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PROBLEMS, SOLUTIONS AND EXPERIENCE OF THE FIRST INTERNATIONAL STUDENT'S OLYMPIAD IN CRYPTOGRAPHY[1]

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A detailed overview of the problems, solutions and experience of the first international student's Olympiad in cryptography, NSUCRYPTO'2014, is given. We start with rules of participation and description of rounds. All 15 problems of the Olympiad and their solutions are considered in detail. There are discussed solutions of the mathematical problems related to cipher constructing such as studying of differential characteristics of S-boxes, S-box masking, determining of relations between cyclic rotation and additions modulo 2 and 2^n, constructing of special linear subspaces in $\mathbb{F}_2^n$; problems about the number of solutions of the equation $F(x)+F(x+a)=b$ over the finite field $\mathbb{F}_{2^n}$ and APN functions. Some unsolved problems in symmetric cryptography are also considered.

Keywords: *cryptography, block ciphers, boolean functions, AES, Olympiad, NSUCRYPTO.*

Introduction

The First Siberian Student's Olympiad in Cryptography with International participation — NSUCRYPTO'2014 was held on November 2014. There exist several school competitions in cryptography and information security, but this one is the first cryptographic Olympiad for students and professionals. The aim of the Olympiad was to involve students and young researchers in solving of curious and hard scientific problems of the modern cryptography. From the very beginning the concept was not to stop on the training olympic tasks but include unsolved research problems at intersection of mathematics and cryptography.

In this article we give a detailed overview of the Olympiad. We start with rules of participation and description of rounds. Then in two big sections we discuss 15 problems of the Olympiad and their solutions. There are both some amusing tasks based on historical ciphers and hard mathematical problems between them. We consider mathematical problems related to cipher constructing such as studying of differential characteristics of S-boxes, S-box masking, determining of relations between cyclic rotation and additions modulo 2 and 2^n, constructing of special linear subspaces in $\mathbb{F}_2^n$. Problems about the number of solutions of the equation $F(x) + F(x + a) = b$ over the finite field $\mathbb{F}_{2^n}$ and APN functions are discussed. Some unsolved problems are proposed such as the problem about the special watermarking cipher. All problems were developed by Programm committee of the Olympiad. Solution check was also its duty.

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Organizers of the Olympiad are Novosibirsk State University, Sobolev Institute of Mathematics (Novosibirsk), Tomsk State University, Belarusian State University and University of Leuven (KU Leuven, Belgium). Programm committee was formed by G. Agibalov, S. Agievich, N. Kolomeec, S. Nikova, I. Pankratova, B. Preneel, V. Rijmen and N. Tokareva. Local organizing committee from Novosibirsk consisted of A. Gorodilova, N. Kolomeec, G. Shushuev, V. Vitkup, D. Pokrasenko and S. Filiyzin. N. Tokareva was the general chair of the Olympiad.

More than 450 participants from 12 countries were registered on the website of the Olympiad, `www.nsucrypto.nsu.ru`. Fifteen participants of the first round and eleven teams of the second round became winners and received the prizes. The list of winners can be found in the last section of this paper.

NSUCRYPTO will be a regular annual Olympiad held on November. This year it starts on November, 15. We invite pupils, students and professionals to participate!

1. Organization and rules of the Olympiad

Here we briefly formulate key points of the Olympiad.

Rounds of the Olympiad. There were two independent Internet rounds. The First round (duration 4 hours 30 minutes) was individual and consisted of two sections: school and student's. It was held on November, 16. Theoretical problems in mathematics of cryptography were offered to participants. The second, team, round (duration 1 week; November, 17–24) was devoted to hard research and programming problems of cryptography.

Everybody can participate! To become a participant of the Olympiad it was necessary and sufficient to register on the website `www.nsucrypto.nsu.ru`. There were no restrictions on status and age of the participants. It means that senior pupils, students and all the others who are interested in cryptography were able to participate. Participants from any countries were welcome. During the registration every participant had to choose his category: "senior pupil", "student" or "other / professional " and the section of the first round: "school" or "student's". The second round was common for all the participants.

Language of the Olympiad. All problems were given in English. But solutions could be written in English or Russian.

Format of the solutions. We accepted solutions in any electronic format (pdf, jpg, txt, rtf, docx, tex, etc). For example a participant was able to write his solutions on a paper and send us a picture of it. Solutions should be written with all necessary details.

Prizes. There were several groups of prizes:

- for senior pupils — winners of the school section of the first round;
- for students — winners of the student's section of the first round;
- for participants in category "other / professional" — winners of the student's section of the first round;
- for participants (for every category separately) — winners of the second round;
- special prizes from the Programm committee for unsolved problems.

Interesting moments. Sometimes we were asked: "The Olympiad is via Internet. Are not you afraid that participants will use everything: supercomputers, books, articles, websites on cryptography?" In fact, we only welcome such an active mobilization of all possible resources in purpose of solving the tasks! We hope that in a future such a brainstorm will help to solve really hard cryptographic problems.

2. Problem structure of the Olympiad

There were 15 problems on the Olympiad. Some of them were included in both rounds.

Thus, the school section of the first round consisted of 6 problems, whereas the student's section contained 8 problems. The first three problems were the same in each section.

Problems of the first round (school section)

N	Problem title	Maximal scores
1	A hidden message	4
2	A crypto room	4
3	The musical notation	4
4	Boolean cubes	4
5	A broken cipher machine	4
6	The Snowflake cipher	4

Problems of the first round (student's section)

N	Problem title	Maximal scores
1	A hidden message	4
2	A crypto room	4
3	The musical notation	4
4	Linear subspaces	12
5	Number of solutions	8
6	A special parameter	10
7	S-box masking	8
8	Add–Rotate–Xor	10

The second round was composed of 11 problems; it was common for all the participants. Three problems presented on the second round are unsolved (with declared special prizes from the Program Committee).

Problems of the second round

N	Problem title	Maximal scores
1	Watermarking cipher	Special prize
2	APN permutation	Special prize
3	The Snowflake cipher	4
4	Number of solutions	8
5	Super S-box	Special prize
6	Boolean cubes	4
7	A special parameter	10
8	A pseudo-random generator	6
9	Add–Rotate–Xor	10
10	Linear subspaces	12
11	The musical notation	4

3. Problems

3.1. Problem "A hidden message" (4 scores)

CrYPtogRapHY iS a ScIEnce Of "seCrET wriTinG". FOr aT Least Two THoUsANd yeaRS ThErE haVE bEeN peOPlE WHo WAnTeD to SEnd MESsaGes WHiCh coUlD oNly bEen rEAd bY tHe pEOPLe FoR whOm tHey were iNteNdeD. a loT oF different MEtHODs FoR coNcEalING mEssageS WerE invENtED stARTING WIth AnCIeNt cIPHerS lIKE "SkytaLE" and "ATBAsH" and ending wiTH MOdErn SymmeTRiC ANd PubliC-kEy enCRYptioN ALGOriTHmS SUch aS AeS and Rsa. the dEVELopMENT Of crYPtOgRaPHy cOntiNueS And NEVER sTopS! decrYPt THe mESsaGe tHat iS hIDdEn in thE teXT oF this TASk! tHE aLphabet FoR THE mEssAGE ConsisTs of ALl tWEnTy six enGliSh letTERS from "a" To "z" ANd Six puNCTuaTIoN MARkS " ", ".", ",", "!", "?", "'".

3.2. Problem "A crypto room" (4 scores)

You are in a crypto room with a secret message in hands. Decrypt it!

3.3. Problem "The musical notation" (4 scores)

Alice and Bob invented a new way for encrypting messages based on musical notations of melodies. They are not very good in musical notations but they know the basic notes "do", "re", "mi", "fa", "sol", "la", "ti", and their places in the staff:

To encrypt a message of length n in English alphabet Alice chooses a melody consisting of n notes. She writes a message under the musical notation of the melody in such a way that each letter of the message corresponds to exactly one note's position in the musical notation. Then for each note ("do", "re", ..., "ti") Alice forms the ordered group of corresponding letters. Further she takes a random integer number k_i, $i = 1, \ldots, 7$, and cyclically shifts letters in the i-th group on k_i positions to the right. After that Alice forms the ciphertext by writing letters of the shifted groups under the musical notation again.

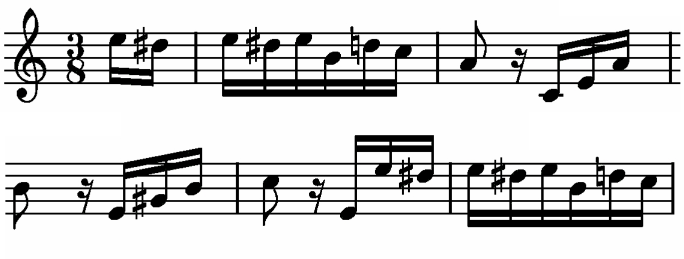
An example. Suppose that Alice wants to send the message `H E L L O`.

The group for "re" is (`E`, `L`); for "mi" — (`H`, `L`, `O`). Alice takes random numbers 2 and 1 for "re" and "mi" respectively. After shifting she gets groups (`E`, `L`) and (`O`, `H`, `L`). Hence the ciphertext for the message is `O E H L L`.

Decrypt the following ciphertext sent to Bob by Alice:

`R O L E L I S E O E E E H T O M V C P B D E F S O N`

It is known that Alice used the musical notation below.



3.4. Problem "Boolean cubes" (4 scores)

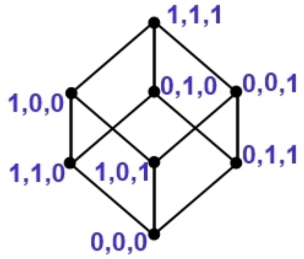
Alice has two cubes E_1 and E_2 of dimension 3 (see the picture below). Their vertices have labels consisting of three integers; for example, (1,0,1) consists of integers 1, 0, 1. Consider an operation A that can be applied for a cube. The operation A contains three steps:

Step 1. Take an arbitrary edge of the cube;

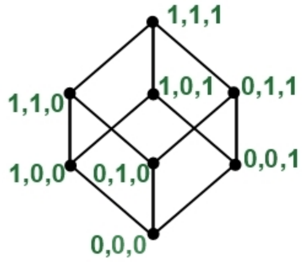
Step 2. Take the number a equal to 1 or -1;

Step 3. Add a to an arbitrary position of the first vertex of the chosen edge. Add a to an arbitrary position of the second vertex of the edge.

Is it possible to get the cube E_2 from the cube E_1 by applying the operation A as many times as necessary? Give your arguments.

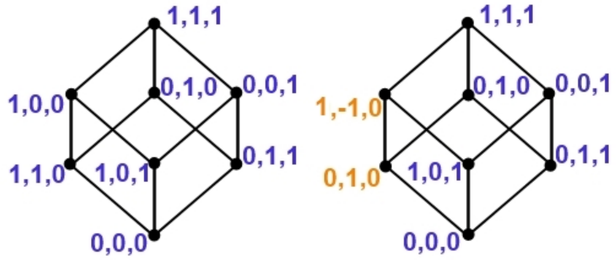


The cube E_1



The cube E_2

An example of applying an operation. Step 1. Take the edge $((1,0,0);(1,1,0))$. Step 2. Let $a=-1$. Step 3. For the vertex $(1,0,0)$ we choose position 2 and for the vertex $(1,1,0)$ we choose position 1; after adding the edge $((1,0,0);(1,1,0))$ becomes $((1,-1,0);(0,1,0))$.



An illustration for the example

3.5. Problem "A broken cipher machine" (4 scores)

Mary works on a cipher machine that encrypts messages like this:

Step 1. It represents a message as the natural number $n=\overline{abcdef\ldots}$;
Step 2. Then it sums all the digits in the number, $S_n=a+b+c+d+e+f+\ldots$;
Step 3. It inverts the order of digits in the number n and gets the number $n'=\overline{\ldots fedcba}$;
Step 4. As a result of the encryption the machine prints the number $m=n'+2\cdot S_n$.

But now the cipher machine is broken: sometimes it works correctly but sometimes it prints random numbers m.

After encryption of her secret number n Mary found out that the result is the power of two, $m=2^k$ for some integer k.

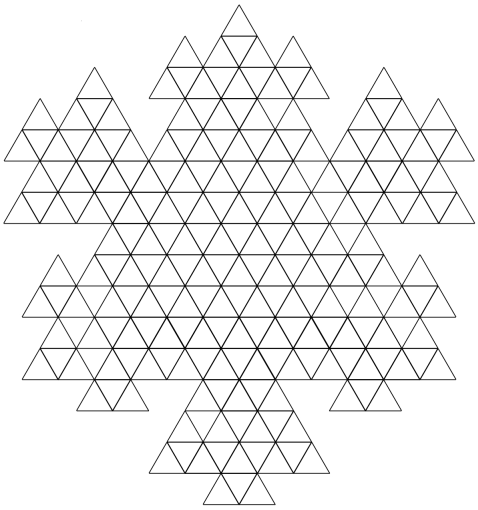
Determine was it the correct encryption in this case?

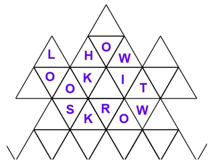
3.6. Problem "The Snowflake cipher" (4 scores)

Alice wants to encrypt some text using the Snowflake cipher. Encryption is described by the following algorithm:

Step 1. Choose an arbitrary small triangle in the snowflake (see below);
Step 2. Put the first letter of your message into this triangle;
Step 3. Write the next letter of the message (without spaces) into an arbitrary empty neighbouring triangle. Neighbouring means having a common edge. Repeat this step until the end of message.
Step 4. After inserting of all the letters, write down the text from snowflake in horizontal order from top to bottom and left to right.

Determine what is the maximal possible length of a message that can be encrypted with the Snowflake cipher?





An example. We want to encrypt the message: `LOOK HOW IT WORKS`. As a result we can get the ciphertext: `LHOWOOKITSKROW`.

3.7. Problem "A pseudo-random generator" (6 scores)

Alice and Bob communicate in Russia through the Internet using some protocol. In the process of communication Bob sends random numbers to Alice. It is known, that Bob's pseudo-random generator works in the following way:

- it generates the binary sequence $u_0, u_1, u_2, \ldots$, where $u_i \in \mathbb{F}_2 = \{0, 1\}$, such that for some secret $c_0, \ldots, c_{15} \in \mathbb{F}_2$ it holds

$$u_{i+16} = c_{15}u_{i+15} \oplus c_{14}u_{i+14} \oplus \ldots \oplus c_0u_i \text{ for all integer } i \geqslant 0;$$

- i-th random number r_i, $i \geqslant 1$, is calculated as

$$r_i = u_{16i} + u_{16i+1}2 + u_{16i+2}2^2 + \ldots + u_{16i+15}2^{15};$$

- Bob initializes $u_0, u_1, \ldots, u_{15}$ using some integer number IV (initial value), where $0 < IV < 2^{16}$, by the same way, i. e.

$$IV = u_0 + u_1 2 + u_2 2^2 + \ldots + u_{15}2^{15};$$

- it is known that as IV Bob uses the number of seconds from January 1, 1970, 00:00 (in his time zone) to his current time (in his time zone too) modulo 2^{16}.

Eva has intercepted the third and the fourth random numbers ($r_3 = 9\,731$ and $r_4 = 57\,586$). She lives in Novosibirsk and knows that Bob has initialized the generator at November 17, 2014, at about 12:05 UTC+6 up to several minutes. The number of seconds from January 1, 1970, 00:00 UTC+6 to November 17, 2014, 12:05 UTC+6 is equal to $1\,416\,225\,900$.

Help Eva to detect Bob's time zone.

3.8. Problem "Number of solutions" (8 scores)

Let $\mathbb{F}_{256}$ be the finite field of characteristic 2 with 256 elements. Consider the function

$$F : \mathbb{F}_{256} \to \mathbb{F}_{256} \text{ such that } F(x) = x^{254}.$$

Since $x^{255} = 1$ for all nonzero $x \in \mathbb{F}_{256}$, we have $F(x) = x^{-1}$ for all nonzero elements of $\mathbb{F}_{256}$. Further, we have $F(0) = 0$.

Alice is going to use the function F as an S-box (that maps 8 bits to 8 bits) in a new block cipher. But before she wants to find answers to the following questions.

- How many solutions may the equation
$$F(x + a) = F(x) + b \tag{1}$$
have for all different pairs of nonzero parameters a and b, where $a, b \in \mathbb{F}_{256}$?
- How many solutions does the equation (1) have for the function $F(x) = x^{2^n-2}$ over the finite field $\mathbb{F}_{2^n}$ for an arbitrary n?

Please, help to Alice!

3.9. Problem "S-box masking" (8 scores)

To provide the security of a block cipher to the side channel attacks, some ideas on masking of elements of the cipher are exploited. Here we discuss masking of S-boxes.

Alice takes a bijective function S (S-box) that maps n bits to n bits. Bob claims that for every such a function S there exist two bijective S-boxes, say S' and S'', mapping n bits to n bits, such that it holds

$$S(x) = S'(x) \oplus S''(x) \text{ for all } x \in \mathbb{F}_2^n.$$

Hence, Alice is able to mask an arbitrary bijective S-box by "dividing it into parts" for realization. But Alice wants to see the proof of this fact. Please help to Bob in giving the arguments.

3.10. Problem "A special parameter" (10 scores)

In differential cryptanalysis of block ciphers a special parameter P is used to measure the diffusion strength. In this problem we study its properties.

Let n, m be positive integer numbers. Let $a = (a_1, \ldots, a_m)$ be a vector, where a_i are elements of the finite field $\mathbb{F}_{2^n}$. Denote by $\mathrm{wt}(a)$ the number of nonzero coordinates a_i, $i = 1, \ldots, m$, and call this number the *weight* of a.

We say that $a, b \in \mathbb{F}_{2^n}^m$ represent *states*. The sum of two states a, b is defined as $a + b = (a_1 + b_1, \ldots, a_n + b_n)$.

Thus, the *special parameter* P of a function $\varphi : \mathbb{F}_{2^n}^m \to \mathbb{F}_{2^n}^m$ is given by

$$P(\varphi) = \min_{a,\, b,\, \text{such that } a \neq b} \{\mathrm{wt}(a + b) + \mathrm{wt}(\varphi(a) + \varphi(b))\}.$$

- Rewrite (simplify) the definition of $P(\varphi)$ when the function φ is linear (recall that a function ℓ is linear if for any x, y it holds $\ell(x + y) = \ell(x) + \ell(y)$).
- Rewrite the definition of $P(\varphi)$ in terms of linear codes, when the linear transformation φ is given by a $m \times m$ matrix M over $\mathbb{F}_{2^n}$, i.e. $\varphi(x) = M \cdot x$.
- Let φ be an arbitrary function. Find a tight upper bound for $P(\varphi)$ as a function of m.
- Can you give an example of the function φ with the maximal possible value of P?

3.11. Problem "Add–Rotate–Xor" (10 scores)

Let $\mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2 = \{0,1\}$. A vector $x \in \mathbb{F}_2^n$ has the form $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{F}_2$. This vector can be interpreted as the integer $x_1 \cdot 2^{n-1} + x_2 \cdot 2^{n-2} + \ldots + x_{n-1} \cdot 2 + x_n$.

Alice can implement by hardware the following functions from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ for all vectors $a, b \in \mathbb{F}_2^n$ and all integers r, $0 < r < n$:

1) $f_a(x) = x \boxplus a$ — addition of vectors x and a as integers modulo 2^n for a fixed a;
2) $g_r(x) = x \lll r$ — cyclic rotation of a vector x to the left by r positions for a fixed positive integer r;
3) $h_b(x) = x \oplus b$ — coordinate-wise sum of vectors x and b modulo 2 for a fixed b.

- Bob asks Alice to construct two devices that compute the functions S_1 and S_2 from $\mathbb{F}_2^2$ to $\mathbb{F}_2^2$ given by their truth table:

x	(00)	(01)	(10)	(11)
$S_1(x)$	(01)	(00)	(10)	(11)
$S_2(x)$	(01)	(11)	(00)	(01)

  Can Alice do it? If "yes", show how it can be done; if "no", give an explanation!
- Generalizing the problem above: can we construct any function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ using only a finite number of compositions of functions f_a, g_r and h_b?
  And what about any permutation over $\mathbb{F}_2^n$?
  Consider at least the cases $n = 2, 3, 4$.
- Is it possible to compute every function h_b using only functions f_a and g_r?

3.12. Problem "Linear subspaces" (12 scores)

Recall several definitions and notions. Each element $x \in \mathbb{F}_2^k$ is a binary vector of length k, i.e. $x = (x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in \mathbb{F}_2$. For two vectors x and y of length k their sum is $x \oplus y = (x_1 \oplus y_1, \ldots, x_k \oplus y_k)$, where $\oplus$ stands for XOR operation. Let $\mathbf{0}$ be the zero element of the vector space, i.e. vector with all-zero coordinates. A nonempty subset $L \subseteq \mathbb{F}_2^k$ is called a *linear subspace* if for any $x, y \in L$ it holds $x \oplus y \in L$. It is easy to see that zero vector belongs to every linear subspace. A linear subspace L of $\mathbb{F}_2^k$ has *dimension* n if it contains exactly 2^n elements.

Problem. For constructing a new secret sharing scheme Mary has to solve the following task on binary vectors. Let n be an integer number, $n \geqslant 2$. Let $\mathbb{F}_2^{2n}$ be a $2n$-dimensional vector space over $\mathbb{F}_2$, where $\mathbb{F}_2 = \{0,1\}$ is a prime field of characteristic 2.

Do there exist subsets $L_1, \ldots, L_{2^n+1}$ of $\mathbb{F}_2^{2n}$ such that the following conditions hold

- L_i is a linear subspace of dimension n for every $i \in \{1, \ldots, 2^n + 1\}$;
- $L_i \cap L_j = \{\mathbf{0}\}$ for all $i, j \in \{1, \ldots, 2^n + 1\}$, $i \neq j$;
- $L_1 \cup \ldots \cup L_{2^n+1} = \mathbb{F}_2^{2n}$?

If "yes", show how to construct these subspaces for an arbitrary integer n.

For example, the case $n = 2$ we consider together. In the vector space $\mathbb{F}_2^4$ we can choose the following 5 required subspaces:

$$L_1 = \{(0000), (0001), (1110), (1111)\};$$
$$L_2 = \{(0000), (0010), (1001), (1011)\};$$
$$L_3 = \{(0000), (0011), (0100), (0111)\};$$
$$L_4 = \{(0000), (0101), (1000), (1101)\};$$
$$L_5 = \{(0000), (0110), (1010), (1100)\}.$$

3.13. Problem "Watermarking cipher" (unsolved)

Here we present an unsolved problem that remains so till now.

It was stated by Gennadiy Agibalov and Irina Pankratova.

Problem. Let X, Y and K be the sets of plaintexts, ciphertexts and keys respectively, where $X = Y = \{0,1\}^n$ and $K = \{0,1\}^m$ for some integer n and m. Recall that two functions $E : X \times K \to Y$ and $D : Y \times K \to X$ are called *an encryption algorithm* and *a decryption algorithm* respectively if for any $x \in X$, $k \in K$ it holds $D(E(x,k),k) = x$. Together E and D form *a cipher*.

Let us call a cipher *watermarking* if for any key $k \in K$ and any subset $I \subseteq \{1, 2, \ldots, n\}$ there exists a key k_I such that for any $x \in X$ it holds

$$D(E(x,k), k_I) = x',$$

where x' is obtained from x by changing all bits with coordinates from I.

A simple example of such a cipher. Let $m = n$ and encryption and decryption algorithms be the following: $E(x,k) = x \oplus k$ and $D(y,k) = y \oplus k$. For any set I and any key k we can easily get the key k_I that is obtained from k by changing all bits with coordinates from I. The main disadvantage of such a cipher that every key should be used only once.

How can we use a watermarking cipher? Suppose you own some digital products (for example, videos), which you want to sell. Let x represent a binary code of a product. For each customer of x you choose the unique set I of coordinates and send to him the encrypted with the key k copy y and the correspondent key k_I. Then after receiving y and k_I the customer decrypts y and gets x'. The difference between the original x and x' is not significant; thus the customer does not know about it. If someone illegally spreads on the Internet bought by him product, you can easily understand who do it because you choose the unique set I for each customer!

Summarize the ideas we need to construct a cipher that has to put into the video something like a "watermark". Lets try! So, the problem is to construct a watermarking cipher. Please think about easy usage of it for an owner and a customer.

3.14. Problem "APN permutation" (unsolved)

Suppose we have a mapping F from $\mathbb{F}_2^n$ to itself (recall that $\mathbb{F}_2^n$ is the vector space of all binary vectors of length n). This mapping is called a *vectorial Boolean function in n variables*. Such functions are used, for example, as S-boxes in block ciphers and should have special cryptographic properties. In this problem we consider the following two properties and the problem of combining them.

- A function F in n variables is a *permutation* if for all distinct vectors $x, y \in \mathbb{F}_2^n$ it has distinct images, i. e. $F(x) \neq F(y)$.
- A function F in n variables is called *Almost Perfect Nonlinear* (APN) if for any nonzero vector $a \in \mathbb{F}_2^n$ and any vector $b \in \mathbb{F}_2^n$ an equation $F(x) \oplus F(x \oplus a) = b$ has at most 2 solutions. Here $\oplus$ is the coordinate-wise sum of vectors modulo 2.

Try to find an APN permutation in 8 variables or prove that it doesn't exist.

History of the problem. The question "Does there exist an APN permutation in even number of variables?" has been studied for more that 20 years. If the number of variables is odd, APN permutations exist as it was proved by K. Nyberg in 1994, see [3]. It is known that for 2 and 4 variables the answer is "No". But for 6 variables K. Browning, J. F. Dillon, M. McQuistan, and A.J. Wolfe have found such a function in 2009, see [1]. You can see it bellow:

$$G = \left(\begin{array}{llllllllllllllll} 0 & 54 & 48 & 13 & 15 & 18 & 53 & 35 & 25 & 63 & 45 & 52 & 3 & 20 & 41 & 33 \\ 59 & 36 & 2 & 34 & 10 & 8 & 57 & 37 & 60 & 19 & 42 & 14 & 50 & 26 & 58 & 24 \\ 39 & 27 & 21 & 17 & 16 & 29 & 1 & 62 & 47 & 40 & 51 & 56 & 7 & 43 & 44 & 38 \\ 31 & 11 & 4 & 28 & 61 & 46 & 5 & 49 & 9 & 6 & 23 & 32 & 30 & 12 & 55 & 22 \end{array} \right).$$

This function is presented as the list of its values, i. e. $G(0) = 0$, $G(4) = 15$, $G(16) = 59$ and so on. For brevity we use integers instead of binary vectors. A binary vector $x = (x_1, \ldots, x_n)$ corresponds to an integer $k_x = x_1 \cdot 2^{n-1} + x_2 \cdot 2^{n-2} + \ldots + x_{n-1} \cdot 2 + x_n$.

Thus, you are welcome to study the next case, $n = 8$.

3.15. Problem "Super S-box" (unsolved)

The next unsolved problem is directly related to AES construction. J. Daemen and V. Rijmen, the designers of AES (Rijndael), have introduced the Super-Sbox representation of two rounds of AES in order to study differential properties, see [2]. The function G from the problem can be considered as a simplified Super-Sbox model of two rounds of AES. To study resistance of AES to differential cryptanalysis, we welcome you to start with differential characteristics of the function G.

Problem. Let $\mathbb{F}_{256}$ be the finite field of 256 elements and α be a primitive element (it means that for any nonzero $x \in \mathbb{F}_{256}$ there exists $i \in \mathbb{N}$ such that $x = \alpha^i$). Let $\mathbb{F}_{256}^4$ be the vector space of dimension 4 over $\mathbb{F}_{256}$. Thus, any element $x \in \mathbb{F}_{256}^4$ is $x = (x_1, x_2, x_3, x_4)$, where $x_i \in \mathbb{F}_{256}$. An arbitrary function from $\mathbb{F}_{256}^4$ to $\mathbb{F}_{256}^4$ can be considered as the set of 4 coordinate functions from $\mathbb{F}_{256}^4$ to $\mathbb{F}_{256}$. Define the following auxiliary functions $F_4, M : \mathbb{F}_{256}^4 \to \mathbb{F}_{256}^4$:

$$F_4(x_1, x_2, x_3, x_4) = (x_1^{254}, x_2^{254}, x_3^{254}, x_4^{254});$$

$$M(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \times \begin{bmatrix} \alpha+1 & 1 & 1 & \alpha \\ \alpha & \alpha+1 & 1 & 1 \\ 1 & \alpha & \alpha+1 & 1 \\ 1 & 1 & \alpha & \alpha+1 \end{bmatrix}.$$

Consider the function $G : \mathbb{F}_{256}^4 \to \mathbb{F}_{256}^4$ that is a combination of F_4 and M:

$$G(x_1, x_2, x_3, x_4) = F_4(M(F_4(x_1, x_2, x_3, x_4))).$$

Find the number of solutions of the equation $G(x + a) = G(x) + b$, where parameters a and b run all nonzero values from $\mathbb{F}_{256}^4$.

Foundation of the problem. J. Daemen and V. Rijmen, the designers of AES (Rijndael), have introduced the Super-Sbox representation of two rounds of AES in order to study differential properties. The function G can be considered as a simplified Super-Sbox model of two rounds of AES. To study resistance of AES to differential cryptanalysis, we welcome you to start with differential characteristics of the function G.

4. Solutions of the problems

Here we would like to discuss solutions of the problems. Our special attention is payed to solutions of the participants (right/wrong and beautiful).

4.1. Problem "A hidden message" (4 scores)

Solution. The famous Bacon's cipher was used here. First of all everyone can get from the text that the alphabet of the message consists of 32 symbols (26 English letters and 6 punctuation marks) and notice that there are upper and lower case letters in the text. Such observations suggest that a binary code was used and a letter case means either 0 or 1. Since cardinality of the alphabet is $32 = 2^5$, a group of five 0s and 1s are needed to code each letter of a secrete message. Thus, if we delete all spaces and punctuation marks and divide the sequence obtained into 5 letters groups, we get the following:

CrYPt	ogRap	HYiSa	ScIEn	ceOfs	eCrET	wriTi	nGFOr	aTLea	stTwo
THoUs	ANdye	aRSTh	ErEha	VEbEe	NpeOP	lEWHo	WAnTe	DtoSE	ndMES
saGes	WHiCh	coUlD	oNlyb	EenrE	AdbYt	HepEO	PLeFo	RwhOm	tHeyw
ereiN	teNde	DaloT	oFdif	feren	tMEtH	ODsFo	RcoNc	EalIN	GmEss
ageSW	erEin	vENtE	DstAR	TINGW	IthAn	CIeNt	cIPHe	rSlIK	ESkyt
aLEan	dATBA	sHand	endin	gwiTH	MOdEr	nSymm	eTRiC	ANdPu	bliCk
EyenC	RYpti	oNALG	OriTH	mSSUc	haSAe	SandR	sathe	dEVEL	opMEN
TOfcr	YPtOg	RaPHy	cOnti	NueSA	ndNEV	ERsTo	pSdec	rYPtT	HemES
saGet	HatiS	hIDdE	ninth	EteXT	oFthi	sTASk	tHEaL	phabe	tFoRT
HEmEs	sAGEC	onsis	TsofA	LltWE	nTysi	xenGl	iShle	tTERS	froma
TozAN	dSixp	uNCTu	aTIoN	MARkS					

If upper case is coded by 0 and lower case by 1, we get the strange beginning of the text "`J,FJ,`" and so on. Hence, use 0 and 1 for lower and upper cases respectively. In this way you get "`We welcome you to the first Siberian student's Olympiad in cryptography with international participation!`". We supposed that letters are coded in alphabetic order with numbers from 0 to 25, and punctuation marks with numbers from 26 to 31.

This problem was completely solved by 34 participants.

Some ideas and typical mistakes of participants are listed below:

- the most part of those participants who sent their solutions for this problem succeeded in solving; as a rule they wrote a program in order to decode the message more faster;
- some participants mentioned in their solutions that Bacon's cipher was used;
- a few participants used the standard decoder BASE32 for decoding the binary string but they did not substituted the punctuation marks from the alphabet of this problem;
- the most surprising comment for this problem was "Somehow Mendeleyev's periodic table was used there!".

4.2. Problem "A crypto room" (4 scores)

The answer to the question "Who is the author?" is Anton Pavlovich Chekhov, a Russian classical writer. The encrypted message hides the name of his famous play "Three sisters".

Solution. Let us consider the steps of solution. Information contained in the picture allows to find the key needed for decryption. At first we reflect from right to left the text written on the blackboard. It says "Use all information you can find here to get the key. The first part of it is `AC`". It becomes clear that the key starts with `AC`. "The next part is `IWP`", as it is written below in the picture. In the upper right corner there is the sentence in which each word starts with the last letter of the original correct word. The result of the letters permutation to the initial positions is the sentence "Then use letters on the photos from up to down". Let us do it. We gets the third part of the key `RRVM`. The final part of the key is `JOV` cause of "`JOV` completes the key".

As a result we obtain the following key `ACIWPRRVMJOV`. Observe that the length of the key is equal to 12 letters, that is exactly equal to the length of the encrypted message.

Vigenere cipher was used for the encryption. Each letter has its position number in the alphabet (`A` — 1, `B` — 2, ..., `Z` — 26 or 0). Using the example of ciphering above the portrait of Julius Caesar we determine the rule of encryption:

$$\mathtt{A} + \mathtt{D} = \mathtt{E};\ \text{that is } 1 + 4 = 5 \pmod{26};$$

$$\mathtt{X} + \mathtt{D} = \mathtt{B};\ \text{that is } 24 + 4 = 2 \pmod{26}.$$

To decipher the ciphertext you need to subtract the key from the ciphertext under the same rule. Deciphering of two first letters of the message is given here:

$$\mathtt{U} - \mathtt{A} = \mathtt{T} \text{ since } 21 - 1 = 20 \pmod{26};$$

$$\mathtt{K} - \mathtt{C} = \mathtt{H} \text{ since } 11 - 3 = 8 \pmod{26}.$$

Thus, we obtain the secret message `THREESISTERS`. This problem were completely solved by one pupil and 30 students. And only one participant has read the last strange lettering in the picture: it was a word "algorithm" written in Malayalam.

This problem was completely solved by 33 participants. The most detailed solution with step-by-step pictures was proposed by O. Smirnov (Saratov State University).

Typical mistakes were:

• inclusion of symbols `5` and `*` into the third part of the key (actually they are not letters);

• using the Caesar cipher with key equal to 4, although it was only a helpful (or unhelpful) example.

4.3. Problem "The musical notation" (4 scores)

Solution. This is a classical permutation cipher. At first we should form the ordered groups corresponding to each note ("do", "re", ..., "ti") according to the ciphertext and the used musical notation:

do								E		E							V									N
re		O		E			S													B		E			O	
mi	R		L		L						E			T				C	P		D		F			
fa																										
sol															O											
la									O			E														
ti						I							H			M								S		

Our task is to find the right shifts for all the groups to form a readable message. The number of all possible variants is equal to $9 \cdot 6 \cdot 4 \cdot 4 \cdot 2 \cdot 1 = 1728$. Fortunately, we do not need to check all these variants as far as if we start with two the biggest groups ("mi" and "re"), we find that the begin and the end could be "`LBEET*O`" and "`DFOREL*S*`". To find the remaining part of the message we cyclically shift smaller groups. Thus, we get the secrete message: `L. Beethoven composed` "`For Elise`".

This problem was in both rounds and was solved by 33 participants. It is interesting to note some creative ways of solving. The extraordinary majority of solutions were programming. Participants generated all possible variants and then filtered them according to different rules:

• tests for invalid combination of three or four consecutive alphabetic letters. Some participants used the known such combinations, but one participant, D. Zajcev (Saratov State University) generated invalid combinations by himself using the novel "War and peace" by Leo Tolstoy in English. The team of P. Hvoryh and V. Laptev (winners of the second round of NSUCRYPTO in category "students") used a formula for naturalness of the language;

• one participant (S. Shabelnikov from Saratov State University) wrote a program that puts the next variant into the Internet for a search and then considers the number of results obtained.

There were also solutions made by hand. The list of some ideas is here:

• as well as in solution presented above some participants began with determining of possible shifts for the biggest groups "mi" and "re". For example, it was done in details by the team of pupils S. Derevyanchenko and E. Klochkova (Specialized Educational Scientific Center of NSU);

• we are pleased to point that many participants have recognized the musical notation and its composer L. V. Beethoven; such participants tried to find shifts in order to form any word from the list "Beethoven", "Elise", "composed". Thus, the team of V. Marchuk, D. Emelyanov and A. Gusakova (Belarusian State University) mentioned that the mistake of Alice was in taking the famous musical fragment.

4.4. Problem "Boolean cubes" (4 scores)

Solution. In the first cube let us assign vertices $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$ with "+" and vertices $(1,1,1),(1,1,0)$, $(0,1,1),(1,0,1)$ with "$-$". If we sum up all the coordinates of vertices in both groups "+" and "$-$" with plus and minus respectively, we obtain the following results: in the group "+" we have $0+1+1+1=3$ (since there is one zero vector and three vectors each with only one nonzero coordinate), in the group "$-$": $-3-2-2-2=-9$, then total sum is -6. Consider the operation A. We can see that A adds 1 or -1 in both sums of groups "+" and "$-$" simultaneously, then the total sum is invariant under multiple applying of operation A and is equal to -6. Let us consider the second cube. If we repeat the assignment of vertices in matching layers, we will see that the total sum is equal to 0. Thus, we can not get the cube E_2 from the cube E_1 by applying the operation A any number of times.

In the first round two pupils solved this problem with maximal scores, both used the idea of the sum's invariant property. Thus, solution of N. Dobronravov (Lyceum 130, Novosibirsk) was very compact and logical.

This problem was also included in the second round and was solved properly by 18 teams. We want to outline some interesting non-trivial ideas of solutions. The team of G. Beloshapko, A. Taranenko and E. Fomenko (Novosibirsk State University) introduced the following value for a cube C:

$$\lambda(C) = \left| \sum_{v \in C} (-1)^{|x(v)|} wt(v) \right|, \text{ where } wt(v) \text{ is the weight of vector } v.$$

They proved that this value does not depend on which vertex is considered to be the zero vertex. After this proof the team underlined that $\lambda(C)$ is invariant under operation A and since $\lambda(E_1) = 6$ and $\lambda(E_2) = 0$, then we can not obtain E_2 from E_1.

The most nontrivial and clear solution belongs to the team of A. Udovenko (Saint Petersburg). The participant divided vertices into 4 groups where there are no edges inside each group and all edges are between neighboring groups:

1. (0,0,0); 2. (1,1,0), (1,0,1), (0,1,1); 3. (1,0,0), (0,1,0), (0,0,1); 4. (1,1,1).

Any application of operation A adds or subtracts 1 from sums of two neighboring groups. We can write this operations as vectors: (1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1). Then participant proved that none of the vector of differences between sums of the groups for all possible positions of (0,0,0) in E_2 can be obtained as a linear combination of operation's vectors. So it is impossible to get the cube E_2 from E_1 using operation A.

The team of S. Skresanov, A. Miloserdov and D. Kirin (Novosibirsk State University) found a short and nice solution via colorings of the vertices.

O. Smirnov, P. Razumovsky and A. Ripinen (Saratov State University) transformed the problem to the task about two special segments and showed that one cannot be obtained from the other.

4.5. Problem "A broken cipher machine" (4 scores)

Solution. It is well known that any positive integer n and the sum of its digits are congruent modulo 3, i. e. $n = S_n \pmod 3$. It follows from the algorithm that $m = 0 \pmod 3$, since $n = n' \pmod 3$. Consequently, $m = n' + 2S_n = n + 2n = 0 \pmod 3$. Since Mary got the result as the power of two, we see that she obtained an incorrect encrypted message.

This problem was completely solved by six pupils. Five of them have sent solutions that are very close to the presented one. For example, such a solution was proposed by the youngest participant A. Dorokhin (Novosibirsk school 159) — the winner of the first round of NSUCRYPTO in category "senior pupil". One participant solved the problem through the implicit proof of the property of congruence between an integer and its sum of digits.

4.6. Problem "The snowflake cipher" (4 scores)

Solution. We may see that the problem is equivalent to finding the longest path through triangles with common edge such that every triangle can be found in this path only once. Let us consider nine *ledges* each consisting of three triangles with only one common edge with the rest of snowflake. Clearly, if our path crosses one of these ledges, then it can be only the beginning or the end of the path. So, the path may contain not more than 4 triangles from 2 chosen ledges.

Let us paint triangles in the rest part of snowflake in black and white colours such that neighbouring triangles have different colours. Among them there are 93 black triangles and 114 white, but since colours of triangles in the path should alternate, not more than 93 black triangles and 94 white triangles can be in the longest path. So, we can say that length of a path in the snowflake is not more than $93 + 94 + 4 = 191$. We present an example of such a path constructed by the team of G. Beloshapko, A. Taranenko and E. Fomenko (Novosibirsk State University).

Unfortunately, pupils did not cope with this problem on the highest score, nevertheless, three of them got some scores due to right but incomplete ideas of estimation.

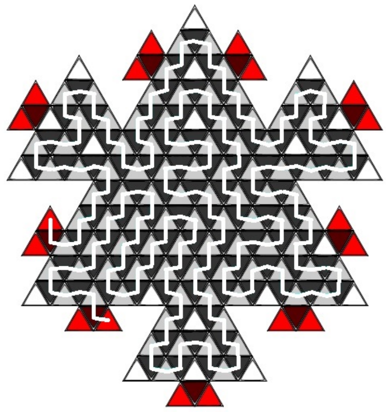
In the second round six teams have solved this problem with full possible scores.

Let us consider an interesting solution proposed by A. Udovenko (Saint Petersburg).

• At the first stage there were introduced two kinds of ledges, the first one P1 is the same as in the previous solution, consisting of three triangles, but there are only 6 such ledges in the group, because the rest of them are supposed to be part of three second type bigger ledges P2. We may see that ledges of the type P2 consist of 24 triangles. Define the P3 as the main big triangle, obtained by deleting all P1 and P2 ledges from the original snowflake.

• At the second stage the author also painted P3 in black and white such that any moving between triangles is possible only if they have different colours. Then he got an estimate that the longest path in P3 will skip at least 11 triangles.

• At third, author considered all possible pairs "start-end" and counted for each pair minimal number of skipped triangles. He obtained that the lower bound on this number is 43 and gave right example of path with 191 length.



The team of V. Marchuk, D. Emelyanov and A. Gusakova (Belarusian State University) proposed very nice solution by transforming the problem to the following one: find a Hamilton cycle in the special graph corresponding to the snowflake.

4.7. Problem "A pseudo-random generator" (6 scores)

Solution. It is a simple task. First of all, the binary sequence from r_3 and r_4 is `11000000011001000100111100000111`. Because of length of the generator is 16, linear complexity of the sequence is not more than 16. Next, we have 32 consecutive bits, and can restore recurrent relation in unique way, for example, using Berlekamp — Massey algorithm. Recurrent relation of the sequence is

$$u_{i+16} = u_{i+5} \oplus u_{i+3} \oplus u_{i+1} \oplus u_i.$$

Since we know recurrent relation, obtain IV: 58 390. And we also know, if the generator were initialized at November 17, 2014, 12:05 UTC+6, IV would be 58 476. In Russia there are time zones from UTC+2 to UTC+12. So, Bob initialized the generator $58\,476 - 58\,390 = 86$ seconds to November 17, 2014, 12:05 UTC+6, i. e. at $12:03:34$. Therefore, both Bob and Eva live in Novosibirsk time.

There were 18 solutions from the teams, and only one was wrong, the most of teams just solved a system of linear equations and have not used Berlekamp — Massey algorithm.

4.8. Problem "Number of solutions" (8 scores)

Solution. Consider a general case, i. e. $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$, $F(x) = x^{2^n-2}$ (and $F(x) = x$ for $n = 1$). We should determine how many solutions the equation $F(x + a) = F(x) + b$ may have for all different pairs of nonzero parameters a and b, where $a, b \in \mathbb{F}_{2^n}$. Since characteristic of $\mathbb{F}_{2^n}$ is 2, operations "−" and "+" coincide.

First of all, note that for $n = 1$ there exists only one pair $(a, b) = (1, 1)$, in this case there are 2 solutions of the equation. In what follows let $n > 1$.

Note that the function $F_a(x) = F(x) + F(x + a)$ has some symmetry: $F_a(x) = F_a(x + a)$, it means that 2 divides the number of solutions and at least for 2^{n-1} distinct $b \in \mathbb{F}_{2^n}$ it holds $F_a(x) \neq b$ for all $x \in \mathbb{F}_{2^n}$.

Therefore, for all $n > 1$ there exist $b \neq 0$ and $a \neq 0$, such that the equation has no solutions. That is why the number of solutions of the equation $F(x+a) = F(x) + b$ can be 0. Consider other possibilities.

Simplify the given equality: suppose that $x \neq 0$ and $x \neq a$. Note also that $y^{2^n-1} = 1$ for all $y \in \mathbb{F}_{2^n}^*$. Then

$$\begin{aligned} (x+a)^{2^n-2} + x^{2^n-2} &= b, && \mid \cdot\, x(x+a) \\ x(x+a)(x+a)^{2^n-2} + x(x+a)x^{2^n-2} &= bx(x+a), \\ bx^2 + abx + a &= 0, && \mid \cdot\, a^{-2}b^{-1} \\ x^2/a^2 + x/a + (ab)^{-1} &= 0. \end{aligned}$$

Note that the number of solutions of $x^2/a^2 + x/a + (ab)^{-1} = 0$ depends only on ab and $x = 0$, $x = a$ are not its solutions. Rewrite this equality as

$$z^2 + z + (ab)^{-1} = 0,$$

where $z = x/a$. Then we have two cases:

- $x = 0$ and $x = a$ are solutions of $F(x) + F(x+a) = b$. Then it should be $a = b^{-1}$, i.e. $ab = 1$. We already have 2 solutions. Next, solve the equality $z^2 + z + 1 = 0$. Both roots of the polynomial $z^2 + z + 1$ belong to the field $\mathbb{F}_{2^2}$. This field is contained in $\mathbb{F}_{2^n}$ if and only if n is even. So, in case if n is even the equation $F(x) + F(x+a) = b$ has exactly 4 solutions. In case when n is odd there are exactly 2 solutions of the equation $F(x) + F(x+a) = b$.
- $x = 0$ and $x = a$ are not the solutions of $F(x) + F(x+a) = b$. Therefore, $ab \notin \mathbb{F}_2$. Note that equation $z^2 + z = z(z+1) = (ab)^{-1}$ have 0 or 2 solutions. Since $(ab)^{-1}$ can be an arbitrary element from $\mathbb{F}_{2^n} \setminus \mathbb{F}_2$, at least for $2^{n-1} - 2$ distinct ab there are exactly two solutions: so, if $n > 2$, it can be 2 solutions in this case. If $n = 2$, then $z^2 + z \in \mathbb{F}_2$, i.e. for $ab \notin \mathbb{F}_2$ there is no solution.

The answer is the following. For $n = 1$ there are always 2 solutions; for $n = 2$ there can be 0 or 4 solutions; for odd n ($n > 1$) there can be 0 or 2 solutions; for even n ($n > 2$) there can be 0, 2 or 4 solutions.

This problem was completely solved during the second round. Teams of P. Hvoryh and V. Laptev (Omsk State Technical University), A. Udovenko (Saint Petersburg), A. Oblaukhov (Novosibirsk State University), S. Belov and G. Sedov (Moscow State University) proposed right and complete solutions.

4.9. Problem "S-box masking" (8 scores)

Solution. We would like to give a solution proposed by Qu L. et al. in the first variant of the paper [4]. Let us represent an arbitrary bijective function $S : \mathbb{F}_2^n \to \mathbb{F}_2^n$ as $S'(x) \oplus S''(x)$, where $S', S'' : \mathbb{F}_2^n \to \mathbb{F}_2^n$ are bijective too. Consider $\mathbb{F}_2^n$ as $\mathbb{F}_{2^n}$. Let $\alpha \in \mathbb{F}_{2^n}$ and $\alpha \neq 0, 1$. If $n > 1$ such an element α does exist. It is clear that $S(x) = \alpha S(x) + (\alpha+1)S(x)$. Note that $\alpha S(x)$ and $(\alpha+1)S(x)$ are bijective since each of them is a composition of two bijective mappings. So, for $n > 1$ the required representation exists. If $n = 1$ there are only two bijective function $S(x) = x$ and $S(x) = x \oplus 1$; their sum is a constant and hence there is no the required representation in this case.

There were two complete combinatorial solutions proposed by S. Godzhaev (Moscow State University) and A. Udovenko (Saint Petersburg).

4.10. Problem "A special parameter" (10 scores)

A special parameter that we consider in this problem is called the *differential branch number* of a transformation, see for example book [2]. In differential cryptanalysis of block ciphers this parameter is used to measure the diffusion strength of a cipher. Some properties of it are discussed in the problem.

Solution. Suppose $\varphi : \mathbb{F}_{2^n}^m \to \mathbb{F}_{2^n}^m$ and $a, b \in \mathbb{F}_{2^n}^m$.

• Since φ is linear, i. e. $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathbb{F}_{2^n}^m$, and the condition $a \neq b$ is equivalent to $a + b \neq 0$, we can rewrite the definition of $P(\varphi)$ in the following way, where by c we denote $a + b$:

$$P(\varphi) = \min_{c \neq 0}\{\mathrm{wt}(c) + \mathrm{wt}(\varphi(c))\}.$$

• Let us consider vectors $(x, \varphi(x)) = (x, M \cdot x)$ of length $2m$, where $x \in \mathbb{F}_{2^n}^m$. Then the set $C = \{(x, M \cdot x) \mid x \in \mathbb{F}_{2^n}^m\}$ is a linear code. Since we have $\mathrm{wt}(a+b)+\mathrm{wt}(\varphi(a)+\varphi(b)) = \mathrm{wt}((a + b, \varphi(a) + \varphi(b))) = \mathrm{dist}((a, \varphi(a)),(b, \varphi(b)))$, where $\mathrm{dist}(x, y)$ means the Hamming distance between vectors x and y, the parameter $P(\varphi)$ is equal to the minimal distance between distinct codewords of the code C.

• Since $a \neq b$, the minimal value of $\mathrm{wt}(a + b)$ is equal to 1. Also the maximal possible value $\mathrm{wt}(\varphi(a) + \varphi(b))$ is m by the definition. Thus, the maximal possible value of $P(\varphi)$ is not more than $m + 1$.

• One can construct an example of such a transformation using a Maximal Distance Separable code with parameters $[2m, m, m + 1]$, for example Reed — Solomon code.

This problem was completely solved by two teams: P. Hvoryh and V. Laptev (Omsk State Technical University), K. Kogos and S. Kyazhin (Moscow Engineering Physics Institute).

4.11. Problem "Add–Rotate–Xor" (10 scores)

The complicated algebraic solution of this problem was given by T. Zieschang in 1997, see [5]. Here we introduce a simple solution proposed by the participants of the Olympiad.

Solution.

- $S_1(x) = g_1(f_1(g_1(f_2(x))))$. It is obvious that all f_a, g_r and h_b are bijective, therefore, any its composition is bijective too. S_2 is not bijective, so, it can not be represented as a composition of them.
- Only permutations on $\mathbb{F}_{2^n}$ can be constructed in this way. It is well known that compositions of function f_1 (a cycle of length 2^n) and transpositions of adjacent elements in the cycle give us all permutations on $\mathbb{F}_{2^n}$. The following construction gives a certain transposition τ: $\tau(x) = g_1(f_{2^n-1}(g_{n-1}(f_2(x))))$. Indeed, $g_{n-1}(y) = 2^{n-1}y_n + \lfloor y/2 \rfloor$; and if $x < 2^n - 2$, then $f_2(x) = x + 2$, so,

$$g_{n-1}(f_2(x)) = g_{n-1}(x + 2) = 2^{n-1}(x + 2)_n + \lfloor (x + 2)/2 \rfloor = \\ 2^{n-1}x_n + \lfloor x/2 \rfloor + 1 = (x_n, x_1, \ldots, x_{n-1}) + 1.$$

  Therefore, τ transposes $2^n - 1$ with $2^n - 2$ and does not change all other elements. Next, f_{2^n-1} eliminates "+1" and g_1 cyclically rotates $(x_n, x_1, \ldots, x_{n-1})$ to the left by one position. Also $\tau(2^n - 2) = 2^n - 1$: $f_2(2^n - 2) = 0$, $g_{n-1}(0) = 0$, $f_{2^n-1}(0) = 2^n - 1$, $g_1(2^n - 1) = 2^n - 1$.
- Yes. The third item is obvious, since we can construct any permutation on $\mathbb{F}_{2^n}$ using the mentioned functions.

In the second round there were 7 right solutions. The most of teams have found a full cycle and a transposition. The solution given above is based on the solution of G. Beloshapko, A. Taranenko and E. Fomenko team (Novosibirsk State University).

Very clear and simple solution was proposed by the team of R. Zhang and A. Luykx (KU Leuven, Belgium): they constructed all transpositions in an explicit way.

4.12. Problem "Linear subspaces" (12 scores)

Here the solution from the program committee is given.

Solution. Consider $\mathbb{F}_2^{2n}$ as 2-dimensional vector space over $\mathbb{F}_{2^n}$, where $\mathbb{F}_{2^n}$ is the Galois field of order 2^n. Denote this vectorial space as V.

Define the following family of sets:

$$L_\alpha = \{(x, \alpha x) \mid x \in \mathbb{F}_{2^n}\}, \text{ where } \alpha \in \mathbb{F}_{2^n}; \quad L_{2^n+1} = \{(0, y) \mid y \in \mathbb{F}_{2^n}\}.$$

It is obvious that every such a set is a linear subspace in V and contains exactly 2^n elements. Let us show that an arbitrary element $(x, y) \in V$ is covered by the union of these subspaces. If $x = 0$ it is covered by L_{2^n+1}. Otherwise, $(x, y) = (x, (y/x)x)$ belongs to the subspace $L_{y/x}$. Note that every two subspaces have only one common element $\mathbf{0}$, since cardinality of V is exactly $2^{2n} = (2^n + 1)(2^n - 1) + 1$. Thus, the answer for the problem is "yes" and the system is constructed.

Another approach (but still using Galois fields) was proposed by the team of P. Hvoryh and V. Laptev (Omsk State Technical University) during the second round. They constructed linear subspaces in the following way:

$$L_0 = \{0, \alpha^{0\cdot(2^n+1)}, \alpha^{1\cdot(2^n+1)}, \ldots, \alpha^{(2^n-2)\cdot(2^n+1)}\} \text{ and}$$
$$L_i = \{\alpha^i x \mid x \in L_0\}, \; i = 1, \ldots, 2^n,$$

where α is any generative element of $\mathbb{F}^*_{2^{2n}}$.

There was also a nice right solution from the team of S. Skresanov, A. Miloserdov and D. Kirin (Novosibirsk State University), it is more close to the given one.

Several teams proposed algorithms for constructing subspaces, but there were several mistakes in this way.

4.13. Problem "Watermarking cipher" (unsolved)

The most deep analysis of this problem was proposed by R. Zhang and A. Luykx (winners of the second round of NSUCRYPTO in category "professional"), but nobody has introduced a concrete solution. May be you can do it?

Some details on this problem will be considered in the talk of G. P. Agibalov on the conference Sibecrypt-2015.

4.14. Problem "APN permutation" (unsolved)

This is another unsolved problem of the Olympiad; it is the well known long standing problem of cryptography. Some ideas on it were proposed by the team of G. Beloshapko, A. Taranenko and E. Fomenko (winners of the second round of NSUCRYPTO in category "students"). They proved some basic properties of APN functions; namely, if a permutation is an APN function, then its inverse function is APN too.

One participant, D. Svitov from NSU, has proposed an online service for distributed search of APN permutations, see [6].

But till now this problem is unsolved.

4.15. Problem "Super S-box" (unsolved)

The problem is still unsolved. Only one team of G. Beloshapko, A. Taranenko and E. Fomenko from Novosibirsk State University has sent a solution with an analysis of the problem for smaller field. They considered $\mathbb{F}_{16}$ and found the exact number of pairs a, b for each number of solutions. It seems that any even number between 0 and 44 can be the number of solutions. They proposed a hypothesis that for $\mathbb{F}_{256}$ it is true the same result: it may be any even number of solutions bounded by some number.

5. Awarding

Awarding of the winners was held in December, 2014 in Novosibirsk State University.

6. Winners of the Olympiad-2014

In this section we publish the names and information about winners of NSUCRYPTO-2014. There are 15 winners in the first round and 11 teams in the second one.

6.1. First round

Winners of the first round in school section ("senior pupils")

Place	Name	Country	City	School	Class	Scores
1	Alexander Dorokhin	Russia	Novosibirsk	MOU 159	8	12
2	Nikita Dobronravov	Russia	Novosibirsk	Lyceum 130	9	10
3	Artem Uskov	Russia	Novosibirsk	Gymnasium 3	11	8
3	Egor Dobronravov	Russia	Novosibirsk	Lyceum 130	9	8

Winners of the first round in student's section (in category "students")

Place	Name	City	University	Department	Course	Scores
1	George Beloshapko	Novosibirsk, Russia	Novosibirsk State University	Mechanics and Mathematics	6	23
2	Roman Lebedev	Novosibirsk, Russia	Novosibirsk State University	Physics	2	14
3	Dmitry Zajcev	Saratov, Russia	Saratov State University	Computer Science and Information Technology	5	12
3	Gleb Shalyganov	Saratov, Russia	Saratov State University	Computer Science and Information Technology	4	12
3	Samir Godzhaev	Moscow, Russia	Moscow State University	Mechanics and Mathematics	1	12
3	Alexander Shein	Saratov, Russia	Saratov State University	Computer Sciences and Information Technologies	5	12
3	Pavel Grachev	Saratov, Russia	Saratov State University	Computer Sciences and Information Technologies	5	12
3	Sergey Shabelnikov	Saratov, Russia	Saratov State University	Computer Sciences and Information Technologies	5	12
3	Angelina Sadohina	Saratov, Russia	Saratov State University	Computer Sciences and Information Technologies	4	12
3	Alexander Tkachev	Novosibirsk, Russia	Novosibirsk State University	Information Technologies	2	12

Winners of the first round in student's section (in category "professionals")

Place	Name	Country	City	Organization	Scores
1	Alexey Udovenko	Russia	Saint-Petersburg	—	12

6.2. Second round

Winners of the second round (in category "senior pupils")

Place	Name	Country	City	School	Class	Scores
1	Stepan Derevyanchenko, Elizaveta Klochkova	Russia	Novosibirsk	Specialized Educational Scientific Center of NSU	11	6

Winners of the second round (in category "students")

Place	Names	City	University	Department	Course	Scores
1	George Beloshapko, Anna Taranenko, Evarist Fomenko	Novosibirsk, Russia	Novosibirsk State University	Mechanics and Mathematics	5-6	55
1	Pavel Hvoryh, Vladimir Laptev	Omsk, Russia	Omsk State Technical University	Information Technologies and Computer Systems, RTF	3	55
2	Saveliy Skresanov, Alexey Miloserdov, Denis Kirin	Novosibirsk, Russia	Novosibirsk State University	Mechanics and Mathematics	2	36
3	Alexey Oblaukhov	Novosibirsk, Russia	Novosibirsk State University	Mathematics and Mechanics	3	33
3	Vadim Marchuk, Dmitry Emelyanov, Anna Gusakova	Minsk, Belarus	Belarusian State University	Applied Mathematics and Computer Science	6	33
3	Oleg Smirnov, Peter Razumovsky, Alexey Ripinen	Saratov, Russia	Saratov State University	Computer Science and Information Technology	4	29
3	Sergey Belov, Grigory Sedov	Obninsk, Moscow, Russia	Moscow State University	Computational Mathematics and Cybernetics	5	28

Winners of the second round (in category "professional")

Place	Names	Country	City	Organization	Scores
1	Ren Zhang, Atul Luykx	Belgium	Leuven	KU Leuven, COSIC	65
2	Konstantin Kogos, Sergey Kyazhin	Russia	Moscow	Moscow Engineering Physics Institute	41
3	Alexey Udovenko	Russia	Saint-Petersburg	—	37

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