On Property Testing of the Binary Rank

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Abstract

Let M be an $n \times m$ $(0,1)$-matrix. We define the s-binary rank, denoted as $\mathrm{br}_s(M)$, of M as the minimum integer d such that there exist d monochromatic rectangles covering all the 1-entries in the matrix, with each 1-entry being covered by at most s rectangles. When $s = 1$, this corresponds to the binary rank, denoted as $\mathrm{br}(M)$, which is well-known in the literature and has many applications.

Let $R(M)$ and $C(M)$ denote the sets of rows and columns of M, respectively. Using the result of Sgall [10], we establish that if M has an s-binary rank at most d, then $|R(M)| \cdot |C(M)| \le \binom{d}{\le s} 2^d$, where $\binom{d}{\le s} = \sum_{i=0}^{s} \binom{d}{i}$. This bound is tight, meaning that there exists a matrix M' with an s-binary rank of d, for which $|R(M')| \cdot |C(M')| = \binom{d}{\le s} 2^d$.

Using this result, we present novel one-sided adaptive and non-adaptive testers for $(0,1)$-matrices with an s-binary rank at most d (and exactly d). These testers require $\tilde{O}\left(\binom{d}{\le s} 2^d/\epsilon\right)$ and $\tilde{O}\left(\binom{d}{\le s} 2^d/\epsilon^2\right)$ queries, respectively.

For a fixed s, this improves upon the query complexity of the tester proposed by Parnas et al. in [9] by a factor of $\tilde{\Theta}(2^d)$.

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1 Introduction

Let M be an $n \times m$ $(0,1)$-matrix. We define the s-binary rank, denoted as $\mathrm{br}_s(M)$, of M as the minimum integer d such that there exist d sets (rectangles) $I_k \times J_k$, where $I_k \subseteq [n] := \{1, \ldots, n\}$ and $J_k \subset [m]$ for $k \in [d]$ that satisfy the conditions: for every $(i,j) \in [n] \times [m]$ where $M[i,j] = 1$, there is at least one and at most s integer $t \in [d]$ such that $(i,j) \in I_t \times J_t$ (each 1-entry in M is covered by at least one and at most s monochromatic rectangles). Additionally, $M[i,j] = 1$ for all $(i,j) \in I_k \times J_k$ for $k \in [d]$ (monochromatic rectangles).

When $s = 1$, the s-binary rank $\mathrm{br}_1(M)$ is known as the binary rank, denoted as $\mathrm{br}(M)$. When $s = \infty$, the s-binary rank $\mathrm{br}_\infty(M)$ is referred to as the Boolean rank. Both of these concepts are well-known in the literature. You can explore many applications of these concepts by referring to notable sources such as Amilhastre and Vigneron [1], Chalermsook et al. [3], and Gregory et al. [5]. These references, along with the internal citations, provide an extensive collection of related works with additional applications.

The binary rank can also be defined as follows: the binary rank of an $n \times m$ $(0,1)$-matrix M is equal to the minimum d such that there exist an $n \times d$ $(0,1)$-matrix N and a $d \times m$ $(0,1)$-matrix L satisfying $M = NL$. The binary rank can also be interpreted as the minimum number of bipartite cliques required to partition all the edges of a bipartite graph with adjacency matrix M. Similarly, the s-binary rank of M is the minimum number of bipartite cliques needed to cover all the edges of a bipartite graph with adjacency matrix M, with each edge being covered by at most s bipartite cliques. In [3], Chalermsook et al, show that approximating the binary rank within a factor of $n^{1-\delta}$ for any given δ is NP-hard.

A *property-testing* algorithm, also known as a *tester*, for the s-binary rank [9], takes as input $0 < \epsilon < 1$, integers d, n, and m, and has query access to the entries of an $n \times m$ $(0,1)$-matrix M. If M has an s-binary rank at most d (or exactly d), the tester accepts with

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probability at least $2/3$. If M is ϵ-far from having an s-binary rank at most d (or exactly d), meaning that modifying more than an ϵ-fraction of the entries of M is required to obtain a matrix with an s-binary rank at most d (or exactly d), then the tester rejects with probability at least $2/3$. If the tester accepts matrices with an s-binary rank at most d (or exactly d) with probability 1, it is referred to as a *one-sided error tester.* In *adaptive testing*, the queries can depend on the answers to previous queries, while in *non-adaptive testing*, all queries are predetermined by the tester in advance. The objective is to construct a tester that makes the fewest number of queries possible.

The testability of the s-binary rank at most d of $(0,1)$-matrices was studied in [8, 9]. In [8], Nakar and Ron presented a non-adaptive one-sided error tester for $s = 1$ that makes $\tilde{O}(2^{4d}/\epsilon^4)$ queries. In [9], Parnas et al. gave non-adaptive and adaptive one-sided error testers for $s = 1$ that makes $O(2^{2d}/\epsilon^2)$ and $O(2^{2d}/\epsilon)$ queries, respectively. The results presented in [9] also hold to the s-binary rank at most d.

In this paper, we establish the following theorems for the testability of the s-binary rank at most d (or exactly d):

▶ **Theorem 1.** *There exists an adaptive one-sided error tester for s-binary rank of $n \times m$ $(0,1)$-matrices that makes $\tilde{O}\left(\binom{d}{\le s}2^d/\epsilon\right)$ queries.*

▶ **Theorem 2.** *There exists a non-adaptive one-sided error tester for s-binary rank of $n \times m$ $(0,1)$-matrices that makes $\tilde{O}\left(\binom{d}{\le s}2^d/\epsilon^2\right)$ queries.*

For fixed s, this improves the query complexity of Parnas et al. in [9] by a factor of $\tilde{O}(2^d)$.

1.1 Our Approach

The tester of Parnas et al. [9] uses the fact that if M' is a $k \times k$ sub-matrix of M and M' is of s-binary rank at most d, then the following properties hold:

1. M' has at most 2^d distinct rows and at most 2^d distinct columns.
2. If M is ϵ-far from having s-binary rank at most d, then extending M' by one more uniformly at random row and column from M yields a $(k+1) \times (k+1)$ sub-matrix M'' of M that, with probability at least $\Omega(\epsilon)$, satisfies: either the number of distinct rows in M'' is greater by one than the number of distinct rows in M', or the number of distinct columns in M'' is greater by one than the number of distinct columns in M'.

So, their adaptive tester starts from an empty matrix $M' = (\ )$ and then runs $O(2^d/\epsilon)$ iterations. At every iteration, if M' is of size $(2^d+1) \times (2^d+1)$ it rejects. Otherwise, it extends M' by uniformly at random one row and one column. Let M'' be the resulting sub-matrix. If the s-binary rank of M'' is greater than d, the tester rejects. If the number of distinct rows or columns in M'' is greater than the number in M', then it continues to the next iteration with $M' \leftarrow M''$. Otherwise, it continues to the next iteration with M'. If, after $O(2^d/\epsilon)$ iterations, M' has s-binary rank at most d, the tester accepts.

If the s-binary rank of M is at most d, then every sub-matrix of M has an s-binary rank at most d, and the tester accepts. If M is ϵ-far from having s-binary rank at most d, then: since, at each iteration, with probability at least $\Omega(\epsilon)$, the number of distinct rows or columns of M' is increased by one, and since matrices of s-binary rank d have at most 2^d distinct rows and at most 2^d distinct columns, with high probability, we either obtain M' with an s-binary rank greater than d, or M' reaches to the dimension of $(2^d+1) \times (2^d+1)$. In both cases, the tester rejects. The query complexity of the tester is $O(2^{2d}/\epsilon)$, which is the worst-case number of the entries of the matrix M', $O(2^{2d})$, times the number of trials $O(1/\epsilon)$ for extending M' by one row and one column.

We now give our approach. Call a sub-matrix M' of M *perfect* if it has distinct rows and distinct columns. Our adaptive tester uses the fact that if M' is a perfect $k \times k'$ sub-matrix of M of s-binary rank d, then

1. $kk' \le \binom{d}{\le s}2^d$.
2. If M is ϵ-far from having s-binary rank at most d, then at least one of the following occurs[1]
   a. With probability at least $\Omega(\epsilon)$, extending M' by one uniformly at random column of M, gives a perfect $k \times (k'+1)$ sub-matrix M'' of M.
   b. With probability at least $\Omega(\epsilon)$, extending M' by one uniformly at random row of M, gives a perfect $(k+1) \times k'$ sub-matrix M'' of M.
   c. With probability at least $\Omega(\epsilon)$, extending M' by one uniformly at random column and one uniformly at random row of M, gives a perfect[2] $(k+1) \times (k'+1)$ sub-matrix M'' of M.

Item 1 follows from Sgall's result in [10] (See Section 3), and item 2 is Claim 10 in Parnas et al [9]. Now, the tester's strategy is as follows. If $k \le k'$, the tester first tries to extend M' with a new column. If it succeeds, it moves to the next iteration. Otherwise, it tries to extend M' with a new row. If it succeeds, it moves to the next iteration. Otherwise, it tries to extend M' with a new row and a new column. If it succeeds, it moves to the next iteration. If it fails, it accepts. If $k' < k$, it starts with the row, then the column, and then both.

Using this strategy, we show that the query complexity will be, at most, the order of the size $kk' \le \binom{d}{\le s}2^d$ of M' times the number of trials, $\tilde{O}(1/\epsilon)$, to find the new row, column, or both. This achieves the query complexity in Theorem 1.

For the non-adaptive tester, the tester, uniformly at random, chooses $t = \tilde{O}\left(\binom{d}{\le s}2^d/\epsilon^2\right)$ rows $r_1, \ldots, r_t \in [n]$ and t columns $c_1, \ldots, c_t \in [m]$ and queries all $M[r_i, c_j]$ for all $i \cdot j \le t$ and puts them in a table. Then it runs the above non-adaptive tester. When the non-adaptive tester asks for uniformly at random row or column, it provides the next element r_i or c_j, respectively. The queries are then answered from the table. We show that the adaptive algorithm does not need to make queries that are not in the table before it halts. This achieves the query complexity in Theorem 2.

1.2 Other Rank Problems

The *real rank* of a $n \times m$-matrix M over any field F is the minimal d, such that there is a $n \times d$ matrix N over F and a $d \times m$ matrix L over F such that $M = NL$. The testability of the real rank was studied in [2, 6, 7]. In [2], Balcan et al. gave a non-adaptive tester for the real rank that makes $\tilde{O}(d^2/\epsilon)$ queries. They also show that this query complexity is optimal.

The Boolean rank (∞-binary rank) was studied in [8, 9]. Parnas et al. in [9] gave a non-adaptive tester for the Boolean rank that makes $\tilde{O}(d^4/\epsilon^4)$ queries[3].

[1] Note that what we have in a-c is not precisely what we use in the algorithm and its proof of correctness. For the exact statement, please refer to Claim 13. It can be observed that both statements are equivalent, allowing for a change in the constant within the Ω notation.

[2] It may happen that events (a) and (b) do not occur and (c) does

[3] The query complexity in [9] is $\tilde{O}(d^4/\epsilon^6)$. We've noticed that Lemma 3 in [9] is also true when we replace $(\epsilon^2/64)n^2$ with $(\epsilon/4)n^2$. To prove that, in the proof of Lemma 3, replace Modification rules 1 and 2 with the following modification: Modify to 0 all beneficial entries. This gives the result stated here, [4].

2 Definitions and Preliminary Results

Let M be a $n\times m$ $(0,1)$-matrix. We denote by $R(M)$ and $C(M)$ the set of rows and columns of M, respectively. The number of distinct rows and columns of M are denoted by $r(M)=|R(M)|$ and, $c(M)=|C(M)|$, respectively. The *binary rank* of a $n\times m$-matrix M, $\mathrm{br}(M)$, is equal to the minimal d, where there is a $n\times d$ $(0,1)$-matrix N and a $d\times m$ $(0,1)$-matrix L such that $M=NL$.

We define the s-binary rank, $\mathrm{br}_s(M)$, of M to be the minimal integer d such that there are d sets (rectangles) $I_k\times J_k$ where $I_k\subseteq[n]:=\{1,\ldots,n\}, J_k\subset[m], k\in[d]$ such that $M[i,j]=1$ for all $(i,j)\in I_k\times J_k$, $k\in[d]$ (monochromatic rectangles) and for every $(i,j)\in[n]\times[m]$ where $M[i,j]=1$ there are at least one and at most s integers $t\in[d]$ such that $(i,j)\in I_t\times J_t$ (each 1-entry in M is covered by at least one and at most s monochromatic rectangles).

We now prove.

▶ **Lemma 3.** *Let M be a $n\times m$ $(0,1)$-matrix. The s-binary rank of M, $\mathrm{br}_s(M)$, is equal to the minimal integer d, where there is a $n\times d$ $(0,1)$-matrix N and a $d\times m$ $(0,1)$-matrix L such that: For $P=NL$,*

1. *For every $(i,j)\in[n]\times[m]$, $M[i,j]=0$ if and only if $P[i,j]=0$.*
2. *For every $(i,j)\in[n]\times[m]$, $P[i,j]\le s$.*

Proof. If M is of s-binary rank d, then there are rectangles $\{I_k\times J_k\}_{k\in[d]}$, $I_k\subseteq[n], J_k\subset[m], k\in[d]$ such that $M[i,j]=1$ for all $(i,j)\in I_k\times J_k$, $k\in[d]$ and for every $(i,j)\in[n]\times[m]$ where $M[i,j]=1$ there are at least one and at most s integers $t\in[d]$ such that $(i,j)\in I_t\times J_t$. Define row vectors $a^{(k)}\in\{0,1\}^n$ and $b^{(k)}\in\{0,1\}^m$ where $a^{(k)}_i=1$ iff (if and only if) $i\in I_k$, and $b^{(k)}_j=1$ iff $j\in J_k$. Then define[4] $P=a^{(1)\prime}b^{(1)}+\cdots+a^{(d)\prime}b^{(d)}$. It is easy to see that $(a^{(k)\prime}b^{(k)})[i,j]=1$ iff $(i,j)\in I_k\times J_k$. Therefore, $P[i,j]=0$ iff $M[i,j]=0$ and $P[i,j]\le s$ for all $(i,j)\in[n]\times[m]$. Define the $n\times d$ matrix $N=\left[a^{(1)\prime}|\cdots|a^{(d)\prime}\right]$ and the $d\times m$ matrix $L=\left[b^{(1)\prime}|\cdots|b^{(d)\prime}\right]'$. It is again easy to see that $P=NL$.

The other direction can be easily seen by tracing backward in the above proof. ◀

We now prove the following,

▶ **Lemma 4.** *Let P be a $n\times m$ matrix. Let N and L be $n\times d$ $(0,1)$-matrix and $d\times m$ $(0,1)$-matrix, respectively, such that $P=NL$. Then $r(P)\le r(N)$ and $c(P)\le c(L)$.*

Proof. We prove the result for r. The proof for c is similar. Let $r_1,\ldots,r_n$ be the rows of N and $p_1,\ldots,p_n$ be the rows of P. Then $p_i=r_iL$. Therefore, if $r_i=r_j$, then $p_i=p_j$. Thus, $r(P)\le r(N)$. ◀

Let M be a $n\times m$ matrix. For $x\in X\subseteq[n]$, $y\in Y\subseteq[m]$, we denote by $M[X,Y]$ the $|X|\times|Y|$ sub-matrix of M, $(M[x',y'])_{x'\in X,y'\in Y}$. Denote by $M[X,y]$ the column vector $(M[x',y])_{x'\in X}$ and by $M[x,Y]$ the row vector $(M[x,y'])_{y'\in Y}$.

For $x\in[n]$ (resp. $y\in[m]$) we say that $M[X,y]$ is a *new column* (resp. $M[x,Y]$ is a *new row*) to $M[X,Y]$ if it is not equal to any of the columns (resp. rows) of $M[X,Y]$.

▶ **Lemma 5.** *Let M be a $n\times m$ matrix, $x\in[n], X\subseteq[n]$, $y\in[m]$, and $Y\subseteq[m]$. Suppose $M[x,Y]$ is not a new row to $M[X,Y]$, and $M[X,y]$ is not a new column to $M[X,Y]$. Then $M[x,Y\cup\{y\}]$ is not a new row to $M[X,Y\cup\{y\}]$ if and only if $M[X\cup\{x\},y]$ is not a new column to $M[X\cup\{x\},Y]$.*

[4] Here x' is the transpose of x.

Proof. If $M[x, Y\cup\{y\}]$ is not a new row to $M[X, Y\cup\{y\}]$, then there is $x'\in X$ such that $M[x, Y\cup\{y\}] = M[x', Y\cup\{y\}]$. Since $M[X, y]$ is not a new column to $M[X, Y]$, there is $y'\in Y$ such that $M[X, y] = M[X, y']$. Since $M[x, Y\cup\{y\}] = M[x', Y\cup\{y\}]$, we have $M[x', y'] = M[x, y']$ and $M[x, y] = M[x', y]$. Since $M[X, y] = M[X, y']$, we have $M[x', y] = M[x', y']$. Therefore, $M[x, y] = M[x, y']$ and $M[X\cup\{x\}, y] = M[X\cup\{x\}, y']$. Thus, $M[X\cup\{x\}, y]$ is not a new column to $M[X\cup\{x\}, Y]$.

Similarly, the other direction follows. ◀

3 Matrices of s-Binary Rank d

In this section, we prove the following two Lemmas.

▶ **Lemma 6.** *For any $n\times m$ $(0,1)$-matrix M of s-binary rank at most d, we have*

$$r(M)\cdot c(M)\le \binom{d}{\le s}2^d.$$

▶ **Lemma 7.** *There is a $(0,1)$-matrix M' of s-binary rank d that satisfies $r(M')\cdot c(M') = \binom{d}{\le s}2^d$.*

To prove Lemma 6, we use the following Sgall's lemma.

▶ **Lemma 8.** *[10]. Let $\mathcal{A},\mathcal{B}\subseteq 2^{[d]}$ be such that for every $A\in\mathcal{A}$ and $B\in\mathcal{B}$, $|A\cap B|\le s$. Then $|\mathcal{A}|\cdot|\mathcal{B}|\le\binom{d}{\le s}2^d$.*

We now prove Lemma 6.

Proof. Since the s-binary rank of M is at most d, by Lemma 3, there is a $n\times d$ $(0,1)$-matrix N and a $d\times m$ $(0,1)$-matrix L such that, for $P = NL$

1. For every $(i,j)\in[n]\times[m]$, $M[i,j] = 0$ if and only if $P[i,j] = 0$.
2. For every $(i,j)\in[n]\times[m]$, $P[i,j]\le s$.

Obviously, $r(M)\le r(P)$ and $c(M)\le c(P)$. Consider $\mathcal{A} = \{A_1,\ldots,A_n\}\subseteq 2^{[d]}$ and $\mathcal{B} = \{B_1,\ldots,B_m\}\subseteq 2^{[d]}$, where $A_i = \{j|N_{i,j} = 1\}$ and $B_k = \{j|L_{j,k} = 1\}$. Since the entries of $P = NL$ are at most s, for every $i\in[n]$ and $k\in[m]$, $|A_i\cap B_k|\le s$.

By Lemma 4 and 8,

$$r(M)\cdot c(M)\le r(P)\cdot c(P)\le r(N)\cdot c(L) = |\mathcal{A}|\cdot|\mathcal{B}|\le\binom{d}{\le s}2^d.$$ ◀

We now prove Lemma 7.

Proof of Lemma 7. Let N be a $2^d\times d$ $(0,1)$-matrix where its rows contain all the vectors in $\{0,1\}^d$. Let L be a $d\times\binom{d}{\le s}$ matrix where its columns contain all the vectors in $\{0,1\}^d$ of weight at most s. Obviously, $P = NL$ is $2^d\times\binom{d}{\le s}$ with entries that are less than or equal to s. Define a $2^d\times\binom{d}{\le s}$ $(0,1)$-matrix M' where $M'[i,j] = 0$ if and only if $P[i,j] = 0$. Then, by Lemma 3, M' is of s-binary rank at most d. We now show that $r(M')\cdot c(M') = \binom{d}{\le s}2^d$.

Since the identity $d\times d$ matrix I_d is a sub-matrix of L, we have that $NI_d = N$ is $(0,1)$-matrix and a sub-matrix of P and therefore of M'. Therefore, $r(M')\ge r(N) = 2^d$. Since I_d is a sub-matrix of N, by the same argument, $c(M')\ge c(L) = \binom{d}{\le s}$. Therefore $r(M')\cdot c(M')\ge\binom{d}{\le s}2^d$. Thus, $r(M')\cdot c(M') = \binom{d}{\le s}2^d$.

We now show that M' has s-binary rank d. Suppose the contrary, i.e., M' has binary rank $d' < d$. Then there are $2^d\times d'$ $(0,1)$-matrix N and $d'\times\binom{d}{\le s}$ $(0,1)$-matrix L such that $P = NL$ and $M'[i,j] = 0$ iff $P[i,j] = 0$. Now by Lemma 4, $r(M')\le r(P)\le r(N)\le 2^{d'} < 2^d$, which gives a contradiction. ◀

4 Testing The s-Binary Rank

In this section, we present the adaptive and non-adaptive testing algorithms for s-binary rank at most d. We first give the adaptive algorithm and prove Theorem 1.

4.1 The Adaptive Tester

Adaptive-Test-Rank$(d, s, M, n, m, \epsilon)$
Input: Oracle that accesses the entries of $n \times m$ $(0,1)$-matrix M.
Output: Either "Accept" or "Reject"

$X \leftarrow \{1\}$; $Y \leftarrow \{1\}$; $t = 9d/\epsilon$.
While $|X| \cdot |Y| \le \binom{d}{\le s} 2^d$ **do**
  If the s-binary rank of $M[X, Y]$ is greater than d, **then** Reject.
  $Finish \leftarrow False$; $X' \leftarrow \emptyset$; $Y' \leftarrow \emptyset$. /* X' and Y' are multi-sets.
  If $|X| \ge |Y|$ **then**
    While (NOT $Finish$) AND $|X'| < t$
      Draw uniformly at random $x \in [n] \backslash X$; $X' \leftarrow X' \cup \{x\}$;
      If $M[x, Y]$ is a new row to $M[X, Y]$
        then $X \leftarrow X \cup \{x\}$; $Finish \leftarrow True$.
    If (NOT $Finish$) **then**
      While (NOT $Finish$) AND $|Y'| < t$
        Draw uniformly at random $y \in [m] \backslash Y$; $Y' \leftarrow Y' \cup \{y\}$.
        If $M[X, y]$ is new column to $M[X, Y]$
          then $Y \leftarrow Y \cup \{y\}$; $Finish \leftarrow True$.
  Else $(|X| < |Y|)$
    While (NOT $Finish$) AND $|Y'| < t$
      Draw uniformly at random $y \in [m] \backslash Y$; $Y' \leftarrow Y' \cup \{y\}$;
      If $M[X, y]$ is a new column to $M[X, Y]$
        then $Y \leftarrow Y \cup \{y\}$; $Finish \leftarrow True$.
    If (NOT $Finish$) **then**
      While (NOT $Finish$) AND $|X'| < t$
        Draw uniformly at random $x \in [n] \backslash X$; $X' \leftarrow X' \cup \{x\}$
        If $M[x, Y]$ is a new row to $M[X, Y]$
          then $X \leftarrow X \cup \{x\}$; $Finish \leftarrow True$.
  While (NOT $Finish$) AND $X' \ne \emptyset$ **do**
    Draw uniformly at random $x \in X'$ and $y \in Y'$
      If $M[x, Y \cup \{y\}]$ is a new row to $M[X, Y \cup \{y\}]$ OR, equivalently,
        $M[X \cup \{x\}, y]$ is a new column to $M[X \cup \{x\}, Y]$
          then $X \leftarrow X \cup \{x\}; Y \leftarrow Y \cup \{y\}$; $Finish \leftarrow True$.
          else $X' \leftarrow X' \backslash \{x\}; Y' \leftarrow Y' \backslash \{y\}$.
  If (NOT $Finish$) **then** Accept
Reject

Figure 1 An adaptive tester for s-binary rank at most d.

In this section, we prove Theorem 1.

Consider the tester **Adaptive-Test-Rank** in Figure 1. The tester, at every iteration of the main While-loop (step 2) has a set X of rows of M and a set Y of columns of M. If $|X| \geq |Y|$ (step 5), the tester first tries to extend $M[X,Y]$ with a new column (steps 6-8). If it succeeds, it moves to the next iteration. Otherwise, it tries to extend $M[X,Y]$ with a new row (steps 9-12). If it succeeds, it moves to the next iteration. Otherwise, it tries to extend $M[X,Y]$ with a new row and a new column (steps 21-26). If it succeeds, it moves to the next iteration. If it fails, it accepts (step 27). If $|X| < |Y|$ (step 13), it starts with the row of $M[X,Y]$ (steps 14-16), then the column (steps 18-20), and then both (steps 21-26). If it fails, it accepts (step 27).

If $|X| \cdot |Y| > \binom{d}{\leq s}2^d$ (step 2 and then step 28) or the s-binary rank of $M[X,Y]$ is greater than d (step 3), then it rejects.

We first prove

▶ **Lemma 9.** *Let $t = 9d/\epsilon$. Tester* **Adaptive-Test-Rank** *makes at most $2\binom{d}{\leq s}2^d t = \tilde{O}\left(\binom{d}{\leq s}2^d\right)/\epsilon$ queries.*

Proof. We prove by induction that at every iteration of the main While-loop (step 2), the tester knows the entries of $M[X,Y]$, and the total number of queries, $q_{X,Y}$, is at most $2|X||Y|t$. Since the While-loop condition is $|X||Y| \leq \binom{d}{\leq s}2^d$, the result follows.

At the beginning of the algorithm, no queries are made, and $|X| = |Y| = 1$. Then $2|X||Y|t = 2t > 0 = q_{X,Y}$. Suppose, at the kth iteration, the tester knows the entries of $M[X,Y]$ and $q_{X,Y} \leq 2|X||Y|t$. We prove the result for the $(k+1)$th iteration.

We have the following cases (at the $(k+1)$th iteration)

Case I. $|X| \geq |Y|$ (step 5) and, for some x, $M[x,Y]$ is a new row to $M[X,Y]$ (step 8).
In that case, *Finish* becomes *true*, and no other sub-while-loop is executed. Therefore, the number of queries made at this iteration is at most $|Y|t$ (to find all $M[x,Y]$), and one element x is added to X. Then, the tester knows all the entries of $M[X \cup \{x\}, Y]$ and

$$q_{X\cup\{x\},Y} = q_{X,Y} + |Y|t \leq 2|X||Y|t + |Y|t \leq 2|X \cup \{x\}| \cdot |Y|t,$$

and the result follows.

Case II. $|X| \geq |Y|$ (step 5), for all $x' \in X'$, $M[x',Y]$ is not a new row to $M[X,Y]$ (step 8), and for some y, $M[X,y]$ is a new column to $M[X,Y]$ (step 12).
In that case, *Finish* becomes *true*, and no other sub-while-loop is executed after the second sub-while-loop (step 10).
Therefore, in this case, the number of queries made at this iteration is at most $|Y|t + |X|t$. $|X|t$ queries in the first sub-while-loop (to find $M[x,Y]$ for all $x \in X'$), and at most $|Y|t$ queries in the second sub-while-loop (to find $M[X,y']$ for all $y' \in Y'$). Then one element y is added to Y. Therefore, the tester knows the entries of $M[X, Y \cup \{y\}]$ and, since $|Y| \leq |X|$,

$$q_{X,Y\cup\{y\}} = q_{X,Y} + |X|t + |Y|t \leq 2|X||Y|t + 2|X|t = 2|X| \cdot |Y \cup \{y\}|t,$$

and the result follows.

Case III. $|X| \geq |Y|$, for all $x' \in X'$, $M[x',Y]$ is not a new row to $M[X,Y]$, for all $y' \in Y'$, $M[X,y']$ is not a new column to $M[X,Y]$, and for some $x \in X', y \in Y'$, $M[x, Y \cup \{y\}]$ is a new row to $M[X, Y \cup \{y\}]$ (step 23).

In this case, $|X'| = |Y'| = t$, the number of queries is $|X|t + |Y|t + t$. Exactly $|X|t$ queries in the first sub-while-loop, $|Y|t$ queries in the second sub-while-loop, and at most[5] t queries in the sub-while-loop in step 21. Then one element x is added to X, and one element y is added to Y. Then the tester knows the entries of $M[X \cup \{x\}, Y \cup \{y\}]$ and

$$q_{X\cup\{x\},Y\cup\{y\}} = q_{X,Y} + |X|t + |Y|t + t \le 2|X|\cdot|Y|t + |X|t + |Y|t + t \le 2|X\cup\{x\}|\cdot|Y\cup\{y\}|t.$$

Case IV. $|X| \ge |Y|$, for all $x' \in X'$, $M[x', Y]$ is not a new row to $M[X, Y]$, for all $y' \in Y'$, $M[X, y']$ is not a new column to $M[X, Y]$, and for all the drawn pairs $x \in X', y \in Y'$, $M[x, Y \cup \{y\}]$ is not a new row to $M[X, Y \cup \{y\}]$ (step 23).
In this case, $Finish$ will have value $False$, and the tester accepts in step 27.
The analysis of the case when $|X| < |Y|$ is similar to the above analysis. ◀

We now prove the completeness of the tester.

▶ **Lemma 10.** *If M is a $n \times m$ $(0,1)$-matrix of s-binary rank at most d, then the tester* **Adaptive-Test-Rank** *accepts with probability* 1.

Proof. The tester rejects if and only if one of the following occurs,
1. $M[X, Y]$ has s-binary rank greater than d.
2. $|X| \cdot |Y| > \binom{d}{\le s} 2^d$.

If $M[X, Y]$ has s-binary rank greater than d, then M has s-binary rank greater than d. This is because, if $M = NL$, then $M[X, Y] = N[X, [d]] \cdot L[[d], Y]$. So item 1 cannot occur.
Before we show that item 2 cannot occur, we prove the following:

▷ Claim 11. The rows (resp. columns) of $M[X, Y]$ are distinct.

Proof. The steps in the tester where we add rows or columns are steps 8, 12 16, 20, and 23. In steps 8, 12 16, 20 it is clear that a row (resp. column) is added only if it is a new row (resp. column) to $M[X, Y]$. Consider step 23 and suppose, w.l.o.g $|X| \ge |Y|$. This step is executed only when $Finish = False$. This happens when $|X'| = |Y'| = t$, for every $x \in X'$, $M[x, Y]$ is not a new row to $M[X, Y]$, and for every $y \in Y'$, $M[X, y]$ is not a new column to $M[X, Y]$. Then x and y are added to X and Y, respectively, if $M[x, Y \cup \{y\}]$ is a new row to $M[X, Y \cup \{y\}]$. Then, by Lemma 5, $M[X \cup \{x\}, y]$ is a new column to $M[X \cup \{x\}, Y]$. So, the rows (and columns) in $M[X \cup \{x\}, Y \cup \{y\}]$ are distinct. This implies the result. ◁

Suppose, to the contrary, $|X| \cdot |Y| > \binom{d}{\le s} 2^d$. Since $M' = M[X, Y]$ satisfies $r(M')c(M') = |X| \cdot |Y| > \binom{d}{\le s} 2^d$, by Lemma 6, the s-binary rank of M', and therefore of M, is greater than d. A contradiction. ◀

We now prove the soundness of the tester.
We first prove the following.

▷ Claim 12. Let M be a $n \times m$ $(0,1)$-matrix, $X \subseteq [n]$, and $Y \subseteq [m]$. Suppose there are two functions, $' : [n] \to X$ and $'' : [m] \to Y$, such that
1. For every $x \in [n]$, $M[x, Y] = M[x', Y]$.
2. For every $y \in [m]$, $M[X, y] = M[X, y'']$.
3. For every $x \in [n]$ and $y \in [m]$, $M[x, y] = M[x', y'']$.

Then M has at most $|X|$ distinct rows and $|Y|$ distinct columns, and its s-binary rank is the s-binary rank of $M[X, Y]$.

[5] This is because, for $x \in X', y \in Y'$, the tester already knows $M[x, Y]$ and $M[X, y]$ from the first and second sub-while-loop and only needs to query $M[x, y]$.

Proof. Let $x \in [n]\backslash X$. For every y, $M[x,y] = M[x',y''] = M[x',y]$. Therefore, row x in M is equal to row x'. Similarly, column y in M is equal to column y''.

Since adding equal columns and rows to a matrix does not change the s-binary rank[6], we have $\mathrm{br}_s(M[X,Y]) = \mathrm{br}_s(M[X,[m]]) = \mathrm{br}_s(M)$. ◁

The following Claim is proved in [9] (Claim 10). Here, we give the proof for completeness.

▷ Claim 13. Let M be a $(0,1)$-matrix that is ϵ-far from having s-binary rank at most d. Let $X \subseteq [n]$ and $Y \subseteq [m]$, such that $\mathrm{br}_s(M[X,Y]) \le d$, the columns of $M[X,Y]$ are distinct, and the rows of $M[X,Y]$ are distinct. Then one of the following must hold:

1. The number of rows $x \in [n]$ where $M[x,Y]$ is a new row to $M[X,Y]$ is at least $n\epsilon/3$.
2. The number of columns $y \in [m]$ where $M[X,y]$ is a new column to $M[X,Y]$ is at least $m\epsilon/3$.
3. The number pairs (x,y), $x \notin X$, $y \notin Y$, where, $M[x,Y] = M[x',Y]$ for some $x' \in X$, $M[X,y] = M[X,y'']$ for some $y'' \in Y$, and $M[x,y] \ne M[x',y'']$, is at least $mn\epsilon/3$.

Proof. Assume, to the contrary, that none of the above statements holds. Change every row x in M where $M[x,Y]$ is a new row to $M[X,Y]$ to a zero row. Let X' be the set of such rows. Change every column y in M where $M[X,y]$ is a new row to $M[X,Y]$ to a zero column. Let Y' be the set of such columns. For every other entry (x,y), $x \notin X$, $y \notin Y$ that is not changed to zero and $M[x,y] \ne M[x',y'']$, change $M[x,y]$ to $M[x',y'']$. Let M' be the matrix obtained from the above changes.

The number of entries (x,y) where $M[x,y] \ne M'[x,y]$ is less than $(n\epsilon/3)m + (m\epsilon/3)n + mn\epsilon/3 = \epsilon mn$. Therefore, M' is ϵ-close to M. By claim 13, $\mathrm{br}_s(M') = \mathrm{br}_s(M[[n]\backslash X', [m]\backslash Y'])$ $= \mathrm{br}_s(M[X,Y]) \le d$. A contradiction. ◁

We now prove the soundness of the tester.

▶ **Lemma 14.** *If M is ϵ-far from having s-binary rank d, then with probability at least $2/3$,* **Adaptive-Test-Rank** *rejects.*

Proof. Consider the while-loop in step 2 at some iteration i. If $\mathrm{br}_s(M[X,Y]) > d$, then the tester rejects in step 3. We will now show that if $\mathrm{br}_s(M[X,Y]) \le d$, then, with probability at most $3e^{-2d}$, the tester accepts at iteration i.

To this end, let $\mathrm{br}_s(M[X,Y]) \le d$. Then, by Claim 13, one of the following holds.

1. The number of rows $x \in [n]$ where $M[x,Y]$ is a new row to $M[X,Y]$ is at least $n\epsilon/3$.
2. The number of columns $y \in [m]$ where $M[X,y]$ is a new column to $M[X,Y]$ is at least $m\epsilon/3$.
3. The number pairs (x,y), $x \notin X$, $y \notin Y$, where, $M[x,Y] = M[x',Y]$ for some $x' \in X$, $M[X,y] = M[X,y'']$ for some $y'' \in Y$, and $M[x,y] \ne M[x',y'']$, is at least $mn\epsilon/3$.

Now at the ith iteration, suppose w.l.o.g, $|X| \ge |Y|$ (the other case $|Y| < |X|$ is similar). If item 1 occurs, then with probability at least $p = 1-(1-\epsilon/3)^t \ge 1-e^{-2d}$, the tester finds a new row to $M[X,Y]$ and does not accept at iteration i. If item 2 occurs, then if it does not find a new row to $M[X,Y]$, with probability at least p, the tester finds a new column to $M[X,Y]$ and does not accept. If item 3 occurs, and it does not find a new row or column to $M[X,Y]$, then with probability at least p, it finds such a pair and does not accept. Therefore, with probability at most $3(1-p) \le 3e^{-2d}$, the tester accepts at iteration i.

[6] If we add a column to a matrix that is equal to column y, then the rectangles that cover column y can be extended to cover the added column.

Since the while-loop runs at most $|X|+|Y| \le 2|X||Y| \le 2\binom{d}{\le s}2^d \le 2^{2d+1}$ iterations, with probability at most $3e^{-2d}2^{2d+1} \le 1/3$, the tester accepts in while-loop. Therefore, with probability at least $2/3$, the tester does not accept in the while-loop. Thus, it either rejects because $\mathrm{br}_s(M[X,Y]) > d$ or rejects in step 28. ◀

4.2 The Non-Adaptive Tester

In this section, we prove Theorem 2.

Non-Adaptive-Test-Rank$(d, s, M, n, m, \epsilon)$
Input: Oracle that accesses the entries of $(0,1)$-matrix M.
Output: Either "Accept" or "Reject".

1. $T \leftarrow \frac{324 \cdot d^2 \binom{d}{\le s} 2^d}{\epsilon^2}$.
2. Draw uniformly at random $x^{(1)}, \ldots, x^{(T)} \in [n]$.
3. Draw uniformly at random $y^{(1)}, \ldots, y^{(T)} \in [m]$.
4. For every $i \in [T]$ and $j \in [T]$ such that $i \cdot j \le T$
5. $\quad D[i,j] \leftarrow$ Query $M[x^{(i)}, y^{(j)}]$
6. $u = 1; w = 1$.
7. Run **Adaptive-Test-Rank**$(d, s, M, n, m, \epsilon)$
   When the tester asks for a uniform at random x - return $x^{(u)}$; $u \leftarrow u+1$
   When the tester asks for a uniform at random y - return $y^{(w)}$; $w \leftarrow w+1$
   When the tester makes the Query $M[x^{(i)}, y^{(j)}]$ - return $D[i,j]$

Figure 2 A non-adaptive tester for s-binary rank at most d.

First, consider **Adaptive-Test-Rank** in Figure 1. Consider steps 7,11,15, and 19, where it draws a new column or row. We prove.

▶ **Lemma 15.** *Let $t = 9d/\epsilon$. At each iteration of* **Adaptive-Test-Rank***, the total number of uniformly at random rows $x \in [n]$ drawn is at most $(|X| + \min(|X|, |Y|-1))t$, and the number of uniformly at random rows $y \in [m]$ drawn is at most $(|Y| + \min(|X|, |Y|))t$.*

Proof. We prove by induction that at every iteration of the main While-loop (step 2), the total number of random rows drawn by the tester, $n_{X,Y}$, is at most $(|X|+\min(|X|, |Y|-1))t$, and the total number of random columns drawn, $m_{X,Y}$, is at most $(|Y| + \min(|X|, |Y|))t$.

At the beginning, $|X| = |Y| = 1$, and the number of columns and rows is 1. In that case,[7], $n_{X,Y} = 1 \le t$ and $m_{X,Y} = 1 \le 2t$. Suppose, at the kth iteration, the induction statement is true. We prove the result for the $(k+1)$th iteration.

At the $(k+1)$th iteration, we have the following cases.

Case I. $|X| \ge |Y|$ (step 5) and, for some x, $M[x, Y]$ is a new row to $M[X, Y]$ (step 8). In that case, *Finish* becomes *true*, and no other sub-while-loop is executed. Therefore, the number of rows drawn at this iteration is at most t, and one element x is added to X. No columns are drawn. Then,

$$n_{X\cup\{x\},Y} \le n_{X,Y}+t \le (|X|+\min(|X|, |Y|-1)+1)t \le (|X\cup\{x\}|+\min(|X\cup\{x\}|, |Y|-1))t,$$

[7] We assume that the first column/row drawn is column/row one

and

$$m_{X\cup\{x\},Y} = m_{X,Y} \le (|Y| + \min(|X|, |Y|))t \le (|Y| + \min(|X \cup \{x\}|, |Y|))t.$$

Thus, the result follows for this case.

Case II. $|X| \ge |Y|$ (step 5), for all $x' \in X'$, $M[x', Y]$ is not a new row to $M[X, Y]$ (step 8), and for some y, $M[X, y]$ is a new column to $M[X, Y]$ (step 12).
In that case, *Finish* becomes *true*, and no other sub-while-loop is executed after the second sub-while-loop (step 10).
Therefore, in this case, the number of rows drawn at this iteration is t, one element y is added to Y, and the number of columns drawn is at most t. Then

$$\begin{aligned} n_{X,Y\cup\{y\}} = n_{X,Y} + t &\le (|X| + \min(|X|, |Y| - 1) + 1)t \\ &= (|X| + |Y|)t = (|X| + \min(|X|, |Y \cup \{y\}| - 1))t, \end{aligned}$$

and

$$m_{X,Y\cup\{y\}} \le m_{X,Y} + t \le (|Y| + \min(|X|, |Y|) + 1)t \le (|Y \cup \{y\}| + \min(|X|, |Y \cup \{y\}|))t.$$

Thus, the result follows for this case.

Case III. $|X| < |Y|$ (step 13), and for some y, $M[X, y]$ is a new column to $M[X, Y]$ (step 16). In that case, *Finish* becomes *true*, and no other sub-while-loop is executed. Therefore, the number of columns drawn at this iteration is at most t, and one element y is added to Y. No rows are drawn. Then,

$$n_{X,Y\cup\{y\}} = n_{X,Y} \le (|X| + \min(|X|, |Y| - 1))t \le (|X| + \min(|X|, |Y \cup \{y\}| - 1))t,$$

and

$$m_{X,Y\cup\{y\}} \le m_{X,Y} + t \le (|Y| + \min(|X|, |Y|) + 1)t = (|Y \cup \{y\}| + \min(|X|, |Y \cup \{y\}|))t.$$

Thus, the result follows for this case.

Case IV. $|X| < |Y|$ (step 13), for all $y' \in Y'$, $M[X, y']$ is not a new row to $M[X, Y]$, and for some x, $M[x, Y]$ is a new column to $M[X, Y]$ (step 20). In that case, *Finish* becomes *true*, and no other sub-while-loop is executed after the fourth sub-while-loop (step 18). In this case, the number of rows drawn at this iteration is t, one element x is added to X, and the number of columns drawn is at most t. Then

$$\begin{aligned} n_{X\cup\{x\},Y} = n_{X,Y} + t &\le (|X| + \min(|X|, |Y| - 1) + 1)t \\ &\le (|X \cup \{x\}| + \min(|X \cup \{x\}|, |Y| - 1))t \\ m_{X\cup\{x\},Y} \le m_{X,Y} + t &\le (|Y| + \min(|X|, |Y|) + 1)t = (|Y| + \min(|X \cup \{x\}|, |Y|))t. \end{aligned}$$

Thus, the result follows for this case.

Case V. For all $x' \in X'$, $M[x', Y]$ is not a new row to $M[X, Y]$, for all $y' \in Y'$, $M[X, y']$ is not a new column to $M[X, Y]$, and for some $x \in X', y \in Y'$, $M[x, Y \cup \{y\}]$ is a new row to $M[X, Y \cup \{y\}]$ (step 23).
In this case, the number of rows drawn at this iteration is t, the number of columns drawn is t, one element x is added to X, and one element y is added to Y. Then

$$\begin{aligned} n_{X\cup\{x\},Y\cup\{y\}} = n_{X,Y} + t &\le (|X| + \min(|X|, |Y| - 1) + 1)t \\ &\le (|X \cup \{x\}| + \min(|X \cup \{x\}|, |Y \cup \{y\}| - 1))t. \end{aligned}$$

$$\begin{aligned} m_{X\cup\{x\},Y\cup\{y\}} = m_{X,Y} + t &\le (|Y| + \min(|X|, |Y|) + 1)t \\ &\le (|Y \cup \{y\}| + \min(|X \cup \{x\}|, |Y \cup \{y\}|))t. \end{aligned}$$ ◀

We are now ready to prove Theorem 2.

Proof. By Lemma 15, the total number of rows and columns drawn in **Adaptive-Test-Rank** up to iteration t is at most $n' := 9(|X| + \min(|X|, |Y| - 1))d/\epsilon \leq 18|X|d/\epsilon$ and $m' := 9(|Y| + \min(|X|, |Y|)d/\epsilon \leq 18|Y|d/\epsilon$, respectively. We also have $|X| \cdot |Y| \leq \binom{d}{\leq s}2^d$. So

$$n' \cdot m' \leq 324|X||Y|d^2/\epsilon^2 \leq T := \frac{324 \cdot d^2\binom{d}{\leq s}2^d}{\epsilon^2}.$$

Consider the tester **Non-Adaptive-Test-Rank** in Figure 2. The tester draws T rows $x^{(1)}, \ldots, x^{(T)} \in [n]$, and columns $y^{(1)}, \ldots, y^{(T)} \in [m]$ and queries all $M[x^{(i)}, y^{(j)}]$ where $ij \leq T$ and puts the result in the table D. Then it runs **Adaptive-Test-Random** using the above-drawn rows and columns. We now show that all the queries that **Adaptive-Test-Random** makes can be fetched from the table D.

At any iteration, the number of rows drawn is at most n', and the number of rows drawn is at most m'. Therefore, the tester needs to know (in the worst case) all the entries $M[x^{(i)}, y^{(j)}]$ where $i \leq n'$ and $j \leq m'$. Since $ij \leq n'm' \leq T$, the result follows.

The number of queries that the tester makes is

$$\sum_{i=1}^{T} \frac{T}{i} = O(T \ln T) = \tilde{O}\left(\frac{\binom{d}{\leq s}2^d}{\epsilon^2}\right). \qquad \blacktriangleleft$$

5 Testing the Exact s-Binary Rank

We first prove the following.

▶ **Lemma 16.** *Let M and M' be $n \times m$ $(0,1)$-matrices that differ in one row (or column). Then $|\mathrm{br}_s(M) - \mathrm{br}_s(M')| \leq 1$.*

Proof. Suppose $\mathrm{br}_s(M) = d$ and M' differ from M in row k. Let N and L be $n \times d$ $(0,1)$-matrix and $d \times m$ $(0,1)$-matrix, respectively, such that $P = NL$, for every $(i,j) \in [n] \times [m]$, $P[i,j] \leq s$, and $P[i,j] = 0$ if and only if $M[i,j] = 0$. Add to N a column (as a $(d+1)$th column) that all its entries are zero except the k-th entry, which equals 1. Then change $N[k,j]$ to zero for all $j \in [d]$. Let N' be the resulting matrix. Add to L another row (as a $(d+1)$th row) equal to the k-th row of M'. Let L' be the resulting matrix. Let $P' = N'L'$. It is easy to see that $P'[i,j] = P[i,j]$ for all $i \neq k$ and j, and the kth row of P' is equal to the kth row of M'. Then, for every $(i,j) \in [n] \times [m]$, $P'[i,j] \leq s$, and $P'[i,j] = 0$ if and only if $M'[i,j] = 0$. Therefore, $\mathrm{br}_s(M') \leq d + 1 = \mathrm{br}_s(M) + 1$. In the same way, $\mathrm{br}_s(M) \leq \mathrm{br}_s(M') + 1$. ◀

▶ **Lemma 17.** *Let $\eta = d^2/(nm)$. Let M be $n \times m$ $(0,1)$-matrix. If M is ϵ-close to having s-binary rank at most d, then M is $(\epsilon + \eta)$-close to having s-binary rank d.*

Proof. We will show that for every $n \times m$ $(0,1)$-matrix H of s-binary rank at most $d-1$, there is a $n \times m$ $(0,1)$-matrix G of s-binary rank d that is η-close to H. Therefore, if M is ϵ-close to having s-binary rank at most d, then it is $(\epsilon + \eta)$-close to having s-binary rank d.

Define the $n \times m$ $(0,1)$-matrices G_k, $k \in [d] \cup \{0\}$, where $G_0 = H$ and for $k \geq 1$, $G_k[i,j] = H[i,j]$ if $j > k$ or $i > d$, and $G_k[[d],[k]] = I_d[[d],[k]]$ where I_d is the $d \times d$ identity matrix. Since $G_d[[d],[d]] = I_d$, we have $\mathrm{br}_s(G_d) \geq d$. It is clear that for every $k \in [d] \cup \{0\}$, G_k is (d^2/nm)-close to H. If $\mathrm{br}_s(G_d) = d$, then take $G = G_d$, and we are done. Otherwise, suppose $\mathrm{br}_s(G_d) > d$.

Now consider a sequence $H=G_0,G_1,G_2,\dots,G_d$. By Lemma 16, we have $\mathrm{br}_s(G_{i-1})-1\le \mathrm{br}_s(G_i)\le \mathrm{br}_s(G_{i-1})+1$. Now since $\mathrm{br}_s(G_0)=\mathrm{br}_s(H)\le d-1$ and $\mathrm{br}_s(G_d)>d$, by the discrete intermediate value theorem, there must be $k\in[d]$ such that $\mathrm{br}_s(G_k)=d$. Then take $G=G_k$, and we are done. ◀

Now, the tester for testing the s-binary rank d runs as follows. If $mn<2d^2/\epsilon$, then find all the entries of M with $mn<2d^2/\epsilon$ queries. If $\mathrm{br}_s(M)=d$, then accept. Otherwise, reject. If $mn\ge 2d^2/\epsilon$, then run **Adaptive-Test-Rank**$(d,s,M,n,m,\epsilon/2)$ (for the non-adaptive, we run **Non-Adaptive-Test-Rank**$(d,s,M,n,m,\epsilon/2)$) and output its answer.

We now show the correctness of this algorithm. If M is of s-binary rank d, then it is of s-binary rank at most d, and the tester accepts.

Now, suppose f is ϵ-far from having s-binary rank d. If $mn<2d^2/\epsilon$, the tester rejects. If $mn\ge 2d^2/\epsilon$, then, by Lemma 17, f is $(\epsilon-\eta)$-far from having s-binary rank at most d, where $\eta=d^2/(nm)$. Since $\eta=d^2/(nm)\le\epsilon/2$, the function f is $(\epsilon/2)$-far from having s-binary rank at most d, and therefore the tester, with probability at least $2/3$, rejects.

6 Concluson and Open Problems

In this work, we introduced the notion of s-binary rank for $(0,1)$-matrices, extending the concept of binary rank. We established a tight upper bound on the size of matrices with s-binary rank at most d, and showed the existence of matrices achieving this bound. Using this result, we presented novel one-sided adaptive and non-adaptive testers for $(0,1)$-matrices with s-binary rank at most d, significantly improving the query complexity compared to prior work. The adaptive tester requires $\tilde O\left(\binom{d}{\le s}2^d/\epsilon\right)$ queries, while the non-adaptive tester requires $\tilde O\left(\binom{d}{\le s}2^d/\epsilon^2\right)$ queries.

The following are open problems that are worth investigating:

Tighter Bounds on Query Complexity: Investigate whether the query complexity of the testers for $(0,1)$-matrices with s-binary rank at most d can be further improved. Specifically, explore alternative approaches or refinements that can reduce the dependence on $\binom{d}{\le s}$ and 2^d in the query complexity bounds.

Generalization Beyond $(0,1)$-Matrices: Extend the concept of s-binary rank to other types of matrices, such as integer-valued matrices or matrices with entries from a larger alphabet. Study the properties, computational aspects, and property testing of these generalizations.

Addressing these open problems will lead to a more profound understanding of the s-binary rank, provide further insights into the structure of matrices, and potentially lead to improved algorithmic techniques and applications in various fields.

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