Some Open Problems in Information-Theoretic Cryptography

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Abstract

Information-theoretic cryptography is full of open problems with a communication-complexity flavor. We will describe several such problems that arise in the study of private information retrieval, secure multi-party computation, secret sharing, private simultaneous messages (PSM) and conditional disclosure of secrets (CDS). In all these cases, there is a huge (exponential) gap between the best known upper and lower bounds. We will also describe the connections between these problems, some old and some new.

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1 Introduction

Information-theoretic cryptography deals with problems of secure communication and secure computation *against computationally unbounded adversaries.* While much of cryptography relies on unproven computational assumptions (and in particular, provides only conditional security), information-theoretic cryptography provides absolute guarantees that are independent of any computational assumption. As such, in the field of information-theoretic cryptography, one could hope to gain a complete understanding of the landscape of secure communication and computation, namely, classify which tasks are possible and which are not, and precisely quantify the computational and communication cost of security.

Indeed, Shannon's celebrated work [33] gave us such a complete picture for *secure communication* against unbounded adversaries: namely that the one-time pad is essentially the best one can do. While several influential works extended the basic model of secure communication to leverage environmental noise [35] or quantum effects [6] to accomplish information-theoretically secure communication in a larger range of settings, Shannon's characterizations gave us a clean and satisfying answer to the basic question.

The situation in *secure computation* turns out to be very different. Broadly speaking, secure computation [36, 22, 5, 10] deals with settings where two or more parties wish to communicate and compute a joint function f on their private inputs while revealing nothing to each other except the output of the computation. The primary complexity measure of secure computation protocols that we care about is their *communication complexity.*

Secure computation is replete with primitives and settings where there is an exponential gap between the known upper and lower bounds on communication complexity. In the basic setting of information theoretically secure 3-party computation that we describe in more

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detail below, the best protocols to compute an arbitrary function incur communication cost $2^{O(n)}$ where n is the total bit-length of the inputs of all parties, whereas the best lower bounds are linear in n. This leads us to ask:

What is the true communication overhead of secure computation?

Furthermore, all known approaches for information-theoretically secure computation incur a communication cost that is proportional to the computational cost of the function (in some computational model, say as a Boolean or arithmetic circuit). Thus, these approaches get stuck at the so-called *circuit-size barrier*. Yet another fundamental question in the foundations of information-theoretic cryptography is:

> *Does the communication cost of secure computation depend on the computational cost?*

It is worth noting here that a simple counting argument establishes the *existence* of functions that require exponentially large circuits, but a similar statement is not known for the communication cost. That is, we do not even know whether there *exist* functions that require super-linear communication to securely compute.

In the rest of this extended abstract, we describe a number of objects of interest in information-theoretic cryptography – private information retrieval, secure multiparty computation, private simultaneous messages, conditional disclosure of secrets, and secret sharing – and the relations among them, as well as the open problems associated to these objects.

2 Information-Theoretic Primitives and their Problems

Private Information Retrieval (PIR). PIR is a protocol among one or more non-communicating servers each holding the same database D, thought of as an N-bit string, and a user holding an index $i \in [N]$. The user wishes to retrieve the i-th bit $D[i]$ from the server(s), without revealing any information about i. Clearly, the server(s) can rather inefficiently accomplish this by sending the entire string D to the user. The objective of PIR, then, is to achieve this goal while communicating (significantly) less than N bits. Such PIR schemes are deemed non-trivial.

Chor, Goldreich, Kushilevitz and Sudan [11], who first defined PIR, also showed that non-trivial *single-server* PIR schemes (with communication less than N bits) require computational assumptions. One line of research that resulted from this work showed several constructions of single-server PIR with decreasing communication complexity under various cryptographic assumptions [27, 9, 28, 7, 20, 19, 8], culminating in [8] that achieves the asymptotically optimal communication complexity of $O(\log N + \lambda)$ bits where λ is the cryptographic security parameter.

Chor, Goldreich, Kushilevitz and Sudan also proposed the natural setting of multi-server PIR where two or more *non-communicating* servers each holding the same database D interact with the client holding an index i. The client is guaranteed that its index is private as long as the servers do not collude with each other.

In the spirit of our questions, let us mention here that the best two-server PIR protocols have total communication complexity $2^{\tilde{O}(\sqrt{\log N})}$ [15] while the lower bound is "merely" $(5 - o(1)) \log N$ [31, 34]. We refer the reader to the excellent survey of Yekhanin and the bibliography maintained by Gasarch [37, 17] for pointers to the long line of work on information-theoretically secure multi-server PIR protocols. Curiously, such PIR schemes turn out to be equivalent in a precise sense to locally decodable codes, an object that does not refer to privacy at all [26].

Secure Multiparty Computation (MPC). In the setting of MPC, a collection of k parties wish to collaborate to compute a publicly known function f on their respective inputs $x_1, x_2, \ldots, x_k$ without leaking any information to each other except the output $f(x_1, \ldots, x_k)$. Such a protocol should be secure against collusions of t corrupted parties. It is well-known that 2-party secure computation of even simple functions requires computational assumptions even against a semi-honest adversary, thus for our purposes, the simplest setting to think about is the 3-party setting with a single corrupted party.

Canonical ways of achieving MPC go through some explicit computational representation of the function f either as a circuit [5, 10] or as a branching program [23]. Consequently, there are functions f for which such MPC protocols have communication exponential in the (total) input length simply because, by a counting argument, there are functions f which require exponentially large circuits (resp., branching programs). Must this be the case?

The lower bounds on the communication complexity of MPC are few and far between. To the best of our knowledge, the state of the art is the work of Data, Prabhakaran and Prabhakaran [13] who show a $1.5n - o(n)$ lower bound for 3-party secure computation (in the presence of a single corrupted party) where n is the total input length of all the parties. Since insecure computation can be achieved by communicating just n bits, this shows that *achieving security has its price*. However, here again, there is a large gap in communication between known protocols and lower bounds.

Interestingly, the two problems we just discussed, namely PIR and MPC, turn out to be equivalent to each other. A beautiful result of Ishai and Kushilevitz [24] tells us that any k-server PIR protocol gives us a $(k+1)$-party secure computation protocol tolerating a single corruption with nearly the same communication complexity, and vice versa. In particular, in the case of 3 parties, the PIR protocol of [15] gives us a 3-party computation protocol for arbitrary functions with communication $2^{\tilde{O}(\sqrt{n})}$ where n is the total input size.

Private Simultaneous Messages (PSM). Feige, Kilian and Naor [16] considered multiparty computation in a very structured and clean model called the private simultaneous messages (PSM) model (inspired by the simultaneous messages model of Babai, Kimmel and Lokam [2]). In the two-party PSM setting, Alice has an input x and Bob has input y, and they share a common random string which is unknown to the outside world. They send a single message each to a referee called Charlie. In turn, after receiving these messages, Charlie should be able to learn $f(x, y)$ (for a fixed, publicly known, function f) but nothing else about x or y. Since neither Bob nor Alice receive any additional information in the course of the protocol, they learn nothing about each other's input.

Again, there are large gaps between lower and upper bounds in this model. Feige, Kilian and Naor showed that there are functions that require $1.5n - o(n)$ bits of communication in this model (where n is the total input length of Alice and Bob). In a recent work, Beimel, Ishai, Kumaresan and Kushilevitz [4] showed that every function f has a PSM protocol with communication $2^{n/4}$ where n again is the total input length. Narrowing this gap is an important open problem.

Conditional Disclosure of Secrets (CDS). Two-party conditional disclosure of secrets (CDS) first defined by Gertner, Ishai, Kushilevitz and Malkin [21] is an important special case of PSM: two parties want to disclose a secret to a third party if and only if their respective inputs satisfy some fixed predicate ϕ. Concretely, Alice holds x, Bob holds y and in addition, they both hold a secret $m \in \{0, 1\}$ (along with some additional private randomness w). Charlie knows both x and y but not m; Alice and Bob want to disclose m to Charlie iff

$\phi(x, y) = 1$. How many bits do Alice and Bob need to communicate to Charlie?

This is a very simple and natural model where non-private computation requires very little communication (just a single bit), whereas the best upper bound for private computation is exponential. Indeed, in the non-private setting, Alice or Bob can send m to Charlie, upon which Charlie computes $\phi(x, y)$ and decides whether to output m or $\perp$. This trivial protocol with one-bit communication is not private because Charlie learns m even when the predicate is false. In contrast, in the private setting, we have a big gap between upper and lower-bounds. The best upper bound we have for CDS for general predicates ϕ requires that Alice and Bob each transmits $2^{\tilde{O}(\sqrt{n})}$ bits [29]. This upper bound works by translating a special type of PIR protocol into a CDS scheme. Indeed, the communication complexity of $2^{\tilde{O}(\sqrt{n})}$ is closely related to that of the Dvir-Gopi 2-server PIR scheme.

The best known lower bound is $\Omega(\log n)$ [18, 1] which is a double-exponential factor away from the upper bound! A central open problem is to narrow this gap; a concrete question in this direction is to improve the lower bound to $\Omega(n)$ even for a non-explicit function. On this note, we remark that [18] show an $\Omega(\sqrt{n})$ lower bound for the inner product predicate for special type of CDS protocols where Charlie's reconstruction algorithm is a linear function computed on the messages of Alice and Bob.

One could of course consider multi-party versions of both PSM and CDS. Recently, [30] showed a CDS protocol that achieves the same complexity of $2^{\tilde{O}(\sqrt{n})}$ even for arbitarily many parties.

Secret Sharing. The classical problem of secret sharing [32], more precisely non-threshold secret sharing [25], is closely related to multiparty CDS (see, e.g., [30] for more details on this connection). For general non-threshold secret-sharing schemes, the best upper bounds on the (individual) share size are exponential in the number of parties n, namely $2^{n-o(n)}$, whereas the best lower bounds are nearly linear [12], namely $\Omega(n/\log n)$ (see Beimel's survey [3] for more details).

In summary, there is a rich landscape of problems in information-theoretic cryptography, all closely related to secure computation, where there is a large gap between known upper and lower bounds on their communication complexity. Furthermore, there are non-trivial relations between all these problems. For example, MPC and PIR are equivalent modulo computational considerations [24]; PSM is a special type of MPC with a restricted communication pattern; multi-server PIR protocols of a special form give us CDS protocols with an equivalent communication complexity [29]; and multi-party CDS protocols are closely related to secret sharing. Despite recent progress [4, 14, 1, 29, 30], much remains to be uncovered in this world, eventually leading us to a better understanding of the question: *What is the true communication overhead of secure computation?*

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