Understanding the Correlation Gap for Matchings

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Abstract

Given a set of vertices V with $|V| = n$, a weight vector $w \in (\mathbb{R}^+ \cup \{0\})^{\binom{V}{2}}$, and a probability vector $x \in [0,1]^{\binom{V}{2}}$ in the matching polytope, we study the quantity

$$\frac{\mathbb{E}_G[\nu_w(G)]}{\sum_{(u,v)\in\binom{V}{2}} w_{u,v}x_{u,v}}$$

where G is a random graph where each edge e with weight w_e appears with probability x_e independently, and let $\nu_w(G)$ denotes the weight of the maximum matching of G. This quantity is closely related to correlation gap and contention resolution schemes, which are important tools in the design of approximation algorithms, algorithmic game theory, and stochastic optimization.

We provide lower bounds for the above quantity for general and bipartite graphs, and for weighted and unweighted settings. The best known upper bound is 0.54 by Karp and Sipser, and the best lower bound is 0.4 for bipartite graphs and 0.33 for general graphs. We show that it is at least 0.47 for unweighted bipartite graphs, at least 0.45 for weighted bipartite graphs, and at least 0.43 for weighted general graphs. To achieve our results, we construct local distribution schemes on the dual which may be of independent interest.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems, G2.2 Graph Theory

Keywords and phrases Matchings, Randomized Algorithms, Contention Resolution Schemes

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2017.32

1 Introduction

We study the size (weight) of the maximum matching of a random graph sampled from various random graph models. Let V be the set of vertices with $|V| = n$. Given the probability vector $x \in [0,1]^{\binom{V}{2}}$ and the weight vector $w \in (\mathbb{R}^+ \cup \{0\})^{\binom{V}{2}}$, let $\mathcal{D}^{\mathsf{G}}_{n,w,x}$ be the distribution of random graphs with n vertices such that each pair $e \in \binom{V}{2}$ becomes an edge with probability x_e independently. If it becomes an edge, its weight is w_e. For bipartite graphs, let V_1 and V_2 be the set of left and right vertices with $|V_1| = |V_2| = n$. Given the probability vector $x \in [0,1]^{V_1 \times V_2}$ and the weight vector $w \in (\mathbb{R}^+ \cup \{0\})^{V_1 \times V_2}$, let $\mathcal{D}^{\mathsf{B}}_{n,w,x}$ be the distribution of random bipartite graphs with $2n$ vertices such that each pair $e \in V_1 \times V_2$ becomes an edge with probability x_e independently. If it becomes an edge, its weight is w_e. We use $\mathcal{D}^{\mathsf{B}}_{n,x}$ (resp. $\mathcal{D}^{\mathsf{G}}_{n,x}$) for the unweighted case ($w = (1, 1, \ldots, 1)$).

* Supported in part by Anupam Gupta's NSF awards CCF-1319811 and CCF-1536002.
† Supported by a Samsung Fellowship and Venkat Guruswami's NSF CCF-1526092.

37th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2017).
Editors: Satya Lokam and R. Ramanujam; Article No. 32; pp. 32:1–32:15
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

We focus on the case when the probability vector x is in the matching polytope of the complete (bipartite) graph. Recall that for bipartite graphs, $x \in [0,1]^{V_1 \times V_2}$ is in the matching polytope if each $v \in V_1 \cup V_2$ satisfies $\sum_u x_{u,v} \leqslant 1$. For general graphs, $x \in [0,1]^{\binom{V}{2}}$ is in the matching polytope if each $v \in V$ satisfies $\sum_u x_{u,v} \leqslant 1$ and each odd set $S \subseteq V$ satisfies $\sum_{\{u,v\} \subseteq S} x_{u,v} \leqslant \lfloor (|S|-1)/2 \rfloor$.[1]

Given a weighted graph G, let $\nu_w(G)$ be the weight of the maximum weight matching of G. If G is unweighted, $\nu(G)$ denotes the cardinality of the maximum matching of G. For any $x \in [0,1]^{\binom{V}{2}}$ and $w \in (\mathbb{R}^+ \cup \{0\})^{\binom{V}{2}}$, we have $\mathbb{E}_{G \sim \mathcal{D}^{\mathsf{G}}_{n,w,x}}[\nu_w(G)] \leqslant \sum_{(u,v) \in \binom{V}{2}} w_{u,v} x_{u,v}$, simply because the probability that (u,v) is included in the maximum matching is at most $x_{u,v}$. The analogous statement also holds for bipartite graphs.

If x is in the matching polytope[2], we can prove that $\mathbb{E}_G[\nu_w(G)] \geqslant \kappa \cdot \sum w_{u,v} x_{u,v}$ for some constant $0 < \kappa < 1$. For the general graph model, κ is known to be at least $(1-1/e)^2 \sim 0.40$ for every w [6]. For the bipartite graph model, κ is known to be at least 0.4 for every w [5]. Karp and Sipser [11] showed an upper bound of 0.54 for both bipartite and general graphs, by demonstrating it for the unweighted models where every edge appears with equal probability. Our main results are the following improved lower bounds on κ. Our first theorem concerns the unweighted bipartite model.

▶ **Theorem 1.1.** *Let $|V_1| = |V_2| = n$ and $x \in [0,1]^{V_1 \times V_2}$ be in the matching polytope of the complete bipartite graph on $V_1 \cup V_2$. Then*

$$\frac{\mathbb{E}_{G \sim \mathcal{D}^{\mathsf{B}}_{n,x}}[\nu(G)]}{\sum_{(u,v) \in V_1 \times V_2} x_{u,v}} \geqslant 0.476. \tag{1}$$

We also obtain a slightly weaker result on the weighted bipartite model.

▶ **Theorem 1.2.** *Let $|V_1| = |V_2| = n$ and $x \in [0,1]^{V_1 \times V_2}$ be in the matching polytope of the complete bipartite graph on $V_1 \cup V_2$. Then for any $w \in (\mathbb{R}^+ \cup \{0\})^{V_1 \times V_2}$,*

$$\frac{\mathbb{E}_{G \sim \mathcal{D}^{\mathsf{B}}_{n,w,x}}[\nu_w(G)]}{\sum_{(u,v) \in V_1 \times V_2} w_{u,v} x_{u,v}} \geqslant \left(1 - \frac{3}{2e}\right) \geqslant 0.4481.$$

Finally, we prove an improved bound on the weighted general graph model.

▶ **Theorem 1.3.** *Let $|V| = n$ and $x \in [0,1]^{\binom{V}{2}}$ be in the matching polytope of the complete graph on $V_1 \cup V_2$. Then for any $w \in (\mathbb{R}^+ \cup \{0\})^{\binom{V}{2}}$,*

$$\frac{\mathbb{E}_{G \sim \mathcal{D}^{\mathsf{G}}_{n,w,x}}[\nu_w(G)]}{\sum_{(u,v) \in \binom{V}{2}} w_{u,v} x_{u,v}} \geqslant \frac{e^2-1}{2e^2} \geqslant 0.4323.$$

1.1 Applications and Related Work

Contention Resolution Schemes and Correlation Gap

Our work is inspired by and related to the rounding algorithms studied in approximation algorithms. Given a downward-closed family $\mathcal{I} \subseteq 2^E$ defined over a ground-set E and a

[1] Our result for general graphs, Theorem 1.3 holds even when x satisfies the first type of constraints.
[2] If x is not in the matching polytope, one can construct examples where $\kappa = \Omega(n)$.

submodular function $f : 2^E \to \mathbb{R}^+$, Chekuri et al. [5] considered the problem of finding $\max_{S \in \mathcal{I}} f(S)$ and introduced *contention resolution schemes* (CR schemes) to obtain improved approximation algorithms for numerous problems. Let $P_{\mathcal{I}}$ be the convex combination of all incidence vectors $\{1_S\}_{S \in \mathcal{I}}$. A c-CR scheme π for $x \in P_{\mathcal{I}}$ is a procedure that, when R is a random subset of E with $e \in R$ independently with probability x_e, returns $\pi(R) \subseteq R$ such that $\pi(R) \in \mathcal{I}$ with probability 1 and $\Pr[e \in \pi(R)] \geqslant c$ for all $e \in E$.

To construct a CR scheme, they introduced the notion of *correlation gap of a polytope*, inspired by [1].[3] Formally, the correlation gap of $\mathcal{I}$ is defined as

$$\kappa(\mathcal{I}) := \inf_{x \in P_{\mathcal{I}},\, w \geqslant 0} \frac{\mathbb{E}_{R \sim \mathcal{D}_x}[\max_{S \subseteq R, S \in \mathcal{I}} \sum_{e \in S} w_e]}{\sum_{e \in E} x_e w_e}, \tag{2}$$

where $\mathcal{D}_x$ is the distribution where each element e appears in R with probability x_e independently. It is easy to see that the existence of c-CR scheme for all $x \in P_{\mathcal{I}}$ implies $\kappa(\mathcal{I}) \geqslant c$. Chekuri et al. [5] proved the converse that every $x \in P_{\mathcal{I}}$ admits a $\kappa(\mathcal{I})$-CR scheme.

By setting E to be the set of all possible edges of a complete (bipartite) graph, and $\mathcal{I}$ to be the set of all matchings of a complete graph, our Theorem 1.2 and Theorem 1.3 for weighted bipartite graphs and weighted general graphs imply that there exist 0.4481-CR scheme and 0.4323-CR scheme for bipartite matching polytopes and general matching polytopes respectively. Note that these lower bounds hold when E' is the set of edges and $\mathcal{I}'$ is a matching polytope of an arbitrary graph G', since

$$\begin{aligned}\kappa(\mathcal{I}) &= \inf_{x \in P_{\mathcal{I}},\, w \geqslant 0} \frac{\mathbb{E}_{R \sim \mathcal{D}_x}[\max_{S \subseteq R, S \in \mathcal{I}} \sum_{e \in S} w_e]}{\sum_{e \in E} x_e w_e} \\ &\leqslant \inf_{x|_{E'} \in P_{\mathcal{I}'},\, w|_{E'} = 0} \frac{\mathbb{E}_{R \sim \mathcal{D}_x}[\max_{S \subseteq R, S \in \mathcal{I}} \sum_{e \in S} w_e]}{\sum_{e \in E} x_e w_e} = \kappa(\mathcal{I}').\end{aligned}$$

Maximum Matching of Random Graphs

The study of maximum matchings in random graphs has a long history. It was pioneered by the work of Erdős and Rényi [8, 7], where they proved that a random graph $G_{n,p}$ has a perfect matching with high probability when $p = \Omega(\frac{\ln n}{n})$. The case for sparse graphs was investigated by Karp and Sipser [11] who gave an accurate estimate of $\nu(G)$ for $G_{n,p}$ where $p = \frac{c}{n-1}$ for some constant $c > 0$.

After these two pioneering results, subsequent work has addressed two aspects. The Karp-Sipser algorithm is a simple randomized greedy algorithm, and the first line of works extend the range of models where this algorithm (or its variants) returns an almost maximum matching. Aronson et al. [2] and Chebolu et al. [4] augmented the Karp-Sipser algorithm to achieve tighter results in the standard $G_{n,p}$ model. Bohman and Frieze [3] considered a new model where a graph is drawn uniformly at random from the collection of graphs with a fixed degree sequence and gave a sufficient condition where the Karp-Sipser algorithm finds an almost perfect matching.

The second line of work is based on the following observation: the standard $G_{n,p}$ model, $p = \Omega(\frac{\ln n}{n})$ is required to have a perfect matching, because otherwise there will be an isolated vertex. This naturally led to the question of finding a natural and *sparser* random graph model with a perfect matching. The considered models include a random regular graph, and a $G_{n,p}$ with prescribed minimal degree. We refer the reader to the work of Frieze and Pittel [10] and Frieze [9] and references therein.

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[3] [1] defined the correlation gap of a set function $f : 2^E \to \mathbb{R}^+$. Our results apply to this definition too when f denotes the weight of the maximum matching.

1.2 Organization

Our main technical contribution is lower bounding correlation gaps via local distribution schemes for dual variables, which are used to prove Theorem 1.1 and Theorem 1.2 for unweighted and weighted bipartite graphs. We present this framework in Section 2 and prove our bounds for unweighted bipartite graphs (Section 3) and weighted bipartite graphs (Section 4). Our result for weighted general graphs is presented in Section 5.

2 Techinques for Bipartite Graphs

Let $V = V_1 \cup V_2$ be the set of vertices with $|V_1| = |V_2| = n$, $E := V_1 \times V_2$. Fix $w \in (\mathbb{R}^+ \cup \{0\})^E$ and $x \in [0,1]^E$ in the bipartite matching polytope of (V, E).

Our proofs for Theorem 1.1 and 1.2 for bipartite graphs follow the following general framework. Let $G = (V, E(G))$ be a sampled from the distribution where each potential edge $e \in E$ appears with probability x_e independently (recall that $E = V_1 \times V_2$ is the set of all potential edges and $E(G)$ is the edges of one sample G). Let $y(G) \in (\mathbb{R}^+ \cup \{0\})^V$ be an optimal *fractional vertex cover* such that for every $e = (u, v) \in E(G)$, $y_u(G) + y_v(G) \geqslant w_e$. By König-Egerváry theorem, $\|y(G)\|_1 = \nu(G)$.

Given G, consider the situation where initially each vertex v has *mass* $y_v(G)$, and each potential edge has mass $y_e(G) = 0$ (we slightly abuse notation and consider $y(G) \in (\mathbb{R}^+ \cup \{0\})^{V \cup E}$). We construct *local distribution schemes* $F_G : (V \cup E) \times (V \cup E) \to \mathbb{R}$ where $F_G(a, b)$ indicates the amount of mass sent from a to b. We require that $F_G(a, a) = 0$, but we allow $F_G(a, b) \neq -F_G(b, a)$ for $a \neq b$ (the net flow from a to b in this case is $F_G(a, b) - F_G(b, a)$). Let $t(G) \in \mathbb{R}^{V \cup E}$ denote the mass of each vertex and edge after the distribution.

$$t_a(G) := y_a(G) + \sum_{b \in V \cup E} F_G(b, a) - \sum_{b \in V \cup E} F_G(a, b).$$

We choose F_G so that it ensures $t_v(G) \geqslant 0$ for every $v \in V$. This implies

$$\sum_{e \in E} t_e(G) \leqslant \sum_{a \in V \cup E} t_a(G) = \sum_{a \in V \cup E} y_a(G) = \sum_{v \in V} y_v(G) = \nu(G).$$

Therefore, if we prove that for each potential edge $e \in E$

$$\mathbb{E}_G[t_e(G)] \geqslant \alpha \cdot w_e x_e, \tag{3}$$

for some $\alpha > 0$, it implies that

$$\mathbb{E}_G[\nu(G)] \geqslant \alpha \cdot \sum_{e \in E} \mathbb{E}_G[t_e(G)] \geqslant \alpha \cdot \sum_{e \in E} w_e x_e.$$

For weighted and unweighted cases, we construct different local distribution schemes $\{F_G\}_G$ that prove (3) with different values of α.

Weighted Bipartite Graphs

Given a sample $G = (V, E(G))$ and a fractional vertex cover $y \in (\mathbb{R}^+ \cup \{0\})^V$, our $F_G(v, e) = y_v(G)/\deg_G(v)$ if $e \in E(G)$ is an edge incident on $v \in V$, and 0 otherwise. Intuitively, each vertex v *distributes* its mass $y_v(G)$ evenly to its incident edges in G. This clearly satisfies

$t_v(G) \geqslant 0$ for every $v \in V$, and for each $e = (u, v) \in E$, we use the following approximation:

$$\begin{aligned}
\mathbb{E}_G[t_e(G)] &= \Pr[e \in G] \cdot \mathbb{E}_G\left[t_e(G) | e \in G\right] \\
&= x_e \, \mathbb{E}_G\left[\frac{y_u(G)}{\deg_G(u)} + \frac{y_v(G)}{\deg_G(v)} | e \in G\right] \\
&\geqslant x_e \, \mathbb{E}_G\left[(y_u(G) + y_v(G)) \frac{1}{\max(\deg_G(u), \deg_G(v))} | e \in G\right] \\
&\geqslant x_e w_e \, \mathbb{E}_G\left[\frac{1}{\max(\deg_G(u), \deg_G(v))} | e \in G\right].
\end{aligned}$$

Therefore, to prove Theorem 1.2, it suffices to prove that for every potential edge $e \in E$,

$$\mathbb{E}_{G \sim \mathcal{D}^{\mathsf{B}}_{n,w,x}}\left[\frac{1}{\max(\deg_G(u), \deg_G(v))} | e \in G\right] \geqslant 0.4481,$$

when G is sampled from $\mathcal{D}^{\mathsf{B}}_{n,w,x}$ with x in the matching polytope. Experimentally trying several extreme cases indicates that the worst case for $e = (u, v) \in E$ happens when $x_e = \varepsilon$ for very small ε, u has only one other edge e_u with $x_{e_u} = 1 - \varepsilon$, and v is incident on $n - 1$ edges $e_{v_1}, \ldots, e_{v_{n-1}}$ with $x_{e_{v_i}} = \frac{1-\varepsilon}{n-1}$. As ε approaches to 0, $\mathbb{E}_G[\frac{1}{\max(\deg_G(u), \deg_G(v))} | e \in G]$ converges to $\mathbb{E}[\frac{1}{1+Y_U}]$ as n grows, where Y_U is drawn from a binomial distribution $B(n-1, \frac{1}{n-1})$. Section 4 formally proves that this is indeed the worst case.

Unweighted Bipartite Graphs

One simple but important observation is that in the above example where $\mathbb{E}_G[t_e(G)] \approx \mathbb{E}[\frac{1}{1+Y_U}]x_e$, e is an edge with very small $x_e = \varepsilon$, and it is adjacent to a large edge e_u with $x_{e_u} = 1 - \varepsilon$. From the persepctive of x_{e_u}, the expected number of adjacent edges is at most 2ε, so $\mathbb{E}_G[t_{e_u}(G)] \approx x_{e_u} \approx 1$. Since e_u gets much more than what it needs ($\mathbb{E}[t_{e_u}] \geqslant 0.476$ suffices to prove Theorem 1.1), it is natural to take some value from $t_{e_u}(G)$ to increase $t_e(G)$.

Formally, given $G = (V, E(G))$, our new local distribution scheme $F_G : (V \cup E) \times (V \cup E) \to \mathbb{R}$ is defined as follows. Let c be an universal constant that will be determined later.

$$F_G(a, b) = \begin{cases} \frac{y_a(G)}{\deg_G(a)} & \text{if } a \in V, b \in E(G), a \in b \\ c x_a^2 x_b & \text{if } a \neq b \in E, a \cap b \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Intuitively, on top of the old local distribution scheme for weighted graphs, each edge e pays $c x_e^2 x_f$ to every adjacent edge f with probability 1 (this quantity does not depend on G). Because this term quadratically depends on the x value of the sender, this payment penalizes edges with large x values to help edges with small x values. For a fixed edge $e = (u, v) \in E$ with fixed $x_e = \varepsilon$, Theorem 3.1 shows that the worst case is when both u and v have $n - 1$ other edges of whose x values are equal to $\frac{1-\varepsilon}{n}$. Finally, Lemma 3.2 shows that $\mathbb{E}[t_e] \geqslant 0.476$ for every $\varepsilon \in (0, 1]$, proving Theorem 1.1.

3 Unweighted Bipartite Graphs

We prove Theorem 1.1 for unweighted bipartite graphs. Given $G = (V, E(G))$, consider the local distribution scheme $F_G : (V \cup E) \times (V \cup E) \to \mathbb{R}$ given in (4). This implies that the mass after this new distribution scheme for an edge $e = (u, v)$ is given by

$$t_e(G) = \alpha_e(G) + \sum_{f \in E \setminus \{e\} : f \ni u} c(x_e x_f^2 - x_e^2 x_f) + \sum_{g \in E \setminus \{e\} : g \ni v} c(x_g^2 x_e - x_e^2 x_g),$$

where $\alpha_e(G) := y_u(G)/\deg_G(u) + y_v(G)/\deg_G(v)$ denotes the mass after the old distribution scheme used for weighted bipartite graphs. We define $\beta_e(x)$ to be the following.

$$\begin{aligned}\beta_e(x) &:= \mathbb{E}_{G \sim \mathcal{D}^{\mathsf{B}}_{n,x}}[t_e(G)] \\ &= \mathbb{E}_{G \sim \mathcal{D}^{\mathsf{B}}_{n,x}}[\alpha_e(G)] + \sum_{f \in E \setminus \{e\} : f \ni u} c(x_e x_f^2 - x_e^2 x_f) + \sum_{g \in E \setminus \{e\} : g \ni v} c(x_g^2 x_e - x_e^2 x_g)\end{aligned}$$

To prove Theorem 1.1, it suffices to prove that $\beta_e(x) \geqslant 0.476 x_e$ for each e. Fix $e = (u, v)$. Let $e_{u_1}, \ldots, e_{u_{n-1}}$ be $n-1$ other edges incident on u and $e_{v_1}, \ldots, e_{v_{n-1}}$ be $n-1$ other edges incident on v. $\mathbb{E}_{G \sim \mathcal{D}^{\mathsf{B}}_{n,x}}[\alpha_e(G)]$ is lower bounded by $x_e \, \mathbb{E}_G[\frac{1}{\max(\deg_G(u), \deg_G(v))} | e \in G]$ as before. Define $F(x_0, y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1})$ by

$$\begin{aligned}F(x_0, y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1}) := x_0 \, \mathbb{E}[\frac{1}{1 + \max(Y, Z)}] + \sum_{i=1}^{n-1} c(x_0 y_i^2 - x_0^2 y_i) \\ + \sum_{i=1}^{n-1} c(x_0 z_i^2 - x_0^2 z_i),\end{aligned}$$

where $Y := Y_1 + \cdots + Y_{n-1}$ and $Z := Z_1 + \cdots + Z_{n-1}$ and each Y_i (resp. Z_i) is an independent Bernoulli random variable with $\mathbb{E}[Y_i] = y_i$ (resp. $\mathbb{E}[Z_i] = z_i$). By construction, $\beta_e(x) \geqslant F(x_e, x_{e_{u_1}}, \ldots, x_{e_{u_{n-1}}}, x_{e_{v_1}}, \ldots, x_{e_{v_{n-1}}})$. Given fixed $\sum_{i=1}^{n-1} x_{e_{u_i}}$ and $\sum_{i=1}^{n-1} x_{v_{u_i}}$, the following theorem shows that F is minimized when $x_{e_{u_1}} = \cdots = x_{e_{u_{n-1}}}$ and $x_{e_{v_1}} = \cdots = x_{e_{v_{n-1}}}$.

▶ **Theorem 3.1.** *For $x_0, y_1, \ldots, y_m, z_1, \ldots, z_m \in [0, 1]$ where $y_s := \sum_{i=1}^m y_i \leqslant 1 - x_0$ and $z_s := \sum_{i=1}^m z_i \leqslant 1 - x_0$,*

$$F(x_0, y_1, \ldots, y_m, z_1, \ldots, z_m) \geqslant F(x_0, \frac{y_s}{m}, \ldots, \frac{y_s}{m}, \frac{z_s}{m}, \ldots, \frac{z_s}{m}).$$

Proof. Without loss of generality, assume $y_1 \geqslant \ldots \geqslant y_m$. We will show that if $y_1 > y_m$,

$$\frac{\partial F}{\partial y_m} - \frac{\partial F}{\partial y_1} \leqslant 0. \tag{5}$$

This implies that as long as $y_1 > y_m$, decreasing y_1 and increasing y_m by the same amount will never increase F while maintaining $y_1 + \cdots + y_m = y_s$, so F is minimized when $y_1 = \cdots = y_m = \frac{y_s}{m}$. The same argument for $z_1, \ldots, z_m$ will prove the theorem.

Let $Y := Y_1 + \cdots + Y_m$ and $Z := Z_1 + \cdots + Z_m$, where each Y_i (resp. Z_i) is an independent Bernoulli random variable with $\mathbb{E}[Y_i] = y_i$ (resp. $\mathbb{E}[Z_i] = z_i$). To prove (5), we first compute $\frac{\partial \, \mathbb{E}[\frac{1}{1+\max(Y,Z)}]}{\partial y_m} - \frac{\partial \, \mathbb{E}[\frac{1}{1+\max(Y,Z)}]}{\partial y_1}$. Let $Y' := Y_2 + \cdots + Y_{m-1}$. We decompose $\mathbb{E}[\frac{1}{1+\max(Y,Z)}]$ as follows.

$$
\begin{aligned}
&\mathbb{E}[\frac{1}{1+\max(Y,Z)}]\\
=&\sum_{i=0}^{m}\sum_{j=0}^{m}\left(\Pr[Y=i]\cdot\Pr[Z=j]\cdot\frac{1}{1+\max(i,j)}\right)\\
=&\sum_{i=0}^{m}\left(\Pr[Y'=i]\cdot\Pr[Z\leqslant i](\frac{(1-y_1)(1-y_m)}{1+i}+\frac{y_1(1-y_m)+(1-y_1)y_m}{2+i}+\frac{y_1y_m}{3+i})\right)\\
+&\sum_{i=0}^{m}\left(\Pr[Y'=i]\cdot\Pr[Z=i+1](\frac{1-y_1y_m}{2+i}+\frac{y_1y_m}{3+i})\right)\\
+&\sum_{i=0}^{m}\Pr[Y'=i]\cdot\Pr[Z\geqslant i+2]\cdot\frac{1}{3+i}
\end{aligned}
$$

Therefore, the directional derivative can be written as

$$
\begin{aligned}
&(\frac{\partial}{\partial y_m}-\frac{\partial}{\partial y_1})\,\mathbb{E}[\frac{1}{1+\max(Y,Z)}]\\
=&(y_1-y_m)\sum_{i=0}^{m}\left(\Pr[Y'=i]\cdot\Pr[Z\leqslant i](\frac{1}{1+i}-\frac{2}{2+i}+\frac{1}{3+i})\right)\\
+&(y_1-y_m)\sum_{i=0}^{m}\left(\Pr[Y'=i]\cdot\Pr[Z=i+1](-\frac{1}{2+i}+\frac{1}{3+i})\right)\\
\leqslant&(y_1-y_m)\sum_{i=0}^{m}\left(\Pr[Y'=i]\cdot\Pr[Z\leqslant i](\frac{1}{1+i}-\frac{2}{2+i}+\frac{1}{3+i})\right)\\
\leqslant&(y_1-y_m)\sum_{i=0}^{m}\left(\Pr[Y'=i]\cdot\Pr[Z\leqslant i](\frac{1}{1+i}-\frac{2}{2+i}+\frac{1}{3+i})\right)\\
\leqslant&\frac{y_1-y_m}{3},
\end{aligned}
$$

where the last inequality follows from the fact that

$$
(\frac{1}{1+i}-\frac{2}{2+i}+\frac{1}{3+i})=\frac{2}{(1+i)(2+i)(3+i)}\leqslant\frac{1}{3}.
$$

Finally,

$$
\begin{aligned}
&(\frac{\partial}{\partial y_m}-\frac{\partial}{\partial y_1})F\\
=&(\frac{\partial}{\partial y_m}-\frac{\partial}{\partial y_1})(x_e\,\mathbb{E}[\frac{1}{1+\max(Y,Z)}]+cx_ey_1^2-cx_e^2y_1+cx_ey_m^2-cx_e^2y_m)\\
\leqslant&\frac{x_e(y_1-y_m)}{3}-2cx_e(y_1-y_m)=0.
\end{aligned}
$$

By taking $c=\frac{1}{6}$. ◀

Therefore, for any $e\in E$, $\beta_e(x)\geqslant F(x_e,\frac{y_s}{n-1},\ldots,\frac{y_s}{n-1},\frac{z_s}{n-1},\ldots,\frac{z_s}{n-1})$ for some $y_s\leqslant 1-x_e$ and $z_s\leqslant 1-x_e$. Let

$$\begin{aligned}G(x_e, y_s, z_s) :=& F(x_e, \frac{y_s}{n-1}, \dots, \frac{y_s}{n-1}, \frac{z_s}{n-1}, \dots, \frac{z_s}{n-1})\\ =& x_e\, \mathbb{E}[\frac{1}{1+\max(Y,Z)}] + (n-1)c(x_e(\frac{y_s}{n-1})^2 - x_e^2(\frac{y_s}{n-1}))\\ &+ (n-1)c(x_e(\frac{z_s}{n-1})^2 - x_e^2(\frac{z_s}{n-1}))\\ =& x_e\, \mathbb{E}[\frac{1}{1+\max(Y,Z)}] + cx_e y_s((\frac{y_s}{n-1}) - x_e) + cx_e z_s((\frac{z_s}{n-1}) - x_e)\\ \geqslant& x_e\, \mathbb{E}[\frac{1}{1+\max(Y,Z)}] - 2cx_e^2\end{aligned}$$

where $Y \sim \mathsf{Binomial}(n-1, \frac{y_s}{n-1})$, $Z \sim \mathsf{Binomial}(n-1, \frac{z_s}{n-1})$. Note that the final quantity is minimized when $y_s = z_s = 1 - x_e$. Finally, let

$$H_{n-1}(x_e) := x_e\, \mathbb{E}[\frac{1}{1+\max(Y,Z)}] - 2cx_e^2,$$

where $Y, Z \sim \mathsf{Binomial}(n-1, \frac{1-x_e}{n-1})$.

▶ **Lemma 3.2.** *For any $m \in \mathbb{N}$ and $x_e \in [0,1]$, $H_m(x_e) \geqslant 0.476x_e$.*

Proof. Since the binomial distribution is approximated by the Poisson distribution in the limit, we use this to ease the calculation. Let $Y, Z \sim \mathsf{Poisson}(1-x)$. Let $H(x) := x\,\mathbb{E}[\frac{1}{1+\max(Y,Z)}] - x^2/3$ (we substitute $c = 1/6$ into the earlier equation). In particular, we write the expectation in full to get

$$\begin{aligned}\mathbb{E}[\frac{1}{1+\max(Y,Z)}] &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{1+\max(j,k)} e^{-2(1-x)} \frac{(1-x)^{j+k}}{j!k!}\\ &= \frac{1}{e^{2(1-x)}} \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \frac{1}{1+\max(j,k-j)} \frac{1}{j!(k-j)!} \Big) (1-x)^k\end{aligned}$$

Let $P_t(x)$ denote the above sum truncated at $k = t$. I.e.

$$P_t(x) := \frac{1}{e^{2(1-x)}} \sum_{k=0}^{t} \Big(\sum_{j=0}^{k} \frac{1}{1+\max(j,k-j)} \frac{1}{j!(k-j)!} \Big) (1-x)^k$$

This is a degree t-polynomial in $(1-x)$ with a normalizing factor of $e^{-2(1-x)}$ and note that $\mathbb{E}[\frac{1}{1+\max(Y,Z)}] \geqslant P_t(x)$ for any $t \in \mathbb{N}$.

Truncating this polynomial with $t = 15$, we can see that this has a minimum value of 0.476 for all values of $x \in [0,1]$. we can see that $\mathbb{E}[\frac{1}{1+\max(Y,Z)}] - x/3 \geqslant P_15(x) - x/3$. In the interval $x \in [0,1]$, this function achieves its minimum at $x = 0$ achieving a minimum of 0.476. ◀

4 Weighted Bipartite Graphs

We prove Theorem 1.2 for weighted bipartite graphs. As explained in Section 2, it suffices to prove that for each $e = (u,v) \in E$,

$$\mathbb{E}_{G \sim \mathcal{D}^{\mathsf{B}}_{n,w,x}} \left[\frac{1}{\max(\deg_G(u), \deg_G(v))} | e \in G \right] \geqslant 0.4481.$$

Fix $e = (u, v)$ and assume $V = \{v, v_1, \ldots, v_{n-1}\} \cup \{u, u_1, \ldots, u_{n-1}\}$. Let $Y = \deg_G(u) - 1$ and $Z = \deg_G(v) - 1$. Given $e \in G$, Y and Z can be represented as $Y = \sum_{i=1}^{n-1} Y_i$ and $Z = \sum_{i=1}^{n-1} Z_i$, where Y_i indicates where $(u, v_i) \in E(G)$ and Z_i indicates where $(v, u_i) \in E(G)$. This construction ensures that

$$\mathbb{E}_G \left[\frac{1}{\max(\deg_G(u), \deg_G(v))} | e \in G \right] = \mathbb{E}_{Y,Z} \left[\frac{1}{1 + \max(Y, Z)} \right].$$

Note that $Y_1, \ldots, Y_{n-1}, Z_1, \ldots, Z_{n-1}$ are mutually independent, and $\mathbb{E}[Y], \mathbb{E}[Z] \leqslant 1$. By monotonicity, assuming $\mathbb{E}[Y] = \mathbb{E}[Z] = 1$ never increases the lower bound. The following theorem shows that the worst case happens when one of Y, Z is consistently 1 and the other is drawn from $\mathsf{Binomial}(n-1, \frac{1}{n-1})$.

▶ **Theorem 4.1.** *Let $Y = Y_1 + \cdots + Y_m$ and $Z = Z_1 + \cdots + Z_m$, where $Y_1, \ldots, Y_m, Z_1, \ldots, Z_m$ are mutually independent Bernoulli random variables with $\mathbb{E}[Y] = \mathbb{E}[Z] = 1$. Then,*

$$\mathbb{E} \left[\frac{1}{1 + \max(Y, Z)} \right] \geqslant \mathbb{E} \left[\frac{1}{1 + Y_U} \right],$$

where Y_U is drawn from $\mathsf{Binomial}(m, \frac{1}{m})$.

Proof. We decompose $\mathbb{E}[\frac{1}{1+\max(Y,Z)}]$ as follows.

$$\begin{aligned}
&\mathbb{E} \left[\frac{1}{1 + \max(Y, Z)} \right] \\
&= \sum_{i=0}^{m} \sum_{j=0}^{m} \left(\Pr[Y = i] \cdot \Pr[Z = j] \cdot \frac{1}{1 + \max(i, j)} \right) \\
&= \sum_{i=0}^{m} \Pr[Y = i] \left[\Big(\sum_{j=0}^{i} \Pr[Z = j] \Big) \cdot \frac{1}{1 + i} + \Big(\sum_{j=i+1}^{m} \Pr[Z = j] \cdot \frac{1}{1 + j} \Big) \right] \\
&= \sum_{i=0}^{m} \Pr[Y = i] \cdot \frac{1}{1 + i} - \sum_{i=0}^{m} \Pr[Y = i] \left[\sum_{j=i+1}^{m} \Pr[Z = j] \Big(\frac{1}{1 + i} - \frac{1}{1 + j} \Big) \right] \\
&= \sum_{i=0}^{m} \Pr[Y = i] \cdot \frac{1}{1 + i} - \sum_{j=1}^{m} \Pr[Z = j] \left[\sum_{i=0}^{j-1} \Pr[Y = i] \Big(\frac{1}{1 + i} - \frac{1}{1 + j} \Big) \right].
\end{aligned}$$

Let $t_j := \sum_{i=0}^{j-1} \Pr[Y = i] \cdot \big(\frac{1}{1+i} - \frac{1}{1+j} \big)$. We prove the following facts about t_j's.

▶ **Lemma 4.2.** *For all $j \geqslant 3$, $\frac{t_2}{2} \geqslant \frac{t_j}{j}$.*

Proof. Fix $j \geqslant 3$. By the definition of t_2 and t_j,

$$
\begin{aligned}
&\frac{t_2}{2} - \frac{t_j}{j} \\
&= \frac{1}{2}\left(\Pr[Y=0](1-\frac{1}{3}) + \Pr[Y=1](\frac{1}{2}-\frac{1}{3})\right) - \frac{1}{j}\left(\sum_{i=0}^{j-1} \Pr[Y=i](\frac{1}{1+i} - \frac{1}{1+j})\right) \\
&= \frac{1}{3}\Pr[Y=0] + \frac{1}{12}\Pr[Y=1] - \frac{1}{j}\left(\sum_{i=0}^{j-1} \Pr[Y=i](\frac{1}{1+i} - \frac{1}{1+j})\right) \\
&= (\frac{1}{3} - \frac{1}{1+j})\Pr[Y=0] + (\frac{1}{12} - \frac{j-1}{2j(j+2)})\Pr[Y=1] \\
&\quad - \frac{1}{j}\left(\sum_{i=2}^{j-1} \Pr[Y=i](\frac{1}{1+i} - \frac{1}{1+j})\right) \\
&\geqslant \left(\frac{1}{3} - \frac{1}{1+j} - \frac{1}{j}\sum_{i=2}^{j-1}(\frac{1}{1+i} - \frac{1}{1+j})\right)\Pr[Y=0] + (\frac{1}{12} - \frac{j-1}{2j(j+2)})\Pr[Y=1],
\end{aligned}
$$

where the inequality follows from $\Pr[Y=0] \geqslant \Pr[Y=i]$ for $i \geqslant 2$. To prove $\frac{t_2}{2} - \frac{t_j}{j} \geqslant 0$, it suffices to prove that $\frac{1}{3} - \frac{1}{1+j} - \frac{1}{j}\sum_{i=2}^{j-1}\left(\frac{1}{1+i} - \frac{1}{1+j}\right) \geqslant 0$, and $\frac{1}{12} - \frac{j-1}{2j(j+2)} \geqslant 0$. It is easy to verify the latter for $j \geqslant 3$. The former can be proved as

$$
\begin{aligned}
&\frac{1}{3} - \frac{1}{1+j} - \frac{1}{j}\sum_{i=2}^{j-1}(\frac{1}{1+i} - \frac{1}{1+j}) \\
=&\frac{1}{3} + \frac{j-2}{j(1+j)} - (\frac{1}{1+j} + \frac{1}{j}\sum_{i=2}^{j-1}\frac{1}{1+i}) \\
\geqslant&\frac{1}{3} + \frac{j-2}{j(1+j)} - (\frac{1}{1+j} + \frac{j-2}{3j}) \\
=&(\frac{1}{3} - \frac{j-2}{3j}) + (\frac{j-2}{j(1+j)} - \frac{1}{1+j}) \\
=&\frac{2}{3j} - \frac{2}{j(1+j)} \geqslant 0,
\end{aligned}
$$

where the first inequality follows from $\frac{1}{1+i} \leqslant \frac{1}{3}$ for $i \geqslant 2$ and the last inequality follows from $j \geqslant 3$. ◀

We prove the theorem by considering the following two cases.

Case 1: $2\Pr[Y=0] \geqslant \Pr[Y=1]$ **or** $2\Pr[Z=0] \geqslant \Pr[Z=1]$

Without loss of generality, assume that $2\Pr[Y=0] \geqslant \Pr[Y=1]$. It is equivalent to

$$
\begin{aligned}
&\Pr[Y=0] \geqslant \frac{2}{3}\Pr[Y=0] + \frac{1}{6}\Pr[Y=1] \\
\Leftrightarrow\quad & t_1 \geqslant \frac{t_2}{2}.
\end{aligned}
$$

By Lemma 4.2, it implies that $t_1 \geqslant \frac{t_j}{j}$ for all $j \geqslant 2$. Then, since $\mathbb{E}[Z] = \sum_{j=1}^m j \cdot \Pr[Z = j] = 1$,

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1+\max(Y,Z)}\right] &= \sum_{i=0}^m \Pr[Y=i] \cdot \frac{1}{1+i} - \sum_{j=1}^m \Pr[Z=j] t_j \\
&\geqslant \sum_{i=0}^m \Pr[Y=i] \cdot \frac{1}{1+i} - t_1 \sum_{j=1}^m j \cdot \Pr[Z=j] \\
&= \sum_{i=0}^m \Pr[Y=i] \cdot \frac{1}{1+i} - t_1 \\
&= \mathbb{E}[\frac{1}{1+\max(Y,1)}].
\end{aligned}$$

The following lemma proves the theorem in the case $t_1 \geqslant \frac{t_2}{2}$.

▶ **Lemma 4.3.** $\mathbb{E}[\frac{1}{1+\max(Y,1)}] \geqslant \mathbb{E}[\frac{1}{1+\max(Y_U,1)}]$.

Proof. Note that $Y = Y_1 + \cdots + Y_m$, and each Y_i is a Bernoulli random variable. Let $y_i := \mathbb{E}[Y_i]$. Without loss of generality, assume $y_1 \geqslant \ldots \geqslant y_m$. We will show that if $y_1 > y_m$,

$$\frac{\partial\, \mathbb{E}[\frac{1}{1+\max(Y,1)}]}{\partial y_m} - \frac{\partial\, \mathbb{E}[\frac{1}{1+\max(Y,1)}]}{\partial y_1} \leqslant 0. \tag{6}$$

This implies that as long as $y_1 > y_m$, decreasing y_1 and increasing y_m by the same amount will never increase $\mathbb{E}[\frac{1}{1+\max(Y,1)}]$ while maintaining $y_1 + \cdots + y_m = 1$, so the expectation is minimized when $y_1 = \cdots = y_m$, or $Y = Y_U$. Consider the following decomposition of $\mathbb{E}[\frac{1}{1+\max(X,Y)}]$.

$$\begin{aligned}
\mathbb{E}_Y\left[\frac{1}{1+\max(1,Y)}\right] &= \Pr[Y=0] \cdot \frac{1}{2} + \sum_{i=1}^m \Pr[Y=i] \cdot \frac{1}{1+i} \\
&= \frac{1}{2}(1 - \sum_{i=1}^m \Pr[Y=i]) + \sum_{i=1}^m \Pr[Y=i] \cdot \frac{1}{1+i} \\
&= \frac{1}{2} - \sum_{i=2}^m \Pr[Y=i] \cdot (\frac{1}{2} - \frac{1}{1+i}) \\
&= \frac{1}{2} - \sum_{i=2}^m \Pr[Y \geqslant i] \cdot (\frac{1}{i} - \frac{1}{1+i}).
\end{aligned}$$

To prove (6), it suffices to prove that for all $i \geqslant 2$,

$$\frac{\partial \Pr[Y \geqslant i]}{\partial y_m} - \frac{\partial \Pr[Y \geqslant i]}{\partial y_1} \geqslant 0.$$

Let $Y' = Y_2 + \cdots + Y_{m-1}$, and fix $i \geqslant 3$.

$$\begin{aligned}
\Pr[Y \geqslant i] &= \Pr[Y' = i-2] y_1 y_m + \Pr[Y' = i-1]\big(y_1(1-y_m) + (1-y_1)y_m + y_1 y_m\big) \\
&\quad + \Pr[Y' \geqslant i] \\
\frac{\partial \Pr[Y \geqslant i]}{\partial y_1} &= \Pr[Y' = i-2] y_m + \Pr[Y' = i-1]\big(1 - y_m\big)
\end{aligned}$$

Therefore,

$$\frac{\partial \Pr[Y \geqslant i]}{\partial y_m} - \frac{\partial \Pr[Y \geqslant i]}{\partial y_1} = \Pr[Y' = i-2](y_1 - y_m) + \Pr[Y' = i-1]\big(y_m - y_1\big)$$
$$= (y_1 - y_m)\big(\Pr[Y' = i-2] + \Pr[Y' = i-1]\big).$$

Finally, it remains to show that $\Pr[Y' = j] \geqslant \Pr[Y' = j+1]$ for all $j \geqslant 0$. The case $j = 0$ is true since $\Pr[Y' = 0] = \prod_{k=2}^{m-1}(1 - y_k)$ and

$$\Pr[Y' = 1] = \sum_{k=2}^{m-1} \Pr[Y' = 0] \cdot \frac{y_k}{1 - y_k} \leqslant \sum_{k=2}^{m-1} \Pr[Y' = 0] \frac{y_k}{1 - y_2}$$
$$= \frac{\Pr[Y' = 0]}{1 - y_2} \sum_{i=2}^{m-1} y_k \leqslant \Pr[Y' = 0],$$

where the last line follows from $\sum_{k=2}^{m-1} y_i \leqslant 1 - y_1 \leqslant 1 - y_2$ since y_1 is the biggest element. The case $j \geqslant 1$ follows from the fact the sequence $(\Pr[Y' = j])_j$ has one mode or two consecutive modes, and at least one of them occurs at $j = 0$ ($\mathbb{E}[Y'] < 1$ implies $\Pr[Y' = 0] > \Pr[Y' = j]$ for all $j \geqslant 2$). ◀

Case 2: $2\Pr[Y = 0] \leqslant \Pr[Y = 1]$ **and** $2\Pr[Z = 0] \leqslant \Pr[Z = 1]$

Since $\sum_{i=0}^{m} \Pr[Z = i] = 1$ and $\mathbb{E}[Z] = \sum_{i=1}^{m} i \cdot \Pr[Z = i] = 1$, we have $\Pr[Z = 0] = \sum_{i=2}^{m}(i-1)\Pr[Z = i]$. Together with the fact $2\Pr[Z = 0] \leqslant \Pr[Z = 1]$, it implies

$$1 - \Pr[Z = 1] = \Pr[Z = 0] + \sum_{i=2}^{m} \Pr[Z = i] \leqslant 2\Pr[Z = 0] < \Pr[Z = 1],$$

so $\Pr[Z = 1] \geqslant \frac{1}{2}$. Finally,

$$\mathbb{E}\left[\frac{1}{1 + \max(Y, Z)}\right] = \sum_{i=0}^{m} \Pr[Y = i] \cdot \frac{1}{1+i} - \sum_{j=1}^{m} \Pr[Z = j] \cdot t_j$$
$$= \sum_{i=0}^{m} \Pr[Y = i] \cdot \frac{1}{1+i} - \Pr[Z = 1] \cdot t_1 - \sum_{j=2}^{m} \Pr[Z = j] \cdot t_j$$
$$\geqslant \sum_{i=0}^{m} \Pr[Y = i] \cdot \frac{1}{1+i} - \Pr[Z = 1] \cdot t_1 - \sum_{j=2}^{m} j \cdot \Pr[Z = j] \cdot \frac{t_2}{2}$$
$$= \sum_{i=0}^{m} \Pr[Y = i] \cdot \frac{1}{1+i} - \Pr[Z = 1] \cdot t_1 - \frac{t_2}{2}(1 - \Pr[Z = 1])$$
$$\geqslant \sum_{i=0}^{m} \Pr[Y = i] \cdot \frac{1}{1+i} - \frac{t_1}{2} - \frac{t_2}{4} = \mathbb{E}\left[\frac{1}{1 + \max(Y, Y_H)}\right],$$

where Y_H is drawn from $\mathsf{Binomial}(2, \frac{1}{2})$. The first inequality follows from Lemma 4.2, and the second inequality follows from $\Pr[Z = 1] \geqslant 0.5$ and $t_1 \leqslant \frac{t_2}{2}$.

Since Y_H satisfies $2\Pr[Y_H = 0] = \Pr[Y_H = 1]$, the analysis for Case 1 shows that $\mathbb{E}[\frac{1}{1+\max(Y,Y_H)}] \geqslant \mathbb{E}[\frac{1}{1+\max(1,Y_U)}]$. ◀

The following lemma finishes the proof of Theorem 1.2.

▶ **Lemma 4.4.** *For any $m \in \mathbb{N}$, if $Y \sim \mathsf{Binomial}(m, \frac{1}{m})$,*

$$\mathbb{E}\left[\frac{1}{1+\max(1,Y)}\right] \geqslant 0.4481$$

Proof. Since the binomial distribution is approximated by the Poisson distribution in the limit, we use this to ease the calculation. Let $Y \sim \mathsf{Poisson}(1)$.

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1+\max(1,Y)}\right] &= \sum_{k=2}^{\infty} \frac{1}{k+1} \Pr[Y' = k] + \frac{1}{2} \Pr[Y < 2] \\
&= \sum_{k=2}^{\infty} \frac{1}{k+1} \frac{1}{k! \cdot e} + \frac{1}{2}(\frac{1}{e} + \frac{1}{e}) \\
&= \frac{1}{e}(\sum_{k=0}^{\infty} \frac{1}{k!} - 1 - 1 - \frac{1}{2}) + \frac{1}{2}(\frac{1}{e} + \frac{1}{e}) \\
&= (e - \frac{5}{2})\frac{1}{e} + \frac{1}{e} \\
&\geqslant 0.4481
\end{aligned}$$

◀

5 General Graphs

In this section, we prove Theorem 1.3 for weighted general graphs. Our proof methods here closely follow that of Lemma 4.9 of Chekuri et al. [5] that lower bounds the correlation gap for monotone submodular functions by $1 - 1/e$. The only difference is that Lemma 5.1 holds for matching with a weaker guarantee (if ν was a monotone submodular function, Lemma 5.1 would hold with $2\nu(G)$ replaced by $\nu(G)$).

Proof. Fix weights $w \in (\mathbb{R}^+ \cup \{0\})^E$. Define $F : [0,1] \to (\mathbb{R}^+ \cup \{0\})$ as $F(x) := \mathbb{E}_{G \sim \mathcal{D}^{\mathsf{G}}_{n,w,x}}[\nu(G)]$. Now, fix $x \in [0,1]^E$ in the matching polytope. We will show $F(x) \geqslant 0.43 \sum_{e \in E} w_e x_e$.

Consider the function $\phi(t) := F(tx)$ for $t \in [0,1]$.

$$\frac{d\phi}{dt} = x \cdot \nabla F(tx) = \sum_{e \in E} x_e \frac{\partial F}{\partial x_e}\bigg|_{tx} \tag{7}$$

For each $e \in E$,

$$\begin{aligned}
\frac{\partial F}{\partial x_e}\bigg|_{tx} &= \frac{\partial\, \mathbb{E}_{G \sim \mathcal{D}^{\mathsf{G}}_{n,w,tx}}[\nu(G)]}{\partial x_e}\Bigg|_{tx} \\
&= \mathbb{E}_{G \sim \mathcal{D}^{\mathsf{G}}_{n,w,tx}}[\nu(G) | e \in G] - \mathbb{E}_{G \sim \mathcal{D}^{\mathsf{G}}_{n,w,tx}}[\nu(G) | e \notin G] \\
&= \mathbb{E}_{G \sim \mathcal{D}^{\mathsf{G}}_{n,w,tx}}[\nu(G \cup \{e\}) - \nu(G \setminus \{e\})],
\end{aligned}$$

where $G \cup \{e\}$ (resp. $G \setminus \{e\}$) denotes the graph $(V, E(G) \cup \{e\})$ (resp. $(V, E(G) \setminus \{e\})$.

▶ **Lemma 5.1.** *For any fixed graph G with weights $\{w_e\}$ and any point x in the matching polytope,*

$$\sum_{e \in E} x_e\big(\nu(G \cup \{e\}) - \nu(G \setminus \{e\})\big) + 2\nu(G) \geqslant \sum_{e \in E} x_e w_e.$$

Proof. Let $M \subseteq E(G)$ be a maximum weight matching of G. Note that

$$\begin{aligned}
&\sum_{e\in E} x_e\big(\nu(G\cup\{e\}) - \nu(G\setminus\{e\})\big) + 2\nu(G)\\
&\geqslant \sum_{e\in E} x_e\big(\nu(G\cup\{e\}) - \nu(G)\big) + 2\sum_{f\in M} w_f\\
&\geqslant \sum_{e\in E} x_e\big(\nu(G\cup\{e\}) - \nu(G)\big) + \sum_{f\in M}\sum_{e\in E: e\sim f} x_e w_f \qquad (8)
\end{aligned}$$

where $f \sim e$ indicates that two edges f and e share an endpoint. To prove the lemma, it suffices to show that for each $e \in E$, the coefficient of of x_e in (8) is at least w_e. We consider the following cases.

- If $M \cup \{e\}$ is a matching, $\nu(G\cup\{e\}) \geqslant \nu(G) + w_e$ and $\nu(G\setminus\{e\}) \leqslant \nu(G)$, so $\nu(G\cup\{e\}) - \nu(G\setminus\{e\}) \geqslant w_e$.
- If e intersects exactly one edge $f \in M$, the coefficient of x_e is $\nu(G\cup\{e\}) - \nu(G) + w_f$. If $w_f \geqslant w_e$, it is at least w_e. If $w_f < w_e$, $M\cup\{e\}\setminus\{f\}$ is a matching of weight $\nu(G)+w_e-w_f$. It implies that $e \notin E(G)$ and $\nu(G\cup\{e\})-\nu(G) \geqslant w_e - w_f$, so $\nu(G\cup\{e\})-\nu(G)+w_f \geqslant w_e$.
- If e intersects two edges $f, g \in M$, the coefficient of x_e is $\nu(G\cup\{e\}) - \nu(G) + w_f + w_g$. If $w_f + w_g \geqslant w_e$, it is at least w_e. If $w_f + w_g < w_e$, $M \cup \{e\} \setminus \{f, g\}$ is a matching of weight $\nu(G) + w_e - w_f - w_g$. It implies that $e \notin E(G)$ and $\nu(G\cup\{e\}) - \nu(G) \geqslant w_e - w_f - w_g$, so $\nu(G\cup\{e\}) - \nu(G) + w_f + w_g \geqslant w_e$. ◀

Combining (7) and Lemma 5.1,

$$\begin{aligned}
\frac{d\phi}{dt} &= \sum_{e\in E} x_e \frac{\partial F}{\partial x_e}\bigg|_{tx}\\
&= \sum_{e\in E} \mathbb{E}_{G\sim\mathcal{D}^{\mathsf{G}}_{n,w,tx}}[\nu(G\cup e) - \nu(G\setminus e)]\\
&\geqslant \sum_{e\in E} x_e w_e - 2\,\mathbb{E}_{G\sim\mathcal{D}^{\mathsf{G}}_{n,w,tx}}[\nu(G)]\\
&= \sum_{e\in E} x_e w_e - 2\phi(t).
\end{aligned}$$

which implies that,

$$\frac{d}{dt}(e^{2t}\phi(t)) = 2e^{2t}\phi(t) + e^{2t}\frac{d\phi}{dt} \geqslant e^{2t}\sum_{e\in E} x_e w_e.$$

Since $\phi(0) = 0$,

$$e^2\phi(1) \geqslant \sum_{e\in E} x_e w_e \int_0^1 e^{2t}dt = \frac{e^2-1}{2}\sum_{e\in E} x_e w_e,$$

which proves the theorem. ◀

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