reduction is strongly convergent.*
- (2) *There does not exist a term t with $t \rightarrow t$.*
- (3) *Every weakly convergent reduction has countable length.*
- (4) *There exists an ordinal α such that all weakly convergent reductions are of length $< \alpha$.*

Proof. $1 \Rightarrow 2$ follows as existence of a term $t \rightarrow t$ yields the weakly, but not strongly, convergent reduction $t \rightarrow t \rightarrow t \rightarrow \dots t$. $2 \Rightarrow 1$ is Corollary 3.11. $2 \Leftrightarrow 4$ is Corollary 3.14, and $2 \Leftrightarrow 3$ is Corollary 3.15. ■

4. Uniform convergence and uniform normalization

We now show uniform normalization under both weak and strong convergence for orthogonal iTRSs with a finite number of rules; this is the main result of the paper.

Theorem 4.1. *The following are equivalent for an orthogonal iTRS R with a finite number of rules:*

- (1) $EXT_w^\infty(R)$ and R admits no cycle of length 1
- (2) $SN_w^\infty(R)$
- (3) $WN_w^\infty(R)$
- (4) $WN^\infty(R)$
- (5) $SN^\infty(R)$

Proof. We prove $(2) \Leftrightarrow (1) \Leftrightarrow (5) \Leftrightarrow (4) \Leftrightarrow (3)$.

$(2) \Leftrightarrow (1)$ follows by Theorem 3.16. $(1) \Rightarrow (5)$ follows from Corollary 3.11. For $(5) \Rightarrow (1)$ reason as follows: $SN^\infty(R) \Rightarrow EXT_w^\infty(R)$ follows by noting that $SN^\infty(R)$ implies that all maximal reductions end in a normal form [19], and that there are no divergent reductions, hence that all weakly continuous reductions are strongly continuous. If R admitted a cycle of length 1, say $s \rightarrow s$, then the reduction $s \rightarrow s \rightarrow s \rightarrow \dots$ of length ω constructed by iterating the step performed in $s \rightarrow s$ is strongly continuous, but not strongly convergent, contradicting $SN^\infty(R)$. $(4) \Leftrightarrow (5)$ is the content of Theorem 1.1. $(3) \Rightarrow (4)$ follows from Proposition 2.13. $(4) \Rightarrow (3)$ follows by observing that if every term s reduces to a normal form by a strongly convergent reduction, that reduction is also weakly convergent. ■

Note that Example 3.9 shows that the assumption of finiteness of the rule set cannot be omitted in the implications $((1) \vee (2) \vee (3)) \Rightarrow (4)$, respectively $((1) \vee (2) \vee (3)) \Rightarrow (5)$.

An immediate consequence of Theorem 4.1 is the following confluence result in weakly convergent rewriting:

Corollary 4.2. *Let R be an orthogonal iTRS with a finite number of rules such that at least one of the following holds*

- (1) $WN_w^\infty(R)$
- (2) R admits no cycle of length 1

Then $CR_w^\infty(R)$.

Proof. If $WN_w^\infty(R)$, Theorem 4.1 implies that R admits no cycle of length 1. Thus, it suffices to prove that $CR_w^\infty(R)$ if R admits no cycle of length 1. By Corollary 3.11, every weakly convergent reduction is also strongly convergent. As R admits no cycle of length 1, R cannot contain a collapsing rule $l \rightarrow x$ as it would give rise to a cycle of length 1: Letting the term s be the fixed point of $s = l\{x \mapsto s\}$, we obtain $s \rightarrow s$. The result now follows as orthogonal, non-collapsing iTRSs R satisfy $CR^\infty(R)$ [11]. ■

As pointed out by a referee, part (1) of the above corollary can also be proved directly from existing results: If $WN_{\mathbb{W}}^{\infty}(R)$ for an orthogonal iTRS R , then by [19], we have $WN^{\infty}(R)$ and hence $SN^{\infty}(R)$, and as orthogonal iTRSs have unique normal forms under strong convergence, we obtain $CR_{\mathbb{W}}^{\infty}(R)$.

5. Conclusion and conjectures

We have used acyclicity to study the differences between weak and strong convergence in iTRSs R with a finite number of rules. In particular, we established a necessary and sufficient criterion for every weakly convergent reduction to also be strongly convergent: That R admits no cycle of length 1. This criterion was employed to show equivalence of (the infinitary analogues of) weak and strong normalization in both weakly and strongly convergent rewriting. As a further consequence, we have derived new results concerning normalization and confluence in the setting of weakly convergent rewriting.

The results of the paper strongly suggest the following conjecture:

Conjecture 5.1. Let R be an orthogonal iTRSs such that (i) R has a finite set of rules, and (ii) R is almost non-collapsing (that is, R has at most one collapsing rule $l \rightarrow x$ and the only variable occurring in l is x). Then, $CR_{\mathbb{W}}^{\infty}(R)$.

If (ii) does not hold, then $CR_{\mathbb{W}}^{\infty}(R)/\sim_h$ where \sim_h is equivalence modulo identification of hypercollapsing subterms (see for example [11, 9] for definitions).

The assumption of finiteness of the rule set is crucial in the above conjecture, as witnessed by the counterexamples of [25].

A more modest conjecture that could perhaps be proved more easily is:

Conjecture 5.2. Let R be a non-collapsing iTRS having a finite set of rules. Then $CR_{\mathbb{W}}^{\infty}(R)/\sim_c$ where \sim_c is equivalence modulo identification of *cyclic* subterms.

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