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On the Approximative Optimality of Treating Interference as Noise in Multi-User Networks

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Dissertation zur Erlangung des Grades eines Doktor-Ingenieurs der Fakultät für Elektrotechnik und Informationstechnik an der Ruhr-Universität Bochum

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Bochum, 2016

Tag der Promotion:	30. September 2016
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To my beloved wife, mother, father, sisters,
and my sweet daughter Lena.

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Acknowledgement

Immeasurable thanks and appreciation are extended to everyone whose help and support in one way or another made this study possible.

I wish to express my deepest gratitude to my supervisor Prof. Dr.-Ing. Aydin Sezgin for offering me the opportunity to pursue my PhD degree in his group. Additionally, his continuous academic support with patience and motivation and insightful comments helped me improve the quality of the thesis.

Besides my supervisor, I would like to thank Prof. Dr. Giuseppe Caire and Prof. Dr. Petar Popovski for reviewing the thesis.

Thanks to the German Research Foundation, Deutsche Forschungsgemeinschaft (DFG) for supporting this work financially.

Further thanks go to Dr.-Ing Anas Chaaban for his useful discussions and comments.

Last but not the least, I wish to express my gratitude to my parents, my sisters, and my wife to whom I dedicate this work, for their understanding, love and endless spiritual support throughout writing this thesis and my life in general.

Bochum, September 20, 2016
Soheil Gherekhloo

Kurzfassung

Die zunehmende Nachfrage nach höheren Datenraten einerseits, sowie beschränkte Ressourcen wie Zeit und Frequenz andererseits, stellt jedoch die Systementwickler vor die Herausforderung der *Interferenzhandhabung.* Durch Nutzung des Unterschiedes zwischen Interferenz und Rauschen als Grundkonzept wurden in den letzten Jahren effiziente Methoden zur Handhabung der Interferenz entwickelt. Dieser Unterschied besteht hauptsächlich darin, dass die Interferenz, anders als das Rauschen, Träger von Informationen ist. Durch Decodierung dieser ungewünschten Information ist der Empfänger in der Lage, die Interferenz zu rekonstruieren und anschließend aufzuheben. Diese Vorgehensweise benötigt jedoch nicht nur ein perfekt synchronisiertes System, sondern auch genaue Kanalkenntnisse, sowohl am Sender als auch am Empfänger. Um diese Anforderungen zu erfüllen, sollen gröẞe Verwaltungsdaten wie Pilotsequenzen übertragen werden. Diese Übertragungen erhöhen selbst Leistungsverbrauch, Datenverkehr sowie die Komplexität der Algorithmen. Trotz der generellen Effizienz der Interferenzdecodierung wird sie aufgrund der vorgenannten Tatsache kaum in den praktischen Realisierungen angewandt. Stattdessen wird die auf der Behandlung von Interferenz als Rauschen (engl.: Treating Interference as Noise, TIN) basierende Herangehensweise angewandt. Vergleicht man dies mit den Anforderungen an eine Kommunikation über einen Punkt-zu-Punkt Kanal, setzt diese Vorgehensweise (TIN) keine weiteren Anforderungen voraus. Darüber hinaus gewinnt man durch das Zeigen der Optimalität von TIN einen großen Vorteil, da keine aufwändige Herangehensweise zur Steigerung der Datenraten mehr nötig ist. All dies motivierte die Forschung, sich mit der Optimalität von TIN in unterschiedlichen Kanälen auseinander zu setzen. In der Literatur ist die folgende Optimalitätsbedingung für TIN bekannt: *In einem Interferenzkanal ist TIN optimal, falls die Interferenzverbindungen gegenüber den gewünschten Verbindungen besonders schwach ausfallen.* In diesem Zusammenhang ergeben sich weitere Fragestellungen. Zum einen kann untersucht werden, ob die bereits genannte Optimalitätsbedingung von TIN für alle möglichen Kanäle erweiterbar ist. Zum anderen stellt sich die Frage, ob TIN notwendigerweise suboptimal ist, wenn die Optimalitätsbedingung nicht erfüllt ist. Um diese Fragen zu beantworten, wird in dieser Arbeit die Optimalität von TIN in einem sehr fundamentalen Netzwerk untersucht. In diesem Netzwerk operieren ein Mehrfachzugriffskanal und ein Punkt-zu-Punkt Kanal über die gleichen Kommunikationsressourcen. Im Folgenden bezeichnen wir diesen Kanal als PIMAC (engl.: Point-to-Point Channel Interfering with a Multiple Access Channel). Die daraus gewonnenen Erkenntnisse helfen uns, die Optimalität von TIN in größeren Netzwerken zu untersuchen.

Die folgenden Schritte sind notwendig, um die Optimalität von TIN im PIMAC

zu zeigen. Der erste Schritt ist die Bestimmung der maximal erreichbaren Rate, die durch Ignorieren der Interferenz am Empfänger und Verwendung eines Gaußschen Codebuchs am Sender erreichbar ist. Im zweiten Schritt werden obere Schranken für die Kapazität charakterisiert. Dann werden die Optimalitätsbedingungen für TIN durch einen Vergleich der erreichbaren Raten mit den oberen Schranken charakterisiert. Um die Analyse zu vervollständigen, wird außerdem die Suboptimalität von TIN für den Fall, dass die Optimalitätsbedingungen nicht erfüllt sind, gezeigt. Unsere Kenntnisse über die Optimalität für TIN in Bezug auf die Kapazität ist nur auf eine kleine Menge der Kanäle begrenzt. Dies liegt hauptsächlich daran, dass die Kapazitätscharakterisierung generell eine schwierige Aufgabe darstellt und einige Zwischenschritte benötigt. Ein wichtiger Schritt dafür ist die Charakterisierung der sogenannten GDoF (engl.: Generalized Degrees of Freedom). Falls ein hohes Signal-zu-Geräusch-Verhältnis vorliegt kann das GDoF-Maß als Näherung für die Kanalkapazität verwendet werden.

Das Ziel dieser Arbeit ist die Charakterisierung der Menge der Kanalparameter, in der das TIN in Bezug auf die GDoF optimal ist. Um dieses Ziel zu erreichen, wird zunächst ein vereinfachtes Modell von PIMAC untersucht. In diesem Modell ist der Zusammenhang zwischen den Ausgängen und Eingängen des PIMAC mit Hilfe einer linearen deterministischen Funktion definiert. Aus diesem Grund wird das Modell als LD-Model (engl.: Linear Deterministic) bezeichnet. Anschließend wird dann die Kapazität des LD-PIMAC ins GDoF-Maß übersetzt.

Aus der vollständigen Untersuchung der Optimalität von TIN im PIMAC in Bezug auf die GDoF, lassen sich interessante Schlussfolgerungen ziehen. So stellt man zum einen fest, dass in einigen Kanälen das TIN suboptimal sein kann, obwohl die Interferenzverbindungen sehr schwach sind. Zum anderen, dass das TIN selbst dann optimal sein kann, wenn die Interferenzverbindungen gegenüber den gewünschten Verbindungen nicht ausreichend schwach sind. Die Gründe dafür liegen nicht nur in der Struktur und dem Informationsfluss des PIMAC, sondern auch in einigen neuen oberen Schranken, die in der vorliegenden Arbeit hergeleitet werden. Die Erkenntnisse sind dabei jedoch nicht nur auf den PIMAC beschränkt, sondern teilweise auch auf größere Netzwerke erweiterbar. Um das zu zeigen wird als nächstes die GDoF-Optimalität von TIN im $M \times 2$ X-Kanal untersucht. Der $M \times 2$ X-Kanal besteht aus M Sendern und 2 Empfängern, in dem jeder Sender zwei unabhängige Nachrichten an beide Empfänger schicken will. Die aus dieser Untersuchung hergeleiteten Bedingungen für die Optimalität von TIN charakterisieren eine größere Menge von Kanalparametern als bisher in der einschlägigen Literatur zu finden ist. Die Notwendigkeit der Bedingungen wird weiterhin für eine bestimmte Art von TIN, in der der $M \times 2$ X-Kanal auf einen 2-Nutzer-Interferenzkanal reduziert wird, hergeleitet.

Abbreviations and Notation

Abbreviations

AWGN	Additive White Gaussian Noise
BC	Broadcast channel
CSI	Channel state information
DoF	Degrees of freedom
GDoF	Generalized degrees of freedom
i.i.d.	Independent and identically distributed
ITLinQ	Information-theoretic link scheduling
IA-CP	Interference alignment with common and private signaling
IC	Interference channel
INR	Interference-to-noise ratio
LD	Linear deterministic
MAC	Multiple access channel
MIMO	Multiple-input multiple-output
PA-CP	Phase alignment with common and private signaling
PIMAC	Point-to-point channel interfering with a multiple access channel
P2P	Point-to-point
Rx	Receiver
SNR	Signal-to-noise ratio
SISO	Single-input single-output
TDMA	Time division multiple access
Tx	Transmitter
TIN	Treating interference as noise

Notations

The following table summarizes the notations used in this thesis.

Lower-case bold: $\boldsymbol{x}$	column vector
Upper-case bold: $\boldsymbol{X}$	matrix
Calligraphic: $\mathcal{X}$	set
x^n	$(x(1), x(2), \ldots, x(n))$
$\boldsymbol{x}_{[a:b]}$	vector formed by the a-th to b-th element of vector $\boldsymbol{x}$
$\boldsymbol{X}_{[a:b]}$	matrix formed by the a-th to b-th rows of a matrix $\boldsymbol{X}$
$\mathbb{F}$	binary field
$\mathbb{N}$	natural numbers
$\mathbb{N}^0$	$\mathbb{N} \cup \{0\}$
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbf{0}_q$	zero-vector of length q
$\mathbf{0}_{l,m}$	zero matrix of size $l \times m$
$\boldsymbol{I}_q$	$q \times q$ identity matrix
$\text{Prob}(E)$	Probability of an event E
$\mathbb{C}\mathcal{N}(\mu, \sigma^2)$	circularly symmetric complex normal distribution with mean μ and variance σ^2
$H(X)$	entropy of X
$H(X, Y)$	joint entropy of X and Y
$H(X\|Y)$	entropy of X given Y
$h(X)$	differential entropy of X
$h(X, Y)$	joint differential entropy of X and Y
$h(X\|Y)$	differential entropy of X given Y
$I(X; Y)$	mutual information between X and Y
$\oplus$	modulo 2 addition
$\lfloor x \rfloor$	integer part of x
$\lceil x \rceil$	$\lfloor x \rfloor + 1$
x^+	$\max\{x, 0\}$
$\|x\|$	$x^+ + (-x)^+$
x^*	complex conjugate of x
$\|\mathcal{X}\|$	cardinality of set $\mathcal{X}$
$\boldsymbol{X}^T$	transpose of $\boldsymbol{X}$

1 Introduction

The role of wireless communications as the required infrastructure of future technologies such as internet of things, car to car communications, etc., becomes more and more important. However, increase in the number of devices and the demand for higher data rate on one hand; and limited resources such as time and frequency on the other hand face the traditional communication networks with a big challenge namely “interference management”.

In recent years, great progresses have been achieved in interference management from theoretical point of view. Novel transmission techniques which achieve approximately the capacity limits of interference networks have been proposed. Most of these techniques use the fact that the interference carries some information. Hence, by decoding and cancelling the interference a cleaner observation of the received signal will be available [HK81]. These types of techniques require perfect channel state information and fully coordinated synchronous systems. This coordination can enhance the performance of a transmission, at the expense of increasing its complexity and power consumption. In most communication systems existing nowadays, the communicating nodes have several practical constraints. One such constraint is the limited computational capability of the communicating nodes. This limitation demands communication schemes with low complexity and consequently, low power consumption. Hence, the traditional schemes which are based on ignoring and avoiding the interference, are still the most applicable approaches in dealing with interference in the practical scenarios. In these approaches, the decoding at the receivers is restricted to a simple technique namely “treating interference as noise” (TIN).

By using TIN, the Gaussian channel coding as in the point-to-point (P2P) channel with power allocation can be used at the transmitter side while the receivers ignore the interference. Hence, compared to the P2P channel, no additional computation is required neither at the transmitters nor at the receivers. Showing the optimality of TIN from a capacity perspective adds to these advantages that no more complex schemes are required. The results on the optimality of TIN can be motivated from different aspects. Some of them are listed bellow.

- **Developing more efficient scheduling mechanisms:** By utilizing the results on the optimality of TIN, more efficient mechanisms on resource allocation and scheduling can be developed. For instance, a recently proposed spectrum sharing mechanism in [NA14], referred to as information-theoretic link scheduling (ITLinQ) uses the information theoretic conditions on optimality of TIN as a metric for grouping the links (sender receiver pairs) in a network into so-called not-detrimental links. The not-detrimental links cause sufficiently low interference to each other. Hence, they are allowed to be active simultaneously over the same frequency band. Through the numerical analysis in [NA14], it is shown that ITLinQ outperforms significantly the previous mechanisms such as FlashLinQ [WTS$^+$10] with respect to the sum-rate. The scheduling mechanism of ITLinQ has been further improved in [YC15] by using a new selection criteria and taking also the power control at the transmitters into account.

- **No loss for secrecy:** Studying the optimality of TIN is also interesting from secrecy point of view. Roughly speaking, if a system operates in the regime in which TIN performs approximately capacity optimal, the secrecy constraint does not lead to a significant loss in the capacity. This fact has been investigated in [GJ15] for the K-user interference channel (IC).

Due to the above mentioned facts, studying the regimes in which TIN performs (approximately) optimally attracted more focus in recent years. The optimality of TIN has been studied for different channels such as interference channel and X-channel. It is shown in [GNAJ15] that TIN is asymptotically optimal in the interference channel which operates in very weak interference regime. This regime is defined formally by

$$\mathrm{SNR}|_{\mathrm{dB}} \geq \max\{\text{observed INR}|_{\mathrm{dB}}\} + \max\{\text{caused INR}|_{\mathrm{dB}}\}, \tag{1.1}$$

where SNR and INR represent signal-to-noise ratio and interference-to-noise ratio, respectively. In words, in the very weak interference regime, the sum of the powers of the strongest interference caused by a user plus the strongest interference it receives is less than or equal to the power of its desired signal, on a logarithmic scale. Roughly speaking, in this regime, the interference links need to be sufficiently weaker than desired links.

Now, one question which arises is whether this condition can be extended to any channel. Is TIN asymptotically optimal in any channel in which interference links are very weaker than the desired links? Or is there any channel in which TIN is suboptimal although the interference links are very weak compared to the desired links? Another question is what happens when we are outside the very-weak interference regime. Is TIN necessarily suboptimal if the interference links are not sufficiently weak? Or perhaps, there exists a regime in which TIN is optimal although some interference links are strong?

In order to answer these questions, we study the optimality of using TIN at receivers with Gaussian codebooks employed at transmitters in an elementary channel. This channel consists of a multiple access channel (MAC) with two transmitters and a point-to-point (P2P) channel which share the same communication medium. Hence, we have a P2P channel interfering with a MAC, known as PIMAC. The simple and compact presentation of the problem for this elemental network allows us to understand the optimality of TIN deeply from different features. Studying this setup captures new facts which answer the aforementioned questions. Moreover, the insights obtained from studying this fundamental setup is useful in understanding the performance of TIN in larger networks.

Showing the optimality of TIN from the capacity point of view consists of two main steps. The first step is to characterize the maximum achievable sum-rate by using TIN at the receiver side, while the transmitters are allowed to use a Gaussian codebook. The second step is to assess the optimality of TIN by establishing some upper bounds on the capacity of the network which serve as benchmark. If for a range of channel parameters, the achievable sum-rate of TIN coincides with the established upper bound, we conclude that no other scheme can outperform TIN and thus in that regime TIN is capacity optimal. Despite continuous attempts in finding the capacity of different elemental networks, except some few exceptions, the complete capacity characterization of many of them is still open. This highlights the fact that the capacity characterization of multi-user networks is in general difficult and requires some intermediate steps. An important step towards characterizing the capacity is to study the so-called generalized degrees of freedom (GDoF) which is an approximation of the capacity at high signal-to-noise ratio (SNR). Studying the GDoF provides also some insights which can be helpful in obtaining the capacity of a setup in finite SNR. The main focus of this thesis is to study the optimality of TIN from the GDoF perspective. In the regimes in which TIN is GDoF optimal, the capacity of the channel is asymptotically achievable at high SNR by using TIN. Additionally, showing the optimality of TIN from the GDoF perspective can lead us to a more general result which is the constant gap optimality of TIN. This result states that TIN achieves the capacity of the channel for all values of SNR within a gap which does not exceed a constant number of bits.

By studying the GDoF optimal regime of TIN in PIMAC, we obtain a surprising result which shows that the GDoF optimal regime of TIN is not restricted to very weak interference regime as in (1.1). Interestingly, there might be cases in which, although (1.1) is not satisfied, TIN is still GDoF optimal. The extension of the GDoF optimal regime of TIN is not only due to the structure of the PIMAC but mainly due to the new established upper bounds in this thesis. In order to complete our study on the GDoF optimality of TIN in PIMAC, we show its suboptimality as long as PIMAC operates outside the GDoF optimal regime of TIN. To do this, we propose a scheme which uses interference decoding and achieves a higher GDoF than that of TIN. Interestingly, it turns out that there are some cases in which TIN is suboptimal, although the observed interference signals at all receivers are very weak compared to the desired signals. It is shown that even in this case, a more sophisticated schemes which incorporate interference alignment might outperform TIN.

In fact, the obtained insights from studying the optimality of TIN in the PIMAC are very helpful in understanding the performance of TIN in more general setups. To show this, the optimality of TIN is also studied in a so-called $M \times 2$ X-channel in which each of the M transmitters wants to communicate independent messages to both receivers. For this channel, the achievable GDoF of TIN with power allocations at the transmitters is studied. It turns out that the transmit power allocation maximizing the achievable GDoF is given by on-off signaling as long as the receivers use TIN. This leads to different variants of TIN scheme whose GDoF optimal regimes are characterized. The characterized GDoF optimal regime of TIN for the $M \times 2$

X-channel expands the GDoF optimal regime established in the literature [GSJ15] significantly.

1.1 Organization of the Thesis

In the following chapter, we present the related works on the optimality of TIN and some preliminary concepts which are required in the thesis. The focus of Chapter 3 is on the optimality of TIN in the PIMAC. In Chapter 4, the optimality of TIN in an $M \times 2$ X-channel is studied. Finally, Chapter 5 concludes the thesis. The content of each chapter is summarized in what follows.

1.1.1 Chapter 2

In the first part of this chapter, we give an overview of the results presented in the literature on the optimality of TIN. The focus of this chapter is not only on understanding the links between the results, but also providing the reader an intuitive explanation about the key ideas behind. To this end, we start the discussion by giving an overview on the results done on the capacity of the interference channel. The importance of the interference channel is not only since it has attracted many researcher as an elemental network which captures main features of interference but also since the results obtained for the interference channel had important impacts on further works on different networks. Next, we discuss the optimality of TIN in X-channel as a general setup which is also studied in this thesis. Finally, we present the PIMAC and related results.

In the next part of this chapter, we present some preliminaries which will be required in the thesis. Since our approach towards the analysis of TIN in PIMAC is based on studying the so-called linear deterministic model, a section is dedicated to this concept. Finally, we present the preliminaries of lattice codes since it will be used in some transmission schemes introduced in the thesis.

1.1.2 Chapter 3

In this chapter, the GDoF of the PIMAC is studied with main focus on the optimality of TIN. Depending the strategy at the transmitter side, different variants of TIN are introduced. While in the simple variant of TIN, namely naive-TIN, all transmitters send with full power simultaneously, in the so-called time division multiple access and TIN (TDMA-TIN), the transmitters divide up the time between each other. Hence, in TDMA-TIN, the PIMAC is reduced to two 2-user interference channels and a point-to-point channel which operate over orthogonal time slots. The performance of each variant of TIN is studied separately. To this end, we consider first the linear deterministic PIMAC which is an approximation of the Gaussian PIMAC. The necessary and sufficient conditions on the optimality of TIN from the capacity point of view are established for the linear deterministic model. This provides us a complete characterization of the capacity optimal and suboptimal regimes of TIN.

Next, the insights obtained from the linear deterministic PIMAC are extended to the Gaussian counterpart. In doing so, the capacity results for the deterministic model are translated to the GDoF result for the Gaussian setup. Hence, we obtain the necessary and sufficient conditions on the GDoF optimality of TIN and furthermore its complete GDoF (sub)optimal regime. Moreover, it is shown that as long as TIN is GDoF optimal, it achieves the capacity of the PIMAC within a constant gap. The material of this chapter has been published partially in the following works.

- [GCDS16] S. Gherekhloo, A. Chaaban, C. Di, and A. Sezgin, "(Sub-)optimality of treating interference as noise in the cellular uplink with weak interference," *IEEE Transactions on Information Theory,* vol. 62, no. 1, pp. 322-356, Jan 2016.

- [GCS16a] S. Gherekhloo, A. Chaaban, and A. Sezgin, "The information-theoretic constant-gap optimality of treating interference as noise in interference networks," in *Utschick, Wolfgang, ed. (Hrsg.): Communications in Interference Limited Networks. Springer International Publishing, 2016* pp. 75-96.

- [GCS14a] S. Gherekhloo, A. Chaaban, and A. Sezgin, "Coordination gains in the cellular uplink with noisy interference," in *Proc. of the 15th IEEE International Workshop on Signal Processing Advances for Wireless Communications (SPAWC)*, pp. 244-248, Toronto, Canada, June 2014.

1.1.3 Chapter 4

In this chapter, the results on the necessary and sufficient conditions of the GDoF optimality of TIN for the PIMAC are extended to a more general network, namely the $M \times 2$ X-channel. To this end, the achievable GDoF of TIN with power allocations at the transmitters is studied. It turns out that the transmit power allocation maximizing the achievable GDoF is given by on-off signaling as long as the receivers use TIN. This leads to two variants of TIN; namely, P2P-TIN and 2-IC-TIN. While in the first variant the $M \times 2$ X-channel is reduced to a point-to-point (P2P) channel, in the second variant the setup is reduced to a 2-user interference channel in which the receivers use TIN. The optimality of these two variants are studied separately. To this end, novel genie-aided upper bounds on the capacity of the X-channel are established. Additionally, it is shown that the addressed conditions on the GDoF optimality of 2-IC-TIN are not only sufficient but also necessary. The introduced GDoF optimal regimes of TIN subsumes and extends the known regime in the literature. The material of this chapter has been published partially in the following works.

- [GCS16b] S. Gherekhloo, A. Chaaban, and A. Sezgin, "Expanded GDoF-optimality regime of treating interference as noise in the $M \times 2$ X-Channel," submitted to *IEEE Transactions on Information Theory,* 2016.

- [GCS14b] S. Gherekhloo, A. Chaaban, and A. Sezgin, "Extended generalized DoF optimality regime of treating interference as noise in the X channel," in *Proc. of the 11th IEEE International Symposium on Wireless Communications Systems (ISWCS)*, pp. 971-975, Barcelona, Spain, Aug. 2014.

1.2 Contributions Outside the Scope of the Thesis

Contributions on other topics which are not part of the thesis are listed below.

- [GS16] S. Gherekhloo, and A. Sezgin, "Latency-limited broadcast channel with cache-equipped helpers," submitted to *IEEE Transactions on Wireless Communications,* 2016.
- [GCS15a] S. Gherekhloo, A. Chaaban, and A. Sezgin, "Cooperation for interference management: A GDoF perspective," submitted to *IEEE Transactions on Information Theory,* 2015.
- [GCS15b] S. Gherekhloo, A. Chaaban, and A. Sezgin, "Optimality of treating interference as noise in the IRC: A GDOF perspective," in *Proc. of Asilomar Conference on Signals, Systems, and Computers,* Pacific Grove, CA, Nov. 2015.
- [GCS14c] S. Gherekhloo, A. Chaaban, and A. Sezgin, "Resolving entanglements in topological interference management with alternating connectivity," in *Proc. of IEEE International Symposium on Information Theory (ISIT)*, Honolulu, HI, USA, July, 2014.
- [GCS14d] S. Gherekhloo, A. Chaaban, and A. Sezgin, "Topological interference management with alternating connectivity: The Wyner-type three user interference channel," in *International Zurich Seminar on Communications (IZS)*, Zurich, Switzerland, Feb., 2014.
- [GCS13] S. Gherekhloo, A. Chaaban, and A. Sezgin, "The generalized degrees of freedom of the interference relay channel with strong interference," in *51st Annual Allerton Conference on Communication, Control and Computing*, 2013.
- [CGS13] A. Chaaban, S. Gherekhloo, and A. Sezgin, "Relays for interference management: Feedback, amplification and neutralization," in *Proc. of the 14th IEEE International Workshop on Signal Processing Advances for Wireless Communications (SPAWC)*, Darmstadt, Germany, Jun., 2013.

2 History and Preliminaries

In 1948, Claude Shannon has published his seminal paper [Sha48] in which he founded the area of information theory that is used as the concept for studying the fundamental limits of communication systems. In this work, he introduced channel capacity as the highest transmission rate for reliable communication between a source and a destination. Since then, finding the channel capacity became the main challenge in studying communication networks from information theory point of view. While in the traditional Gaussian point-to-point channel, additive noise imposes the main restriction on the communication capacity, in networks with multiple senders and receivers, interference might strongly affect the capacity. Due to the fact that interference has a structure, receivers are able to decode and remove it partially from the received signal. In doing so, receivers obtain a relatively cleaner version of the desired signal. Although this way of interference management improves the transmission rate in many cases, it is not used in practical scenarios due to the complexity of encoding and decoding. Alternative to decoding the interference is to ignore it completely, i.e., treating interference as noise (TIN) at the undesired receiver. In this scheme, the receivers decode their desired message in the same way as if they do not observe any interference. This is a common way to deal with interference in practical communication scenarios, due to the low complexity and low requirements on coordination between different nodes. The simplicity of TIN has attracted the attention of many researchers to study its optimality in different channels. In what follows, we give an overview of these results.

2.1 TIN in Interference Channel

The interference channel (IC) consists of multiple point-to-point channels which use the same resources such as time and frequency concurrently. Studying this channel as a basic setup which captures the effects of interference is not only important from theoretical point of view but also from practical aspects. For instance, in cellular networks, the users located close to the edge of the cell might suffer from strong interference caused by the base station of the adjacent cell. This can be modelled as an interference channel. The practical scenarios of interference channel are not restricted to wireless networks but also to wired channels. For instance, the IC is an appropriate setup for representing the crosstalk phenomenon in telephone lines.

The smallest possible interference channel is a 2-user IC which consists of two transmitters (Tx1 and Tx2) and two receivers (Rx1 and Rx2) in which Txi (with $i \in \{1,2\}$) wants to communicate its message W_i to Rxi and causes interference at Rxi' (with $i' \in \{1,2\}$ and $i' \neq i$). The Gaussian 2-user IC is shown in Fig. 2.1.

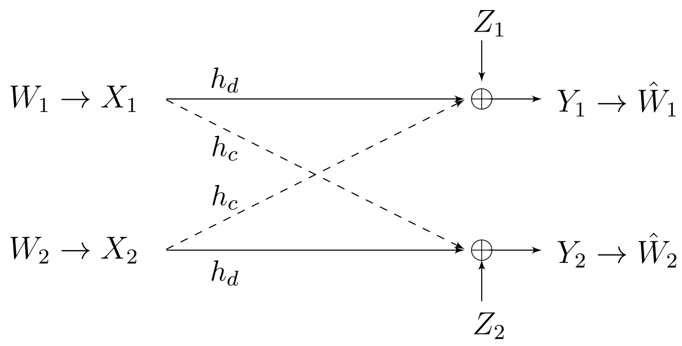


Figure 2.1: The system model of a symmetric Gaussian 2-user interference channel.

The capacity region of 2-user IC has been first studied by Ahlswede in [Ahl74] by introducing fundamental inner and outer bounds. In [Car75], Carleial has introduced the so-called very strong interference regime, in which interference does not reduce the capacity regime of two parallel point-to-point channels. The capacity region of the Gaussian interference channel in the strong interference regime has been obtained later in [Sat81] and [HK81]. Additionally, it has been shown in [Sat81] that the capacity of the Gaussian 2-user IC in strong interference regime is the intersection of the capacity region of two multiple access channels (MAC). The inputs of both multiple access channels are the sent signals by the transmitters, i.e., X_1 and X_2 while the outputs are either Y_1 (for Rx1) or Y_2 (for Rx2). The inner bounds for the capacity of the 2-user IC in the strong interference regime is based on rate splitting and successive interference cancellation which was extensively improved by Han and Kobayashi in [HK81]. Based on the Han-Kobayashi scheme, the message of each transmitter is split into two sub-message namely private and common message. While the private message is decoded only at the desired receiver, the common message is intended to both receivers. Although the Han-Kobayashi scheme is the best known inner bound on the capacity of the interference channel, no outer-bound could justify its optimality in the low interference regime for decades. After more than 20 years, Kramer established in [Kra04] two outer bounds on the capacity of the 2-user Gaussian interference channel. The main idea presented in this work was a genie-aided approach in which a genie provides the receivers some additional information. The side information was chosen such that each receiver were able to decode both messages. The bounds in [Kra04] developed the already known upper bounds of Sato [Sat77] and Carleial [Car83] and optimized them for the weak interference regimes. However, these bounds did not coincide with the achievable sum-rate in the weak interference regime. In 2008, Etkin, Tse, and Wang have shown in their well-known work [ETW08] that the Han-Kobayashi scheme achieves the capacity of the Gaussian 2-user IC within one bit gap. Besides this fundamental result, the work done by Etkin *et. al.* had significant impact on subsequent works from different aspects listed below.

- **Introducing new metrics:** The authors of [ETW08] were aware of the difficulty of capacity characterization in general. Due to this, they introduced the so-called generalized degrees of freedom (GDoF) metric which serves as a stepping stone towards characterizing the capacity. The GDoF metric generalizes the metric degrees of freedom (DoF). The notion of DoF originates from the capacity of the multi-antenna Gaussian channel [Tel99] which is later used for representing the maximum available number of independent point-to-point streams in a network at high signal-to-noise ration (SNR). To be more precise, we introduce the DoF metric for the symmetric Gaussian 2-user IC shown in Fig 2.1. The channels h_d and h_c represent the real valued desired and interference links, respectively. Moreover, the signal power is P and the noise power is 1. Similar to [CJ08], the DoF of this setup is defined by

$$d = \lim_{\text{SNR}\to\infty} \frac{C_\Sigma(\text{SNR})}{\frac{1}{2}\log_2(\text{SNR})}, \tag{2.1}$$

where SNR is the received signal-to-noise ratio and C_Σ represents the sum-capacity of the channel. Notice that in the definition of DoF, the interference-to-noise ratio (INR) is not considered. In fact, this definition does not capture the affect of different channel strengths in a network. Hence, it was not appropriate for studying the capacity of the IC in the weak interference regime where $|h_c| < |h_d|$. To solve this problem, Etkin *et. al.* introduced a new parameter for representing the channel strength, i.e.,

$$\alpha = \frac{\log_2(P|h_c|^2)}{\log_2(P|h_d|^2)}. \tag{2.2}$$

In fact, the parameter α represents the capacity of the point-to-point interference link with respect to that of the desired link at high SNR. By including this parameter into the definition of the DoF, we obtain the GDoF metric which can be applied to more general set of networks. The GDoF is defined as follows [ETW08]

$$d(\alpha) = \lim_{\text{SNR}\to\infty} \frac{C_\Sigma(\text{SNR},\alpha)}{\frac{1}{2}\log_2(\text{SNR})}. \tag{2.3}$$

Intuitively, the metric GDoF determines the maximum number of interference-free and independent streams available in a network at high SNR while the capacity of each stream is the same as the capacity of a reference point-to-point (P2P) channel, i.e., $\frac{1}{2}\log_2(\text{SNR})$. Hence, the aforementioned definition of the GDoF is equivalent to

$$C_\Sigma(\text{SNR},\alpha) = d(\alpha)\log_2(\text{SNR}) + o(\log_2(\text{SNR})), \tag{2.4}$$

where $\frac{o(\log_2(\text{SNR}))}{\log_2(\text{SNR})} \to 0$ as $\text{SNR} \to \infty$.

- **TIN optimality:** The first step towards showing the optimality of treating interference as noise (TIN) in the IC has been made in [ETW08]. Notice

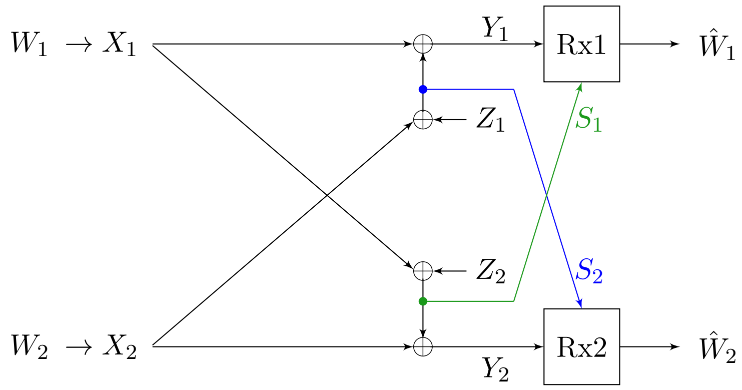


Figure 2.2: In the Genie-aided bound which shows the GDoF optimality of TIN in very-weak interference regime [ETW08], S_1 and S_2 are provided to Rx1 and Rx2, respectively. The sum-capacity of this channel serves as an upper bound for the original 2-user IC.

that the Han-Kobayshi scheme reduces to TIN, by just setting the rate of the common messages to zero. Erkin *et. al.* have shown that there exists a range of channel parameters in which TIN achieves the sum-capacity of the Gaussian 2-user IC within a constant gap. To show this, they obtained a new upper bound which serves as an inspiration of some upper bounds established in the subsequent works on the optimality of TIN. This upper bound is a type of genie-aided bound proposed by Kramer in [Kra04]. To establish this bound, the genie provides to each receiver the noisy version of the interference signal received at the other receiver (see Fig. 2.2). Since after providing additional information to receivers, they are more capable in decoding their desired messages, the capacity of the obtained channel serves as an upper bound for the capacity of the original interference channel. By using this bound, it is shown that in a symmetric Gaussian 2-user IC with signal power P and noise power 1 TIN is approximately capacity optimal in very-weak interference regime which is defined as follows

$$P|h_c|^2 \leq \sqrt{P|h_d|^2}. \tag{2.5}$$

By using the parameter α defined in (2.2), the very-weak interference regime can be defined by

$$\alpha \leq \frac{1}{2}. \tag{2.6}$$

This result on optimality of TIN in interference channel has been refined in three independent works [AV09, SKC09, MK09]. In these works, it was shown that TIN achieves the sum-capacity of the 2-user IC in the so-called noisy interference regime introduced in [AV09].

The noisy interference regime is a smaller regime than the obtained regime from extending the characterized very-weak interference regime in [ETW08] to moderate and low SNR (defined in (2.5)). The results on the optimality of TIN from capacity point of view have been further extended in [BSP08], [SCKP10], and [AV11] for the multiple antenna case of Gaussian IC. Moreover, the optimality of TIN from the capacity point of view has been also investigated for parallel Gaussian interference channels in [SCKP11].

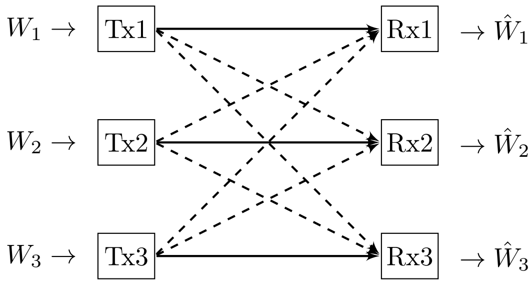
At this point it is worth mentioning that even characterizing the achievable rate region of TIN in the IC was not trivial and required optimizing the transmit power. It has been shown in [CSHP12] that without involving time-sharing, the achievable rate region of TIN is in general non-convex. In addition to this, the difficulty of establishing tight upper bounds on the capacity in closed form has motivated the researchers to focus their study more on the optimality of TIN from the GDoF perspective. The GDoF optimality of TIN in [ETW08] has been extended to larger networks by Jafar and Vishwanath in [JV10]. Interestingly, it turned out that the GDoF optimal regime of TIN in the symmetric K-user IC with $K > 2$ is independent from the number of users K. Hence, TIN is GDoF optimal in symmetric K-user IC as long as $\alpha \leq \frac{1}{2}$. Additionally, it is shown that if a symmetric K-user IC operates outside the very-weak interference regime ($\alpha \leq \frac{1}{2}$), TIN will be outperformed by a smarter transmission scheme.

The optimality of TIN in the fully-connected fully-asymmetric K-user IC has been studied in [GNAJ15]. In this work, it has been shown that TIN achieves the capacity region of the K-user IC within a constant gap if for all transmitter-receiver pairs

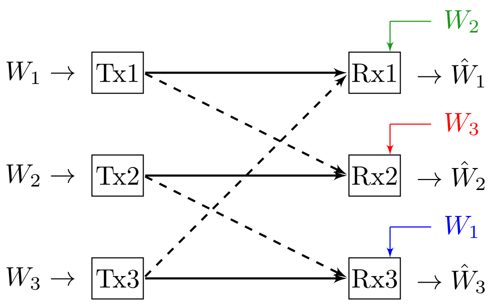
$$\mathrm{SNR}|_{\mathrm{dB}} \geq \max\{\text{observed } \mathrm{INR}|_{\mathrm{dB}}\} + \max\{\text{caused } \mathrm{INR}|_{\mathrm{dB}}\} \tag{2.7}$$

holds. Note that the obtained condition by applying (2.7) to the symmetric K-user IC is equivalent to the result in [JV10]. To show the GDoF optimality of TIN, they have established the outer bound in two steps. First, they reduced the K-user IC to a cyclic IC. Next, by using the result of [ZY13] on the capacity region of a K-user cyclic Gaussian IC, they have established an upper bound for the asymmetric K-user IC. In fact the upper bound presented in [GNAJ15] is inspired from the bounds established in aforementioned works. In order to highlight this fact, in what follows we present an intuitive explanation of the upper bound for the 3-user IC shown in Fig. 2.3a. Here, we enhance the 3-user IC in two steps. In each step, a genie provides some side information to the receivers [Kra04]. This cannot decrease the capacity of the 3-user IC. Hence, the capacity of resulting channel serves as an upper bound for the original 3-user IC. The provided messages are chosen in a way such that the original channel is reduced to a cyclic IC. An example for the obtained cyclic IC is shown in Fig. 2.3b. Next, as it is shown in Fig. 2.3c, we provide the noisy version of the interference caused by each transmitter to its desired receiver. Notice that the side information in this step is very similar to the side information provided in 2-user IC in very-weak interference regime [ETW08].

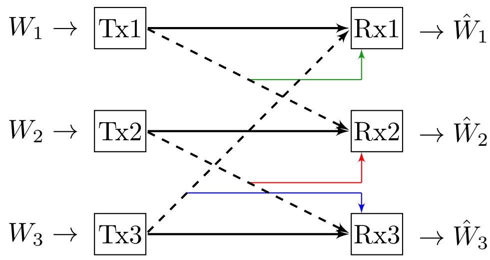
The optimality of TIN in the IC in terms of the approximation of the capacity has been also studied for the multiple antenna case. For instance, while in [GJ11]



(a) An asymmetric interfernece channel



(b) Giving the side information W_2, W_3, and W_1 to Rx1, Rx2, and Rx3, respectively, we obtian the cyclic 3-user IC.



(c) The noisy version of the interferenece caused by each transmitter is provided to the desired receiver.

Figure 2.3: The steps for obtaining an upper bound for 3-user IC inspired by [GNAJ15].

the capacity of the single-input multiple-output IC is studied within a constant gap, the approximation for the capacity of the multiple-input multiple-output IC is investigated in [KV12, KV13]. Interestingly, in all these works, it is shown that TIN does not perform necessarily optimal if the network operates in the very weak interference regime.

2.2 TIN in X-channel

The interference channel can be extended to a more general setup, namely X-channel, in which each transmitter has an independent message for each receiver [MAMK08, JS08]. The X-channel is an appropriate network for modelling the scenarios in which multiple base stations serve the same users. Studying the capacity of the X-channel as a rich network which can be reduced to many channels such as interference channel, multiple access channel, broadcast channel, etc, is interesting from both theoretical and practical points of view.

Since the X-channel is reduced to an interference channel by setting the rates of some messages to zero, all achievable some rates in the interference channel are also achievable in the X-channel. Moreover, the capacity of an X-channel serves as an upper bound for the capacity of its enclosed interference channels. Notice that from the statements above, we can not follow that whenever TIN is suboptimal in an IC, it is also suboptimal in the corresponding X-channel. On the other hand, from the optimality of TIN in the interference channel, we cannot necessarily conclude the optimality of TIN in the X-channel. The main point which we want to highlight by this discussion is that although we obtain some deep insights from understanding the performance of TIN in the IC which can be also useful or even might be extendable to the X-channel, studying the optimality of TIN in the X-channel is a distinct problem which requires individual analysis.

Exploiting the fact that the 2×2 X-channel is a combination of two broadcast channels and two multiple access channels, different schemes for maximizing the multiplexing gain have been proposed for the multiple-input multiple-output (MIMO) X-channel by Maddah-Ali, Motahari, and Kandani in [MaMK06a, MaMK06b]. Later in [MAMK08], it was shown that the DoF of $\lfloor \frac{4r}{3} \rfloor$ is achievable in the MIMO 2×2 X-channel with r antennas at each node. By using the concept of symbol extension introduced by Jafar and Shamai in [JS08], the achievable DoF is increased upto $\frac{4r}{3}$ in [JS08]. This achievable DoF coincides with the upper bound on the DoF of the MIMO X-channel which is also established in [JS08]. The next step towards obtaining the capacity of the X-channel was characterizing its GDoF. This has been accomplished by Huang, Cadambe, and Jafar for the single-input single-output (SISO) 2×2 X-channel in [HCJ12]. Moreover, in this work the GDoF optimal regime of TIN is characterized for the 2×2 X-channel. It turned out that as long as TIN is GDoF optimal in the 2-user IC [ETW08], it is also optimal in the corresponding X-channel. In order to show this, they established an upper bound on the GDoF of the X-channel. To this end, some side information is provided to receivers. The provided information to each receiver are the undesired messages which are not included in the IC in addition to the same side information used in the genie-aided bound of IC in very weak interference regime [ETW08] (see Fig. 2.2). The enhanced channel after providing this side information to the receivers of the X-channel are illustrated in Fig. 2.4. In this figure, W_{ji} represents the message of the ith transmitter which is desired at the jth receiver. By comparing this figure with Fig. 2.2, we see that in the X-channel in addition to S_1 and S_2, the messages W_{21} and W_{12} are also provided as side information to the first and the second receiver, respectively. Roughly speaking, by providing these messages to the receivers, each receiver is interfered by only one transmitter. In this sense, the setup becomes similar to the 2-user IC and hence, its capacity can be bounded as in the 2-user IC by providing S_j to Rxj with $j \in \{1, 2\}$.

It is worth noting that since one can consider two different IC's in a 2×2 X-channel, two distinct regimes for GDoF optimality of TIN are addressed in [HCJ12]. Furthermore, in [HCJ12], the conditions on capacity optimality of TIN are established for the 2×2 X-channel. Interestingly, they have shown that that as long as TIN is capacity optimal in the 2-user IC [AV09, SKC09, MK09], it is also optimal in the corresponding X-channel.

The optimality of TIN in larger X-channels has been investigated in [GSJ15] from the GDoF perspective. Interestingly, it is shown in this work that as long as the optimality conditions of TIN presented in [GNAJ15] are satisfied for a K-user IC,

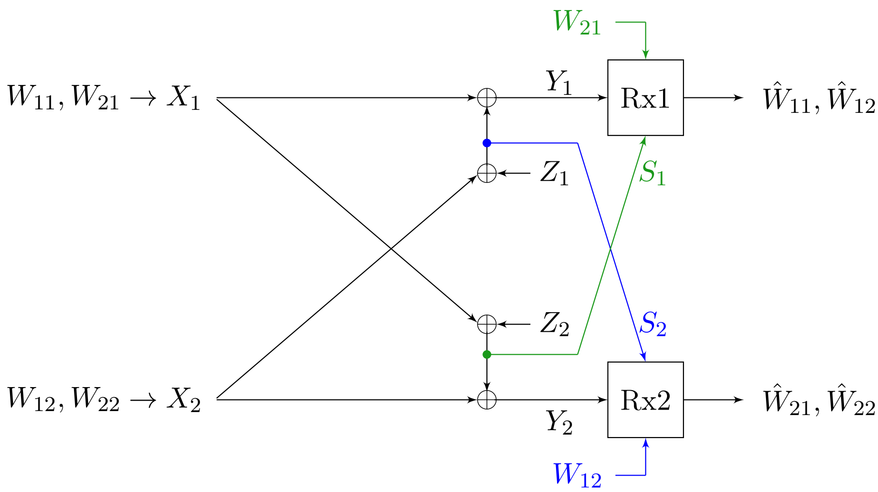


Figure 2.4: The enhanced channel after providing the side information to the receivers of the 2×2 X-channel. The obtained upper bound on the GDoF of this enhanced channel is used in [HCJ12] to show the GDoF optimality of TIN in the X-channel.

TIN is also GDoF optimal in the corresponding $K \times K$ X-channel. In other words, if TIN is GDoF optimal in a K-user IC based on the conditions in [GNAJ15], increasing the message set cannot increase the GDoF. To show this, they established a tight genie-aided upper bound on the GDoF. To establish this bound, they extended the insights obtained from [GNAJ15] to the X-channel. An intuitive explanation of the bound in [GSJ15] is as follows. Consider the corresponding K-user IC which satisfies the conditions in [GNAJ15]. Based on this IC, we use the terms (un)desired transmitters or receivers. In their genie-aided bound, a subset of undesired messages are provided to each receiver. Roughly speaking, this subset of the messages reduces the fully connected X-channel to a setup with a cyclic connectivity. In addition to this, similar to [GNAJ15], they provided to each receiver the noisy version of the strongest interference which is caused by the desired transmitter. Using this genie-aided bound, they obtained an upper bound on the GDoF similar to the bound proposed for the K-user IC.

The authors of [GSJ15] extended their results to an $M \times N$ X-channel with $M \neq N$. It turned out that reducing an $M \times N$ X-channel to a K-user IC, with $K = \min\{M, N\}$, and using TIN at the receivers is GDoF optimal as long as the optimality conditions in [GNAJ15] are satisfied. However, as the authors of [GSJ15] have also commented, their addressed conditions on optimality of TIN are sufficient but not necessary. In this thesis, we extend their GDoF optimal regime of TIN for the 3×2 X-channel.

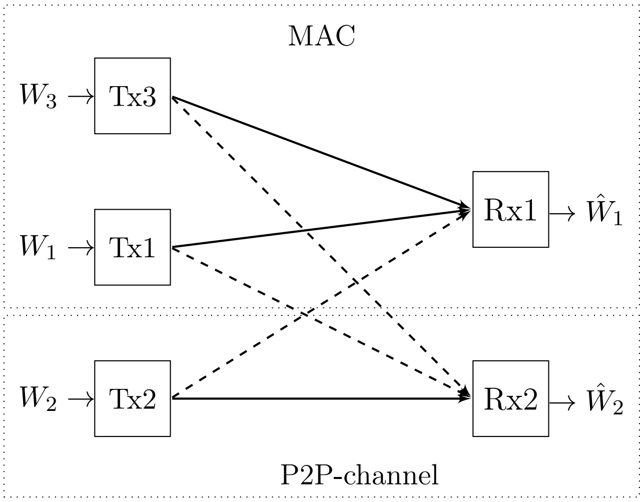


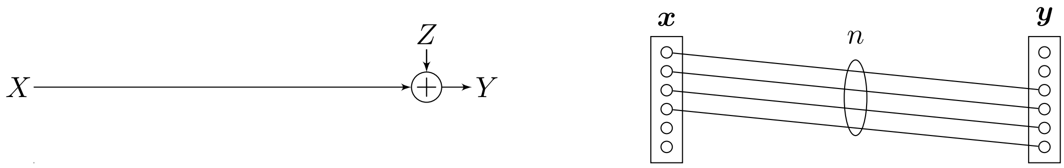
Figure 2.5: The system model of the PIMAC. Tx1 and Tx3 are the MAC transmitters. They communicate their messages to Rx1. Moreover, Tx2 communicates with Rx2 over the same medium.

2.3 Point-to-Point Channel Interfering with a Multiple Access Channel (PIMAC)

By considering a point-to-point (P2P) channel which interferes a multiple access channel (MAC) with two transmitters, the PIMAC [CS11c] is generated. This network is shown in Fig. 2.5. Such a setup arises when a P2P communication system uses the same communication medium as a cellular uplink for instance. In fact, the PIMAC merges two important channels, namely the multiple access channel (MAC) and 2-user IC into a setup.

The capacity of a partially connected PIMAC (known as Z-PIMAC), in which only the MAC transmitters cause interference to the undesired receiver, has been studied in [ZSCP14]. In this work, an achievable rate region based on superposition coding and joint decoding was provided. This scheme achieves the capacity of the Z-PIMAC with very strong interference. Moreover, it is shown that in the weak interference regime and with power constraints, treating interference as noise achieves the capacity within a half of bit.

The more general setup has been considered in [CS11c]. In this work, the capacity region of the PIMAC is characterized for some parameter ranges mainly in the strong interference regime. Moreover, using the similar genie aided approach as in [AV09], a new outer bound for weak interference regime is established. It is claimed intuitively that if the interference power is sufficiently low (lower than a threshold), then treating interference as noise achieves a sum rate which is close to the upper bound. Interestingly, Chaaban and Sezgin have shown in [CS12] that the achievable sum-rate of the so-called naive-TIN in which all transmitters send with full power cannot achieve the capacity. To show this, they proposed a differed variant of TIN



(a) The Gaussian P2P channel with capacity $C = \log_2(\text{SNR})$ at high SNR.

(b) The approximated linear deterministic model for the P2P channel.

Figure 2.6: In the LD-P2P channel, only the n top-most bits of the vector $\boldsymbol{x}$ are received at the receiver side, where $n = \lceil \log_2(\text{SNR}) \rceil$.

in which the transmitters of the MAC (Tx3 and Tx1 in Fig. 2.5) share the time between each other. Hence, the PIMAC is reduced to two 2-user IC's which operate over orthogonal time slots. It turns out that this type of TIN outperforms naive-TIN which shows the suboptimality of naive-TIN.

The capacity of the linear deterministic [ADT11] PIMAC was also studied in [BW12] under a symmetry condition on the channel parameters. The condition is as follows: The powers of the interference caused by the MAC transmitters at the P2P receiver are equal. For this case, the sum-capacity of the deterministic PIMAC is derived and it has been shown that it is larger than that of the deterministic IC. In addition to the mentioned works, the capacity of PIMAC under different cognition assumptions have been studied in [SBSV+08, CS11b, CS11a].

The complete characterization of the GDoF optimal regime of TIN in PIMAC is a focus of this thesis.

2.4 Deterministic Model

Our approach towards the analysis of optimality of TIN is starting by studying the so-called *deterministic model.* The main idea of the deterministic model is to eliminate the randomness of noise from the setup in order to simplify the analysis. The general deterministic model has been first proposed by El Gamal and Costa in 1982 for the 2-user interference channel [GC82]. Using this idea, the linear deterministic model (LD model) has been introduced by Avestimehr *et al.* in [ADT11]. The LD model is an approximation for a Gaussian wireless network in which the output of the channel is a linear deterministic function of the input. In this model, the transmitters send bit vectors. Depending on the channel strength, a certain number of bits sent by each transmitter is received at the receiver side. In Fig. 2.6, the Gaussian and the LD model of a P2P channel are illustrated.

While in the Gaussian channel (shown in Fig. 2.6a), the sent signal X is added with a random noise Z, in the LD model, there is no randomness in the channel. In the LD model, only the top-most n bits of the input vector[1] $\boldsymbol{x} \in \mathbb{F}_2^q$ (with $q \geq n$) are

[1]With a slight notational abuse, we use the boldface lower-case for denoting the random vectors.

received, where $n = \lceil C \rceil$ and C denotes the capacity of the Gaussian P2P channel at high SNR, i.e., $C = \log_2(\text{SNR})$. More precisely, the input-output relationship of the LD-P2P channel is given as follows

$$\boldsymbol{y} = \boldsymbol{S}^{q-n}\boldsymbol{x}, \tag{2.8}$$

where $\boldsymbol{y} \in \mathbb{F}_2^q$ is the output vector and $\boldsymbol{S}$ is the shift matrix given by

$$\boldsymbol{S} = \begin{pmatrix} \mathbf{0}_{q-1}^T & 0 \\ \boldsymbol{I}_{q-1} & \mathbf{0}_{q-1} \end{pmatrix}. \tag{2.9}$$

In other words, the impact of the noise in the Gaussian channel is reflected by the shifting operator of the transmit signal vector in the LD model. The larger the capacity in the Gaussian setup, the more bits are conveyed in the LD-model.

Characterizing the capacity for the LD model is easier than its Gaussian counterpart. However, the insights obtained from the LD model does not only improve our understanding from the Gaussian setup but also can be used directly for establishing an approximation for the capacity of the Gaussian channel. This fact has been shown first in [BT08] for the two-user IC and later in [BPT10] for many-to-one and one-to-many IC. In addition to the IC, the LD model has been studied for many setups such as wireless networks with arbitrary number of relays [ADT11], multiple access channel interfering with a point to point link [BW11], [BW12], and the X-channel [HCJ12].

It is worth noting that the LD model is not the only deterministic model which is available in the literature. Niesen and Maddah-Ali proposed a so-called lower triangular deterministic model in [NMA13] which is more appropriate than LD model for explaining the transmission schemes in which interference alignment [JS08, MAMK08] is involved. However, studying the triangular deterministic model is more complicated and involving than the LD model. Moreover, if interference alignment is not required, studying the LD model is sufficient for understanding the transmission scheme in the Gaussian setup. Interestingly, it turned out in [SJ16] that the details which are not captured by the LD model but in the lower triangular deterministic model are irrelevant for TIN. Due to these facts, in this thesis, we do the analysis based on the LD model.

2.5 Lattice Codes

In order to extend the transmission schemes proposed for a linear deterministic channel to its Gaussian counterpart a lattice code is sometimes required. A lattice code as a code with linear structure becomes unavoidable, especially if a linear superposition of independent codewords needs to be decoded or reconstructed. This fact has reinforced the importance of application of lattices in communication [NG11, SJV+08, BPT10, WNPS10, NCL11]. In what follows, we present some required preliminaries on lattices. For more details on lattices, the interested reader is referred to [Zam09, NG11].

An n-dimensional lattice Λ is a set of points in $\mathbb{R}^n$ which satisfies the following condition. For any arbitrary λ_1 and λ_2 in Λ, we have

$$-\lambda_1 \in \Lambda \tag{2.10}$$

$$\lambda_1 + \lambda_2 \in \Lambda. \tag{2.11}$$

Due to the conditions above, we conclude that the origin is always in Λ. The fundamental Voronoi region $\mathcal{V}(\Lambda)$ of the lattice Λ is the set of all points in $\mathbb{R}^n$ which have the origin as the closest $\lambda \in \Lambda$. For instance, for one dimensional lattice $\Lambda = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$, the Voronoi region is all real x in interval $\left[-\frac{1}{2}, -\frac{1}{2}\right]$.

In this thesis, some proposed schemes are based on the so-called nested-lattice code. In order to define the nested-lattice code, we consider two lattices Λ_f and Λ_c, which represent the fine and the coarse lattice, respectively. The course lattice is a subset of the fine lattice, i.e., $\Lambda_c \subset \Lambda_f$. Using the nested-lattice code a message W is encoded into a length-n codeword λ, where

$$\lambda \in \Lambda_f \cap \mathcal{V}(\Lambda_c). \tag{2.12}$$

Hence, the number of possible codewords is the number of lattice points in Λ_f which are in the Voronoi region of Λ_c. This is used for characterizing the rate of the nested lattice code which is given by

$$R = \frac{1}{n} \log_2 |\Lambda_f \cap \mathcal{V}(\Lambda_c)|. \tag{2.13}$$

Moreover, the power of a nested lattice code is the second moment of the Voronoi region of Λ_c. The achievable rate by using nested lattice code over a point-to-point additive white Gaussian noise (AWGN) channel is summarized in the following lemma.

Lemma 1 ([EZ04]). *Nested lattice code achieves the capacity of the point-to-point AWGN channel.*

Apart from the lower complexity of nested lattice codes compared to random coding and its achievable rate which coincides the capacity of the AWGN channel, the nested lattice code offers certain properties which make it a suitable choice for a general network with multiple users [NG07]. In what follows, we present the main aspects which will be required in this thesis.

Suppose that two transmitters, Tx1 and Tx2 use the same nested-lattice codebook (Λ_f, Λ_c) with power P and rate R to encode their messages W_1 and W_2 into length-n codewords λ_1 and λ_2, respectively. Then, Txi (with $i \in \{1, 2\}$) generates a real signal x_i^n as follows

$$x_i^n = (\lambda_i - d_i) \mod \Lambda_c, \tag{2.14}$$

where d_i is an n-dimensional random dither vector.

Now, suppose that there is a receiver (Rx) which knows the dither vector d_i and is interested in the sum of the codewords $(\lambda_1 + \lambda_2) \mod \Lambda_c$. The observation at the receiver is given by

$$y^n = x_1^n + x_2^n + z^n, \tag{2.15}$$

where z^n is a sequence of real valued additive with Gaussian noise with variance σ_n^2. The following lemma summarizes the conditions on reliable decoding of the sum of the codewords.

Lemma 2 ([WNPS10]). *Given observation y^n (cf. (2.15)), the sum $(\lambda_1 + \lambda_2) \mod \Lambda_c$ is reliably decodable as long as*

$$R \leq \frac{1}{2}\left[\log_2\left(\frac{1}{2} + \frac{P}{\sigma_n^2}\right)\right]^+. \tag{2.16}$$

This lemma shows that by using the nested lattice code, the sum of two codewords is reliably decodable as long as the given rate constraint is satisfied. Interestingly, the gap between the achievable rate of nested lattice code to the capacity of the point-to-point channel does not exceed half of bit. This gap is negligible in high signal-to-noise ratio ($\sigma_n^2 \ll P$). Hence, it does not have any impact on the achievable (generalized) degrees of freedom.

3 TIN in the PIMAC

In this chapter, we study the impact of introducing one more transmitter (without introducing a new receiver) to the 2-user interference channel which operates in very weak interference regime [ETW08] on the optimality of TIN. This setup is equivalent to the point-to-point channel interfering with a multiple access channel known as PIMAC. We present both necessary and sufficient conditions under which TIN achieves the capacity of the PIMAC within a constant gap. To do this, first we study the performance of TIN from the capacity point of view in the linear deterministic PIMAC. Using the insights obtained from the linear deterministic setup, we extend the results to the Gaussian PIMAC.

3.1 System Model of the PIMAC

The system we consider consists of a point-to-point (P2P) channel interfering with a multiple access channel (MAC), which is called PIMAC. As shown in Fig. 3.1, each transmitter has a message to be sent to one receiver. Namely, transmitters 1 (Tx1) and transmitter 3 (Tx3) want to send the messages W_1 and W_3, respectively, to receiver 1 (Rx1), and transmitter 2 (Tx2) wants to send the message W_2 to receiver 2 (Rx2). The message W_i is a random variable, uniformly distributed over the message set

$$\mathcal{W}_i = \left\{1, \cdots, \lfloor 2^{nR_i} \rfloor\right\}, \tag{3.1}$$

where R_i denotes the rate of the message.

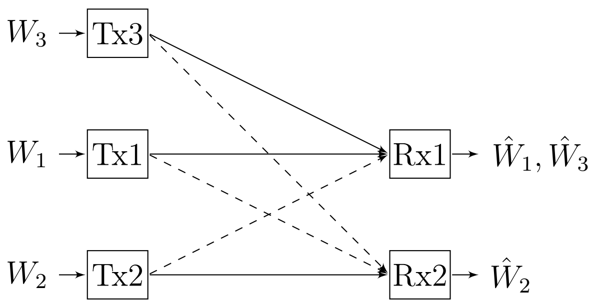


Figure 3.1: The message flow in the PIMAC where the solid arrows indicate desired message flow and dashed arrows indicate interference.

To send its message, Txi (with $i \in \{1, 2, 3\}$) uses an encoding function f_i to map its message W_i into a codeword of length n symbols $X_i^n \in \mathbb{C}^n$, i.e.,

$$X_i^n = f_i(W_i). \tag{3.2}$$

After the transmission of all n symbols of the codewords, Rxj (with $j \in \{1,2\}$) obtains Y_j^n. Rx1 decodes W_1 and W_3 from Y_1^n by using a decoding function g_1. Hence, it obtains

$$(\hat{W}_1, \hat{W}_3) = g_1(Y_1^n). \tag{3.3}$$

Similarly, Rx2 receives Y_2^n and decodes W_2 by using a decoding function g_2, i.e.,

$$\hat{W}_2 = g_2(Y_2^n). \tag{3.4}$$

The messages sets, encoding functions, and decoding functions constitute a code for the channel which is denoted by an $\left(\left(2^{nR_1}, 2^{nR_2}, 2^{nR_3}\right), n\right)$ code.

An error E_i occurs if for some $i \in \{1,2,3\}$, $\hat{W}_i \neq W_i$. We assume that the message triple (W_1, W_2, W_3) is uniformly distributed over $[1 : 2^{nR_1}] \times [1 : 2^{nR_1}] \times [1 : 2^{nR_3}]$. A code for the PIMAC induces an average error probability $\mathbb{P}^{(n)}$ defined as

$$\mathbb{P}^{(n)} = \text{Prob}\left((\hat{W}_1, \hat{W}_2, \hat{W}_3) \neq (W_1, W_2, W_3)\right). \tag{3.5}$$

Reliable communication takes place if this error probability can be made arbitrarily small by increasing n. This can occur if the rate triple (R_1, R_2, R_3) satisfies some achievability constraints which need to be found.

The achievability of a rate triple (R_1, R_2, R_3) is defined as the existence of a reliable coding scheme which achieves these rates. In other words, a rate triple (R_1, R_2, R_3) is said to be achievable if there exists a sequence of $\left(\left(2^{nR_1}, 2^{nR_2}, 2^{nR_3}\right), n\right)$ codes such that $\mathbb{P}^{(n)} \to 0$ as $n \to \infty$. The set of all achievable rate triples is the capacity region of the PIMAC denoted by $\mathcal{C}$. Further, the sum-capacity defined as the maximum achievable sum-rate, i.e.,

$$C_\Sigma = \max_{(R_1, R_2, R_3) \in \mathcal{C}} R_\Sigma, \tag{3.6}$$

where $R_\Sigma = \sum_{i=1}^{3} R_i$. The focus of the rest of this chapter is on the sum-capacity of the PIMAC.

3.1.1 Gaussian PIMAC (G-PIMAC)

The setup of the PIMAC can be interpreted as a 2-user interference channel with an additional transmitter. In more details, consider a 2-user asymmetric IC consisting of two transmitters Tx1 and Tx2 which want to communicate with their desired receivers Rx1 and Rx2, respectively. Now, by adding an additional transmitter (Tx3) which wants to communicate only with Rx1, we generate a PIMAC. The system model of the Gaussian PIMAC is shown in Fig. 4.1. In the Gaussian PIMAC, the

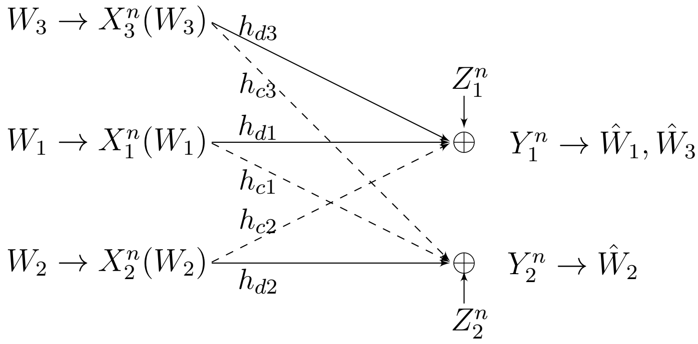


Figure 3.2: System model of the Gaussian PIMAC.

received signals of the two receivers at time index $t \in \{1, \cdots, n\}$ (denoted by $y_1[t]$ and $y_2[t]$) can be written as[1]

$$y_1[t] = h_{d1}x_1[t] + h_{c2}x_2[t] + h_{d3}x_3[t] + z_1[t], \tag{3.7}$$
$$y_2[t] = h_{c1}x_1[t] + h_{d2}x_2[t] + h_{c3}x_3[t] + z_2[t], \tag{3.8}$$

where $x_i[t]$, $i \in \{1, 2, 3\}$, is a realization of the random variable X_i which represents the transmit symbol of Txi, and $z_j[t]$, $j \in \{1, 2\}$, is a realization of the random variable $Z_j \sim \mathbb{CN}(0, 1)$ which represents the additive white Gaussian noise (AWGN), and the constants h_k, $k \in \{d1, d2, c1, c2, d3, c3\}$ represent the complex (static) channel coefficients. We assume that global channel state information (CSI) is available at all nodes. Note that the noises Z_1 and Z_2 are independent from each other and are both independently and identically distributed (i.i.d.) over time. The transmitters of the Gaussian PIMAC have power constraints P which must be satisfied by their transmitted signals. Namely, the condition

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}[|X_i[t]|^2] = P_i \leq P,$$

must be satisfied for all $i \in \{1, 2, 3\}$.

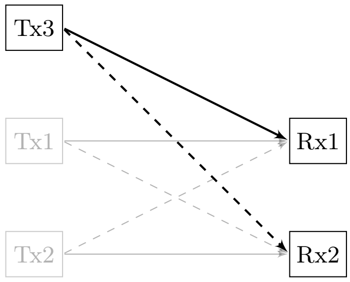
We consider the interference limited scenario, and hence, we assume that all signal-to-noise and interference-to-noise ratios are larger than 1, i.e.,

$$\min\{|h_{d1}|^2, |h_{c1}|^2, |h_{d2}|^2, |h_{c2}|^2, |h_{d3}|^2, |h_{c3}|^2\}P > 1. \tag{3.9}$$

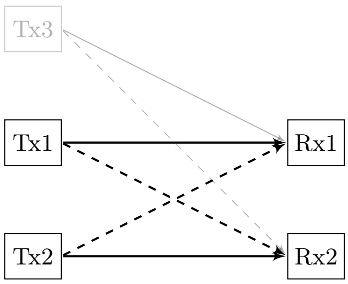
Definition 1. *The strength of the channels are represented by the following parameters*

$$\alpha_k = \frac{\log_2(P|h_k|^2)}{\log_2(\rho)}, \quad \textit{where } k \in \{d1, c1, d2, c2, d3, c3\}, \tag{3.10}$$

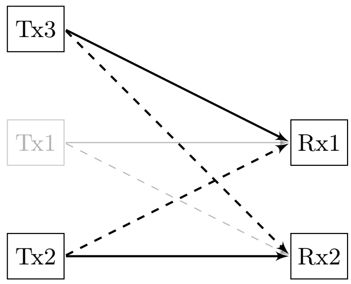
[1]The time index t will be suppressed henceforth for clarity unless necessary.



(a) All transmitters except Tx3 are inactive. In this case, PIMAC is reduced to a P2P channel.



(b) Tx1 communicates with Rx1 and Tx2 communicates with Rx2. In this case, PIMAC is reduced to a 2-user IC.



(c) Tx3 communicates with Rx1 and Tx2 communicates with Rx2. In this case, PIMAC is reduced to a 2-user IC.

Figure 3.3: In TDMA-TIN, we use time sharing between the users. In each time slot, the channel can operate as in a P2P channel (in (a)) or 2-user IC's (in (b) and (c)). In all figures, the solid and the dashed lines represent the desired and interference channels, respectively. In (b) and (c), while Tx2 is always active, Tx1 and Tx3 which are MAC transmitters, share the transmission time between themselves. Hence, in these cases, PIMAC is reduced to two 2-user IC's.

where $1 < \rho$ is the received SNR for the reference P2P channel. Moreover, we define the generalized degrees of freedom (GDoF) of the PIMAC as follows

$$d_\Sigma(\boldsymbol{\alpha}) = \lim_{\rho\to\infty} \frac{C_{\mathrm{G},\Sigma}(\rho, \boldsymbol{\alpha})}{\log_2(\rho)}, \tag{3.11}$$

where $\boldsymbol{\alpha} = (\alpha_{d1}, \alpha_{c1}, \alpha_{d2}, \alpha_{c2}, \alpha_{d3}, \alpha_{c3})$ and $C_{\mathrm{G},\Sigma}(\rho, \boldsymbol{\alpha})$ denotes the sum-capacity of the Gaussian PIMAC.

This definition is equivalent to

$$C_{\mathrm{G},\Sigma}(\rho, \boldsymbol{\alpha}) = d_\Sigma(\boldsymbol{\alpha}) \log_2(\rho) + o(\log_2(\rho)),$$

where $\frac{o(\log_2(\rho))}{\log_2(\rho)} \to 0$ as $\rho \to \infty$. In this chapter, the focus is on analysing the (sub-

)optimality of simple (in terms of computation and decoding complexity) transmission schemes. To this end, we study the optimality of two types of TIN, namely naive-TIN and TDMA-TIN which are defined as follows.

Definition 2 (**Naive-TIN:**). *This is the simplest variant of TIN in which all transmitters send simultaneously with their maximum power during the whole transmission. Note that in this type of TIN, no coordination between the Tx's is required. At the receiver side, each receiver decodes its desired message as in the interference free channel by treating the interference as noise. Interestingly, despite of the simplicity of this scheme, it is optimal in some regimes of many networks such as the 2-user IC [ETW08, AV09, SKC09, MK09], the K-user IC [SKC08], and the 2×2 X channel [HCJ12].*

Definition 3 (**TDMA-TIN:**). *In this type of TIN, we allow some coordination between the transmitters in order to have a smarter variant of TIN. This might lead to a more capable scheme than the naive-TIN. To do this, we have a time division between three types of channels (one P2P channel and two 2-user IC's) operating over orthogonal time slots. In the assigned time slots to the P2P channel, Tx3 sends with full power while other Tx's are inactive (See Fig. 3.3a). In the remaining time slots, Tx1 and Tx3 which are both communicating with Rx1, coordinate their transmission by sharing the transmission time between themselves. These two users send with their maximum allowed power only in their assigned time slots. Moreover, Tx2 sends always with the maximum power (See Fig. 3.3b and 3.3c). Note that no power control is addressed in this scheme and the only coordination between Tx's is for time scheduling. Similar to naive-TIN, in this scheme, the receivers decode their desired message by treating the interference as noise.*

Remark 1. *Let the received signal-to-interference-plus-noise power ratio at a receiver be denoted as* $\mathrm{SINR}=\frac{P_{\mathrm{des}}}{1+P_{\mathrm{int}}}$*, where* P_{des} *and* P_{int} *represent the received power from desired and interference signals, respectively. The achievable rate using treating interference as noise at the receiver is given by*

$$R_{\mathrm{TIN}} = \log_2(1 + \mathrm{SINR}).$$

Our approach towards the performance analysis of different types of TIN in the Gaussian PIMAC starts with the linear-deterministic (LD) approximation of the wireless network introduced by Avestimehr et al. in [ADT11]. Next, we introduce the linear deterministic PIMAC (LD-PIMAC).

3.1.2 Linear Deterministic PIMAC (LD-PIMAC)

The Gaussian PIMAC shown in Fig. 4.1 can be approximated by the LD model as follows. An input symbol at Txi is given by a binary vector $\boldsymbol{x}_i \in \mathbb{F}_2^q$ where $q = \max\{n_{d1}, n_{c1}, n_{d2}, n_{c2}, n_{d3}, n_{c3}\}$ and the integer n_k, $k \in \{d1, c1, d2, c2, d3, c3\}$ represents the Gaussian channel coefficients as follows

$$n_k = \left\lceil \log_2 \left(P|h_k|^2 \right) \right\rceil^+. \tag{3.12}$$

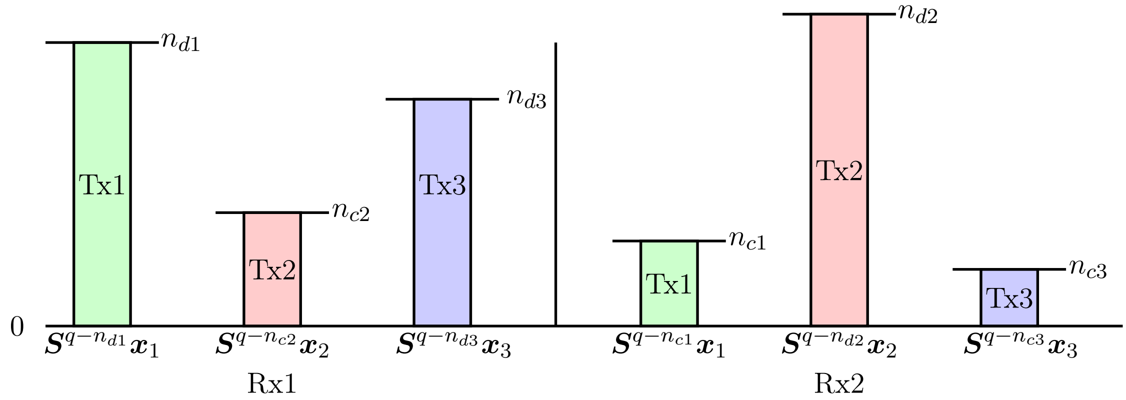


Figure 3.4: Block representation of received signal

The output symbol $\boldsymbol{y}_j \in \mathbb{F}_2^q$ at Rxj is given by a deterministic function of the inputs given by

$$\begin{aligned} \boldsymbol{y}_1 &= \boldsymbol{S}^{q-n_{d1}}\boldsymbol{x}_1 \oplus \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2 \oplus \boldsymbol{S}^{q-n_{d3}}\boldsymbol{x}_3, \\ \boldsymbol{y}_2 &= \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1 \oplus \boldsymbol{S}^{q-n_{d2}}\boldsymbol{x}_2 \oplus \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3, \end{aligned} \tag{3.13}$$

where $\boldsymbol{S} \in \mathbb{F}_2^{q\times q}$ is a down-shift matrix defined as

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{0}_{q-1}^T & 0 \\ \boldsymbol{I}_{q-1} & \boldsymbol{0}_{q-1} \end{pmatrix}. \tag{3.14}$$

These input-output equations approximate the input-output equations of the Gaussian PIMAC given in (3.7) and (3.8) in the high SNR regime. A graphical representation of the received vectors $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ is shown in Fig. 3.4, showing the three (shifted) transmitted vectors (shown as rectangular blocks) whose sum constitutes the received vector. This block representation will be used in the sequel for graphical illustration of various schemes.

We denote the sum-capacity of the LD-PIMAC by $C_{\text{det},\Sigma}$.

3.2 TIN in the LD-PIMAC

In this section, we focus on regimes of the PIMAC where the interference parameters caused by Tx1 and Tx2 are small. Notice that if we remove Tx3 from PIMAC, the remaining network resembles an asymmetric IC. For this IC, the noisy interference regime is defined as the regime where

$$n_{c1} + n_{c2} \leq \min\{n_{d1}, n_{d2}\}. \tag{3.15}$$

In this regime, treating interference as noise (TIN) is optimal in the IC [GNAJ15]. Adding Tx3 leads to some changes in the channel where TIN might not be the optimal scheme any more, even if the interference caused by Tx3 is very weak. However, in some special cases of the LD-PIMAC (specific ranges of the channel parameters), the TIN schemes can achieve the sum-capacity as we shall show next.

In order to evaluate the performance of TIN in the LD-PIMAC, we present the achievable sum-rate of TIN in Section 3.2.1. The upper bounds on the sum-capacity are presented in Section 3.2.2. Next, we evaluate the optimality of TIN in the LD-PIMAC with respect to the capacity in Section 3.2.3.

3.2.1 Achievable Sum-rate of TIN

In this section, we introducing different types of TIN in more details and we study their achievable sum-rate in the LD-PIMAC.

Naive-TIN

First consider the naive-TIN scheme for the LD-PIMAC. In this variant of TIN, the transmitters send over the interference free components of the received signal at their corresponding receivers. Namely, transmitters 1 and 3 share the top-most $(\max\{n_{d1}, n_{d3}\} - n_{c2})^+$ bits of $\boldsymbol{y}_1$ and transmitter 2 sends over the top-most $(n_{d2} - \max\{n_{c1}, n_{c3}\})^+$ bits of $\boldsymbol{y}_2$. We call this variant naive-TIN. An example of this scheme for the case in which $n_{d3} < n_{d1}$ and $n_{c3} < n_{c1}$ is illustrated in Fig. 3.5. We observe that, the top-most $n_{d1} - n_{c2}$ levels received at Rx1 are free of interference. These bits are shared between Tx1 and Tx3. In this example, Tx1 sends $n_{d1} - n_{d3}$ bits and Tx3 sends $n_{d3} - n_{c2}$ bits. Notice that the whole number of information bits sent by Tx1 and Tx3 ($\boldsymbol{x}_1$ and $\boldsymbol{x}_3$) cannot exceed $n_{d1} - n_{c2}$. Moreover, at Rx2, the top-most $n_{d2} - n_{c1}$ levels of Tx2 are observed interference free. Therefore, the number of information bits in $\boldsymbol{x}_2$ is $n_{d2} - n_{c1}$. Supposing that the transmitted bits are independent from each other, we achieve

$$R_{\Sigma,\text{Naive}-\text{TIN}} = \max\{n_{d1}, n_{d3}\} - n_{c2} + (n_{d2} - \max\{n_{c1}, n_{c3}\})^+. \tag{3.16}$$

In general, the achievable sum-rate of naive-TIN is given in the following proposition.

Proposition 1 (Naive-TIN). *As long as* (3.15) *is satisfied in an LD-PIMAC, the naive-TIN achieves any sum-rate* $R_\Sigma \leq R_{\Sigma,\text{Naive}-\text{TIN}}$.

TDMA-TIN

By careful examination of Naive-TIN, it can be seen that one can do better by using a smarter variant of TIN. Namely, consider the case when $n_{d3} > n_{d1}$ and $n_{c3} < n_{c1}$. In this case, it would be better to keep Tx1 silent and operate the PIMAC as an IC with transmitters 2 and 3 active, thus achieving $R_\Sigma = n_{d3} - n_{c2} + n_{d2} - n_{c3}$ which is clearly greater than (3.16) for this case.

To take this fact into account, we combine the TIN scheme with TDMA to obtain the TDMA-TIN scheme. In this scheme, we switch off Tx1 and Tx2 in a τ_1 fraction

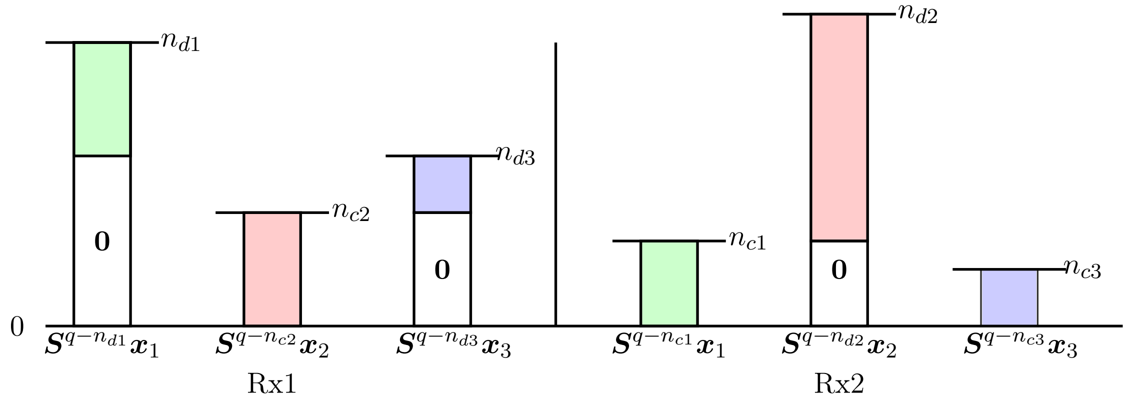


Figure 3.5: An example for Naive-TIN where $n_{d3} < n_{d1}$ and $n_{c3} < n_{c1}$.

of time while Tx3 is active[2]. In the remaining $(1-\tau_1)$ fraction of time, Tx1 and Tx3 share the time in a way such that Tx1 transmits in a fraction of τ_2 of the time, and Tx3 transmits in a fraction of τ_3 of the time, while Tx2 is kept active. Note that $\tau_2+\tau_3 = 1-\tau_1$. Regardless of the strategy at the transmitter side, the receivers use always TIN for decoding their desired signals. This scheme transforms the PIMAC into a P2P channel and two 2-user IC's operating over orthogonal time slots. This achieves

$$\begin{aligned} R_{\Sigma,\text{TDMA}-\text{TIN}} = \max_{\tau_1,\tau_2,\tau_3\in[0,1]} \ & \tau_1 n_{d3} + \tau_2[(n_{d1}-n_{c2})^+ + (n_{d2}-n_{c1})^+] \\ & + \tau_3[(n_{d3}-n_{c2})^+ + (n_{d2}-n_{c3})^+] \\ \text{subject to} \quad & \tau_1+\tau_2+\tau_3 = 1. \end{aligned}$$

This optimization problem is linear in τ_1, τ_2, and τ_3 and is solved by setting the optimization variable equal to one of the extremes of the interval $[0,1]$. Namely, the maximization above is achieved by activating the channel which yields the highest sum-rate. The achievable sum-rate of this scheme is given in the following proposition.

Proposition 2 (TDMA-TIN)**.** *As long as* (3.15) *is satisfied in an LD-PIMAC, the TDMA-TIN achieves any sum-rate* $R_\Sigma \le R_{\Sigma,\text{TDMA}-\text{TIN}}$*, where*

$$R_{\Sigma,\text{TDMA}-\text{TIN}} = \max\left\{\begin{matrix} n_{d3} \\ (n_{d1}-n_{c2})+(n_{d2}-n_{c1}) \\ (n_{d3}-n_{c2})^+ + (n_{d2}-n_{c3})^+ \end{matrix}\right\}. \tag{3.17}$$

Remark 2. *The proposed TDMA-TIN scheme is a special case of the TIN with power control where a user is either off or sends with full power. This is very similar to a so-called binary power control. However, some cases of binary power control are excluded*

[2]The main issue which separates TIN from other transmission schemes is the decoding at the receivers. By using TIN, the receivers decode their desired signals as in a P2P channel, regardless whether any interference signal is received or not. Hence, reducing the PIMAC to a P2P channel is also a type of TIN.

*from our proposed TDMA-TIN. These cases are discussed in what follows. Consider the cases when the PIMAC is reduced to the P2P channels where either Tx*1 *or Tx*2 *are active while other Tx's are inactive. Doing this, we cannot achieve more than* $\max\{n_{d1}, n_{d2}\}$*. Due to the condition in* (3.15)*, these P2P channels are outperformed by using TIN in the 2-user IC when Tx*1 *and Tx*2 *are active. Therefore, we exclude these schemes from the TDMA-TIN. Moreover, by switching Tx*2 *off, and letting Tx*1 *and Tx*3 *be active, the channel is reduced into an LD-MAC achieving* $\max\{n_{d1}, n_{d3}\}$ *which cannot outperform the achievable sum-rate in* (3.17)*. Therefore, this case is also excluded from the proposed TDMA-TIN.*

TIN alongside power control

In TDMA-TIN, the achievable sum-rates can be achieved by letting the active transmitters send with full power. A more general strategy would be to allow that each Tx sends with some power less than or equal to P (power control) and the receivers use TIN [CSHP12], [GNAJ15], [GSJ15]. In the following lemma, we summarize the analysis on the performance of the TIN scheme alongside power control at the transmitter side with respect to the achievable sum-rate.

Lemma 3. *In the LD-PIMAC, the achievable sum-rate by using TIN at the receiver side alongside power control at the transmitter side, is upper bounded by the sum-rate in* (3.17)*.*

Proof. First, we write the achievable sum-rate using TIN with power control for the LD-PIMAC. To do this, suppose that Tx's do not send with full power. It means that in the LD-PIMAC, the Tx's do not need to use necessarily the most significant bits. Hence, Txi sends such that only its m_{ji} bits are received at Rxj. For instance, suppose that Tx1 generates the following signal

$$\boldsymbol{x}_1 = \begin{pmatrix} \mathbf{0}_{n_{d1}-m_{11}} \\ \boldsymbol{d}_1 \end{pmatrix}, \tag{3.18}$$

where $\boldsymbol{d}_1 \in \mathbb{F}_2^{q-(n_{d1}-m_{11})}$ represents the information bit vector sent by Tx1 and $m_{11} \leq n_{d1}$. Rx1 obtains the top-most n_{d1} bits of $\boldsymbol{x}_1$. Hence, we can write

$$\boldsymbol{y}_1 = \begin{pmatrix} \mathbf{0}_{q-m_{11}} \\ \boldsymbol{d}_{1,[1:m_{11}]} \end{pmatrix}. \tag{3.19}$$

Similarly, since Rx2 receives the n_{c1} bits of $\boldsymbol{x}_1$, we have

$$\boldsymbol{y}_2 = \begin{pmatrix} \mathbf{0}_{q-n_{c1}+n_{d1}-m_{11}} \\ \boldsymbol{d}_{1,[1:(n_{c1}-n_{d1}+m_{11})^+]} \end{pmatrix}. \tag{3.20}$$

Therefore, if the number of information bits received at Rx1 from Tx1 is $m_{11} \in [0, n_{d1}]$, Rx2 receives

$$m_{21} = (m_{11} - (n_{d1} - n_{c1}))^+ \tag{3.21}$$

information bits from Tx1. Similarly, we can write that

$$m_{12} = (m_{22} - (n_{d2} - n_{c2}))^+, \tag{3.22}$$
$$m_{23} = (m_{13} - (n_{d3} - n_{c3}))^+ \quad \text{if } n_{c3} \leq n_{d3}, \tag{3.23}$$
$$m_{13} = (m_{23} - (n_{c3} - n_{d3}))^+ \quad \text{if } n_{d3} < n_{c3}, \tag{3.24}$$

and $m_{22} \in [0, n_{d2}]$, $m_{13} \in [0, n_{d3}]$, and $m_{23} \in [0, n_{c3}]$. Let $\mathcal{S}_m$ represent the set of all possible $(m_{11}, m_{21}, m_{12}, m_{22}, m_{13}, m_{23})$. Now, by using TIN at the receiver side, the maximum achievable sum-rate is

$$R_{\Sigma,\text{TIN}} = (\max\{m_{11}, m_{13}\} - m_{12})^+ + (m_{22} - \max\{m_{21}, m_{23}\})^+ . \tag{3.25}$$

The goal is to show that there exists no $(m_{11}, m_{21}, m_{12}, m_{22}, m_{13}, m_{23}) \in \mathcal{S}_m$ which provides a higher sum-rate than that of the TDMA-TIN in (3.17). To do this, we will show that for any arbitrary $(m_{11}, m_{21}, m_{12}, m_{22}, m_{13}, m_{23}) \in \mathcal{S}_m$, the achievable sum-rate using TDMA-TIN is larger than or equal to (3.25). First, we present the following properties of $(m_{11}, m_{21}, m_{12}, m_{22}, m_{13}, m_{23}) \in \mathcal{S}_m$

$$m_{11} - m_{21} = \min\{m_{11}, n_{d1} - n_{c1}\} \leq n_{d1} - n_{c1}, \tag{3.26}$$
$$m_{22} - m_{12} = \min\{m_{22}, n_{d2} - n_{c2}\} \leq n_{d2} - n_{c2}, \tag{3.27}$$
$$m_{13} - m_{23} \leq \min\{m_{13}, (n_{d3} - n_{c3})^+\} \leq (n_{d3} - n_{c3})^+. \tag{3.28}$$

These properties can be directly obtained from (3.21)-(3.24). Now, we compare (3.17) with (3.25) by distinguishing between following cases:

- $m_{13} \leq m_{11}$ and $m_{23} \leq m_{21}$: In this case the sum-rate in (3.25) is upper bounded as follows

$$\begin{aligned} R_{\Sigma,\text{TIN}} &= (m_{11} - m_{12})^+ + (m_{22} - m_{21})^+ \\ &\leq \max\{m_{11} - m_{12} + m_{22} - m_{21}, m_{11}, m_{22}\} \\ &\overset{(a)}{\leq} \max\{n_{d1} - n_{c2} + n_{d2} - n_{c1}, n_{d1}, n_{d2}\}, \end{aligned}$$

  where in (a), we used the properties (3.26) and (3.27) and the fact that $m_{11} \leq n_{d1}$, $m_{22} \leq n_{d2}$, and all n-parameters are non-negative. Now, by using the condition in (3.15), we can upper bound the sum-rate as follows

$$R_{\Sigma,\text{TIN}} \leq n_{d1} - n_{c2} + n_{d2} - n_{c1} \leq R_{\Sigma,\text{TDMA}-\text{TIN}}.$$

- $m_{13} \leq m_{11}$ and $m_{21} < m_{23}$: In this case, we upper bound the sum-rate in (3.25) as follows

$$\begin{aligned} R_{\Sigma,\text{TIN}} &= (m_{11} - m_{12})^+ + (m_{22} - m_{23})^+ \\ &\overset{(b)}{\leq} (m_{11} - m_{12})^+ + (m_{22} - m_{21})^+ , \end{aligned}$$

  where in (b), we used the condition of this case $m_{21} < m_{23}$. As it is shown above this expression is upper bounded by $R_{\Sigma,\text{TDMA}-\text{TIN}}$.

- $m_{11} < m_{13}$ and $m_{21} \leq m_{23}$: In this case, the sum-rate in (3.25) is upper bounded by

$$\begin{aligned} R_{\Sigma,\text{TIN}} &= (m_{13} - m_{12})^+ + (m_{22} - m_{23})^+ \\ &\leq \max\{m_{13} - m_{12} + m_{22} - m_{23}, m_{13}, m_{22}\} \\ &\overset{(c)}{\leq} \max\{n_{d3} - n_{c2} + n_{d2} - n_{c3}, n_{d3}, n_{d2}\}, \end{aligned}$$

where in (c), we used the properties (3.27) and (3.28) and the fact that $m_{13} \leq n_{d3}$, $m_{22} \leq n_{d2}$, and all n-parameters are non-negative. By using the condition in (3.15), this sum-rate is upper bounded by

$$R_{\Sigma,\text{TIN}} \leq \max\{n_{d3} - n_{c2} + n_{d2} - n_{c3}, n_{d3}, (n_{d1} - n_{c2}) + (n_{d2} - n_{c1})\} \leq R_{\Sigma,\text{TDMA-TIN}}.$$

- $m_{11} < m_{13}$ and $m_{23} < m_{21}$: In this case, the sum-rate in (3.25) is upper bounded as follows

$$\begin{aligned} R_{\Sigma,\text{TIN}} &= (m_{13} - m_{12})^+ + (m_{22} - m_{21})^+ \\ &\overset{(d)}{\leq} (m_{13} - m_{12})^+ + (m_{22} - m_{23})^+, \end{aligned}$$

where in (d), we used the condition of this case, i.e., $m_{23} < m_{21}$. As we have shown in the previous case, this expression is upper bounded by $R_{\Sigma,\text{TDMA-TIN}}$.

We have shown for any arbitrary $(m_{11}, m_{21}, m_{12}, m_{22}, m_{13}, m_{23}) \in \mathcal{S}_m$ that the achievable sum-rate in (3.25) is upper bounded by $R_{\Sigma,\text{TDMA-TIN}}$. □

From Lemma 3, we conclude that power allocation at the transmitter side while the receivers use TIN cannot outperform TDMA-TIN. Hence, in what follows, we exclude power allocation alongside TIN.

3.2.2 Sum-capacity Upper Bounds

In order to assess the capacity optimality of TIN, we need to compare the achievable sum-rate of TIN with the sum-capacity upper bounds of the LD-PIMAC. In this section, we establish some upper bounds which will be required. The main idea of these bounds is reducing the LD-PIMAC by removing one interferer at Rx2 into a channel that can be treated similar to the IC.

Lemma 4. *The sum-capacity of the LD-PIMAC is upper bounded as follow*

$$C_{\text{det},\Sigma} \leq \max\{n_{d1} - n_{c1}, n_{c2}, n_{d3}\} + \max\{n_{d2} - n_{c2}, n_{c1}\}. \tag{3.29}$$

Proof. The idea of the proof is to create a genie-aided channel where each receiver experiences one and only one interference just as in the IC. By doing this, the resulting channel can be treated in a similar way as the IC [BT08], and the given bound can be obtained. To this end, we give W_3 to Rx2 as side information. This enhances the LD-PIMAC to a channel where Rx1 experiences interference from $\boldsymbol{x}_2$ and Rx2 experiences interference from $\boldsymbol{x}_1$ only. Next, we treat the resulting enhanced channel as an IC and derive a bound similar to that of the IC with noisy interference. Namely, we give the interference caused by Tx1 given by $\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n$ to Rx1 as side information, and we give the interference caused by Tx2 given by $\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n$ to Rx2 as side information. The resulting LD-PIMAC which has been enhanced with side information is more capable than the original LD-PIMAC, and hence the capacity of the former serves as an upper bound for the capacity of the latter. Next, by using Fano's inequality we can bound R_Σ as follows

$$n(R_\Sigma - \epsilon_n) \leq I(W_1, W_3; \boldsymbol{y}_1^n, \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) + I(W_2; \boldsymbol{y}_2^n, \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3), \tag{3.30}$$

where $\epsilon_n \to 0$ as $n \to \infty$. By using the chain rule, and the independence of the different messages, we can rewrite this bound as

$$\begin{aligned} n(R_\Sigma - \epsilon_n) \leq\ & I(W_1, W_3; \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) + I(W_1, W_3; \boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) \\ & + I(W_2; \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|W_3) + I(W_2; \boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3). \end{aligned} \tag{3.31}$$

Now, we treat each of the mutual information terms in (3.31) separately. The first mutual information term can be written as

$$\begin{aligned} I(W_1, W_3; \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) &= H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) - H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n|W_1, W_3) \\ &= H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n), \end{aligned} \tag{3.32}$$

since $H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n|W_1, W_3) = 0$. The second mutual information term in (3.31) satisfies

$$\begin{aligned} I(W_1, W_3; \boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) &= H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) - H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n, W_1, W_3) \\ &= H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) - H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n), \end{aligned} \tag{3.33}$$

since given W_1 and W_3, the only randomness remaining in $\boldsymbol{y}_1$ is that originating from $\boldsymbol{x}_2$. The third mutual information term in (3.31) satisfies

$$\begin{aligned} I(W_2; \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|W_3) &= H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|W_3) - H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|W_2, W_3) \\ &= H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n), \end{aligned} \tag{3.34}$$

which follows since $H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|W_2, W_3) = 0$ and since $\boldsymbol{x}_2$ is independent of W_3. Finally, the last mutual information term in (3.31) satisfies

$$\begin{aligned} I(W_2; \boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3) &= H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3) - H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_2, W_3) \\ &= H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3) - H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n), \end{aligned} \tag{3.35}$$

since given W_2 and W_3, the only randomness in $\boldsymbol{y}_2$ is that of $\boldsymbol{x}_1$. Now by substituting (3.32)-(3.35) in (3.31), we obtain

$$\begin{aligned} n(R_\Sigma - \epsilon_n) &\leq H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) + H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) - H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) + H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) \\ &\quad + H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3) - H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) \\ &= H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) + H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3). \end{aligned}$$

Now, notice that given $\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1[t]$ with $t = 1, \ldots, n$, the top-most n_{c1} components of $\boldsymbol{x}_1[t]$ are known and can be subtracted from $\boldsymbol{y}_1[t]$ leaving $\max\{n_{d1} - n_{c1}, n_{c2}, n_{d3}\}$ random components in $\boldsymbol{y}_1[t]$. The entropy of a binary vector is maximized if its components are i.i.d. with a Bernoulli distribution with probability 1/2, and the maximum entropy is equal to the length of the vector. This leads to

$$\begin{aligned} H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) &= \sum_{t=1}^{n} H(\boldsymbol{y}_1[t]|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n, \boldsymbol{y}_1^{t-1}) \\ &\overset{(a)}{\leq} \sum_{t=1}^{n} H(\boldsymbol{y}_1[t]|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1[t]) \\ &\leq \sum_{t=1}^{n} \max\{n_{d1} - n_{c1}, n_{c2}, n_{d3}\} \\ &= n \max\{n_{d1} - n_{c1}, n_{c2}, n_{d3}\}, \end{aligned}$$

where step (a) follows since conditioning does not increase the entropy. Similarly,

$$H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_3) \leq n \max\{n_{d2} - n_{c2}, n_{c1}\}.$$

Therefore, we can write

$$n(R_\Sigma - \epsilon_n) \leq n(\max\{n_{d1} - n_{c1}, n_{c2}, n_{d3}\} + \max\{n_{d2} - n_{c2}, n_{c1}\}).$$

By dividing the expression by n and letting $n \to \infty$, we get (3.29) which concludes the proof. □

The following is another upper bound on the sum-rate of the LD-PIMAC obtained by removing the interference from Tx1 to Rx2, i.e., giving W_1 to Rx2 as side information. Then, the resulting interference channel is treated as in [BT08].

Lemma 5. *The sum-capacity of the LD-PIMAC is upper bounded as follows*

$$C_{\text{det},\Sigma} \leq \max\{n_{d1}, n_{c2}, n_{d3} - n_{c3}\} + \max\{n_{d2} - n_{c2}, n_{c3}\}. \tag{3.36}$$

Proof. We give $\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n$ and $(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_1)$ as side information to Rx1 and Rx2, respectively. Then, by using Fano's inequality we may write

$$\begin{aligned} n(R_\Sigma - \epsilon_n) &\leq I(W_1, W_3; \boldsymbol{y}_1^n, \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) + I(W_2; \boldsymbol{y}_2^n, \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_1) \\ &\overset{(a)}{=} I(W_1, W_3; \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) + I(W_1, W_3; \boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) \\ &\quad + I(W_2; \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|W_1) + I(W_2; \boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_1) \\ &\overset{(b)}{=} H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) + H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) - H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n, W_1, W_3) \\ &\quad + H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) + H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_1) - H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_2, W_1) \\ &= H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) + H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) - H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) \\ &\quad + H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) + H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_1) - H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) \\ &= H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) + H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, W_1), \end{aligned}$$

where step (a) follows by using the chain rule and the independence of the messages, and step (b) follows from the fact that $\boldsymbol{x}_3$ and $\boldsymbol{x}_2$ can be reconstructed knowing W_3 and W_2, respectively, and since $\boldsymbol{x}_2$ is independent of W_1. Next, by proceeding similar to the proof of Lemma 4, we can show that

$$n(R_\Sigma - \epsilon_n) \leq n(\max\{n_{d1}, n_{c2}, n_{d3} - n_{c3}\} + \max\{n_{d2} - n_{c2}, n_{c3}\}). \quad (3.37)$$

By dividing this inequality by n and letting $n \to \infty$, we get the upper bound in (3.36) which concludes the proof of Lemma 5. □

We need to present two new upper bounds which are established different from the upper bounds in Lemma 4 and Lemma 5. For establishing these two upper bounds the following lemma is required.

Lemma 6. *The difference between the entropies of*

$$\boldsymbol{Y}_A = \boldsymbol{S}^{\ell-\ell_1}\boldsymbol{A} \oplus \boldsymbol{S}^{\ell-\ell_2}\boldsymbol{B} \quad (3.38)$$

$$\boldsymbol{Y}_B = \boldsymbol{S}^{\ell-\ell_1}\boldsymbol{A} \oplus \boldsymbol{S}^{\ell-\ell_3}\boldsymbol{B}, \quad (3.39)$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are two independent $\ell \times n$ random binary matrices with $\ell_1, \ell_2, \ell_3 \in \mathbb{N}^0$, and $\ell_2 \leq \ell_3 - \ell_1$, satisfies[3]

$$H(\boldsymbol{Y}_A) - H(\boldsymbol{Y}_B) \leq 0. \quad (3.40)$$

Proof. To prove this lemma, we define the following matrices

$$\boldsymbol{B}_1 = \boldsymbol{B}_{[1:(\ell_2-\ell_1)^+]}, \quad \boldsymbol{B}_2 = \boldsymbol{B}_{[(\ell_2-\ell_1)^++1:\ell_2]}, \quad \boldsymbol{B}_3 = \boldsymbol{B}_{[\ell_2+1:\ell_3-\ell_1]}, \quad (3.41)$$

$$\boldsymbol{B}_4 = \boldsymbol{B}_{[\ell_3-\ell_1+1:\ell_3]}, \quad \boldsymbol{B}_5 = \boldsymbol{B}_{[\ell_3+1:\ell]}. \quad (3.42)$$

Notice that if $\ell_2 = \ell_3 - \ell_1$, $\ell_2 + 1 > \ell_3 - \ell_1$. Hence, the matrix $\boldsymbol{B}_3$ does not have any component. Moreover, due to the condition $\ell_2 \leq \ell_3 - \ell_1$, the matrices $\boldsymbol{B}_2$ and $\boldsymbol{B}_4$ do not have any common row. Therefore, the matrix $\boldsymbol{B}$ can be written as follows

$$\boldsymbol{B}^T = \begin{bmatrix} \boldsymbol{B}_1^T & \boldsymbol{B}_2^T & \boldsymbol{B}_3^T & \boldsymbol{B}_4^T & \boldsymbol{B}_5^T \end{bmatrix}. \quad (3.43)$$

Moreover, we split the matrix $\boldsymbol{A}$ into

$$\boldsymbol{A}_1 = \boldsymbol{A}_{[1:\ell_1]}, \quad \boldsymbol{A}_2 = \boldsymbol{A}_{[\ell_1+1:\ell]}. \quad (3.44)$$

Therefore, we have

$$\boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{A}_1^T & \boldsymbol{A}_2^T \end{bmatrix}. \quad (3.45)$$

[3] A similar lemma with a slightly different structure than (3.38) and (3.39) was given in [BW12].

Using (3.43) and (3.45), we can write

$$\boldsymbol{S}^{\ell-\ell_1}\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0}_{(\ell-\ell_1),n} \\ \boldsymbol{A}_1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{S}^{\ell-\ell_3}\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0}_{(\ell-\ell_3),n} \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_3 \\ \boldsymbol{B}_4 \end{bmatrix}.$$

Now, we lower bound $H(\boldsymbol{Y}_B)$ as follows

$$\begin{aligned}
H(\boldsymbol{Y}_B) &= H(\boldsymbol{S}^{\ell-\ell_1}\boldsymbol{A} \oplus \boldsymbol{S}^{\ell-\ell_3}\boldsymbol{B}) \\
&= H\left(\begin{bmatrix} \boldsymbol{0}_{(\ell-\ell_3),n} \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_3 \\ \boldsymbol{A}_1 \oplus \boldsymbol{B}_4 \end{bmatrix}\right) \\
&= H\left(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{A}_1 \oplus \boldsymbol{B}_4\right) \\
&\overset{(a)}{=} H(\boldsymbol{B}_1) + H(\boldsymbol{B}_2|\boldsymbol{B}_1) + H(\boldsymbol{B}_3|\boldsymbol{B}_2, \boldsymbol{B}_1) + H(\boldsymbol{A}_1 \oplus \boldsymbol{B}_4|\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3) \\
&\overset{(b)}{\geq} H(\boldsymbol{B}_1) + H(\boldsymbol{B}_2|\boldsymbol{B}_1) + H(\boldsymbol{A}_1 \oplus \boldsymbol{B}_4|\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{B}_4) \\
&= H(\boldsymbol{B}_1) + H(\boldsymbol{B}_2|\boldsymbol{B}_1) + H(\boldsymbol{A}_1|\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{B}_4) \\
&\overset{(c)}{=} H(\boldsymbol{B}_1) + H(\boldsymbol{B}_2|\boldsymbol{B}_1) + H(\boldsymbol{A}_1|\boldsymbol{B}_1, \boldsymbol{B}_2) \\
&\overset{(a)}{=} H(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{A}_1) \\
&\overset{(d)}{\geq} H(f(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{A}_1)) \\
&\overset{(e)}{=} H(\boldsymbol{Y}_A),
\end{aligned}$$

where (a) follows by using the chain rule, (b) follows from the facts that entropy is non-negative and conditioning does not increase the entropy, (c) follows due to the independence of the matrix $\boldsymbol{B}$ of $\boldsymbol{A}_1$, (d) follows since the entropy of function of random variables cannot be larger than the entropy of the random variables, and (e) follows by setting

$$\begin{aligned}
f(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{A}_1) &= \begin{bmatrix} \boldsymbol{0}_{(\ell-\ell_1),n} \\ \boldsymbol{A}_1 \end{bmatrix} \oplus \begin{bmatrix} \boldsymbol{0}_{(\ell-\ell_2),n} \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix} \\
&= \boldsymbol{S}^{\ell-\ell_1}\boldsymbol{A} \oplus \boldsymbol{S}^{\ell-\ell_2}\boldsymbol{B} \\
&= \boldsymbol{Y}_A.
\end{aligned}$$

Therefore, $H(\boldsymbol{Y}_B) \geq H(\boldsymbol{Y}_A)$ which leads to $H(\boldsymbol{Y}_A) - H(\boldsymbol{Y}_B) \leq 0$. □

Now, we are ready to present the other two upper bounds in the following lemma.

Lemma 7. *The sum-capacity of the LD-PIMAC with* $n_{c1}+n_{c2}\le\min\{n_{d1},n_{d2}\}$ *is upper bounded by*

$$C_{\mathrm{det},\Sigma}\le n_{d3}-n_{c3}+\max\{n_{c3},n_{d2}-n_{c2}\}\qquad \text{if } n_{d3}-2n_{c3}\ge n_{d1}-n_{c1} \tag{3.46}$$

$$C_{\mathrm{det},\Sigma}\le n_{d1}-n_{c1}+\max\{n_{c3},n_{d2}-n_{c2}\}\qquad \text{if } n_{d3}-n_{c3}\le n_{d1}-2n_{c1}. \tag{3.47}$$

Proof. First, we establish the upper bound given in (3.46). To do this, we give

$$\boldsymbol{s}_1^n=\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n$$

as side information to Rx1 and $\boldsymbol{s}_2^n=\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n$ to Rx2. Note that the side information provided to Rx1 is the top-most bits of $\boldsymbol{y}_1^n$ upto the first n_{c3} most significant bits of $\boldsymbol{x}_3^n$ (see Fig. 3.6). Obviously, by giving these side information, the resulting

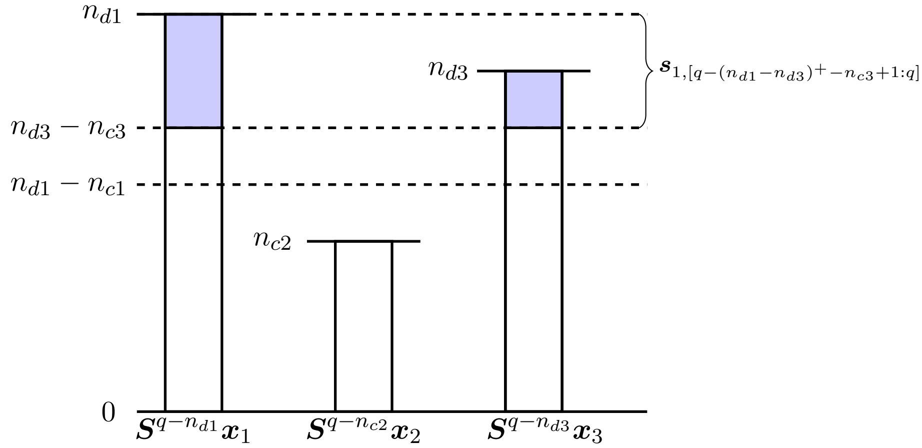


Figure 3.6: The block representation of $\boldsymbol{y}_1$ and the elements $q-(n_{d1}-n_{d3})^+-n_{c3}+1:q$ of $\boldsymbol{s}_1=\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3$.

LD-PIMAC channel is more capable than the original LD-PIMAC. Then, we use Fano's inequality to write

$$n(R_\Sigma-\epsilon_n)\le I(\boldsymbol{x}_1^n,\boldsymbol{x}_3^n;\boldsymbol{y}_1^n,\boldsymbol{s}_1^n)+I(\boldsymbol{x}_2^n;\boldsymbol{y}_2^n,\boldsymbol{s}_2^n), \tag{3.48}$$

where $\epsilon_n\to 0$ as $n\to\infty$. Using the chain rule, we obtain

$$n(R_\Sigma-\epsilon_n)\le I(\boldsymbol{x}_1^n,\boldsymbol{x}_3^n;\boldsymbol{s}_1^n)+I(\boldsymbol{x}_1^n,\boldsymbol{x}_3^n;\boldsymbol{y}_1^n|\boldsymbol{s}_1^n)+I(\boldsymbol{x}_2^n;\boldsymbol{s}_2^n)+I(\boldsymbol{x}_2^n;\boldsymbol{y}_2^n|\boldsymbol{s}_2^n). \tag{3.49}$$

Next, we consider each of the mutual information terms in (3.49) separately. Using the definition of $\boldsymbol{s}_1^n$, the first term can be rewritten as

$$\begin{aligned}I(\boldsymbol{x}_1^n,\boldsymbol{x}_3^n;\boldsymbol{s}_1^n)=&H(\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n)\\&-H(\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n|\boldsymbol{x}_1^n,\boldsymbol{x}_3^n)\\=&H(\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n).\end{aligned} \tag{3.50}$$

Now consider the second mutual information term in (3.49). This term can be rewritten as

$$\begin{aligned}I(\boldsymbol{x}_1^n, \boldsymbol{x}_3^n; \boldsymbol{y}_1^n|\boldsymbol{s}_1^n) =&H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n) - H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n, \boldsymbol{x}_1^n, \boldsymbol{x}_3^n)\\ =&H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n \oplus \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) - H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n), \quad (3.51)\end{aligned}$$

since given $\boldsymbol{x}_1^n$ and $\boldsymbol{x}_3^n$, the remaining randomness in $\boldsymbol{y}_1^n$ is that of $\boldsymbol{x}_2^n$ with $\boldsymbol{x}_2^n$ being independent of $\boldsymbol{x}_1^n$ and $\boldsymbol{x}_3^n$. Note also that $\boldsymbol{s}_1^n$ is independent of $\boldsymbol{x}_2^n$. By using the definition of $\boldsymbol{s}_2^n$, the third mutual information term in (3.49) satisfies

$$\begin{aligned}I(\boldsymbol{x}_2^n; \boldsymbol{s}_2^n) &= H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) - H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|\boldsymbol{x}_2^n)\\ &= H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n), \quad (3.52)\end{aligned}$$

since $H(\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n|\boldsymbol{x}_2^n) = 0$. Finally, the last term in (3.49) is rewritten as

$$\begin{aligned}I(\boldsymbol{x}_2^n; \boldsymbol{y}_2^n|\boldsymbol{s}_2^n) =&H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) - H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n, \boldsymbol{x}_2^n)\\ =&H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) - H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n \oplus \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n), \quad (3.53)\end{aligned}$$

since given $\boldsymbol{x}_2^n$, the only randomness remaining in $\boldsymbol{y}_2^n$ is that of $\boldsymbol{x}_1^n$ and $\boldsymbol{x}_3^n$. Moreover, $\boldsymbol{x}_1^n$ and $\boldsymbol{x}_3^n$ are independent of $\boldsymbol{x}_2^n$. Now, substituting (3.50), (3.51), (3.52), and (3.53) into (3.49), we obtain

$$\begin{aligned}n(R_\Sigma - \epsilon_n) \leq&H(\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n \oplus \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n)\\ &+ H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n \oplus \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n)\\ &+ H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n) - H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n \oplus \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n). \quad (3.54)\end{aligned}$$

Now, we bound the sum of the first and the last terms in (3.54). To do this, we consider the following cases

- $n_{d1} - n_{d3} + n_{c3} \geq 0$: In this case by using Lemma 6 and condition (3.46), we can write

$$H(\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n \oplus \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) - H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n \oplus \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) \leq 0 \quad (3.55)$$

- $n_{d1} - n_{d3} + n_{c3} < 0$: In this case, it holds

$$\begin{aligned}&H(\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n \oplus \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) - H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n \oplus \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n)\\ &= H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) - H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n \oplus \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n) \quad (3.56)\\ &\leq H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n) - H(\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n \oplus \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n|\boldsymbol{x}_1^n) = 0, \quad (3.57)\end{aligned}$$

  where we used the facts that conditioning does not increase the entropy and $\boldsymbol{x}_1^n$ and $\boldsymbol{x}_3^n$ are independent.

Therefore, the expression in (3.54) is upper bounded as follows

$$\begin{aligned}n(R_\Sigma-\epsilon_n) &\leq H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-(n_{d1}-n_{d3}+n_{c3})^+}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n)+H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n)\\ &\overset{(a)}{\leq} H(\boldsymbol{y}_{1,[q-(n_{d3}-n_{c3})+1:q]}^n)+H(\boldsymbol{y}_{2,[q-(n_{d2}-n_{c2})+1:q]}^n)\\ &\overset{(b)}{\leq} n(n_{d3}-n_{c3}+\max\{n_{c3},n_{d2}-n_{c2}\},\end{aligned}$$

where in (a), we use the fact that conditioning does not increase the entropy. Moreover, in (b) we use the fact that the i.i.d. Bernoulli distribution 1/2 maximizes the entropy terms. So by dividing by n and letting $n\to\infty$, $\epsilon\to 0$ and the sum rate is bounded by

$$R_\Sigma\leq n_{d3}-n_{c3}+\max\{n_{c3},n_{d2}-n_{c2}\}, \tag{3.58}$$

as long as the condition in (3.46) holds. This proves (3.46).

The proof for upper bound in (3.47) is similar to (3.46) with a slight difference in defining the side information provided to Rx1. Here,

$$\boldsymbol{s}_1^n=\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-(n_{d3}-n_{d1}+n_{c1})^+}\boldsymbol{x}_3^n \tag{3.59}$$

is provided as side information to Rx1. The provided side information to Rx2 is $\boldsymbol{s}_2^n=\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n$. The sum-capacity of the original LD-PIMAC is upper bounded by the genie-aided LD-PIMAC. By using Fano's inequality, we can write

$$\begin{aligned}n(R_\Sigma-\epsilon_n) &\leq I(\boldsymbol{x}_1^n,\boldsymbol{x}_3^n;\boldsymbol{y}_1^n,\boldsymbol{s}_1^n)+I(\boldsymbol{x}_2^n;\boldsymbol{y}_2^n,\boldsymbol{s}_2^n) &(3.60)\\ &\overset{(c)}{=} I(\boldsymbol{x}_1^n,\boldsymbol{x}_3^n;\boldsymbol{s}_1^n)+I(\boldsymbol{x}_1^n,\boldsymbol{x}_3^n;\boldsymbol{y}_1^n|\boldsymbol{s}_1^n)+I(\boldsymbol{x}_2^n;\boldsymbol{s}_2^n)+I(\boldsymbol{x}_2^n;\boldsymbol{y}_2^n|\boldsymbol{s}_2^n) &(3.61)\\ &= H(\boldsymbol{s}_1^n)-H(\boldsymbol{s}_1^n|\boldsymbol{x}_1^n,\boldsymbol{x}_3^n)+H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n)-H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n,\boldsymbol{x}_1^n,\boldsymbol{x}_3^n)\\ &\quad+H(\boldsymbol{s}_2^n)-H(\boldsymbol{s}_2^n|\boldsymbol{x}_2^n)+H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n)-H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n,\boldsymbol{x}_2^n), &(3.62)\end{aligned}$$

where in (c), we use the chain rule. Using the fact that $H(\boldsymbol{s}_1^n|\boldsymbol{x}_1^n,\boldsymbol{x}_3^n)=0$, $H(\boldsymbol{s}_2^n|\boldsymbol{x}_2^n)=0$, and $H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n,\boldsymbol{x}_1^n,\boldsymbol{x}_3^n)=H(\boldsymbol{s}_2^n)$, we obtain

$$n(R_\Sigma-\epsilon_n)\leq H(\boldsymbol{s}_1^n)+H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n)+H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n)-H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n,\boldsymbol{x}_2^n). \tag{3.63}$$

Now, consider $H(\boldsymbol{s}_1^n)-H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n,\boldsymbol{x}_2^n)$. This expression can be rewritten as follows

$$\begin{aligned}&H(\boldsymbol{s}_1^n)-H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n,\boldsymbol{x}_2^n)\\ &=H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-(n_{d3}-n_{d1}+n_{c1})^+}\boldsymbol{x}_3^n)-H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n).\end{aligned} \tag{3.64}$$

By using Lemma 6 similar to the proof of (3.46), we can write

$$H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-(n_{d3}-n_{d1}+n_{c1})^+}\boldsymbol{x}_3^n)-H(\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n)\leq 0, \tag{3.65}$$

as long as the condition of (3.47) is satisfied. Therefore, we upper bound the expression in (3.63) as follows

$$n(R_\Sigma-\epsilon_n)\leq H(\boldsymbol{y}_1^n|\boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n\oplus\boldsymbol{S}^{q-(n_{d3}-n_{d1}+n_{c1})^+}\boldsymbol{x}_3^n)+H(\boldsymbol{y}_2^n|\boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n). \tag{3.66}$$

Next, by proceeding similar to the proof of (3.46), we can upper bound the expression in (3.66) as follows

$$n(R_\Sigma - \epsilon_n) \le n(n_{d1} - n_{c1} + \max\{n_{d2} - n_{c2}, n_{c3}\}). \tag{3.67}$$

By dividing this inequality by n and letting $n \to \infty$, we get the upper bound in (3.47) which concludes the proof of Lemma 7. □

An upper bound on the sum-capacity of the LD-PIMAC is still required for the special case when $n_{d3} - n_{c3} = n_{d1} - n_{c1}$. This is presented in the following lemma.

Lemma 8. *The sum-capacity of the LD-PIMAC with* $n_{d3} - n_{c3} = n_{d1} - n_{c1}$ *is upper bounded by*

$$C_{\mathrm{det},\Sigma} \le \max\{n_{d1} - n_{c1}, n_{c2}\} + \max\{n_{c1}, n_{c3}, n_{d2} - n_{c2}\}. \tag{3.68}$$

Proof. To establish this upper bound, we give $\boldsymbol{s}_1^n = \boldsymbol{S}^{q-n_{c1}}\boldsymbol{x}_1^n \oplus \boldsymbol{S}^{q-n_{c3}}\boldsymbol{x}_3^n$ and $\boldsymbol{s}_2^n = \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2^n$ to Rx1 and Rx2, respectively. Obviously, The sum-capacity of the generated setup (after providing the side information) provides an upper bound for the sum-capacity of the original LD-PIMAC. Now, we use the Fano's inequality to write

$$\begin{aligned}
n(R_\Sigma - \epsilon_n) \le& I(\boldsymbol{x}_1^n, \boldsymbol{x}_3^n; \boldsymbol{y}_1^n, \boldsymbol{s}_1^n) + I(\boldsymbol{x}_2^n; \boldsymbol{y}_2^n, \boldsymbol{s}_2^n) \\
\stackrel{(a)}{=}& I(\boldsymbol{x}_1^n, \boldsymbol{x}_3^n; \boldsymbol{s}_1^n) + I(\boldsymbol{x}_1^n, \boldsymbol{x}_3^n; \boldsymbol{y}_1^n|\boldsymbol{s}_1^n) + I(\boldsymbol{x}_2^n; \boldsymbol{s}_2^n) + I(\boldsymbol{x}_2^n; \boldsymbol{y}_2^n|\boldsymbol{s}_2^n) \\
\stackrel{(b)}{=}& H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n) + H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n),
\end{aligned}$$

where in (a), we used the chain rule and in (b), we used the fact that $H(\boldsymbol{s}_1^n|\boldsymbol{x}_1^n, \boldsymbol{x}_3^n) = 0$, $H(\boldsymbol{y}_1^n|\boldsymbol{s}_1^n, \boldsymbol{x}_1^n, \boldsymbol{x}_3^n) = H(\boldsymbol{s}_2^n)$, $H(\boldsymbol{s}_2^n|\boldsymbol{x}_2^n) = 0$, and $H(\boldsymbol{y}_2^n|\boldsymbol{s}_2^n, \boldsymbol{x}_2^n) = H(\boldsymbol{s}_1^n)$. Now, notice that $\boldsymbol{s}_1^n$ appears in the top-most $\max\{n_{c1}, n_{c3}\}$ bits of signal vector $\boldsymbol{y}_1^n$. This is due to the condition $n_{d1} - n_{c1} = n_{d3} - n_{c3}$. Hence, knowing $\boldsymbol{s}_1^n$, the randomness of the top-most n_{c1} bits of $\boldsymbol{x}_1^n$ and the top-most n_{c3} bits of $\boldsymbol{x}_3^n$ can be removed from $\boldsymbol{y}_1^n$. Hence, we can write

$$n(R_\Sigma - \epsilon_n) \le n(\max\{n_{d1} - n_{c1}, n_{d3} - n_{c3}, n_{c2}\} + \max\{n_{c1}, n_{c3}, n_{d2} - n_{c2}\}).$$

By dividing the expression by n letting $n \to \infty$ and keeping in mind that $n_{d1} - n_{c1} = n_{d3} - n_{c3}$, we obtain (3.68). □

3.2.3 Optimality and Suboptimality of TIN

Now, we are ready to assess the capacity optimality of TIN in the LD-PIMAC. For the sake of simplicity and clear presentation of the results, we split the channel parameters into several regimes which are presented in the following subsection. Next, we discuss in which regimes TIN performs optimally. Finally, by showing the suboptimality of TIN in the remaining regimes we conclude our analysis on the performance of TIN in the LD-PIMAC.

3.2.3.1 Regimes under Consideration in LD-PIMAC

In this section, we introduce three regimes of the LD-PIMAC which satisfies (3.15). These regimes are determined based on the operational meaning.

Definition 4. *For an LD-PIMAC with $n_{c1}+n_{c2} \leq \min\{n_{d1}, n_{d2}\}$, we define regimes 1 to 3 (shown in Fig. 3.7) as follows:*

- **Regime 1 (Tx3-off):**

$$n_{d3} \leq n_{d1} - n_{c1} \text{ or } n_{d3} - (n_{d1} - 2n_{c1}) \leq n_{c3} \leq n_{d2} - n_{c2} \tag{3.69}$$

- **Regime 2 (Tx1-off):**

$$\min\{n_{c3}, n_{c1}\} + n_{d1} - n_{c1} \leq n_{d3} - n_{c3} \tag{3.70}$$

- **Regime 3 (All Tx's active)***: All remaining cases excluding the special case $n_{d3} - n_{c3} = n_{d1} - n_{c1}$.*

Remark 3. *The case $n_{d3} - n_{c3} = n_{d1} - n_{c1}$ is not included in the considered regimes. This is due to some special properties of the LD-PIMAC in this case. Hence, we study it separately.*

Before, we proceed, it is worth to describe these regimes intuitively. In regime 1, the desired channel of Tx3 to Rx1 is weak while the interference caused by this transmitter to Rx2 might be very strong. Hence, in regime 1, it is optimal to switch Tx3 off. This regime is divided into following sub-regimes as shown in Fig. 3.7

- **Sub-regime 1A**: $n_{d3} \leq n_{d1} - n_{c1}$ and $n_{c3} \leq n_{c1}$,
- **Sub-regime 1B**: $n_{d3} \leq n_{d1} - n_{c1}$ and $n_{c3} > n_{c1}$,
- **Sub-regime 1C**: $n_{d3} > n_{d1} - n_{c1}$, $n_{c3} \leq n_{d2} - n_{c2}$ and $n_{d3} - n_{c3} \leq n_{d1} - 2n_{c1}$.

In Regime 2, the difference between the strength of the desired and interference channels of Tx3 is so larger than that of the Tx1 that it is optimal to switch Tx1 off. This regime consists of following sub-regimes which are illustrated in Fig. 3.7

- **Sub-regime 2A**: $n_{d3} - n_{c3} \geq n_{d1}$ and $n_{c1} \leq n_{c3} \leq n_{d2} - n_{c2}$,
- **Sub-regime 2B**: $n_{d3} - n_{c3} \geq n_{d1}$ and $n_{c3} < n_{c1}$,
- **Sub-regime 2C**: $n_{d3} - 2n_{c3} \geq n_{d1} - n_{c1}$ and $n_{d3} - n_{c3} < n_{d1}$,
- **Sub-regime 2D**: $n_{d3} - n_{c3} \geq n_{d1}$ and $n_{c3} > n_{d2} - n_{c2}$.

In remaining case (regime 3), it is suboptimal to switch a transmitter off. This regime is divided into several sub-regimes (shown in Fig. 3.7) given as follows

- **Sub-regime 3A**: $n_{d1} - n_{c1} < n_{d3} < n_{c3} + n_{d1} - n_{c1}$ and $n_{d2} - n_{c2} < n_{c3}$,

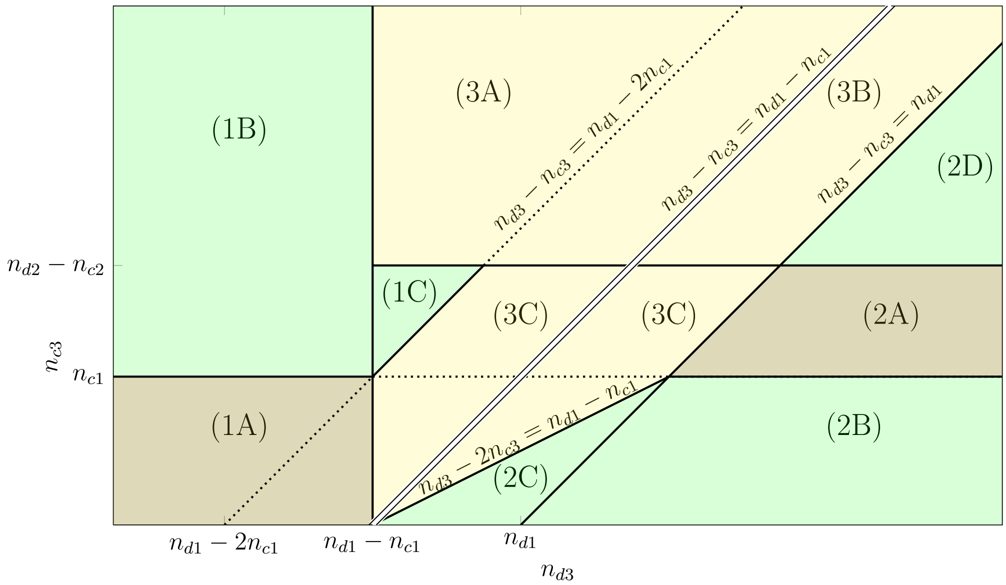


Figure 3.7: The (n_{d3}, n_{c3})-plane of the parameter space of the LD-PIMAC with $n_{c1} + n_{c2} \leq \min\{n_{d1}, n_{d2}\}$ divided into 10 sub-regimes.

- **Sub-regime 3B**: $n_{c3} + n_{d1} - n_{c1} < n_{d3} < n_{d1} + n_{c3}$ and $n_{d2} - n_{c2} < n_{c3}$,
- **Sub-regime 3C**: $\max\{n_{d1} - n_{c1}, n_{d1} - 2n_{c1} + n_{c3}\} < n_{d3} < \min\{n_{d1} - n_{c1} + 2n_{c3}, n_{c3} + n_{d1}\}$ and $n_{c3} \leq n_{d2} - n_{c2}$ and $n_{d3} - n_{c3} \neq n_{d1} - n_{c1}$.

In the following section, we present the optimality of different variants of TIN over these sub-regimes in details.

3.2.3.2 Optimality of TIN

Here, we study the optimality of TDMA-TIN and naive-TIN. First, we show that TDMA-TIN is sum-capacity optimal in regimes 1 and 2. The following theorem characterizes the sum-capacity of the LD-PIMAC in regimes where TDMA-TIN is optimal.

Theorem 1. *TDMA-TIN is capacity optimal for the LD-PIMAC in regimes 1 and 2 defined in Definition 4 (shown in Fig. 3.7). In these regimes the sum-capacity is given by*

$$C_{\text{det},\Sigma} = \begin{cases} n_{d1} - n_{c1} + n_{d2} - n_{c2}, & \textit{regime 1} \\ n_{d3} - n_{c3} + n_{d2} - n_{c2}, & \textit{sub-regimes 2A, 2B, and 2C} \\ n_{d3}, & \textit{sub-regime 2D.} \end{cases} \tag{3.71}$$

Proof. In order to prove this theorem, we need to show that (3.71) is both achievable and optimal. The achievable sum-rate of TIN is studied in Section 3.2.1. The sum-capacity expression given in Theorem 1 can be achieved by using the TDMA-TIN scheme as follows. We start with regime 1. By calculating (3.17) while taking the conditions of regime 1 (given in Definition 4) into account, it can be easily verified that the TDMA-TIN scheme can achieve $R_\Sigma = (n_{d1} - n_{c2}) + (n_{d2} - n_{c1})$ in this regime. This achievable sum-rate coincides with (3.71) in regime 1. For sub-regimes 2A, 2B, and 2C, by calculating (3.17) and taking the conditions of these sub-regimes given in Definition 4 into account, TDMA-TIN achieves $R_\Sigma = (n_{d3} - n_{c2})^+ + (n_{d2} - n_{c3})^+$ which is equal to $n_{d3} - n_{c2} + n_{d2} - n_{c3}$ in these sub-regime. This achievable sum-rate also coincides with (3.71) in sub-regimes 2A, 2B, and 2C. Finally, by calculating (3.17) while taking the conditions of sub-regime 2D (given in Definition 4) into account, we obtain the achievable sum-rate $R_\Sigma = n_{d3}$ by using TDMA-TIN. This coincides with (3.71) for sub-regime 2D.

In conclusion, TDMA-TIN achieves the sum-capacity expression given in Theorem 1 in regimes 1 and 2. This concludes the proof of the achievability.

Now, we need to show the converse of Theorem 1. The converse is based on Lemmas 4, 5, 7 presented in Section 3.2.2. It can be easily checked that the upper bound of Lemma 4 reduces to $(n_{d1} - n_{c1}) + (n_{d2} - n_{c2})$ in sub-regimes 1A and 1B. Therefore, this lemma proves Theorem 1 for these sub-regimes. By examining the upper bound in Lemma 5 for the sub-regimes 2A, 2B, and 2D, it can be easily verified that the upper bound of Lemma 5 reduces to $n_{d3} - n_{c3} + \max\{n_{d2} - n_{c2}, n_{c3}\}$. Therefore, Lemma 5 proves Theorem 1 for the sub-regimes 2A, 2B, and 2D. It remains to prove Theorem 1 for the sub-regimes 1C and 2C. To do this, we consider Lemma 7. The upper bound presented in this lemma reduces to $(n_{d1} - n_{c1}) + (n_{d2} - n_{c2})$ and $(n_{d3} - n_{c3}) + (n_{d2} - n_{c2})$ in sub-regimes 1C and 2C. Therefore, this lemma proves Theorem 1 for these sub-regimes. This concludes the proof of the converse of Theorem 1 for regimes 1 and 2. □

Interestingly, we can notice that TDMA-TIN is optimal in the case that one MAC transmitter causes noisy interference $n_{c1} \leq \min\{n_{d1}, n_{d2}\} - n_{c2}$, and the other causes strong interference $n_{c3} > \max\{n_{d1}, n_{d3}\}$. This can be seen in regime 1. The intuition here is that Tx3 in this case causes strong interference to Rx2, but has a weak channel to its desired receiver Rx1. In this case, Tx3 harms Rx2 while not increasing the achievable sum-rate of the MAC, and hence, it is better to switch it off. The resulting channel is an IC with noisy interference where TIN is optimal.

Corollary 1. *Naive-TIN is capacity optimal for the LD-PIMAC is sub-regimes 1A and 2A.*

Proof. Since the performance of TDMA-TIN and naive-TIN is the same in sub-regimes 1A and 2A , naive-TIN is sum-capacity optimal in these two sub-regimes. □

At this point, it is worth to remark that naive-TIN can achieve the sum-capacity in (3.71) only in sub-regimes 1A and 2A. This can be verified by evaluating (3.16) in regimes 1 and 2 using the conditions given in Definition 4. By doing so, it can be verified that

- $R_{\Sigma,\text{Naive}-\text{TIN}} < n_{d1} - n_{c1} + n_{d2} - n_{c2}$ in sub-regimes 1B and 1C,
- $R_{\Sigma,\text{Naive}-\text{TIN}} < n_{d3} - n_{c3} + n_{d2} - n_{c2}$ in sub-regimes 2B and 2C,
- $R_{\Sigma,\text{Naive}-\text{TIN}} < n_{d3}$ in sub-regime 2D,
- $R_{\Sigma,\text{Naive}-\text{TIN}} = n_{d1} - n_{c1} + n_{d2} - n_{c2}$ in sub-regime 1A, and
- $R_{\Sigma,\text{Naive}-\text{TIN}} = n_{d3} - n_{c3} + n_{d2} - n_{c2}$ in sub-regime 2A.

This shows the inferiority of the naive-TIN scheme in comparison to the smarter TDMA-TIN which is sum-capacity optimal for a wider range of channel parameters.

Now, we need to assess TIN for the special case given in Remark 3, i.e, $n_{d3} - n_{c3} = n_{d1} - n_{c1}$. The optimality of TIN in this case is summarized in the following theorem.

Theorem 2. *On the whole line $n_{d3} - n_{c3} = n_{d1} - n_{c1}$, TDMA-TIN achieves the sum-capacity of the LD-PIMAC which is*

$$C_{\text{det},\Sigma} = n_{d3} - n_{c3} + \max\{n_{c3}, n_{d2} - n_{c2}\}. \tag{3.72}$$

Naive-TIN achieves this sum-capacity only if $n_{c3} \leq n_{d2} - n_{c2}$.

Proof. First consider the converse. The upper bound given in Lemma 8 reduces to (3.72) by applying the conditions (3.15) and $n_{d3} - n_{c3} = n_{d1} - n_{c1}$. Now, we need to show that (3.72) is achievable by using TDMA-TIN. By taking the condition (3.15) and $n_{d3} - n_{c3} = n_{d1} - n_{c1}$ into account, it can be verified that the achievable sum-rate of TDMA-TIN in (3.17) coincides with the sum-capacity (3.72). Hence, we conclude that TDMA-TIN is optimal when $n_{d3} - n_{c3} = n_{d1} - n_{c1}$ holds. Now, consider the achievable sum-rate using naive-TIN in (3.16). This reduces to

$$R_{\Sigma,\text{Naive}-\text{TIN}} = n_{d3} - n_{c2} + (n_{d2} - n_{c3})^+ \text{ if } n_{d3} - n_{c3} = n_{d1} - n_{c1}. \tag{3.73}$$

The expression (3.73) coincides with (3.72) as long as $n_{c3} \leq n_{d2} - n_{c2}$. This shows the optimality of naive-TIN in this case. For the case that $n_{c3} > n_{d2} - n_{c2}$, the sum-capacity in (3.72) is n_{d3} which is strictly larger than the achievable sum-rate of naive-TIN in (3.73). This shows the suboptimality of naive-TIN in this case. □

Consequently, with this, the optimality of TDMA-TIN in the LD-PIMAC for regimes 1, 2 and the special case $n_{d3} - n_{c3} = n_{d1} - n_{c1}$ is shown. To complete the analysis, we need to still evaluate the performance of TDMA-TIN in regime 3. In this regime, TDMA-TIN is not optimal. In fact, in this regime, a combination of common signaling and interference alignment achieves higher rates. This is discussed in the next sub-section.

3.2.3.3 Suboptimality of TIN

Both naive-TIN and TDMA-TIN are suboptimal in regime 3. In order to show this, we propose an alternative scheme which outperforms the presented TIN schemes. The proposed scheme which is called IA-CP, is based on private and common signalling with interference alignment [CJ08]. The following proposition summarizes the achievable sum-rate using the proposed scheme.

Proposition 3. *The following sum-rate is achievable by using IA-CP in a PIMAC with* $n_{c1} + n_{c2} \leq \min\{n_{d1}, n_{d2}\}$*.*

$$R_\Sigma = \min\{n_{d3} + (n_{d2} - n_{c2}), n_{c3} + (n_{d1} - n_{c1})\} \qquad \textit{regime 3A} \qquad (3.74)$$

$$R_\Sigma = \min\{n_{d1} + n_{c3}, (2n_{d3} - n_{c3}) - (n_{d1} - n_{c1})\} \qquad \textit{regime 3B} \qquad (3.75)$$

$$R_\Sigma = \min\left\{2\mu - \nu, n_{d1} - (n_{c1} - n_{c3})^+, n_{d3} - (n_{c3} - n_{c1})^+\right\} + (n_{d2} - n_{c2}) \qquad \textit{regime 3C} \qquad (3.76)$$

where $\mu = \max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}$ *and* $\nu = \min\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}$*.*

Now we describe the scheme that achieves the sum-rate given in this proposition.

Remark 4. *A more sophisticated interference alignment scheme which achieves higher rates for the PIMAC was given in [BW12]. Since our aim here is to show the suboptimality of TIN, the following simple alignment scheme suffices.*

Interference alignment with common and private signaling (IA-CP scheme)

We construct $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ as follows

$$\boldsymbol{x}_1 = \begin{bmatrix} \mathbf{0}_{\ell_1} \\ \boldsymbol{u}_{1,a} \\ \mathbf{0}_{n_{c1}-\ell_1-R_a} \\ \boldsymbol{u}_{1,p} \\ \mathbf{0}_{q-n_c-R_{1,p}} \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} \mathbf{0}_{n_{c2}} \\ \boldsymbol{u}_{2,p1} \\ \mathbf{0}_{R_a} \\ \boldsymbol{u}_{2,p2} \\ \mathbf{0}_{q-n_{c2}-R_{2,p1}-R_{2,p2}-R_a} \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} \boldsymbol{u}_{3,c} \\ \mathbf{0}_{\ell_3} \\ \boldsymbol{u}_{3,a} \\ \mathbf{0}_{n_{c3}-R_{3,c}-\ell_3-R_a} \\ \boldsymbol{u}_{3,p} \\ \mathbf{0}_{q-n_{c3}-R_{3,p}} \end{bmatrix}, \qquad (3.77)$$

where R_a is the length of vectors $\boldsymbol{u}_{1,a}$ and $\boldsymbol{u}_{3,a}$ and the sub-script a refers to the alignment signals, and $R_{1,p}$, $R_{2,p1}$, $R_{2,p2}$, and $R_{3,p}$ are the lengths of the vectors $\boldsymbol{u}_{1,p}$, $\boldsymbol{u}_{2,p1}$, $\boldsymbol{u}_{2,p2}$, and $\boldsymbol{u}_{3,p}$ and the sub-script p refers to private signals. The common signal vector $\boldsymbol{u}_{3,c}$ has a length of

$$R_{3,c} = \min\{[n_{d3} - (n_{d1} - n_{c1})]^+, [n_{c3} - (n_{d2} - n_{c2})]^+\}. \qquad (3.78)$$

For sake of simplicity, the value of $R_{3,c}$ is given in Table 3.1.

The ℓ_1 and ℓ_3 zeros introduced in $\boldsymbol{x}_1$ and $\boldsymbol{x}_3$ are used to shift $\boldsymbol{u}_{1,a}$ and $\boldsymbol{u}_{3,a}$ down appropriately (power allocation). We fix these parameters as follows

$$\ell_1 = (n_{c1} - n_{c3})^+, \qquad \ell_3 = (n_{c3} - n_{c1} - R_{3,c})^+. \qquad (3.79)$$

$R_{3,c}$	$N_{d31} < n_{d3} \leq N_{d32}$	$N_{d32} < n_{d3} < N_{d33}$	$N_{d33} < n_{d3} < N_{d34}$
$n_{d2} - n_{c2} < n_{c3}$	$\min\left\{\begin{matrix} n_{d3} - (n_{d1} - n_{c1}) \\ n_{c3} - (n_{d2} - n_{c2}) \end{matrix}\right\}$	$n_{c3} - (n_{d2} - n_{c2})$	$n_{c3} - (n_{d2} - n_{c2})$
$n_{c3} \leq n_{d2} - n_{c2}$	Out of regime 3	0	0

Table 3.1: $N_{d31} = \min\{n_{d1} - n_{c1}, n_{c3} + n_{d1} - 2n_{c1}\}$, $N_{d32} = \max\{n_{d1} - n_{c1}, n_{c3} + n_{d1} - 2n_{c1}\}$, $N_{d33} = n_{c3} + n_{d1} - n_{c1}$, and $N_{d34} = \min\{n_{c3} + n_{d1}, n_{d1} - n_{c1} + 2n_{c3}\}$.

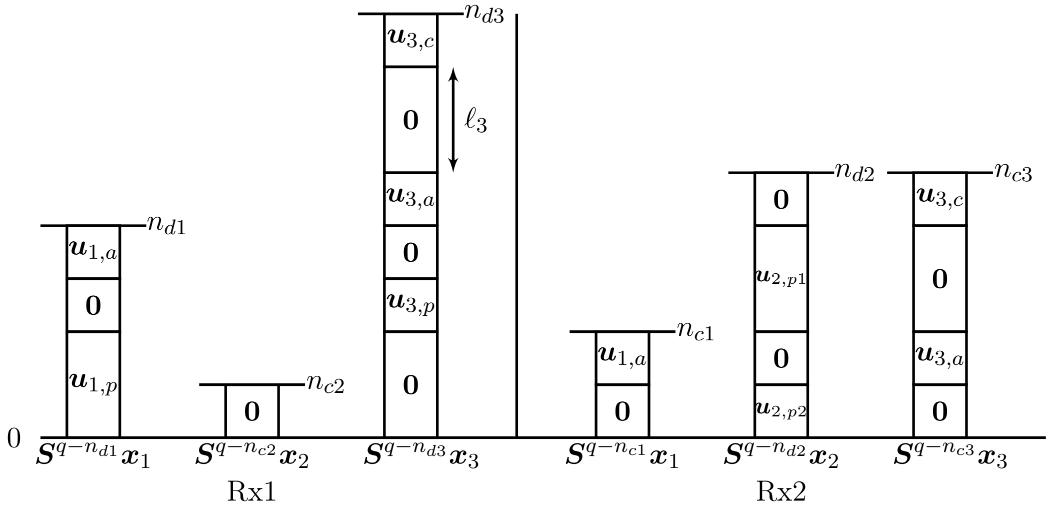


Figure 3.8: A graphical illustration of the received signals at receivers 1 and 2 for the case that $n_{d1} - n_{c1} < n_{d3} - n_{c3}$ and $n_{d2} - n_{c2} < n_{c3}$ when the transmit signals are constructed as in (3.77).

A graphical illustration of the received signals at both receivers is shown in Fig. 3.8 for the case when $n_{d2} - n_{c2} < n_{c3}$. As it is shown in this figure, the private signals are not received at undesired receivers. This can be guaranteed, since the private signals of Tx1 and Tx3 are not allocated to the top most n_{c1} bits of $\boldsymbol{x}_1$ and the top-most n_{c3} bits of $\boldsymbol{x}_3$. Hence, Rx1 can obtain at most $n_{d1} - n_{c1}$ private bits from Tx1 and $n_{d3} - n_{c3}$ private bits from Tx3. Therefore, the private signals of the MAC transmitters are received at the lower-most $\max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}$ bits of $\boldsymbol{y}_1$. Since the private signals from Tx1 and Tx3 are treated as in a multiple access channel at Rx1, their sum-rate is fixed by

$$R_{1,p} + R_{3,p} = \max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}. \tag{3.80}$$

Moreover, the private signal of Tx2 must not be received at Rx1, thus it must be sent at the lowest $q - n_{c2}$ bits of $\boldsymbol{x}_2$.

The main idea of the scheme is to align the vectors $\boldsymbol{u}_{1,a}$ and $\boldsymbol{u}_{3,a}$ at Rx2 while they are received without any overlap at Rx1. To align these vectors at Rx2, the condition

$$n_{c1} - \ell_1 = n_{c3} - R_{3,c} - \ell_3 \tag{3.81}$$

must be satisfied. This condition is indeed satisfied by our choice of $R_{3,c}$ in (3.78) and ℓ_1, ℓ_3 in (3.79). Moreover, the aligned signal $\boldsymbol{u}_{1,a}$ and $\boldsymbol{u}_{3,a}$ must not have an overlap with private signal of Tx2 at Rx2. Due to this, the private signal of Tx2 is split into two parts, $\boldsymbol{u}_{2,p1}$, and $\boldsymbol{u}_{2,p2}$ with sum-rate

$$R_{2,p1} + R_{2,p2} = n_{d2} - n_{c2} - R_a. \tag{3.82}$$

Now, we are ready to discuss the reliability of decoding at the receivers. First, consider Rx2. Since $R_{3,c}$ in (3.78) is chosen such that $\boldsymbol{u}_{3,c}$ does not overlap the private and alignment signals at Rx2, Rx2 is able to decode $\boldsymbol{u}_{3,c}$. Due to the condition in (3.82), Rx2 is able to decode $\boldsymbol{u}_{2,p1}$, $\boldsymbol{u}_{1,a} \oplus \boldsymbol{u}_{3,a}$, and $\boldsymbol{u}_{2,p2}$ as long as

$$R_a \leq n_{c1} - \ell_1. \tag{3.83}$$

Notice that since $R_a \leq n_{c1} \leq n_{d2}-n_{c2}$, the sum $R_{2,p1}+R_{2,p2}$ in (3.82) is non-negative.

Now, consider Rx1. In order to guarantee that the common signal vector $\boldsymbol{u}_{3,c}$ is decodable at Rx1, an overlap between $\boldsymbol{u}_{3,c}$ and the alignment signal vectors ($\boldsymbol{u}_{1,a}$, $\boldsymbol{u}_{3,a}$) and private signal vectors ($\boldsymbol{u}_{1,p}$, $\boldsymbol{u}_{3,p}$) at Rx1 needs to be avoided. An overlap between $\boldsymbol{u}_{3,c}$ and private signal vectors is avoided by the choice of $R_{3,c}$ in (3.78). While an overlap between $\boldsymbol{u}_{3,c}$ and $\boldsymbol{u}_{3,a}$ is avoided by the alignment condition in (3.81), the following condition has to be satisfied

$$R_{3,c} \leq (n_{d3} - (n_{d1} - \ell_1))^+ \quad \text{if } 0 < R_a, \tag{3.84}$$

to guarantee that $\boldsymbol{u}_{3,c}$ does not overlap $\boldsymbol{u}_{1,a}$ at Rx1. Now, we need to guarantee that Rx1 decodes $\boldsymbol{u}_{3,a}$ and $\boldsymbol{u}_{1,a}$ reliably. For decoding these signal vectors, we address a decoding order. The order of decoding these signal vectors depends on the sign of $S = (n_{d3} - n_{c3}) - (n_{d1} - n_{c1})$. If S is positive (see Fig. 3.8), $\boldsymbol{u}_{3,a}$ is received on the top of $\boldsymbol{u}_{1,a}$ at Rx1 and hence, Rx1 decodes $\boldsymbol{u}_{1,a}$ first after decoding $\boldsymbol{u}_{3,a}$ and vice verse.[4] An example for the case when S is negative is illustrated in Fig. 3.9. Regardless of the order of decoding, an overlap between vectors $\boldsymbol{u}_{1,a}$ and $\boldsymbol{u}_{3,a}$ at Rx1 has to be avoided. To this end, we have following conditions

$$R_a \leq \begin{cases} n_{d3} - (\ell_3 + R_{3,c}) - (n_{d1} - \ell_1) & \text{if } S > 0 \\ (n_{d1} - \ell_1) - (n_{d3} - (\ell_3 + R_{3,c})) & \text{if } S < 0 \end{cases}. \tag{3.85}$$

By substituting ℓ_1, ℓ_3 in (3.79) and $R_{3,c}$ in (3.78) into (3.85), and setting $\mu = \max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}$ and $\nu = \min\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}$, we can rewrite the conditions in (3.85) as

$$R_a \leq \mu - \nu. \tag{3.86}$$

[4] The decoding order in the deterministic case is not important. However, it is important in the Gaussian setup. In order to have the same decoding procedure for both models, we use the decoding order also in the deterministic case.

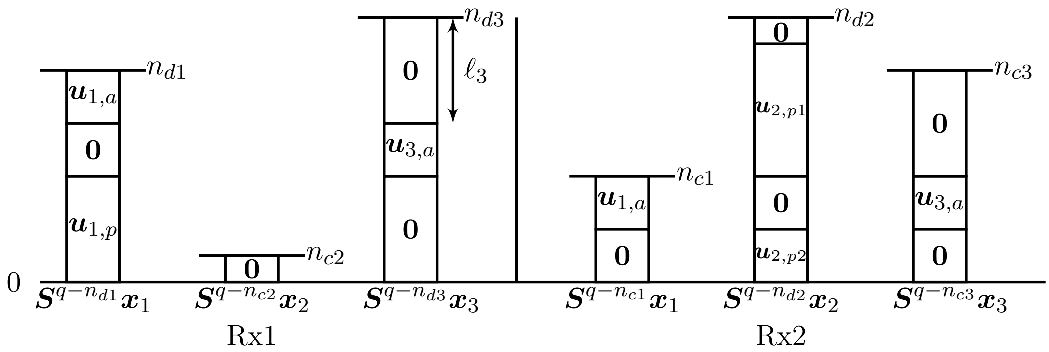


Figure 3.9: A graphical illustration of the received signals at receivers 1 and 2 for the case that $n_{d1} - n_{c1} > n_{d3} - n_{c3}$ and $n_{d2} - n_{c2} > n_{c3}$ when the transmit signals are constructed as in (3.77).

In addition to this, vectors $\boldsymbol{u}_{1,a}$ and $\boldsymbol{u}_{3,a}$ must not have any overlap with private vectors $\boldsymbol{u}_{1,p}$ and $\boldsymbol{u}_{3,p}$. Due to this, the following condition has to be satisfied

$$R_a \leq (\min\{n_{d1} - \ell_1, n_{d3} - (\ell_3 + R_{3,c})\} - \underbrace{\max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}}_{R_{1,p}+R_{3,p}})^+. \quad (3.87)$$

By using (3.78), (3.79), and definition of μ and ν, we rewrite (3.87) as

$$R_a \leq (\min\{n_{d1} - (n_{c1} - n_{c3})^+, n_{d3} - (n_{c3} - n_{c1})^+\} - \mu)^+. \quad (3.88)$$

Combining the condition in (3.83), (3.86) and (3.88), we obtain

$$R_a \leq (\min\{n_{c1} - \ell_1 + \mu, 2\mu - \nu, n_{d1} - (n_{c1} - n_{c3})^+, n_{d3} - (n_{c3} - n_{c1})^+\} - \mu)^+. \quad (3.89)$$

The rate of the aligned signals R_a is given in Table 3.2. Note that both conditions (3.84) and (3.89) are satisfied by chosen R_a and $R_{3,c}$ in Table 3.1 and 3.2.

R_a	$N_{d31} < n_{d3} \leq N_{d32}$	$N_{d32} < n_{d3} < N_{d33}$	$N_{d33} < n_{d3} < N_{d34}$
$n_{d2} - n_{c2} < n_{c3}$	0	0	$\min\begin{Bmatrix} n_{d1} - \mu \\ \mu - \nu \end{Bmatrix}$
$n_{c1} < n_{c3} \leq n_{d2} - n_{c2}$	Out of regime 3	$\min\begin{Bmatrix} \nu - \mu + n_{c1} \\ \mu - \nu \end{Bmatrix}$	$\min\begin{Bmatrix} n_{d1} - \mu \\ \mu - \nu \end{Bmatrix}$
$n_{c3} \leq n_{c1}$	Out of regime 3	$\min\begin{Bmatrix} n_{d3} - \mu \\ \mu - \nu \end{Bmatrix}$	$\min\begin{Bmatrix} \nu - \mu + n_{c3} \\ \mu - \nu \end{Bmatrix}$

Table 3.2: $N_{d31} = \min\{n_{d1} - n_{c1}, n_{c3} + n_{d1} - 2n_{c1}\}$, $N_{d32} = \max\{n_{d1} - n_{c1}, n_{c3} + n_{d1} - 2n_{c1}\}$, $N_{d33} = n_{c3} + n_{d1} - n_{c1}$, and $N_{d34} = \min\{n_{c3} + n_{d1}, n_{d1} - n_{c1} + 2n_{c3}\}$

Remark 5. *One can improve the proposed scheme by choosing a non-zero R_a for the case that $n_{d2} - n_{c2} < n_{c3}$ and $n_{d1} - 2n_{c1} < n_{d3} - n_{c3} < n_{d1} - n_{c1}$. Since our goal is to outperform TDMA-TIN, we avoided using an alignment signal in this case to decrease the complexity of the scheme.*

R_Σ	$N_{d31} < n_{d3} \leq N_{d32}$	$N_{d32} < n_{d3} < N_{d33}$	$N_{d33} < n_{d3} < N_{d34}$
$n_{d2} - n_{c2} < n_{c3}$	$\min\left\{\begin{matrix} n_{d3} + (n_{d2} - n_{c2}) \\ n_{c3} + (n_{d1} - n_{c1}) \end{matrix}\right\}$		$\min\left\{\begin{matrix} n_{d1} + n_{c3} \\ (2n_{d3} - n_{c3}) - (n_{d1} - n_{c1}) \end{matrix}\right\}$
$n_{c3} \leq n_{d2} - n_{c2}$	Out of regime 3	$(n_{d2} - n_{c2}) + \min\left\{\begin{matrix} 2\mu - \nu \\ n_{d1} - (n_{c1} - n_{c3})^+ \\ n_{d3} - (n_{c3} - n_{c1})^+ \end{matrix}\right\}$	

Table 3.3: $N_{d31} = \min\{n_{d1} - n_{c1}, n_{c3} + n_{d1} - 2n_{c1}\}$, $N_{d32} = \max\{n_{d1} - n_{c1}, n_{c3} + n_{d1} - 2n_{c1}\}$, $N_{d33} = n_{c3} + n_{d1} - n_{c1}$, and $N_{d34} = \min\{n_{c3} + n_{d1}, n_{d1} - n_{c1} + 2n_{c3}\}$

Using this scheme, we achieve

$$R_\Sigma = R_{1,p} + R_{3,p} + R_{2,p1} + R_{2,p2} + 2R_a + R_{3,c}. \tag{3.90}$$

By substituting (3.80) and (3.82) into (3.90), we obtain

$$R_\Sigma = \max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\} + (n_{d2} - n_{c2}) + R_a + R_{3,c}. \tag{3.91}$$

Now, by using the chosen R_a and $R_{3,c}$ in Table 3.1 and 3.2, we obtain the achievable sum-rate. This is given in Table 3.3, which completes the proof of Proposition 3.

Comparison with TDMA-TIN

Now, we need to show that the sum-rate in Proposition 3 is higher than the achievable sum-rate using the TDMA-TIN given in (3.17) for regime 3. We show this for sub-regimes 3A, 3B, and 3C separately.

First, consider sub-regime 3A. In this sub-regime, TDMA-TIN achieves

$$\begin{aligned} R_{\Sigma,\text{TDMA-TIN}} &= \max\{n_{d3}, (n_{d1} - n_{c1}) + (n_{d2} - n_{c2}), (n_{d3} - n_{c2})^+ + (n_{d2} - n_{c3})^+\} \\ &= \max\{n_{d3}, (n_{d1} - n_{c1}) + (n_{d2} - n_{c2}), n_{d3} - n_{c2} + (n_{d2} - n_{c3})^+\} \\ &\overset{(a)}{=} \max\{n_{d3}, n_{d1} - n_{c1} + n_{d2} - n_{c2}\}, \end{aligned} \tag{3.92}$$

where step (a) follows since in regime 3A, $n_{d2} - n_{c2} < n_{c3}$. Using the definition of sub-regime 3A and the condition in (3.15), we upper bound the expression in (3.92) by

$$R_{\Sigma,\text{TDMA-TIN}} < \min\{n_{d3} + (n_{d2} - n_{c2}), n_{c3} + (n_{d1} - n_{c1})\}. \tag{3.93}$$

Note that (3.93) is the achievable sum-rate given in Proposition 3 for sub-regime 3A. Therefore, the scheme IA-CP outperforms TDMA-TIN and consequently naive-TIN in this sub-regime. Now, consider sub-regime 3B. Doing similar steps as in (3.92), we can write the achievable sum-rate using the TDMA-TIN scheme for sub-regime 3B as

$$R_{\Sigma,\text{TDMA-TIN}} = \max\{n_{d3}, n_{d1} - n_{c1} + n_{d2} - n_{c2}\}. \tag{3.94}$$

Due to the conditions of sub-regime 3B, we have $(n_{d2} - n_{c2}) + (n_{d1} - n_{c1}) < n_{c3} + (n_{d1} - n_{c1}) < n_{d3}$. Hence, we rewrite (3.94) as

$$R_{\Sigma,\text{TDMA-TIN}} = n_{d3}. \tag{3.95}$$

Since in sub-regime 3B, $n_{c3} + (n_{d1} - n_{c1}) < n_{d3} < n_{d1} + n_{c3}$, the achievable sum-rate in (3.95) is bounded by

$$R_{\Sigma,\text{TDMA-TIN}} < \min\{n_{d1} + n_{c3}, (2n_{d3} - n_{c3}) - (n_{d1} - n_{c1})\}. \tag{3.96}$$

The expression in (3.96) coincides with the achievable sum-rate in Proposition 3 for sub-regime 3B. Hence, we conclude that the scheme IA-CP outperforms TDMA-TIN and naive-TIN in sub-regime 3B. Finally, we consider sub-regime 3C. In this sub-regime TDMA-TIN achieves

$$R_{\Sigma,\text{TDMA-TIN}} = (n_{d2} - n_{c2}) + \max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}. \tag{3.97}$$

In sub-regimes 3C, the achievable sum-rate of IA-CP is

$$R_{\Sigma} = (n_{d2} - n_{c2}) + \min\left\{2\mu - \nu, n_{d1} - (n_{c1} - n_{c3})^+, n_{d3} - (n_{c3} - n_{c1})^+\right\}, \tag{3.98}$$

where $\mu = \max\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}$ and $\nu = \min\{n_{d3} - n_{c3}, n_{d1} - n_{c1}\}$. This can be rewritten as

$$\begin{aligned} R_{\Sigma} &= (n_{d2} - n_{c2}) + \mu + \min\left\{\mu - \nu, n_{d1} - (n_{c1} - n_{c3})^+ - \mu, n_{d3} - (n_{c3} - n_{c1})^+ - \mu\right\} \\ &= R_{\Sigma,\text{TDMA}-\text{TIN}} + \min\left\{\mu - \nu, n_{d1} - (n_{c1} - n_{c3})^+ - \mu, n_{d3} - (n_{c3} - n_{c1})^+ - \mu\right\}. \end{aligned} \tag{3.99}$$

The min expression above is the rate of the aligned signal vector which is given in Table 3.2. Notice that in sub-regime 3C, this min expression is positive. Hence, the sum-rate in (3.99) is higher than the achievable sum-rate using TDMA-TIN in sub-regime 3C. Thus, both TDMA-TIN and naive-TIN are suboptimal in regime 3.

Remark 6. *The expression $\mu - \nu$ is equal to zero, when $n_{d3} - n_{c3} = n_{d1} - n_{c1}$. Note that this is the case which is excluded from regime 3 (cf. Remark 3). In this case, IA-CP achieves the same sum-rate as TDMA-TIN. This is due to the fact that in this special case, the LD-PIMAC can be modelled as an IC with inputs $\tilde{\boldsymbol{x}}_1 = \boldsymbol{S}^{q-n_{d1}}\boldsymbol{x}_1 \oplus \boldsymbol{S}^{q-n_{d3}}\boldsymbol{x}_3$ and $\boldsymbol{x}_2$ and outputs $\boldsymbol{y}_1 = \tilde{\boldsymbol{x}}_1 \oplus \boldsymbol{S}^{q-n_{c2}}\boldsymbol{x}_2$ and $\boldsymbol{y}_2 = \boldsymbol{S}^{n_{d1}-n_{c1}}\tilde{\boldsymbol{x}}_1 \oplus \boldsymbol{S}^{q-n_{d2}}\boldsymbol{x}_2$. Obviously, aligning the interference signals at the undesired receiver while they are separable at the desired receiver is not doable in the 2-user IC. Hence, IA-CP cannot outperform TDMA-TIN.*

3.3 TIN in the Gaussian PIMAC

For the linear deterministic PIMAC, we have shown that the naive-TIN scheme is optimal only in sub-regimes 1A and 2A, while TDMA-TIN is optimal in regimes 1 and 2. In this section, we assess the optimality of naive-TIN and TDMA-TIN in the Gaussian case by finding the gap between the upper bound and the achievable sum-rates in the regimes where this gap can be upper bounded by a constant.

Regimes under Consideration in Gaussian PIMAC

Similar to the LD-PIMAC, in the Gaussian counterpart, we restrict the analysis to the case where TIN achieves the capacity of the IC consisting Tx1, Rx1 and Tx2, Rx2 [ETW08] within a constant gap. Hence, the following condition is always satisfied

$$\min\{P|h_{d1}|^2, P|h_{d2}|^2\} \geq Ph_{c1}^2 Ph_{c2}^2. \tag{3.100}$$

Using Definition 1, this condition can be rewritten as follows

$$\min\{\alpha_{d1}, \alpha_{d2}\} \geq \alpha_{c1} + \alpha_{c2}. \tag{3.101}$$

We divide the parameter space of the Gaussian PIMAC (which satisfies (3.101)) into several regimes defined similar to the deterministic case (Definition 4) with n_k replaced by α_k for $k \in \{d1, c1, d2, c2, d3, c3\}$. The channel parameters in the Gaussian PIMAC are summarized in Table 3.4.

LD-PIMAC	Gaussian PIMAC
n_{d1}	α_{d1}
n_{c1}	α_{c1}
n_{d2}	α_{d2}
n_{c2}	α_{c2}
n_{d3}	α_{d3}
n_{c3}	α_{c3}

Table 3.4: Related channel parameters in the Gaussian and linear deterministic PIMAC.

3.3.1 Achievable Lower Bounds using TIN

In this section, we present the lower bounds which can be achieved by using different variants of TIN in the Gaussian PIMAC. First, we present the achievable sum-rate using TIN. Next, we formulate the achievable GDoF of TIN from the sum-rate expressions.

3.3.1.1 Achievable Sum-rate using TIN

Here, we study the achievable sum-rate of different types of TIN in the Gaussian PIMAC.

Naive-TIN

In naive-TIN, all transmitters encode their messages using Gaussian codebook with full power. This causes interference at the undesired receivers. At the receiver side, the strategy is the same as if there is no interference. Therefore, the receivers decode their desired signals while the interference is treated as noise. Hence, Rx1 decodes

W_1 and W_3 as in a multiple access channel (successive decoding) with noise variance $1 + |h_{c2}|^2 P$, and Rx2 decodes W_2 as in a point-to-point channel with noise variance $1 + |h_{c1}|^2 P + |h_{c3}|^2 P$. Hence, we obtain the following achievable rate.

Proposition 4. *In the Gaussian PIMAC, naive-TIN achieves any sum-rate* $R_\Sigma \leq R_{\Sigma,\text{Naive}-\text{TIN}}$*, where*

$$R_{\Sigma,\text{Naive}-\text{TIN}} = \log_2\left(1 + \frac{\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}}}{1 + \rho^{\alpha_{c2}}}\right) + \log_2\left(1 + \frac{\rho^{\alpha_{d2}}}{1 + \rho^{\alpha_{c1}} + \rho^{\alpha_{c3}}}\right). \tag{3.102}$$

TDMA-TIN

In contrast to naive-TIN, in TDMA-TIN not all transmitters are active at the same time. In this scheme, we divide the transmission time into three fractions, i.e, τ_1, τ_2, and τ_3, where $\tau_1 + \tau_2 + \tau_3 = 1$. While in τ_1 fraction of time only Tx3 is active and hence, we have a point-to-point channel, in the remaining $(1 - \tau_1)$ fraction of time, we have two types of 2-user IC's. The active transmitters in the first type are Tx1 and Tx2. In total, τ_2 fraction of time is assigned to this IC. In the other type of IC, Tx3 and Tx2 are active. We assign τ_3 fraction of time to this type. In this scheme, all transmitters send such that they consume their maximum power P in the whole transmission. In other words, Tx1, Tx2, and Tx3 send $X_1 \sim \mathbb{C}\mathcal{N}(0, \frac{P}{\tau_2})$, $X_2 \sim \mathbb{C}\mathcal{N}(0, \frac{P}{(1-\tau_1)})$, and $X_3 \sim \mathbb{C}\mathcal{N}(0, \frac{P}{\tau_1+\tau_3})$ in τ_2, $(1 - \tau_1)$, and $(\tau_1 + \tau_3)$ fraction of time, respectively. The achievable sum-rate using TDMA-TIN is presented in the following proposition.

Proposition 5. *In the Gaussian PIMAC, TDMA-TIN achieves any sum-rate* $R_\Sigma \leq R_{\Sigma,\text{TDMA}-\text{TIN}}$*, where*

$$\begin{aligned} R_{\Sigma,\text{TDMA}-\text{TIN}} = \max_{\tau_1,\tau_2,\tau_3 \in [0,1]} & \tau_1 \log_2\left(1 + \frac{\rho^{\alpha_{d3}}}{\tau_1 + \tau_3}\right) \qquad (3.103) \\ & + \tau_2 \left[\log_2\left(1 + \frac{\frac{\rho^{\alpha_{d1}}}{\tau_2}}{1 + \frac{\rho^{\alpha_{c2}}}{(1-\tau_1)}}\right) + \log_2\left(1 + \frac{\frac{\rho^{\alpha_{d2}}}{(1-\tau_1)}}{1 + \frac{\rho^{\alpha_{c1}}}{\tau_2}}\right)\right] \\ & + \tau_3 \left[\log_2\left(1 + \frac{\frac{\rho^{\alpha_{d3}}}{\tau_1+\tau_3}}{1 + \frac{\rho^{\alpha_{c2}}}{(1-\tau_1)}}\right) + \log_2\left(1 + \frac{\frac{\rho^{\alpha_{d2}}}{(1-\tau_1)}}{1 + \frac{\rho^{\alpha_{c3}}}{\tau_1+\tau_3}}\right)\right] \\ \text{subject to} \quad & \tau_1 + \tau_2 + \tau_3 = 1. \end{aligned}$$

Naive-TIN versus TDMA-TIN

The difference between TDMA-TIN and naive-TIN is that while all transmitters are simultaneously active in the latter, the same is not true in the former which orthogonalizes the users in time. As we have seen in the LD-PIMAC, naive-TIN can never outperform TDMA-TIN. It is shown in [CS12] that a simpler variant of TDMA-TIN in which only the time sharing between the two 2-user IC's of PIMAC is allowed, leads to a larger sum-rate than that of naive-TIN as long as $\alpha_{d3} - \alpha_{c3} \neq \alpha_{d1} - \alpha_{c1}$. Since the considered TDMA-TIN in this work is more general than the one in [CS12], our proposed TDMA-TIN can also outperform naive-TIN as long as $\alpha_{d3} - \alpha_{c3} \neq \alpha_{d1} - \alpha_{c1}$. Note that the excluded case corresponds to the special case discussed in Remark 3. We study this case later in details.

Remark 7. *A similar behaviour that keeping one or two transmitters silent can improve the achievable sum-rate of TIN, has been also observed in the K-user IC in [GNAJ15, Example 2].*

3.3.1.2 Achievable GDoF using TIN

Since for constant gap optimality, the GDoF optimality is required, it is worth to convert the achievable sum-rate of Naive-TIN and TDMA-TIN into the GDoF expression (cf. (3.11)).

Naive-TIN

Consider the achievable sum-rate of naive-TIN given in (3.102). At high SNR, this sum-rate can be written as follows

$$\begin{aligned}
R_{\Sigma,\mathrm{Naive-TIN}} &\approx \log_2\left(1+\frac{\rho^{\alpha_{d1}}+\rho^{\alpha_{d3}}}{\rho^{\alpha_{c2}}}\right)+\log_2\left(1+\frac{\rho^{\alpha_{d2}}}{\rho^{\alpha_{c1}}+\rho^{\alpha_{c3}}}\right)\\
&= \log_2\left(\rho^0+\rho^{\alpha_{d1}-\alpha_{c2}}+\rho^{\alpha_{d3}-\alpha_{c2}}\right)+\log_2\left(\rho^{\alpha_{c1}}+\rho^{\alpha_{c3}}+\rho^{\alpha_{d2}}\right)\\
&\quad -\log_2(\rho^{\alpha_{c1}}+\rho^{\alpha_{c3}})\\
&\approx \log_2\rho[\max\{0,\alpha_{d1}-\alpha_{c2},\alpha_{d3}-\alpha_{c2}\}+\max\{\alpha_{c1},\alpha_{c3},\alpha_{d2}\}\\
&\qquad -\max\{\alpha_{c1},\alpha_{c3}\}]\\
&\overset{(a)}{\approx} \log_2\rho[\max\{\alpha_{d1},\alpha_{d3}\}-\alpha_{c2}+(\alpha_{d2}-\max\{\alpha_{c1},\alpha_{c3}\})^+]
\end{aligned}$$

The step (a) follows due to the condition in (3.101). Now, by dividing the sum-rate by $\log_2\rho$ and letting $\rho\to\infty$, we obtain the GDoF which is given in the following proposition.

Proposition 6. *Naive-TIN achieves any $d_\Sigma \leq d_{\Sigma,\mathrm{Naive-TIN}}$, where*

$$d_{\Sigma,\mathrm{Naive-TIN}}(\boldsymbol{\alpha}) = \max\{\alpha_{d1},\alpha_{d3}\}-\alpha_{c2}+(\alpha_{d2}-\max\{\alpha_{c1},\alpha_{c3}\})^+. \tag{3.104}$$

TDMA-TIN

We identify first the achievable sum-rate of TDMA-TIN in (3.103) at high SNR

$$
\begin{aligned}
R_{\Sigma,\mathrm{TDMA-TIN}} &\overset{(a)}{\approx} \max_{\tau_1,\tau_2,\tau_3\in[0,1]} \tau_1 \log_2\left(\frac{\rho^{\alpha_{d3}}}{\tau_1+\tau_3}\right) \\
&\quad + \tau_2\left[\log_2\left(\frac{\frac{\rho^{\alpha_{d1}}}{\tau_2}}{\frac{\rho^{\alpha_{c2}}}{(1-\tau_1)}}\right) + \log_2\left(\frac{\frac{\rho^{\alpha_{d2}}}{(1-\tau_1)}}{\frac{\rho^{\alpha_{c1}}}{\tau_2}}\right)\right] \\
&\quad + \tau_3\left[\log_2\left(\frac{\frac{\rho^{\alpha_{d3}}}{\tau_1+\tau_3}}{\frac{\rho^{\alpha_{c2}}}{(1-\tau_1)}}\right) + \log_2\left(\frac{\frac{\rho^{\alpha_{d2}}}{(1-\tau_1)}}{\frac{\rho^{\alpha_{c3}}}{\tau_1+\tau_3}}\right)\right] \\
&= \max_{\tau_1,\tau_2,\tau_3\in[0,1]} \log_2\rho[\tau_1\alpha_{d3} + \tau_2(\alpha_{d1}-\alpha_{c2}+\alpha_{d2}-\alpha_{c1}) \\
&\quad + \tau_3[(\alpha_{d3}-\alpha_{c2})^+ + (\alpha_{d2}-\alpha_{c3})^+]] + \tau_1\log_2\left(\frac{1}{\tau_1+\tau_3}\right),
\end{aligned}
$$

where $\tau_1+\tau_2+\tau_3=1$. Note that (a) is due to the high SNR approximation. Now, by dividing the sum-rate by $\log_2\rho$ and letting $\rho\to\infty$ and keeping the condition $\alpha_{c1}+\alpha_{c2}\le\min\{\alpha_{d1},\alpha_{d2}\}$ in mind, we obtain the following achievable GDoF using TDMA-TIN

$$
\begin{aligned}
d_{\Sigma,\text{TDMA-TIN}}(\boldsymbol{\alpha}) = \max_{\tau_1,\tau_2,\tau_3\in[0,1]} \quad & \tau_1\alpha_{d3} + \tau_2(\alpha_{d1}-\alpha_{c2}+\alpha_{d2}-\alpha_{c1}) \\
& + \tau_3[(\alpha_{d3}-\alpha_{c2})^+ + (\alpha_{d2}-\alpha_{c3})^+] \\
\text{subject to} \quad & \tau_1+\tau_2+\tau_3 = 1.
\end{aligned}
$$

Since this maximization is linear in τ_1, τ_2, and τ_3, we obtain the optimal solution by assigning the whole transmission time to the type which achieves the highest GDoF. Hence, the achievable GDoF of TDMA-TIN can be presented as in the following proposition.

Proposition 7. *TDMA-TIN achieves any* $d_\Sigma \le d_{\Sigma,\mathrm{TDMA-TIN}}$*, where*

$$
d_{\Sigma,\mathrm{TDMA-TIN}}(\boldsymbol{\alpha}) = \max\left\{\begin{matrix} \alpha_{d3} \\ (\alpha_{d1}-\alpha_{c2}) + (\alpha_{d2}-\alpha_{c1}) \\ (\alpha_{d3}-\alpha_{c2})^+ + (\alpha_{d2}-\alpha_{c3})^+ \end{matrix}\right\}. \tag{3.105}
$$

As we have shown for the LD-PIMAC, as long as receivers treat interference as noise, the best power allocation at the transmitter side cannot achieve higher sum-rate than TDMA-TIN. In the following lemma, we extend this result to the Gaussian PIMAC.

Lemma 9. *The achievable GDoF by using TIN at the receiver side alongside power control at the transmitter side is upper bounded by the GDoF achieved by TDMA-TIN given in* (3.105).

Proof. First, we write the achievable GDoF of using TIN at the receiver side with power control at the transmitter side. Suppose that Txi transmits x_i which is a realization of random variable $X_i \sim \mathbb{C}\mathcal{N}(0, P_i)$ with $P_i \leq P$. Doing this, the maximum achievable sum-rate using TIN is given by

$$R_{\Sigma,\mathrm{TIN}} = \log_2\left(1 + \frac{P_1|h_{d1}|^2 + P_3|h_{d3}|^2}{1 + P_2|h_{c2}|^2}\right) + \log_2\left(1 + \frac{P_2|h_{d2}|^2}{1 + P_1|h_{c1}|^2 + P_3|h_{c3}|^2}\right). \tag{3.106}$$

Now, we define

$$\begin{aligned}
\alpha_{11} &= \left(\frac{\log_2(P_1|h_{d1}|^2)}{\log_2\rho}\right)^+, \quad \alpha_{21} = \left(\frac{\log_2(P_1|h_{c1}|^2)}{\log_2\rho}\right)^+,\\
\alpha_{22} &= \left(\frac{\log_2(P_2|h_{d2}|^2)}{\log_2\rho}\right)^+, \quad \alpha_{12} = \left(\frac{\log_2(P_2|h_{c2}|^2)}{\log_2\rho}\right)^+,\\
\alpha_{13} &= \left(\frac{\log_2(P_3|h_{d3}|^2)}{\log_2\rho}\right)^+, \quad \alpha_{23} = \left(\frac{\log_2(P_3|h_{c3}|^2)}{\log_2\rho}\right)^+,
\end{aligned}$$

where given the condition (3.101), they satisfy

$$\begin{aligned}
\alpha_{21} &= (\alpha_{11} - (\alpha_{d1} - \alpha_{c1}))^+\\
\alpha_{12} &= (\alpha_{22} - (\alpha_{d2} - \alpha_{c2}))^+\\
\alpha_{23} &= (\alpha_{13} - (\alpha_{d3} - \alpha_{c3}))^+ \quad \text{if } \alpha_{c3} \leq \alpha_{d3}\\
\alpha_{13} &= (\alpha_{23} - (\alpha_{c3} - \alpha_{d3}))^+ \quad \text{if } \alpha_{d3} < \alpha_{c3}.
\end{aligned}$$

and $\alpha_{11} \in [0, \alpha_{d1}]$, $\alpha_{21} \in [0, \alpha_{c1}]$, $\alpha_{22} \in [0, \alpha_{d2}]$, $\alpha_{12} \in [0, \alpha_{c2}]$, $\alpha_{13} \in [0, \alpha_{d3}]$, and $\alpha_{23} \in [0, \alpha_{c3}]$. Moreover, for any arbitrary P_1, P_2, and P_3, the following conditions are satisfied

$$\begin{aligned}
\alpha_{11} - \alpha_{21} &\leq \alpha_{d1} - \alpha_{c1}\\
\alpha_{22} - \alpha_{12} &\leq \alpha_{d2} - \alpha_{c2}\\
\alpha_{13} - \alpha_{23} &\leq (\alpha_{d3} - \alpha_{c3})^+.
\end{aligned}$$

Now, we can convert the achievable sum-rate in (3.106) to the GDoF expression and write

$$d_{\Sigma,\mathrm{TIN}}(\boldsymbol{\alpha}) = (\max\{\alpha_{11}, \alpha_{13}\} - \alpha_{12})^+ + (\alpha_{22} - \max\{\alpha_{21}, \alpha_{23}\})^+. \tag{3.107}$$

Now, similar to proof of Lemma 3, we can show that the GDoF in (3.107) is outperformed by (3.105). □

Remark 8. *Notice that the achievable GDoF of TDMA-TIN in the Gaussian PIMAC is equal to that of binary power control (i.e., transmitters either send with full power or are completely inactive) alongside TIN. Hence, from Lemma 9, we conclude that the general power allocation problem for maximizing the GDoF in the Gaussian PIMAC while the receivers use TIN is reduced to the binary power allocation problem.*

The rest of this section is dedicated to performance analysis of TDMA-TIN and naive-TIN in the Gaussian PIMAC. To do this, we use the upper bounds on the sum-capacity which are presented in the following sub-section, as the benchmark.

3.3.2 Upper Bounds

Here, we use the insights from linear deterministic PIMAC, in order to establish the upper bounds for the Gaussian case. To do this, the following lemmata are required.

Lemma 10. *Consider two observations*

$$Y^n = f_1 A^n + f_2 B^n + Z_y^n \tag{3.108}$$

$$S^n = g_1 A^n + Z_s^n, \tag{3.109}$$

where f_1, f_2, and g_1 are complex deterministic variables. Moreover, A, B, $Z_y \sim \mathbb{CN}(0,1)$, and $Z_s \sim \mathbb{CN}(0,1)$ are four independent complex random variables. Additionally, suppose that the variance of $A[t]$ and $B[t]$ with $t = 1, \dots, n$ are $P_A[t]$ and $P_B[t]$, respectively, where

$$P_A = \frac{1}{n}\sum_{t=1}^{n} P_A[t] \leq \bar{P}_A \tag{3.110}$$

$$P_B = \frac{1}{n}\sum_{t=1}^{n} P_B[t] \leq \bar{P}_B. \tag{3.111}$$

Then, we can bound

$$\frac{1}{n}\left[h(Y^n|S^n) - h(Z_y^n)\right] \leq \log_2\left(1 + |f_2|^2\bar{P}_B + \frac{|f_1|^2\bar{P}_A}{1+|g_1|^2\bar{P}_A}\right). \tag{3.112}$$

Proof. We start the proof by bounding $h(Y^n|S^n) - h(Z_y^n)$ as follows

$$\begin{aligned} h(Y^n|S^n) - h(Z_y^n) &\overset{(a)}{=} \sum_{t=1}^{n} h(Y[t]|S^n, Y^{t-1}) - \sum_{t=1}^{n} h(Z_y[t]|Z_y^{t-1}) \\ &\overset{(b)}{\leq} \sum_{t=1}^{n} h(Y[t]|S[t]) - \sum_{t=1}^{n} h(Z_y[t]), \end{aligned} \tag{3.113}$$

where in step (a), we used the chain rule. Step (b), follows since conditioning does not increase the entropy and Z_y is i.i.d.. Since the Gaussian input maximizes the conditional differential entropy given a covariance constraint [Tho87], we can write

$$\begin{aligned} h(Y^n|S^n) - h(Z_y^n) &= \sum_{t=1}^{n} h(Y_G[t]|S_G[t]) - \sum_{t=1}^{n} h(Z_y[t]) \\ &= \sum_{t=1}^{n} \Delta[t], \end{aligned} \tag{3.114}$$

where the subscript G indicates that the inputs are i.i.d. and Gaussian distributed, i.e., $A[t] \sim \mathbb{CN}(0, P_A[t])$ and $B[t] \sim \mathbb{CN}(0, P_B[t])$. Additionally, $\Delta[t] = h(Y_G[t]|S_G[t]) - h(Z_y[t])$. Now, we proceed by computing $\Delta[t]$ as follows

$$\begin{aligned}\Delta[t] &= h(Y_G[t], S_G[t]) - h(S_G[t]) - h(Z_y[t]) \\ &= \log_2\left(\frac{\begin{vmatrix}|f_1|^2P_A[t] + |f_2|^2P_B[t] + 1 & f_1g_1^*P_A[t] \\ g_1f_1^*P_A[t] & |g_1|^2P_A[t] + 1\end{vmatrix}}{1 + |g_1|^2P_A[t]}\right) \\ &= \log_2\left(1 + |f_1|^2P_A[t] + |f_2|^2P_B[t] - \frac{|f_1|^2|g_1|^2P_A[t]^2}{1 + |g_1|^2P_A[t]}\right) \\ &= \log_2\left(1 + |f_2|^2P_B[t] + \frac{|f_1|^2P_A[t]}{1 + |g_1|^2P_A[t]}\right). \end{aligned} \tag{3.115}$$

Due to the fact that the expression in (3.115) is concave, we can write that

$$\frac{1}{n}\sum_{t=1}^{n}\Delta[t] \leq \log_2\left(1 + |f_2|^2P_B + \frac{|f_1|^2P_A}{1 + |g_1|^2P_A}\right). \tag{3.116}$$

Now, since the function (3.116) is an increasing function with respect to P_A and P_B and $P_A \leq \bar{P}_A$, $P_B \leq \bar{P}_B$, we have

$$\frac{1}{n}\sum_{t=1}^{n}\Delta[t] \leq \log_2\left(1 + |f_2|^2\bar{P}_B + \frac{|f_1|^2\bar{P}_A}{1 + |g_1|^2\bar{P}_A}\right). \tag{3.117}$$

This completes the proof of the lemma. □

In the following lemma, we bound the difference between the entropies of two (noisy) linearly independent linear combinations of two random variables under some conditions.

Lemma 11. *Let A and B be independent random variables satisfying*

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}[|A[t]|^2] \leq P, \quad \frac{1}{n}\sum_{t=1}^{n}\mathbb{E}[|B[t]|^2] \leq P,$$

and let Z_i, $i \in \{A, B\}$, be distributed as $\mathbb{CN}(0, 1)$. Define Y_A and Y_B as the outputs of the following noisy channels,

$$\begin{aligned} Y_A &= h_1A + h_2B + Z_A \\ Y_B &= h_1A + h_3B + Z_B, \end{aligned}$$

where the constants h_1, h_2, and h_3 are complex-valued. If the conditions

$$P|h_2|^2 \leq \left(\frac{|h_3|}{|h_1|}\right)^2, \tag{3.118}$$

$$1 < P|h_1|^2. \tag{3.119}$$

are satisfied, then the difference between the entropies of Y_A^n *and* Y_B^n *satisfies*

$$h(Y_A^n) - h(Y_B^n) \leq n. \tag{3.120}$$

Proof. We start by upper bounding the expression $h(Y_A^n) - h(Y_B^n)$ as follows

$$\begin{aligned}
&h(Y_A^n) - h(Y_B^n) \\
&= h(Y_A^n) - h(Y_B^n) - h(Z_A^n) + h(Z_B^n) \\
&= I(A^n, B^n; Y_A^n) - I(A^n, B^n; Y_B^n) \\
&\overset{(a)}{=} I(A^n; Y_A^n) + I(B^n; Y_A^n|A^n) - I(B^n; Y_B^n) - I(A^n; Y_B^n|B^n) \\
&\overset{(b)}{\leq} I(A^n; Y_A^n) + I(B^n; Y_A^n|A^n) - I(B^n; Y_B^n) - I(A^n; Y_B^n|B^n) + I(A^n; B^n|Y_A^n) \\
&\overset{(c)}{=} I(A^n; Y_A^n, B^n) + I(B^n; Y_A^n|A^n) - I(B^n; Y_B^n) - I(A^n; Y_B^n|B^n),
\end{aligned} \tag{3.121}$$

where in (a) and (c), we used the chain rule and in (b), we used the non-negativity of mutual information. We proceed by using the chain rule and the independence of A and B, to get

$$\begin{aligned}
&h(Y_A^n) - h(Y_B^n) \\
&\leq I(A^n; Y_A^n|B^n) + I(B^n; Y_A^n|A^n) - I(B^n; Y_B^n) - I(A^n; Y_B^n|B^n) \\
&= I(A^n; h_1A^n + Z_A^n) + I(B^n; h_2B^n + Z_A^n|A^n) - I(B^n; Y_B^n) - I(A^n; h_1A^n + Z_B^n) \\
&\overset{(a)}{=} I(B^n; h_2B^n + Z_A^n) - I(B^n; h_1A^n + h_3B^n + Z_B^n),
\end{aligned} \tag{3.122}$$

where (a) follows since $I(A^n; h_1A^n + Z_A^n) = I(A^n; h_1A^n + Z_B^n)$ since Z_A and Z_B have the same distribution. By defining the random variable $\tilde{A} = \frac{A}{\sqrt{P}}$ which satisfies $\frac{1}{n}\sum_{t=1}^n \mathbb{E}[|\tilde{A}[t]|^2] \leq 1$, we obtain

$$\begin{aligned}
&h(Y_A^n) - h(Y_B^n) \\
&\leq I(B^n; h_2B^n + Z_A^n) - I(B^n; \tilde{A}^n + \frac{h_3}{\sqrt{P}h_1}B^n + \frac{1}{\sqrt{P}h_1}Z_B^n), \\
&\overset{(a)}{\leq} I(B^n; h_2B^n + Z_A^n) - I(B^n; \tilde{A}^n + \frac{h_3}{\sqrt{P}h_1}B^n + Z_B^n) \\
&= I(B^n; h_2B^n + Z_A^n) - h(\tilde{A}^n + \frac{h_3}{\sqrt{P}h_1}B^n + Z_B^n) + h(\tilde{A}^n + Z_B^n),
\end{aligned} \tag{3.123}$$

where step (a) is due to the fact that increasing the noise variance (by $1 - \frac{1}{P|h_1|^2} > 0$, cf. (3.119)) leads to a degraded channel, and hence, decreases the mutual information. Now, since conditioning does not increase the differential entropy, we bound

(3.123) as follows

$$\begin{aligned}
&h(Y_A^n) - h(Y_B^n) \\
&\leq I(B^n; h_2 B^n + Z_A^n) - h(\tilde{A}^n + \frac{h_3}{\sqrt{P}h_1} B^n + Z_B^n | \tilde{A}^n) + h(\tilde{A}^n + Z_B^n) \\
&= I(B^n; h_2 B^n + Z_A^n) - h(\frac{h_3}{\sqrt{P}h_1} B^n + Z_B^n) + h(\tilde{A}^n + Z_B^n) - h(Z_A^n) + h(Z_B^n) \\
&= I(B^n; h_2 B^n + Z_A^n) - I(B^n; \frac{h_3}{\sqrt{P}h_1} B^n + Z_B^n) + h(\tilde{A}^n + Z_B^n) - h(Z_A^n) \\
&= I(B^n; B^n + \frac{1}{h_2} Z_A^n) - I(B^n; B^n + \frac{\sqrt{P}h_1}{h_3} Z_B^n) + h(\tilde{A}^n + Z_B^n) - h(Z_A^n). \quad (3.124)
\end{aligned}$$

Now by increasing the noise variance in the second mutual information term in (3.124) (by $\frac{1}{|h_2|^2} - \frac{P|h_1|^2}{|h_3|^2} \geq 0$, cf. (3.118)), we obtain a degraded channel, and hence this mutual information term decreases. This leads to the following upper bound

$$\begin{aligned}
&h(Y_A^n) - h(Y_B^n) \\
&\leq I(B^n; B^n + \frac{1}{h_2} Z_A^n) - I(B^n; B^n + \frac{1}{h_2} Z_B^n) + h(\tilde{A}^n + Z_B^n) - h(Z_A^n) \\
&= h(\tilde{A}^n + Z_B^n) - h(Z_A^n) \\
&= I(\tilde{A}^n; \tilde{A}^n + Z_B^n) + h(Z_B^n) - h(Z_A^n) \\
&\stackrel{(b)}{\leq} n, \quad (3.125)
\end{aligned}$$

where step (b) follows since the capacity of the Gaussian channel with input $\tilde{A}$ and output $\tilde{A} + Z_B$ is upper bounded by 1 (since $\tilde{A}$ has power 1). □

3.3.2.1 Sum-capacity Upper Bounds

In the following lemma, we establish an upper bound on the sum-capacity of the Gaussian PIMAC. The main idea of this bound is to reduce the PIMAC to an IC and then to bound the sum-capacity of the IC. To do this, we provide the interference caused by Tx3 to Rx2. This cannot decrease the sum-capacity. The sum-capacity of the obtained channel will be further bound as in [ETW08].

Lemma 12. *The sum-capacity of the Gaussian PIMAC is upper bounded by*

$$C_{\mathrm{G},\Sigma} \leq \log_2\left(1 + \rho^{\alpha_{c2}} + \rho^{\alpha_{d3}} + \frac{\rho^{\alpha_{d1}}}{1+\rho^{\alpha_{c1}}}\right) + \log_2\left(1 + \rho^{\alpha_{c1}} + \frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c2}}}\right). \quad (3.126)$$

Proof. In order to establish this upper bound, $S_1^n = h_{c1} X_1^n + Z_2^n$ is given to Rx1 as side information, and $S_2^n = h_{c2} X_2^n + Z_1^n$ and X_3^n are given to Rx2 as side information. Then, by Fano's inequality, we have

$$n(R_\Sigma - \epsilon_n) \leq I(X_1^n, X_3^n; Y_1^n, S_1^n) + I(X_2^n; Y_2^n, S_2^n, X_3^n),$$

where $\epsilon_n \to 0$ as $n \to \infty$. Then, we proceed by using the chain rule to write

$$\begin{aligned} n(R_\Sigma - \epsilon_n) \leq & I(X_1^n, X_3^n; S_1^n) + I(X_1^n, X_3^n; Y_1^n | S_1^n) \\ & + I(X_2^n; X_3^n) + I(X_2^n; S_2^n | X_3^n) + I(X_2^n; Y_2^n | S_2^n, X_3^n). \end{aligned} \tag{3.127}$$

Since X_2 and X_3 are independent, $I(X_2^n; X_3^n) = 0$ and we obtain

$$\begin{aligned} n(R_\Sigma - \epsilon_n) \leq & h(S_1^n) - h(S_1^n | X_1^n, X_3^n) + h(Y_1^n | S_1^n) - h(Y_1^n | X_1^n, X_3^n, S_1^n) \\ & + h(S_2^n | X_3^n) - h(S_2^n | X_3^n, X_2^n) + h(Y_2^n | S_2^n, X_3^n) - h(Y_2^n | S_2^n, X_2^n, X_3^n) \\ \overset{(a)}{=} & h(S_1^n) - h(Z_2^n) + h(Y_1^n | S_1^n) - h(S_2^n) + h(S_2^n) - h(Z_1^n) \\ & + h(h_{d2} X_2^n + h_{c1} X_1^n + Z_2^n | S_2^n) - h(S_1^n) \\ = & h(Y_1^n | S_1^n) + h(h_{d2} X_2^n + h_{c1} X_1^n + Z_2^n | S_2^n) - h(Z_1^n) - h(Z_2^n), \end{aligned} \tag{3.128}$$

where step (a) follows since the random variables X_1, X_2, X_3, Z_1, and Z_2 are independent from each other. Now, by dividing (3.128) by n and letting $n \to \infty$ and applying Lemma 10, we can bound (3.128) as follows

$$\begin{aligned} R_\Sigma \leq & \log_2 \left(1 + |h_{c2}|^2 P + |h_{d3}|^2 P + \frac{|h_{d1}|^2 P}{1 + |h_{c1}|^2 P} \right) \\ & + \log_2 \left(1 + P |h_{c1}|^2 + \frac{P |h_{d2}|^2}{1 + P |h_{c2}|^2} \right) \\ = & \log_2 \left(1 + \rho^{\alpha_{c2}} + \rho^{\alpha_{d3}} + \frac{\rho^{\alpha_{d1}}}{1 + \rho^{\alpha_{c1}}} \right) + \log_2 \left(1 + \rho^{\alpha_{c1}} + \frac{\rho^{\alpha_{d2}}}{1 + \rho^{\alpha_{c2}}} \right) \end{aligned} \tag{3.129}$$

which concludes the proof of (3.126). □

Similarly, we can establish another upper bound by providing the interference caused by Tx1 to Rx2. The obtained upper bound is given in the following lemma.

Lemma 13. *The sum-capacity of the Gaussian PIMAC is upper bounded by*

$$C_{\mathrm{G},\Sigma} \leq \log_2 \left(1 + \rho^{\alpha_{c2}} + \rho^{\alpha_{d1}} + \frac{\rho^{\alpha_{d3}}}{1 + \rho^{\alpha_{c3}}} \right) + \log_2 \left(1 + \rho^{\alpha_{c3}} + \frac{\rho^{\alpha_{d2}}}{1 + \rho^{\alpha_{c2}}} \right) \tag{3.130}$$

Proof. For establishing this upper bound, we provide $S_1^n = h_{c3} X_3^n + Z_2^n$ to Rx1 and $S_2^n = h_{c2} X_2^n + Z_1^n$, X_1^n to Rx2. Then, by proceeding with similar steps as we have already mentioned in the proof of Lemma 12, we obtain the upper bound in (3.130). □

We still need to introduce two upper bounds. In compared to the bounds in Lemmata 12 and 13, in these bounds Rx2 observes both interference signals from Tx1, Tx3. Moreover, the side information provided to Rx1 is a noisy superposition of X_1 and X_3. Additionally, we use Lemma 11 to make these bounds tighter. In the following lemma, the first bound is presented.

Lemma 14. *The sum-capacity of the Gaussian PIMAC is upper bounded by*

$$\begin{aligned} C_{\mathrm{G},\Sigma} \leq & \log_2\left(1+\rho^{\alpha_{c2}}+\frac{\rho^{\alpha_{d1}}+\rho^{\alpha_{d3}}}{1+\rho^{\alpha_{c1}-\alpha_{d1}}(\rho^{\alpha_{d1}}+\rho^{\alpha_{d3}})}\right) \\ & +\log_2\left(1+\rho^{\alpha_{c3}}+\rho^{\alpha_{c1}}+\frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c2}}}\right)+1, \\ & \textit{if}\quad \alpha_{d3}-\alpha_{d1}\leq \alpha_{c3}-2\alpha_{c1}, \end{aligned} \tag{3.131}$$

Proof. In order to establish the upper bound (3.131), the side information $S_1^n = \frac{h_{c1}}{h_{d1}}(h_{d1}X_1^n + h_{d3}X_3^n) + Z^n$ is given to Rx1 and the side information $S_2^n = h_{c2}X_2^n + Z_1^n$ is given to Rx2, where $Z^n \sim \mathbb{CN}(0,1)$ denotes an AWGN which is independent from Z_1^n and Z_2^n and i.i.d. over time. Using Fano's inequality, we obtain

$$n(R_\Sigma - \epsilon_n) \leq I(X_1^n, X_3^n; Y_1^n, S_1^n) + I(X_2^n; Y_2^n, S_2^n), \tag{3.132}$$

where $\epsilon_n \to \infty$ as $n \to \infty$. Then, using the chain rule, we have

$$\begin{aligned} n(R_\Sigma - \epsilon_n) \leq & I(X_1^n, X_3^n; S_1^n) + I(X_1^n, X_3^n; Y_1^n|S_1^n) + I(X_2^n; S_2^n) + I(X_2^n; Y_2^n|S_2^n) \\ = & h(S_1^n) - h(S_1^n|X_1^n, X_3^n) + h(Y_1^n|S_1^n) - h(Y_1^n|S_1^n, X_1^n, X_3^n) \\ & + h(S_2^n) - h(S_2^n|X_2^n) + h(Y_2^n|S_2^n) - h(Y_2^n|S_2^n, X_2^n) \\ \overset{(a)}{=} & h(S_1^n) - h(Z^n) + h(Y_1^n|S_1^n) - h(S_2^n) + h(S_2^n) - h(Z_1^n) \\ & + h(Y_2^n|S_2^n) - h(Y_2^n|S_2^n, X_2^n) \\ \overset{(b)}{=} & h(h_{c1}X_1^n + \frac{h_{c1}}{h_{d1}}h_{d3}X_3^n + Z^n) - h(Z^n) + h(Y_1^n|S_1^n) - h(Z_1^n) \\ & + h(Y_2^n|S_2^n) - h(h_{c1}X_1^n + h_{c3}X_3^n + Z_2^n) \\ \overset{(c)}{\leq} & h(Y_1^n|S_1^n) - h(Z^n) + h(Y_2^n|S_2^n) - h(Z_1^n) + n, \end{aligned} \tag{3.133}$$

where (a) and (b) follow from the fact that the transmitted signals from different Tx's and the additive noise signals are all independent from each other, and (c) follows from Lemma 11. Note that the first condition of Lemma 11 is satisfied given the condition of bound (3.131) which is equivalent to $P\left(\frac{|h_{c1}|}{|h_{d1}|}\right)^2|h_{d3}|^2 \leq \left(\frac{|h_{c3}|}{|h_{c1}|}\right)^2$. This condition corresponds to $n_{d3} - n_{c3} < n_{d1} - 2n_{c1}$ in the linear deterministic model which also defines a border of regime 3. The second condition of Lemma 11 $(1 < P|h_{c1}|^2)$ holds, since we consider the interference limited scenario (3.9).

In order to bound (3.133), we define first two new variables $V[t] = h_{d1}X_1[t] + h_{d3}X_3[t]$ and $U[t] = h_{c3}X_3[t] + h_{c1}X_1[t]$ to write

$$\begin{aligned} n(R_\Sigma - \epsilon_n) \leq & h(V^n + h_{c2}X_2^n + Z_1^n|\frac{h_{c1}}{h_{d1}}V^n + Z^n) - h(Z^n) \\ & + h(h_{d2}X_2^n + U^n + Z_2|h_{c2}X_2^n + Z_1^n) - h(Z_1^n) + n. \end{aligned} \tag{3.134}$$

Now, by using Lemma 10, and considering the fact that

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}\left[|V[t]|^2\right] \leq (|h_{d1}|^2 + |h_{d3}|^2)P \tag{3.135}$$

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}\left[|U[t]|^2\right] \leq (|h_{c3}|^2 + |h_{c1}|^2)P, \tag{3.136}$$

we bound (3.134) as follows

$$\begin{aligned} n(R_\Sigma - \epsilon_n) \leq & n \left[\log_2\left(1 + |h_{c2}|^2 P + \frac{(|h_{d1}|^2 + |h_{d3}|^2)P}{1 + \frac{|h_{c1}|^2}{|h_{d1}|^2}(|h_{d1}|^2 + |h_{d3}|^2)P}\right)\right. \\ & \left. + \log_2\left(1 + |h_{c3}|^2 P + |h_{c1}|^2 P + \frac{|h_{d2}|^2 P}{1 + |h_{c2}|^2 P}\right) + 1\right] \end{aligned} \tag{3.137}$$

Now, by dividing (3.137) by n, letting $n \to \infty$, and using Definition (1), we obtain

$$\begin{aligned} R_\Sigma \leq & \log_2\left(1 + \rho^{\alpha_{c2}} + \frac{\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}}}{1 + \rho^{\alpha_{c1} - \alpha_{d1}}(\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}})}\right) \\ & + \log_2\left(1 + \rho^{\alpha_{c3}} + \rho^{\alpha_{c1}} + \frac{\rho^{\alpha_{d2}}}{1 + \rho^{\alpha_{c2}}}\right) + 1, \end{aligned} \tag{3.138}$$

which completes the proof. □

Similarly, we establish the last upper bound in following lemma.

Lemma 15. *The sum-capacity of the Gaussian PIMAC is upper bounded by*

$$\begin{aligned} C_{\mathrm{G},\Sigma} \leq & \log_2\left(1 + \rho^{\alpha_{c2}} + \frac{\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}}}{1 + \rho^{\alpha_{c3} - \alpha_{d3}}(\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}})}\right) \\ & + \log_2\left(1 + \rho^{\alpha_{c3}} + \rho^{\alpha_{c1}} + \frac{\rho^{\alpha_{d2}}}{1 + \rho^{\alpha_{c2}}}\right) + 1, \\ & \textit{if} \quad \alpha_{d1} - \alpha_{d3} \leq \alpha_{c1} - 2\alpha_{c3}. \end{aligned} \tag{3.139}$$

Proof. For establishing the bound (3.139), the side information $S_1^n = \frac{h_{c3}}{h_{d3}}(h_{d1}X_1^n + h_{d3}X_3^n) + Z^n$ is given to Rx1, and the side information $S_2^n = h_{c2}X_2^n + Z_1^n$ is given to Rx2, where $Z \sim \mathbb{C}\mathcal{N}(0,1)$ denotes an AWGN which is independent from all other random variables and i.i.d. over time. Then by proceeding with similar steps as we have already mentioned in the proof of Lemma 14, we obtain the bound given in (3.139). □

3.3.2.2 GDoF Upper Bounds

Here, we want to translate the sum-capacity upper bounds of the Gaussian PIMAC presented in Subsection 3.3.2.1 into the GDoF upper bounds. To do this, we use the definition of the GDoF given in Definition 1. The GDoF upper bounds are presented in the following lemma.

Lemma 16. *The GDoF of the Gaussian PIMAC is upper bounded by*

$$d_\Sigma \leq \max\{\alpha_{c2}, \alpha_{d3}, \alpha_{d1} - \alpha_{c1}\} + \max\{\alpha_{c1}, \alpha_{d2} - \alpha_{c2}\} \tag{3.140}$$
$$d_\Sigma \leq \max\{\alpha_{c2}, \alpha_{d1}, \alpha_{d3} - \alpha_{c3}\} + \max\{\alpha_{c3}, \alpha_{d2} - \alpha_{c2}\} \tag{3.141}$$
$$d_\Sigma \leq \alpha_{d1} - \alpha_{c1} + \max\{\alpha_{c3}, \alpha_{d2} - \alpha_{c2}\} \text{ if } \alpha_{d3} - \alpha_{d1} \leq \alpha_{c3} - 2\alpha_{c1} \tag{3.142}$$
$$d_\Sigma \leq \alpha_{d3} - \alpha_{c3} + \max\{\alpha_{c3}, \alpha_{d2} - \alpha_{c2}\} \text{ if } \alpha_{d1} - \alpha_{d3} \leq \alpha_{c1} - 2\alpha_{c3} \tag{3.143}$$

Proof. We start with proving the upper bound in (3.140). To do this, consider (3.126). By dividing this expression by ρ and letting $\rho \to \infty$, and keeping in mind that the α-parameters are non-negative (due to the condition in (3.9)), we obtain

$$d_\Sigma \leq \lim_{\rho\to\infty} \frac{1}{\log_2 \rho} \left[\log_2 \left(\rho^{\alpha_{c2}} + \rho^{\alpha_{d3}} + \frac{\rho^{\alpha_{d1}}}{\rho^{\alpha_{c1}}} \right) + \log_2 \left(\rho^{\alpha_{c1}} + \frac{\rho^{\alpha_{d2}}}{\rho^{\alpha_{c2}}} \right) \right] \tag{3.144}$$
$$= \max\{\alpha_{c2}, \alpha_{d3}, \alpha_{d1} - \alpha_{c1}\} + \max\{\alpha_{c1}, \alpha_{d2} - \alpha_{c2}\}. \tag{3.145}$$

Similarly, we can establish the upper bounds in (3.141), (3.142), and (3.143) from (3.130), (3.131), and (3.139), respectively. □

3.3.3 Optimality and Suboptimality of TIN

Now, we are ready to evaluate the optimality of different variants of TIN in the Gaussian PIMAC. By dividing the parameter space of the Gaussian PIMAC into different regimes as done in the LD-PIMAC (Definition 4), the optimality and suboptimality of TIN are presented in the following subsections.

3.3.3.1 Optimality of TIN

First, we present the GDoF optimality of different types of TIN in the Gaussian PIMAC. To this end, we need to compare the achievable GDoF of naive-TIN (in (3.104)) and TDMA-TIN (in (3.105)) with the upper bounds on the GDoF of the Gaussian PIMAC in (3.140)-(3.143). In regimes, where the achievable GDoF of TIN coincides with a GDoF upper bound, TIN performs GDoF optimally.

Notice that by replacing n_k in the capacity bounds of the LD-PIMAC by α_k for $k \in \{d1, c1, d2, c2, d3, c3\}$, we obtain the valid bounds for the Gaussian PIMAC (except for the special case $\alpha_{d1} - \alpha_{c1} = \alpha_{d3} - \alpha_{c3}$). This can be verified by comparing the achievable GDoF of naive-TIN and TDMA-TIN in the Gaussian PIMAC given in (3.104) and (3.105) with the achievable capacity of these schemes in the LD-PIMAC (given in (3.16) and (3.17)). Moreover, by replacing the n-parameters in the upper bounds on the capacity of the LD-PIMAC (given in (3.29), (3.36), (3.46), and (3.47)) with the α-parameters, we obtain the GDoF upper bounds in (3.140)-(3.142). Hence, we conclude that in sub-regimes 1A and 2A, in which naive-TIN performs optimally from the capacity point of view in the LD-PIMAC (cf. Corollary 1), it performs also GDoF optimally in the Gaussian counterpart. Moreover, since TDMA-TIN is capacity optimal in regimes 1 and 2 for the LD-PIMAC, it performs also GDoF optimally in the Gaussian PIMAC. Therefore, TIN performs optimally in regimes 1 and 2. The GDoF of the Gaussian PIMAC in regime 1 and 2 is summarized in the following theorem.

Theorem 3. *In regimes 1 and 2, the GDoF of the Gaussian PIMAC is given by*

$$d_\Sigma = \begin{cases} \alpha_{d1} - \alpha_{c1} + \alpha_{d2} - \alpha_{c2}, & \textit{regime 1} \\ \alpha_{d3} - \alpha_{c3} + \alpha_{d2} - \alpha_{c2}, & \textit{sub-regimes 2A, 2B, and 2C} \\ \alpha_{d3}, & \textit{sub-regime 2D.} \end{cases} \tag{3.146}$$

Proof. The TDMA-TIN scheme achieves the GDoF upper bounds of the Gaussian PIMAC in regimes 1 and 2. By taking the conditions of regimes 1 and 2 into account, we can verify that the achievable GDoF of TDMA-TIN in (3.105) reduces to (3.146). This completes the proof if this theorem. □

The next step after the GDoF characterization of a network is the capacity analysis. A scheme achieves the capacity of a network if there is an upper bound on the sum-capacity which coincides with the achievable sum-rate of that scheme. If the gap between the upper bound and the achievable sum-rate is bounded by a constant which does not scale with the SNR, we know the capacity of the network within a constant gap. A scheme might be suboptimal although it achieves the sum-capacity within a constant gap. For instance, while naive-TIN is always outperformed by TDMA-TIN, it achieves the capacity of the Gaussian PIMAC in sub-regimes 1A and 2A within a constant gap. This result is presented in details in the following corollary.

Corollary 2. *The achievable sum-rate of naive-TIN is within a gap of* $3 + 2\log_2 3$ *bits of the sum-capacity of the Gaussian PIMAC in sub-regimes 1A and 2A.*

Proof. Here, we focus on sub-regimes 1A and 2B. In sub-regime 1A where $\alpha_{d3} \leq \alpha_{d1} - \alpha_{c1}$ and $\alpha_{c3} \leq \alpha_{c1}$, the upper bound given in (3.126) can be further upper bounded as follows

$$\begin{aligned} C_{\mathrm{G},\Sigma} &\leq \log_2\left(1 + \rho^{\alpha_{c2}} + \rho^{\alpha_{d3}} + \frac{\rho^{\alpha_{d1}}}{1+\rho^{\alpha_{c1}}}\right) + \log_2\left(1 + \rho^{\alpha_{c1}} + \frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c2}}}\right) \\ &< \log_2\left(1 + \rho^{\alpha_{c2}} + \rho^{\alpha_{d3}} + \frac{\rho^{\alpha_{d1}}}{\rho^{\alpha_{c1}}}\right) + \log_2\left(1 + \rho^{\alpha_{c1}} + \frac{\rho^{\alpha_{d2}}}{\rho^{\alpha_{c2}}}\right) \\ &< \log_2\left(4\rho^{\alpha_{d1}-\alpha_{c1}}\right) + \log_2\left(3\rho^{\alpha_{d2}-\alpha_{c2}}\right) \\ &= [\alpha_{d1} - \alpha_{c1} + \alpha_{d2} - \alpha_{c2}]\log_2\rho + 2 + \log_2 3, \end{aligned} \tag{3.147}$$

where we used the fact that in sub-regime 1A, $\max\{0, \alpha_{c2}, \alpha_{d3}, \alpha_{d1} - \alpha_{c1}\} = \alpha_{d1} - \alpha_{c1}$, $\max\{0, \alpha_{c1}, \alpha_{d2} - \alpha_{c2}\} = \alpha_{d2} - \alpha_{c2}$, and $\rho^{\alpha_{d1}-\alpha_{c1}}, \rho^{\alpha_{d2}-\alpha_{c2}} > 1$ due to (3.9).

On the other hand, for the achievable rate of naive-TIN, we have

$$\begin{aligned} R_{\Sigma,\mathrm{Naive-TIN}} &= \log_2\left(1 + \frac{\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}}}{1+\rho^{\alpha_{c2}}}\right) + \log_2\left(1 + \frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c1}}+\rho^{\alpha_{c3}}}\right) \\ &> \log_2\left(\frac{\rho^{\alpha_{d1}}}{2\rho^{\alpha_{c2}}}\right) + \log_2\left(\frac{\rho^{\alpha_{d2}}}{3\rho^{\alpha_{c1}}}\right) \\ &= [\alpha_{d1} - \alpha_{c1} + \alpha_{d2} - \alpha_{c2}]\log_2\rho - 1 - \log_2 3, \end{aligned} \tag{3.148}$$

where we used $\rho^{\alpha_{c1}}, \rho^{\alpha_{c2}} > 1$ (cf. (3.9)).

Comparing (3.147) with (3.148) in this regime, we see that naive-TIN is within a gap of $G_{\text{Naive-TIN,1A}} = 3 + 2\log_2 3$ bits to the sum-capacity.

Similarly, for sub-regime 2A where $\alpha_{d3} - \alpha_{c3} \geq \alpha_{d1}$ and $\alpha_{c1} \leq \alpha_{c3} \leq \alpha_{d2} - \alpha_{c2}$, the upper bound (3.130) can be upper bounded as

$$\begin{aligned} C_{\text{G},\Sigma} &\leq \log_2\left(1 + \rho^{\alpha_{c2}} + \rho^{\alpha_{d1}} + \frac{\rho^{\alpha_{d3}}}{1+\rho^{\alpha_{c3}}}\right) + \log_2\left(1 + \rho^{\alpha_{c3}} + \frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c2}}}\right) \\ &< [\alpha_{d3} - \alpha_{c3} + \alpha_{d2} - \alpha_{c2}]\log_2\rho + 2 + \log_2 3, \end{aligned} \tag{3.149}$$

which follows since $\alpha_{d3} - \alpha_{c3} \geq \alpha_{d1}$ and $\alpha_{c3} \leq \alpha_{d2} - \alpha_{c2}$ in this regime, whereas for the achievable sum-rate of the naive-TIN scheme in this regime, we have

$$\begin{aligned} R_{\Sigma,\text{Naive-TIN}} &= \log_2\left(1 + \frac{\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}}}{1+\rho^{\alpha_{c2}}}\right) + \log_2\left(1 + \frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c1}} + \rho^{\alpha_{c3}}}\right). \\ &> [\alpha_{d3} - \alpha_{c3} + \alpha_{d2} - \alpha_{c2}]\log_2\rho - 1 - \log_2 3, \end{aligned}$$

which follows since $\alpha_{c1} < \alpha_{c3}$ in this regime. Therefore, naive-TIN is within a constant gap $G_{\text{Naive-TIN,2A}} = 3 + 2\log_2 3$ bits to the sum-capacity. □

Now, consider TDMA-TIN. This scheme can achieve the capacity of the Gaussian PIMAC within a constant gap in regimes 1 and 2. This is presented in more details in the following corollary.

Corollary 3. *The gap between the achievable sum-rate of TDMA-TIN and the sum-capacity of the Gaussian PIMAC is bounded by $4+\log_2 3$ bits in sub-regimes 1A, 1B, 2A and 2B, and by 7 bits in sub-regimes 1C and 2C and $2+\log_2 3$ bits in sub-regime 2D.*

Proof. First consider sub-regimes 1A, 1B, and 1C. In these sub-regimes, by setting $\tau_2 = 1$ and $\tau_1 = \tau_3 = 0$, the achievable rate of TDMA-TIN in (3.103) satisfies

$$\begin{aligned} R_{\Sigma,\text{TDMA-TIN}} &\geq \log_2\left(1 + \frac{\rho^{\alpha_{d1}}}{1+\rho^{\alpha_{c2}}}\right) + \log_2\left(1 + \frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c1}}}\right) \\ &> \log_2\left(\frac{\rho^{\alpha_{d1}}}{2\rho^{\alpha_{c2}}}\right) + \log_2\left(\frac{\rho^{\alpha_{d2}}}{2\rho^{\alpha_{c1}}}\right) \\ &= [(\alpha_{d1} - \alpha_{c2}) + (\alpha_{d2} - \alpha_{c1})]\log_2(\rho) - 2. \end{aligned} \tag{3.150}$$

Similar to sub-regime 1A (see (3.147)), it can be shown that the upper bound for the capacity in sub-regime 1B is upper bounded by the expression in (3.147). By comparing (3.147) with (3.150), we see that TDMA-TIN is within a constant gap of $G_{\text{TDMA-TIN,1A,1B}} = 4 + \log_2 3$ bits to the sum-capacity in sub-regimes 1A and 1B.

In sub-regime 1C, we relax the upper bound in (3.131) as follows

$$\begin{aligned}
C_{\mathrm{G},\Sigma} \leq& \log_2\left(1+\rho^{\alpha_{c2}}+\frac{\rho^{\alpha_{d1}}+\rho^{\alpha_{d3}}}{1+\rho^{\alpha_{c1}-\alpha_{d1}}(\rho^{\alpha_{d1}}+\rho^{\alpha_{d3}})}\right)\\
&+\log_2\left(1+\rho^{\alpha_{c3}}+\rho^{\alpha_{c1}}+\frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c2}}}\right)+1\\
<& \log_2\left(2\rho^{\alpha_{c2}}+\frac{2\rho^{\max\{\alpha_{d1},\alpha_{d3}\}}}{\rho^{\max\{\alpha_{c1},\alpha_{c1}-\alpha_{d1}+\alpha_{d3}\}}}\right)+\log_2\left(3\rho^{\alpha_{c3}}+\frac{\rho^{\alpha_{d2}}}{\rho^{\alpha_{c2}}}\right)+1\\
<& \log_2\left(4\rho^{\alpha_{d1}-\alpha_{c1}}\right)+\log_2\left(4\rho^{\alpha_{d2}-\alpha_{c2}}\right)+1\\
=&(\alpha_{d1}-\alpha_{c1}+\alpha_{d2}-\alpha_{c2})\log_2\rho+5, \qquad (3.151)
\end{aligned}$$

where we used the fact that in sub-regime 1C, $\alpha_{c3} > \alpha_{c1}$ and $\max\{0, \alpha_{c1}, \alpha_{c3}, \alpha_{d2} - \alpha_{c2}\} = \alpha_{d2} - \alpha_{c2}$. Comparing (3.151) and (3.150), we conclude that TDMA-TIN is within a constant gap of $G_{\mathrm{TDMA-TIN,1C}} = 7$ bits to the sum-capacity in sub-regime 1C.

For sub-regimes 2A, 2B, and 2C, by setting $\tau_3 = 1$ and $\tau_1 = \tau_2 = 0$, the achievable sum-rate of TDMA-TIN in (3.103) satisfies

$$\begin{aligned}
R_{\Sigma,\mathrm{TDMA-TIN}} >& \log_2\left(1+\frac{\rho^{\alpha_{d3}}}{1+\rho^{\alpha_{c2}}}\right)+\log_2\left(1+\frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c3}}}\right)\\
>& (\alpha_{d3}-\alpha_{c2}+\alpha_{d2}-\alpha_{c3})\log_2\rho-2. \qquad (3.152)
\end{aligned}$$

Similar to sub-regime 2A (see (3.149)), the upper bound for the capacity can be relaxed in sub-regime 2B. Doing this, we can show that the capacity in sub-regime 2B is upper bounded by the expression in (3.149). Comparing (3.149) and (3.152), we see that TDMA-TIN achieves a sum-rate within a constant gap of $G_{\mathrm{TDMA-TIN,2A,2B}} = 4 + \log_2 3$ bits to the sum-capacity in sub-regimes 2A and 2B.

For sub-regime 2C, we relax the upper bound given in (3.139) as follows

$$\begin{aligned}
C_{\mathrm{G},\Sigma} \leq& \log_2\left(1+\rho^{\alpha_{c2}}+\frac{\rho^{\alpha_{d1}}+\rho^{\alpha_{d3}}}{1+\rho^{\alpha_{c3}-\alpha_{d3}}(\rho^{\alpha_{d1}}+\rho^{\alpha_{d3}})}\right)\\
&+\log_2\left(1+\rho^{\alpha_{c3}}+\rho^{\alpha_{c1}}+\frac{\rho^{\alpha_{d2}}}{1+\rho^{\alpha_{c2}}}\right)+1\\
<& \log_2\left(2\rho^{\alpha_{c2}}+\frac{2\rho^{\max\{\alpha_{d1},\alpha_{d3}\}}}{\rho^{\max\{\alpha_{c3}-\alpha_{d3}+\alpha_{d1},\alpha_{c3}\}}}\right)+\log_2\left(3\rho^{\alpha_{c1}}+\frac{\rho^{\alpha_{d2}}}{\rho^{\alpha_{c2}}}\right)+1\\
=& \log_2\left(2\rho^{\alpha_{c2}}+2\rho^{\alpha_{d3}-\alpha_{c3}}\right)+\log_2\left(3\rho^{\alpha_{c1}}+\rho^{\alpha_{d2}-\alpha_{c2}}\right)+1\\
<& \log_2\left(4\rho^{\alpha_{d3}-\alpha_{c3}}\right)+\log_2\left(4\rho^{\alpha_{d2}-\alpha_{c2}}\right)+1\\
=&(\alpha_{d3}-\alpha_{c3}+\alpha_{d2}-\alpha_{c2})\log_2\rho+5, \qquad (3.153)
\end{aligned}$$

where we used the facts that in sub-regime 2C, $\alpha_{c1} > \alpha_{c3}$ and $\max\{\alpha_{d1}-\alpha_{c1}, \alpha_{c2}, 0\} = \alpha_{d1} - \alpha_{c1} \leq \alpha_{d3} - \alpha_{c3}$. By comparing (3.152) and (3.153), we see that the rate obtained with TDMA-TIN is within a constant gap of $G_{\mathrm{TDMA-TIN,2C}} = 7$ bits to the sum capacity in sub-regime 2C.

Finally, we consider the sub-regime 2D. In this sub-regime, we set in (3.103) $\tau_1 = 1$ and $\tau_2 = \tau_3 = 0$ to obtain

$$\begin{aligned} R_{\Sigma,\text{TDMA-TIN}} &\geq \log_2 (1 + \rho^{\alpha_{d3}}) \\ &> \alpha_{d3} \log_2 \rho. \end{aligned} \tag{3.154}$$

Now, we relax the sum-capacity in (3.130) as follows

$$\begin{aligned} C_{\mathrm{G},\Sigma} &\leq \log_2 \left(1 + \rho^{\alpha_{c2}} + \rho^{\alpha_{d1}} + \frac{\rho^{\alpha_{d3}}}{1 + \rho^{\alpha_{c3}}}\right) + \log_2 \left(1 + \rho^{\alpha_{c3}} + \frac{\rho^{\alpha_{d2}}}{1 + \rho^{\alpha_{c2}}}\right) \\ &< \log_2 \left(3\rho^{\alpha_{d1}} + \frac{\rho^{\alpha_{d3}}}{\rho^{\alpha_{c3}}}\right) + \log_2 \left(2\rho^{\alpha_{c3}} + \frac{\rho^{\alpha_{d2}}}{\rho^{\alpha_{c2}}}\right) \\ &< \log_2 \left(3\rho^{\alpha_{d1}} + \rho^{\alpha_{d3}-\alpha_{c3}}\right) + \log_2 \left(2\rho^{\alpha_{c3}} + \rho^{\alpha_{d2}-\alpha_{c2}}\right) \\ &< \log_2 \left(4\rho^{\alpha_{d3}-\alpha_{c3}}\right) + \log_2 \left(3\rho^{\alpha_{c3}}\right) \\ &= \alpha_{d3} \log_2 \rho + 2 + \log_2 3 \end{aligned} \tag{3.155}$$

where we used the facts that in regime 2D, $\alpha_{d1} \leq \alpha_{d3} - \alpha_{c3}$ and $\alpha_{d2} - \alpha_{c2} < \alpha_{c3}$. By comparing (3.154) and (3.155), we see that the rate obtained by TDMA-TIN is within a constant gap of $G_{\mathrm{TDMA-TIN,2D}} = 2 + \log_2 3$ bits to the sum-capacity in sub-regime 2D. □

3.3.3.2 Suboptimality of TIN

Although TDMA-TIN always outperforms naive-TIN, it is suboptimal in regime 3. As discussed in Subsection 3.2.3.3, a combination of common and private signaling with interference alignment outperforms TDMA-TIN in regime 3 and hence, TDMA-TIN cannot achieve the capacity of the LD-PIMAC. In this section, we show that TDMA-TIN cannot achieve the capacity of the Gaussian PIMAC within a constant gap in regime 3. To do this, first we show that TDMA-TIN is suboptimal in terms of GDoF. This is shown by proposing the so-called IA-CP (interference alignment with common and private signaling) scheme which achieves a higher GDoF than TDMA-TIN in regime 3. Next, we show that the gap between the achievable sum-rate of TDMA-TIN and capacity increases with SNR. In the following proposition, we present the achievable GDoF of IA-CP.

Proposition 8. *The following GDoF is achievable in regime 3 of the the Gaussian PIMAC using IA-CP*

$$d_{\Sigma,\mathrm{IA-CP}} = d_{3,c} + 2d_a + d_{1,p} + d_{2,p1} + d_{2,p2} + d_{3,p}, \tag{3.156}$$

where

$$
\begin{aligned}
d_{3,c} &\leq \min \begin{Bmatrix} [\alpha_{d3} + r_{3,c} - \max\{C_1, \alpha_{d1} + r_{1,a}, \alpha_{d3} + r_{3,a}, \alpha_{c2} + r_{2,p1}\}]^+ \\ [\alpha_{c3} + r_{3,c} - \max\{C_2, \alpha_{c1} + r_{1,a}, \alpha_{c3} + r_{3,a}, \alpha_{d2} + r_{2,p1}\}]^+ \end{Bmatrix}, \\
d_{1,p} &\leq [\alpha_{d1} + r_{1,p} - \max\{0, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+, \\
d_{3,p} &\leq [\alpha_{d3} + r_{3,p} - \max\{0, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+, \\
d_{3,p} + d_{1,p} &\leq [\max\{\alpha_{d3} + r_{3,p}, \alpha_{d1} + r_{1,p}\} - \max\{0, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+, \\
d_{2,p1} &\leq [\alpha_{d2} + r_{2,p1} - \max\{C_2, \alpha_{c1} + r_{1,a}, \alpha_{c3} + r_{3,a}\}]^+, \\
d_{2,p2} &\leq [\alpha_{d2} + r_{2,p2} - \max\{0, \alpha_{c1} + r_{1,p}, \alpha_{c3} + r_{3,p}\}]^+, \\
d_a &\leq \begin{cases} A_1 & \textit{if } \frac{|h_{d3}|}{|h_{c3}|} < \frac{|h_{d1}|}{|h_{c1}|} \\ A_2 & \textit{if } \frac{|h_{d3}|}{|h_{c3}|} > \frac{|h_{d1}|}{|h_{c1}|} \end{cases},
\end{aligned}
$$

with

$$
\begin{aligned}
C_1 &= \max\{0, \alpha_{d1} + r_{1,p}, \alpha_{d3} + r_{3,p}, \alpha_{c2} + r_{2,p2}\}, \\
C_2 &= \max\{0, \alpha_{c1} + r_{1,p}, \alpha_{c3} + r_{3,p}, \alpha_{d2} + r_{2,p2}\},
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \min \begin{Bmatrix} [\alpha_{c1} + r_{1,a} - C_2]^+ \\ [\alpha_{d1} + r_{1,a} - \max\{C_1, \alpha_{d3} + r_{3,a}, \alpha_{c2} + r_{2,p1}\}]^+ \\ [\alpha_{d3} + r_{3,a} - \max\{C_1, \alpha_{c2} + r_{2,p1}\}]^+\} \end{Bmatrix}, \\
A_2 &= \min \begin{Bmatrix} [\alpha_{c1} + r_{1,a} - C_2]^+ \\ [\alpha_{d3} + r_{3,a} - \max\{C_1, \alpha_{d1} + r_{1,a}, \alpha_{c2} + r_{2,p1}\}]^+ \\ [\alpha_{d1} + r_{1,a} - \max\{C_1, \alpha_{c2} + r_{2,p1}\}]^+ \end{Bmatrix},
\end{aligned}
$$

and for $i \in \{1,3\}$, $r_{i,a}, r_{i,p}, r_{2,p1}, r_{2,p2}, r_{3,c} \leq 0$, $\rho^{r_{1,a}} + \rho^{r_{1,p}} \leq 1$, $\rho^{r_{2,p1}} + \rho^{r_{2,p2}} \leq 1$, $\rho^{r_{3,c}} + \rho^{r_{3,a}} + \rho^{r_{3,p}} \leq 1$ *and* $\alpha_{c1} + r_{1,a} = \alpha_{c3} + r_{3,a}$.

In what follows, we present the IA-CP scheme and the achievability of (3.156).

IA-CP scheme in Gaussian PIMAC

Similar to the presented IA-CP scheme for the LD-PIMAC, this scheme is also in the Gaussian setup based on common and private signaling with interference alignment. First, the transmitters split their messages as follows:

- Tx1 splits its message W_1 into $W_{1,p}$ and $W_{1,a}$ with rates $R_{1,p}$ and R_a, respectively.
- Tx2 splits its message W_2 into $W_{2,p1}$ and $W_{2,p2}$ with rates $R_{2,p1}$ and $R_{2,p2}$, respectively.
- Tx3 splits its message W_3 into $W_{3,c}$, $W_{3,a}$, and $W_{3,p}$ with rates $R_{3,c}$, R_a, and $R_{3,p}$ respectively.

The alignment message $W_{1,a}$ is further split into $W_{1,a}^R$ and $W_{1,a}^I$ with rates R_a^R and R_a^I, with $R_a^R + R_a^I = R_a$. Similarly, $W_{3,a}$ is split into $W_{3,a}^C$, $C = \{R, I\}$, with rate R_a^C. The superscript $C = \{R, I\}$ determines as whether the message is intended for the real part or the imaginary part of the channel. *Encoding:* The alignment messages $W_{1,a}^C$ and $W_{3,a}^C$ are encoded into $x_{1,a}^{C,n}$ and $x_{3,a}^{C,n}$ using nested-lattice codes. Note that Tx1 and Tx3 use the same nested-lattice codebook (Λ_f, Λ_c) with rate R_a and power 1, where Λ_c and Λ_f denote the coarse and fine lattices, respectively. Txi, $i \in \{1, 3\}$, encodes its message $W_{i,a}^C$ into a length-n codeword $\lambda_{i,a}^C$ from the nested-lattice codebook (Λ_f, Λ_c). Then, it constructs the following signal

$$x_{i,a}^{C,n} = \sqrt{\frac{P_{i,a}}{2}}[(\lambda_{i,a}^C - d_{i,a}^C) \bmod \Lambda_c], \quad C = \{R, I\},$$

where $P_{i,a}/2$ is the power of the alignment signal $x_{i,a}^{C,n}$ and $d_{i,a}^C$ is an n-dimensional random dither vector known also at the receivers. Since the length of all sequences in this section is n, we drop the superscript n for the sake of simplicity.

Additionally, the messages $W_{i,p}$, $W_{2,p1}$, $W_{2,p2}$, and $W_{3,c}$ are encoded into $x_{i,p}$, $x_{2,p1}$, $x_{2,p2}$, and $x_{3,c}$ with powers $P_{i,p}$, $P_{2,p1}$, $P_{2,p2}$, and $P_{3,c}$, respectively, using Gaussian random codebooks. Then the transmitters send the signals

$$\begin{aligned} x_3 &= x_{3,c} + \mathrm{e}^{-\mathrm{j}\,\varphi_{c3}}\underbrace{(x_{3,a}^R + \mathrm{j}\,x_{3,a}^I)}_{x_{3,a}} + x_{3,p}, \\ x_1 &= \mathrm{e}^{-\mathrm{j}\,\varphi_{c1}}\underbrace{(x_{1,a}^R + \mathrm{j}\,x_{1,a}^I)}_{x_{1,a}} + x_{1,p}, \\ x_2 &= x_{2,p1} + x_{2,p2}, \end{aligned}$$

where φ_k represents the phase of the channel h_k, where $k \in \{d1, c1, d2, c2, d3, c3\}$. Note that the assigned powers must fulfill the given power constraints, hence,

$$\begin{aligned} P_{3,c} + P_{3,a} + P_{3,p} = P_3 &\le P, \\ P_{1,a} + P_{1,p} = P_1 &\le P, \\ P_{2,p1} + P_{2,p2} = P_2 &\le P. \end{aligned}$$

Using (3.7) and (3.8), we can write the received signals of the receivers as follows

$$\begin{aligned} y_1 &= h_{d1}(\mathrm{e}^{-\mathrm{j}\,\varphi_{c1}}x_{1,a} + x_{1,p}) + h_{d3}(x_{3,c} + \mathrm{e}^{-\mathrm{j}\,\varphi_{c3}}x_{3,a} + x_{3,p}) + h_{c2}(x_{2,p1} + x_{2,p2}) + z_1, \\ y_2 &= h_{d2}(x_{2,p1} + x_{2,p2}) + h_{c1}(\mathrm{e}^{-\mathrm{j}\,\varphi_{c1}}x_{1,a} + x_{1,p}) + h_{c3}(x_{3,c} + \mathrm{e}^{-\mathrm{j}\,\varphi_{c3}}x_{3,a} + x_{3,p}) + z_2. \end{aligned}$$

Recall from our discussion in Subsection 3.2.3.3 that the signals $x_{1,a}$ and $x_{3,a}$ must be aligned at Rx2. Therefore, the powers of these two signals must be adjusted such that

$$|h_{c1}|^2 P_{1,a} = |h_{c3}|^2 P_{3,a}, \tag{3.157}$$

which guarantees that the two alignment signals are received at Rx2 at the same power. Namely, the alignment signals are received at Rx2 as

$$\begin{aligned}
&h_{c1}\mathrm{e}^{-\mathrm{j}\,\varphi_{c1}}x_{1,a} + h_{c3}\mathrm{e}^{-\mathrm{j}\,\varphi_{c3}}x_{3,a} \\
&= |h_{c1}|\sqrt{\frac{P_{1,a}}{2}}\left[(\lambda^R_{1,a} - d^R_{1,a}) \bmod \Lambda_c + \mathrm{j}\left[(\lambda^I_{1,a} - d^I_{1,a}) \bmod \Lambda_c\right]\right] \\
&\quad + |h_{c3}|\sqrt{\frac{P_{3,a}}{2}}\left[(\lambda^R_{3,a} - d^R_{3,a}) \bmod \Lambda_c + \mathrm{j}\left[(\lambda^I_{3,a} - d^I_{3,a}) \bmod \Lambda_c\right]\right] \\
&= |h_{c1}|\sqrt{\frac{P_{1,a}}{2}}\left[(\lambda^R_{1,a} - d^R_{1,a}) \bmod \Lambda_c + (\lambda^R_{3,a} - d^R_{3,a}) \bmod \Lambda_c\right. \\
&\qquad\qquad \left. + \mathrm{j}\left[(\lambda^I_{1,a} - d^I_{1,a}) \bmod \Lambda_c + (\lambda^I_{3,a} - d^I_{3,a}) \bmod \Lambda_c\right]\right].
\end{aligned}$$

Decoding: Since the PIMAC is not symmetric, the decoding process is not the same for both receives. Therefore, we discuss the decoding at the two receivers separately.

First consder Rx1. This receiver decodes first $x_{3,c}$ while all other signals are treated as noise. To do this reliably, the following constraint needs to be satisfied

$$R_{3,c} \leq \log_2\left(1 + \frac{|h_{d3}|^2 P_{3,c}}{1 + |h_{d1}|^2 P_1 + |h_{d3}|^2(P_{3,a} + P_{3,p}) + |h_{c2}|^2 P_2}\right). \tag{3.158}$$

As long as (3.158) is satisfied, Rx1 is able to decode $W_{3,c}$ and hence, it is able to reconstruct and remove $x_{3,c}$ from the received signal y_1. Further decoding process at Rx1 depends on the channel strength. In what follows, we distinguish between two different cases.

- $\frac{|h_{d1}|}{|h_{c1}|} < \frac{|h_{d3}|}{|h_{c3}|}$: In this case, Rx1 proceeds the decoding in the following order $W_{3,a} \to W_{1,a} \to \{W_{1,p}, W_{3,p}\}$. The receiver decodes each of these signals while treating the other signals as noise, then it subtracts the contribution of the decoded signal, and proceeds with decoding the next one. Note that Rx1 multiplies the received signal with $\mathrm{e}^{\mathrm{j}(\varphi_{ci}-\varphi_{di})}$ before decoding the alignment messages $W^C_{i,a}$. Then after removing the contribution of $W^C_{i,a}$ from the received signal, Rx1 multiplies the resulting signal with $\mathrm{e}^{-\mathrm{j}(\varphi_{ci}-\varphi_{di})}$. Since nested-lattice codes achieve the capacity of the point-to-point AWGN channel, the rate constraints for successive decoding of messages $W_{3,a}$ and $W_{1,a}$ at Rx1 are given by

$$R^R_a = R^I_a \leq \frac{1}{2}\log_2\left(1 + \frac{|h_{d3}|^2 \frac{P_{3,a}}{2}}{\frac{1}{2}(1 + |h_{d1}|^2 P_1 + |h_{d3}|^2 P_{3,p} + |h_{c2}|^2 P_2)}\right), \tag{3.159}$$

$$R^R_a = R^I_a \leq \frac{1}{2}\log_2\left(1 + \frac{|h_{d1}|^2 \frac{P_{1,a}}{2}}{\frac{1}{2}(1 + |h_{d1}|^2 P_{1,p} + |h_{d3}|^2 P_{3,p} + |h_{c2}|^2 P_2)}\right). \tag{3.160}$$

  Note the term $\frac{1}{2}$ in the denominator is needed to obtain the fraction of the noise and interference power in the real or the imaginary part. Thus, we obtain

$$R_a \leq \log_2\left(1+\frac{|h_{d3}|^2 P_{3,a}}{1+|h_{d1}|^2 P_1+|h_{d3}|^2 P_{3,p}+|h_{c2}|^2 P_2}\right), \tag{3.161}$$

$$R_a \leq \log_2\left(1+\frac{|h_{d1}|^2 P_{1,a}}{1+|h_{d1}|^2 P_{1,p}+|h_{d3}|^2 P_{3,p}+|h_{c2}|^2 P_2}\right). \tag{3.162}$$

- $\frac{|h_{d3}|}{|h_{c3}|} < \frac{|h_{d1}|}{|h_{c1}|}$: In this case, the decoding order at Rx1 is $W_{1,a} \to W_{3,a} \to \{W_{1,p}, W_{3,p}\}$. Similar to the previous case, we obtain the following rate constraints

$$R_a \leq \log_2\left(1+\frac{|h_{d1}|^2 P_{1,a}}{1+|h_{d1}|^2 P_{1,p}+|h_{d3}|^2 (P_{3,a}+P_{3,p})+|h_{c2}|^2 P_2}\right), \tag{3.163}$$

$$R_a \leq \log_2\left(1+\frac{|h_{d3}|^2 P_{3,a}}{1+|h_{d1}|^2 P_{1,p}+|h_{d3}|^2 P_{3,p}+|h_{c2}|^2 P_2}\right). \tag{3.164}$$

The remaining signals $x_{1,p}$ and $x_{3,p}$ are treated in the same way in both cases. Rx1 decodes $W_{1,p}$ and $W_{3,p}$ as in a multiple access channel while treating $W_{2,p1}$ and $W_{2,p2}$ as noise. Rx1 can decode $W_{1,p}$ and $W_{3,p}$ successfully if the following conditions are satisfied

$$R_{1,p} \leq \log_2\left(1+\frac{|h_{d1}|^2 P_{1,p}}{1+|h_{c2}|^2 P_2}\right), \tag{3.165}$$

$$R_{3,p} \leq \log_2\left(1+\frac{|h_{d3}|^2 P_{3,p}}{1+|h_{c2}|^2 P_2}\right), \tag{3.166}$$

$$R_{3,p}+R_{1,p} \leq \log_2\left(1+\frac{|h_{d3}|^2 P_{3,p}+|h_{d1}|^2 P_{1,p}}{1+|h_{c2}|^2 P_2}\right). \tag{3.167}$$

Now consiser Rx2. The decoding order at Rx2 is $W_{3,c} \to W_{2,p1} \to f(W_{1,a}, W_{3,a}) \to W_{2,p2}$, where $f(W_{1,a}, W_{3,a})$ is a function of $W_{1,a}$ and $W_{3,a}$. Namely, Rx2 decodes the sum of the lattice codewords corresponding to $W_{1,a}^R$ and $W_{3,a}^R$ and also the sum of the lattice codewords corresponding to $W_{1,a}^I$ and $W_{3,a}^I$. First, Rx2 decodes $W_{3,c}$ while the other signals are treated as noise. For reliable decoding of $W_{3,c}$ the following constraint needs to be satisfied

$$R_{3,c} \leq \log_2\left(1+\frac{|h_{c3}|^2 P_{3,c}}{1+|h_{c1}|^2 P_1+|h_{c3}|^2 (P_{3,a}+P_{3,p})+|h_{d2}|^2 P_2}\right). \tag{3.168}$$

Next, Rx2 reconstructs $x_{3,c}$ from $W_{3,c}$ and it removes the interference caused by $x_{3,c}$. Then, it decodes $W_{2,p1}$ while treating the other signals as noise. Therefore, the rate of $W_{2,p1}$ needs to satisfy

$$R_{2,p1} \leq \log_2\left(1+\frac{|h_{d2}|^2 P_{2,p1}}{1+|h_{c1}|^2 P_1+|h_{c3}|^2 (P_{3,a}+P_{3,p})+|h_{d2}|^2 P_{2,p2}}\right). \tag{3.169}$$

Next the receiver decodes the sums $(|h_{c1}|\lambda^R_{1,a} + |h_{c3}|\lambda^R_{3,a}) \bmod \Lambda_c$ and $(|h_{c1}|\lambda^I_{1,a} + |h_{c3}|\lambda^I_{3,a}) \bmod \Lambda_c$. Decoding these sums is possible as long as

$$R^R_a = R^I_a \leq \frac{1}{2}\left[\log_2\left(\frac{1}{2} + \frac{|h_{c1}|^2\frac{P_{1,a}}{2}}{\frac{1}{2}(1+|h_{c1}|^2P_{1,p}+|h_{c3}|^2P_{3,p}+|h_{d2}|^2P_{2,p2})}\right)\right]^+. \quad (3.170)$$

Since $R^R_a = R^I_a$ and $R_a = R^R_a + R^I_a$, we obtain

$$R_a \leq \left[\log_2\left(\frac{1}{2} + \frac{|h_{c1}|^2P_{1,a}}{1+|h_{c1}|^2P_{1,p}+|h_{c3}|^2P_{3,p}+|h_{d2}|^2P_{2,p2}}\right)\right]^+. \quad (3.171)$$

The receiver can then construct the received sum of alignment signals $|h_{c1}|x_{1,a} + |h_{c3}|x_{3,a}$ from the decoded sums of codewords $(|h_{c1}|\lambda^R_{1,a} + |h_{c3}|\lambda^R_{3,a}) \bmod \Lambda_c$ and $(|h_{c1}|\lambda^C_{1,a} + |h_{c3}|\lambda^C_{3,a}) \bmod \Lambda_c$. After reconstructing the sum of alignment signals, its contribution is removed from the received signal and then $W_{2,p2}$ is decoded. Decoding $W_{2,p2}$ is possible reliably as long as

$$R_{2,p2} \leq \log_2\left(1 + \frac{|h_{d2}|^2P_{2,p2}}{1+|h_{c1}|^2P_{1,p}+|h_{c3}|^2P_{3,p}}\right). \quad (3.172)$$

As a result of this decoding process, the following sum-rate is achievable

$$R_{\Sigma,\text{IA-CP}} = R_{3,c} + 2R_a + R_{1,p} + R_{2,p1} + R_{2,p2} + R_{3,p}, \quad (3.173)$$

where the terms above satisfy (3.158)-(3.172). Since, we are interested in an approximation of the sum-rate at high SNR, we translate the achievable sum-rate into the achievable GDoF as follows

$$d_{\Sigma,\text{IA-CP}}(\boldsymbol{\alpha}) \leq d_{3,c} + 2d_a + d_{1,p} + d_{2,p1} + d_{2,p2} + d_{3,p}, \quad (3.174)$$

where

$$d_{3,c} = \frac{R_{3,c}}{\log_2\rho}, \qquad d_a = \frac{R_a}{\log_2\rho}, \qquad d_{1,p} = \frac{R_{1,p}}{\log_2\rho},$$
$$d_{3,p} = \frac{R_{3,p}}{\log_2\rho}, \qquad d_{2,p1} = \frac{R_{2,p2}}{\log_2\rho}, \qquad d_{2,p2} = \frac{R_{2,p2}}{\log_2\rho},$$

and $\rho \to \infty$. We start by defining

$$r_{3,c} = \frac{\log_2\left(\frac{P_{3,c}}{P}\right)}{\log_2\rho}, \quad r_{1,a} = \frac{\log_2\left(\frac{P_{1,a}}{P}\right)}{\log_2\rho}, \quad r_{3,a} = \frac{\log_2\left(\frac{P_{3,a}}{P}\right)}{\log_2\rho}, \quad r_{1,p} = \frac{\log_2\left(\frac{P_{1,p}}{P}\right)}{\log_2\rho},$$
$$r_{3,p} = \frac{\log_2\left(\frac{P_{3,p}}{P}\right)}{\log_2\rho}, \quad r_{2,p1} = \frac{\log_2\left(\frac{P_{2,p1}}{P}\right)}{\log_2\rho}, \quad r_{2,p2} = \frac{\log_2\left(\frac{P_{2,p2}}{P}\right)}{\log_2\rho}.$$

Note that since $P_{i,a}, P_{i,p}, P_{2,p1}, P_{2,p2}, P_{3,c} \leq P$ and $1 < \rho$, then we have

$$r_{i,a}, r_{i,p}, r_{2,p1}, r_{2,p2}, r_{3,c} \leq 0.$$

Furthermore, we impose the constraints

$$\rho^{r_{1,a}} + \rho^{r_{1,p}} \leq 1, \quad \rho^{r_{2,p1}} + \rho^{r_{2,p2}} \leq 1, \quad \rho^{r_{3,c}} + \rho^{r_{3,a}} + \rho^{r_{3,p}} \leq 1,$$

in order to satisfy the power constraints, and

$$\alpha_{c1} + r_{1,a} = \alpha_{c3} + r_{3,a}$$

in order to satisfy (3.157). Now, we substitute these parameters in the rate constraints (3.158)-(3.172) and approximate the expression in the high SNR regime. Consider the constraint (3.158). This can be written as

$$\begin{aligned}
R_{3,c} &\leq \log_2\left(1 + \frac{|h_{d3}|^2 P_{3,c}}{1 + |h_{d1}|^2 P_1 + |h_{d3}|^2 (P_{3,a} + P_{3,p}) + |h_{c2}|^2 P_2}\right) \\
&= \log_2\left(1 + \frac{|h_{d3}|^2 P \frac{P_{3,c}}{P}}{1 + |h_{d1}|^2 P \frac{P_1}{P} + |h_{d3}|^2 P \frac{(P_{3,a} + P_{3,p})}{P} + |h_{c2}|^2 P \frac{P_2}{P}}\right) \\
&= \log_2\left(1 + \frac{|h_{d3}|^2 P \frac{P_{3,c}}{P}}{1 + |h_{d1}|^2 P \frac{P_{1,a} + P_{1,p}}{P} + |h_{d3}|^2 P \frac{(P_{3,a} + P_{3,p})}{P} + |h_{c2}|^2 P \frac{P_{2,p1} + P_{2,p2}}{P}}\right) \\
&= \log_2\left(1 + \frac{\rho^{\alpha_{d3} + r_{3,c}}}{1 + \rho^{\alpha_{d1}}(\rho^{r_{1,a}} + \rho^{r_{1,p}}) + \rho^{\alpha_{d3}}(\rho^{r_{3,a}} + \rho^{r_{3,p}}) + \rho^{\alpha_{c2}}(\rho^{r_{2,p1}} + \rho^{r_{2,p2}})}\right) \\
&\approx \log_2\left(\frac{\rho^{\alpha_{d3} + r_{3,c}}}{1 + \rho^{\alpha_{d1}}(\rho^{r_{1,a}} + \rho^{r_{1,p}}) + \rho^{\alpha_{d3}}(\rho^{r_{3,a}} + \rho^{r_{3,p}}) + \rho^{\alpha_{c2}}(\rho^{r_{2,p1}} + \rho^{r_{2,p2}})}\right) \\
&\approx \log_2(\rho)[\alpha_{d3} + r_{3,c} - \max\{0, \alpha_{d1} + r_{1,a}, \alpha_{d1} + r_{1,p}, \alpha_{d3} + r_{3,a}, \\
&\qquad \alpha_{d3} + r_{3,p}, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+,
\end{aligned}$$

where the approximation follows by considering SNR high enough so that the additive constants can be neglected. By following a similar procedure, we can show that the rate constraints (3.158)-(3.172) translate to

$$\begin{aligned}
d_{3,c} \leq \min\{&[\alpha_{d3} + r_{3,c} - \max\{0, \alpha_{d1} + r_{1,a}, \alpha_{d1} + r_{1,p}, \alpha_{d3} + r_{3,a}, \alpha_{d3} + r_{3,p}, \\
&\alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+, \\
&[\alpha_{c3} + r_{3,c} - \max\{0, \alpha_{c1} + r_{1,a}, \alpha_{c1} + r_{1,p}, \alpha_{c3} + r_{3,a}, \alpha_{c3} + r_{3,p}, \\
&\alpha_{d2} + r_{2,p1}, \alpha_{d2} + r_{2,p2}\}]^+\},
\end{aligned}$$

$$\begin{aligned}
d_{1,p} &\leq [\alpha_{d1} + r_{1,p} - \max\{0, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+, \\
d_{3,p} &\leq [\alpha_{d3} + r_{3,p} - \max\{0, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+, \\
d_{3,p} + d_{1,p} &\leq [\max\{\alpha_{d3} + r_{3,p}, \alpha_{d1} + r_{1,p}\} - \max\{0, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+, \\
d_{2,p1} &\leq [\alpha_{d2} + r_{2,p1} - \max\{0, \alpha_{c1} + r_{1,p}, \alpha_{c1} + r_{1,a}, \alpha_{c3} + r_{3,p}, \\
&\qquad \alpha_{c3} + r_{3,a}, \alpha_{d2} + r_{2,p2}\}]^+, \\
d_{2,p2} &\leq [\alpha_{d2} + r_{2,p2} - \max\{0, \alpha_{c1} + r_{1,p}, \alpha_{c3} + r_{3,p}\}]^+,
\end{aligned}$$

and

$$\begin{aligned} d_a \leq \min\{&[\alpha_{c1} + r_{1,a} - \max\{0, \alpha_{c1} + r_{1,p}, \alpha_{c3} + r_{3,p}, \alpha_{d2} + r_{2,p2}\}]^+, \\ &[\alpha_{d3} + r_{3,a} - \max\{0, \alpha_{d1} + r_{1,p}, \alpha_{d1} + r_{1,a}, \alpha_{d3} + r_{3,p}, \alpha_{c2} + r_{2,p1}, \\ &\qquad \alpha_{c2} + r_{2,p2}\}]^+, \\ &[\alpha_{d1} + r_{1,a} - \max\{0, \alpha_{d1} + r_{1,p}, \alpha_{d3} + r_{3,p}, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+\}, \end{aligned}$$

if $\frac{|h_{d1}|}{|h_{c1}|} < \frac{|h_{d3}|}{|h_{c3}|}$, and

$$\begin{aligned} d_a \leq \min\{&[\alpha_{c1} + r_{1,a} - \max\{0, \alpha_{c1} + r_{1,p}, \alpha_{c3} + r_{3,p}, \alpha_{d2} + r_{2,p2}\}]^+, \\ &[\alpha_{d1} + r_{1,a} - \max\{0, \alpha_{d1} + r_{1,p}, \alpha_{d3} + r_{3,p}, \alpha_{d3} + r_{3,a}, \alpha_{c2} + r_{2,p1}, \\ &\qquad \alpha_{c2} + r_{2,p2}\}]^+, \\ &[\alpha_{d3} + r_{3,a} - \max\{0, \alpha_{d1} + r_{1,p}, \alpha_{d3} + r_{3,p}, \alpha_{c2} + r_{2,p1}, \alpha_{c2} + r_{2,p2}\}]^+\}, \end{aligned}$$

if $\frac{|h_{d3}|}{|h_{c3}|} < \frac{|h_{d1}|}{|h_{c1}|}$. This shows the achievability of the rate given in Proposition 8.

IA-CP versus TDMA-TIN

In order to show the suboptimality of TDMA-TIN in regime 3, we need to show that the achievable GDoF of IA-CP is strictly larger than that of TDMA-TIN. By varying the power allocation parameters (r's) in the IA-CP scheme, different GDoF can be achieved. In order to obtain the highest achievable GDoF of the scheme, one has to optimize over the various power allocations. Next, we show that there exist power allocations that lead to higher achievable GDoF than that of TDMA-TIN in regime 3.

Corollary 4. *TDMA-TIN cannot achieve the GDoF of the Gaussian PIMAC in regime 3.*

Proof. To prove this corollary, we need to find power allocations for the IA-CP scheme (presented in Proposition 8) which lead to higher GDoF than (3.105) in regime 3. First, we fix the power allocation parameters of IA-CP for sub-regime 3A as follows

$$r_{1,p} = -\alpha_{c1}, \quad r_{2,p2} = -\alpha_{c2}, \quad r_{3,c} = 0, \quad r_{1,a} = r_{2,p1} = r_{3,a} = r_{3,p} = -\infty.$$

This is equivalent to setting the powers of the private, common, and alignment signals to $P_{1,p} = \frac{1}{|h_{c1}|^2}$ $P_{2,p2} = \frac{1}{|h_{c2}|^2}$ (note that $\frac{1}{|h_{c1}|^2}, \frac{1}{|h_{c2}|^2} < P$ according to (3.9)), $P_{3,c} = P$, and $P_{1,a} = P_{2,p1} = P_{3,a} = P_{3,p} = 0$. This satisfies the power constraint. Next, we substitute these parameters in Proposition 8 to obtain

$$\begin{aligned} d_{3c} &\leq \min\{\alpha_{d3} - (\alpha_{d1} - \alpha_{c1}), \alpha_{c3} - (\alpha_{d2} - \alpha_{c2})\}, \\ d_{1,p} &\leq \alpha_{d1} - \alpha_{c1}, \\ d_{2,p2} &\leq \alpha_{d2} - \alpha_{c2}, \\ d_{3,p}, d_a, d_{2,p1} &\leq 0, \end{aligned}$$

for an achievable GDoF of

$$d_{\Sigma,\mathrm{IA-CP,3A}}(\boldsymbol{\alpha}) = \min\{\alpha_{d3} + (\alpha_{d2} - \alpha_{c2}), \alpha_{c3} + (\alpha_{d1} - \alpha_{c1})\}. \quad (3.175)$$

Now, similar to the analysis for the LD-PIMAC, by comparing (3.175) with (3.105), we can show that the achievable GDoF using IA-CP is higher than that of the TDMA-TIN in sub-regime 3A.

Now, we prove Corollary 4 for sub-regime 3B. In this sub-regime, we choose the power allocation parameters of IA-CP as follows [5]

$$r_{1,a} = \frac{-2}{\log_2 \rho}, \quad r_{3,a} = \alpha_{c1} - \alpha_{c3} - \frac{2}{\log_2 \rho}, \quad r_{3,c} = \frac{-2}{\log_2 \rho}$$
$$r_{3,p} = \frac{-2}{\log_2 \rho} - \alpha_{c3}, \quad r_{2,p1} = -\alpha_{c2} - \frac{2}{\log_2 \rho}, \quad r_{1,p} = \frac{-2}{\log_2 \rho} - \alpha_{c1},$$

$$r_{2,p2} = \max\{(\alpha_{d3} - \alpha_{c3}) - (\alpha_{d1} - \alpha_{c1}), \alpha_{d1} - (\alpha_{d3} - \alpha_{c3})\} - \alpha_{d2} - \frac{2}{\log_2 \rho}.$$

which corresponds to setting

$$P_{1,a} = \frac{P}{4}, \quad P_{3,a} = \frac{|h_{c1}|^2}{|h_{c3}|^2}\frac{P}{4}, \quad P_{3,c} = \frac{P}{4},$$
$$P_{3,p} = \frac{1}{4|h_{c3}|^2}, \quad P_{2,p1} = \frac{1}{4|h_{c2}|^2}, \quad P_{1,p} = \frac{1}{4|h_{c1}|^2}$$

$$P_{2,p2} = \max\left\{\frac{|h_{d3}|^2|h_{c1}|^2}{4|h_{c3}|^2|h_{d1}|^2|h_{d2}|^2}, \frac{P|h_{d1}|^2|h_{c3}|^2}{4|h_{d3}|^2|h_{d2}|^2}\right\}.$$

This power allocation can satisfy the power constraint and the alignment constraint. By applying this power allocation to Proposition 8 and letting $\rho \to \infty$, we obtain

$$\begin{aligned}
d_a &= \min\{\alpha_{d1} - \alpha_{d3} + \alpha_{c3}, (\alpha_{d3} - \alpha_{c3}) - (\alpha_{d1} - \alpha_{c1})\},\\
d_{3,c} &= \alpha_{c3} - (\alpha_{d2} - \alpha_{c2}),\\
d_{3,p} &\leq \alpha_{d3} - \alpha_{c3},\\
d_{2,p1} &= \alpha_{d2} - \alpha_{c2} - \alpha_{c1},\\
d_{2,p2} &= \max\{\alpha_{d1} - \alpha_{d3} + \alpha_{c3}, (\alpha_{d3} - \alpha_{c3}) - (\alpha_{d1} - \alpha_{c1})\},\\
d_{1,p} &\leq \alpha_{d1} - \alpha_{c1},\\
d_{1,p} + d_{3,p} &= \alpha_{d3} - \alpha_{c3}.
\end{aligned}$$

Hence, we achieve the following GDoF

$$d_{\Sigma,\mathrm{IA-CP,3B}}(\boldsymbol{\alpha}) = (\alpha_{d3} - \alpha_{c3}) + (\alpha_{d2} - \alpha_{c2}) + d_{3,c} + d_a. \quad (3.176)$$

[5] In Appendix 3.A, it is explained how we choose the power allocation using the insight obtained from LD-PIMAC.

Due to the fact that in sub-regime 3B, $d_{3,c} + d_a$ is always positive, the achievable GDoF is strictly larger than $(\alpha_{d3} - \alpha_{c3}) + (\alpha_{d2} - \alpha_{c2})$. Moreover, by substituting $d_{3,c}$ into (3.176), we obtain $d_{\Sigma,\text{IA-CP},3\text{B}}(\boldsymbol{\alpha}) = \alpha_{d3} + d_a$ which is larger than α_{d3} since in sub-regime 3B, d_a is positive. Hence, we conclude that the achievable GDoF of IA-CP is larger than that of TDMA-TIN given in (3.105) in sub-regime 3B.

Finally, we show Corollary 4 for sub-regime 3C. To this end, we choose the power allocation parameters of IA-CP accordingly. The following power allocation parameters can be used to show that IA-CP outperforms TDMA-TIN in terms of GDoF in sub-regime 3C, and thus prove Corollary 4 in this sub-regime.

$$r_{1,a} = -(\alpha_{c1} - \alpha_{c3})^+ - \frac{1}{\log_2 \rho}, \quad r_{1,p} = -\alpha_{c1} - \frac{1}{\log_2 \rho},$$
$$r_{2,p1} = -\alpha_{c2} - \frac{1}{\log_2 \rho}, \quad r_{2,p2} = \max\{d_a^{(1)}, d_a^{(2)}\} - \alpha_{d2} - \frac{1}{\log_2 \rho},$$
$$r_{3,p} = -\alpha_{c3} - \frac{1}{\log_2 \rho}, \quad r_{3,a} = -(\alpha_{c3} - \alpha_{c1})^+ - \frac{1}{\log_2 \rho}, \quad r_{3,c} = -\infty,$$

where $d_a^{(1)}$ and $d_a^{(2)}$ are defined in Table 3.5. These correspond to setting

Cases		$d_a^{(1)}$	$d_a^{(2)}$
$\alpha_{d3} - \alpha_{c3} > \alpha_{d1} - \alpha_{c1}$	$\alpha_{c3} < \alpha_{c1}$	$\alpha_{d3} - \alpha_{c3} - \alpha_{d1} + \alpha_{c1}$	$\alpha_{d1} - \alpha_{c1} - \alpha_{d3} + 2\alpha_{c3}$
	$\alpha_{c1} \leq \alpha_{c3}$	$\alpha_{d3} - \alpha_{c3} - \alpha_{d1} + \alpha_{c1}$	$\alpha_{d1} - \alpha_{d3} + \alpha_{c3}$
$\alpha_{d3} - \alpha_{c3} < \alpha_{d1} - \alpha_{c1}$	$\alpha_{c3} < \alpha_{c1}$	$\alpha_{d1} - \alpha_{c1} - \alpha_{d3} + \alpha_{c3}$	$\alpha_{d3} - \alpha_{d1} + \alpha_{c1}$
	$\alpha_{c1} \leq \alpha_{c3}$	$\alpha_{d1} - \alpha_{c1} - \alpha_{d3} + \alpha_{c3}$	$\alpha_{d3} - \alpha_{c3} - \alpha_{d1} + 2\alpha_{c1}$

Table 3.5: The values of $d_a^{(1)}$ and $d_a^{(2)}$.

$$P_{1,a} = P_{3,a}\frac{|h_{c3}|^2}{|h_{c1}|^2}, \quad P_{3,p} = \frac{1}{2|h_{c3}|^2}, \quad P_{1,p} = \frac{1}{2|h_{c1}|^2}, \quad P_{2,p1} = \frac{1}{2|h_{c2}|^2}, \quad P_{3,c} = 0.$$

The remaining parameters are given in Table 3.6.

		$P_{3,a}$	$P_{2,p2}$
$\frac{\lvert h_{d3}\rvert}{\lvert h_{c3}\rvert} > \frac{\lvert h_{d1}\rvert}{\lvert h_{c1}\rvert}$	$\lvert h_{c3}\rvert^2 < \lvert h_{c1}\rvert^2$	$\frac{P}{2}$	$\max\left\{\frac{\lvert h_{d3}\rvert^2\lvert h_{c1}\rvert^2}{2\lvert h_{d1}\rvert^2\lvert h_{d2}\rvert^2\lvert h_{c3}\rvert^2}, \frac{P\lvert h_{d1}\rvert^2\lvert h_{c3}\rvert^4}{2\lvert h_{d3}\rvert^2\lvert h_{c1}\rvert^2\lvert h_{d2}\rvert^2}\right\}$
	$\lvert h_{c1}\rvert^2 \leq \lvert h_{c3}\rvert^2$	$\frac{P}{2}\frac{\lvert h_{c1}\rvert^2}{\lvert h_{c3}\rvert^2}$	$\max\left\{\frac{\lvert h_{d3}\rvert^2\lvert h_{c1}\rvert^2}{2\lvert h_{d1}\rvert^2\lvert h_{d2}\rvert^2\lvert h_{c3}\rvert^2}, \frac{P\lvert h_{d1}\rvert^2\lvert h_{c3}\rvert^2}{2\lvert h_{d3}\rvert^2\lvert h_{d2}\rvert^2}\right\}$
$\frac{\lvert h_{d3}\rvert}{\lvert h_{c3}\rvert} < \frac{\lvert h_{d1}\rvert}{\lvert h_{c1}\rvert}$	$\lvert h_{c3}\rvert^2 < \lvert h_{c1}\rvert^2$	$\frac{P}{2}$	$\max\left\{\frac{\lvert h_{d1}\rvert^2\lvert h_{c3}\rvert^2}{2\lvert h_{c1}\rvert^2\lvert h_{d3}\rvert^2\lvert h_{d2}\rvert^2}, \frac{P\lvert h_{d3}\rvert^2\lvert h_{c1}\rvert^2}{2\lvert h_{d1}\rvert^2\lvert h_{d2}\rvert^2}\right\}$
	$\lvert h_{c1}\rvert^2 \leq \lvert h_{c3}\rvert^2$	$\frac{P}{2}\frac{\lvert h_{c1}\rvert^2}{\lvert h_{c3}\rvert^2}$	$\max\left\{\frac{\lvert h_{d1}\rvert^2\lvert h_{c3}\rvert^2}{2\lvert h_{c1}\rvert^2\lvert h_{d3}\rvert^2\lvert h_{d2}\rvert^2}, \frac{P\lvert h_{d3}\rvert^2\lvert h_{c1}\rvert^4}{2\lvert h_{c3}\rvert^2\lvert h_{d1}\rvert^2\lvert h_{d2}\rvert^2}\right\}$

Table 3.6: Power allocation parameters ($P_{3,a}$ and $P_{2,p2}$) for IA-CP in sub-regime 3C.

The given power allocation parameters satisfy the power constraint and the alignment constraint. By substituting these power allocation parameters in the constraints in Proposition 8 and letting $\rho \to \infty$, we obtain

$$\begin{aligned} d_{3,c} &= 0 \\ d_{1,p} &\leq \alpha_{d1} - \alpha_{c1} \\ d_{3,p} &\leq \alpha_{d3} - \alpha_{c3} \\ d_{1,p} + d_{3,p} &= \max\{\alpha_{d1} - \alpha_{c1}, \alpha_{d3} - \alpha_{c3}\} \\ d_{2,p1} &= \alpha_{d2} - \alpha_{c2} - \min\{\alpha_{c1}, \alpha_{c3}\} \\ d_{2,p2} &= \max\{d_a^{(1)}, d_a^{(2)}\} \\ d_a &= \min\{d_a^{(1)}, d_a^{(2)}\}, \end{aligned}$$

where $d_a^{(1)}$ and $d_a^{(2)}$ are provided in Table 3.5. Hence, the proposed scheme achieves

$$\begin{aligned} d_{\Sigma,\mathrm{IA-CP},3\mathrm{C}}(\boldsymbol{\alpha}) &= 2d_a + d_{1,p} + d_{3,p} + d_{2,p1} + d_{2,p2} + d_{3,c} \\ &= d_a + \underbrace{\max\{\alpha_{d3} - \alpha_{c3}, \alpha_{d1} - \alpha_{c1}\} + \alpha_{d2} - \alpha_{c2}}_{d_{\Sigma,\text{TDMA-TIN},3\text{C}}(\boldsymbol{\alpha})} \\ &> d_{\Sigma,\text{TDMA-TIN},3\text{C}}(\boldsymbol{\alpha}), \end{aligned}$$

since in sub-regime 3C, d_a is positive.

Therefore, TDMA-TIN is outperformed by IA-CP in regime 3, in terms of GDoF. This shows that TDMA-TIN cannot achieve the GDoF of the Gaussian PIMAC in regime 3 which completes the proof of corollary 4. □

Remark 9. *Similar to the LD-PIMAC, in the Gaussian case the proposed IA-CP scheme cannot outperform TDMA-TIN in terms of GDoF when* $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$. *Surprisingly, while TDMA-TIN achieves the sum-capacity of the LD-PIMAC when* $n_{d3} - n_{c3} = n_{d1} - n_{c1}$*, it cannot achieve the GDoF of the Gaussian PIMAC in the equivalent case, i.e.,* $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$*, except over a subset of channel coefficient values of measure* 0. *Moreover, naive-TIN is also GDoF suboptimal in this case. We show this by introducing a scheme which outperforms TDMA-TIN and naive-TIN in terms of GDoF. Interestingly, in this scheme phase alignment [CJW10] is required. The scheme and its achievable GDoF are presented in Appendix 3.B in details.*

As we have shown, TDMA-TIN is not GDoF optimal in regime 3 and for the special case $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$. Now, we are ready to extend this result and show suboptimality of TDMA-TIN in these ceases. This result is presented in the following Corollary.

Corollary 5. *TIN cannot achieve the sum-capacity of Gaussian PIMAC within a constant gap in regime 3 and for the case* $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$.

Proof. As we have shown in Lemma 9, the achieved GDoF using TIN at the receiver side alongside power control at the transmitter side is upper bounded by the GDoF of TDMA-TIN. Moreover, we have shown that TDMA-TIN is outperformed in terms of GDoF by better schemes in regime 3 and for the case when $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$.

Hence, in these cases, TDMA-TIN and subsequently TIN with power control cannot achieve the GDoF of the Gaussian PIMAC, i.e., $d_\Sigma(\boldsymbol{\alpha})$. Therefore, the gap between the achievable GDoF of TIN and the GDoF of the Gaussian PIMAC is lower bounded by a positive value a. This can be written as

$$d_\Sigma(\boldsymbol{\alpha}) - d_{\Sigma,\mathrm{TIN}}(\boldsymbol{\alpha}) \geq a > 0.$$

Now, by using the definition of the GDoF, we can write the capacity of Gaussian PIMAC and achievable sum-rate of TIN as follows

$$\begin{aligned} C_{\mathrm{G},\Sigma}(\rho,\boldsymbol{\alpha}) &= d_\Sigma(\boldsymbol{\alpha})\log_2(\rho) - o(\log_2(\rho)) \\ R_{\Sigma,\mathrm{TIN}}(\rho,\boldsymbol{\alpha}) &= d_{\Sigma,\mathrm{TIN}}(\boldsymbol{\alpha})\log_2(\rho) - o_{\mathrm{TIN}}(\log_2(\rho)). \end{aligned}$$

Now, by obtaining the difference between the sum-capacity and the achievable sum-rate, we can write

$$C_{\mathrm{G},\Sigma}(\rho,\boldsymbol{\alpha}) - R_{\Sigma,\mathrm{TIN}}(\rho,\boldsymbol{\alpha}) \geq a\log_2(\rho) \underbrace{-o_{\text{IA-CP}}(\log_2(\rho)) - o_{\mathrm{TIN}}(\log_2(\rho))}_{-o_s(\log_2(\rho))}.$$

While the term $-o_s(\log_2(\rho))$ does not scale with ρ as $\rho \to \infty$, the first term $a\log_2(\rho)$ increases by ρ. This shows that the gap between the sum-capacity of the Gaussian PIMAC and the achievable sum-rate of TIN grows as a function of ρ. Hence, TIN cannot achieve the sum-capacity of Gaussian PIMAC within a constant gap. □

3.3.4 Discussion and Numerical Analysis

Discussion

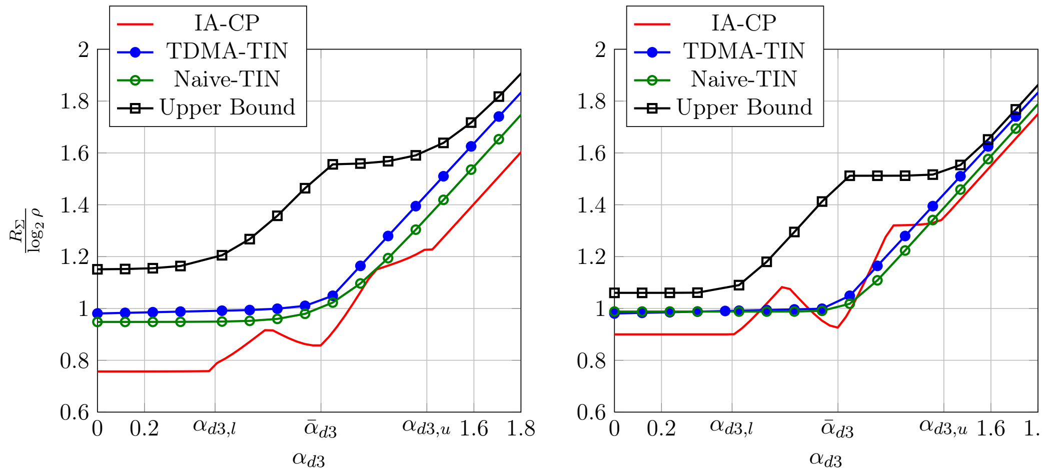
Our analysis shows two interesting results about the GDoF optimality of TIN. Firstly, there are some regimes where TIN is suboptimal, although we have very-weak interference, in the sense that the strongest interference caused by a user plus the strongest interference it receives is less than or equal to the strongest desired channel parameter, i.e.,

$$\max\{\alpha_{c3},\alpha_{c1}\} + \alpha_{c2} \leq \alpha_{d2} \tag{3.177}$$

$$\max\{\alpha_{c3},\alpha_{c1}\} + \alpha_{c2} \leq \max\{\alpha_{d1},\alpha_{d3}\}. \tag{3.178}$$

Secondly, there are some regimes where interference is not very-weak (according to (3.177) and (3.178)), but still TIN is optimal.

Regarding the first point, IA-CP (in regime 3) leads to a better performance than using plain TIN. This conclusion is particularly interesting in regimes where both receivers experience very-weak interference according to conditions (3.177) and (3.178). Note that the 2-user IC which consists of Tx1, Tx2, Rx1, and Rx2, operates in the noisy interference regime ($\alpha_{c1} + \alpha_{c2} \leq \min\{\alpha_{d1},\alpha_{d2}\}$). By adding to this setup a transmitter which has a strong channel to its desired receiver ($\alpha_{d1} < \alpha_{d3}$)



(a) $\rho = 40dB$.

(b) $\rho = 100dB$.

Figure 3.10: Normalized achievable sum-rate as a function of α_{d3} for a PIMAC with $\alpha_{d1} = \alpha_{d2} = 1$, $\alpha_{c3} = 0.45$, $\alpha_{c1} = \alpha_{c2} = 0.5$. Moreover, $\bar{\alpha}_{d3} = \alpha_{d1} - \alpha_{c1} + \alpha_{c3}$, $\alpha_{d3,l} = \max\{\alpha_{d1} - \alpha_{c1}, \alpha_{d1} + \alpha_{c3} - 2\alpha_{c1}\}$, and $\alpha_{d3,u} = \min\{\alpha_{d1} + \alpha_{c3}, \alpha_{d1} - \alpha_{c1} + 2\alpha_{c3}\}$

and which causes very weak interference to the undesired Rx ($\alpha_{c3} < \alpha_{c1}$), we obtain a PIMAC which satisfies (3.177) and (3.178). One would expect that TIN is optimal in this case. However, even in this case interference alignment might outperform TIN although the channel parameters satisfy (3.177) and (3.178). For instance, if $(\alpha_{d1}, \alpha_{c1}, \alpha_{d2}, \alpha_{c2}, \alpha_{d3}, \alpha_{c3}) = (8, 4, 7, 2, 9, 3)$, then while conditions (3.177) and (3.178) are satisfied, the channel is in sub-regime 3C where IA-CP outperforms TDMA-TIN.

Regarding the second point, it can be seen that the interference in the parts of regimes 1, 2 (for instance when $\alpha_{c3} > \alpha_{d2} - \alpha_{c2}$) cannot be characterized as very-weak (according to (3.177) and (3.178)). However, TDMA-TIN is still GDoF optimal in this regime. It is worth mentioning that the GDoF optimality of TIN in this case ($\alpha_{c3} > \alpha_{d2} - \alpha_{c2}$) is mainly due to the structure of the PIMAC. For instance, if Tx2 and Tx3 were allowed to communicate with Rx1 and Rx2, respectively, then TIN would not be optimal in whole sub-regime $\alpha_{c3} > \alpha_{d2} - \alpha_{c2}$. On the other hand, the GDoF optimality of TIN in regimes 1C is mainly due to our new lemma (Lemma 11) and upper bounds developed in Lemmata 14 and 15.

Numerical Analysis

Here, we present some numerical comparison of the achievable rates of the TIN schemes with a sum-rate upper bounds. Fig. 3.10 shows the normalized achievable sum-rate $R_\Sigma / \log_2 \rho$ as a function of α_{d3} for a PIMAC with $\alpha_{d1} = \alpha_{d2} = 1$, $\alpha_{c3} = 0.45$ and $\alpha_{c1} = \alpha_{c2} = 0.5$. While in Fig. 3.10a, $\rho = 40$dB, in Fig. 3.10b, $\rho = 100$dB.

Notice that the 2-user IC consists of Tx1, Tx2, Rx1, and Rx2 operates in very weak interference regime. As it is shown in both figures, for $\alpha_{d3} \leq \alpha_{d3,l}$, the slopes of the achievable sum-rate of TDMA-TIN and Naive-TIN are the same as that of the upper bound. This indicates that these schemes are asymptotically optimal (within a constant gap). However, this optimality will be lost if the link from Tx3 to Rx1 gets stronger. Interestingly, although the receivers of PIMAC still experience very-weak interference, neither naive-TIN nor TDMA-TIN can be optimal in some subsets of sub-regime 3C corresponds to $\alpha_{d3} \in [\alpha_{d3,l}, \alpha_{d3,u}]$, where

$$\alpha_{d3,l} = \max\{\alpha_{d1} - \alpha_{c1}, \alpha_{d1} + \alpha_{c3} - 2\alpha_{c1}\}$$
$$\alpha_{d3,u} = \min\{\alpha_{d1} + \alpha_{c3}, \alpha_{d1} - \alpha_{c1} + 2\alpha_{c3}\}.$$

The better performance of IA-CP than TDMA-TIN at asymptotically high SNR in sub-regime 3C can be seen in Fig3.10b. Now, consider the normalized gap between the upper bound and the achievable sum-rate of TIN and TDMA-TIN (with respect to $\log_2 \rho$) in regimes where these schemes are GDoF-optimal, i.e., $\alpha_{d3} \leq \alpha_{d3,l}$ and $\alpha_{d3} \geq \alpha_{d3,u}$. It can be seen that this gap in Fig. 3.10a is larger than the one observed in Fig. 3.10b. This observation visualizes the statement that the impact of the constant gap is negligible compared to the capacity for high SNR values.

3.4 Summary

We examined the optimality of the simple scheme of treating interference as noise (TIN) in a network consisting of a P2P channel interfering with a MAC (PIMAC). We derived some upper bounds on the sum-rate for both the deterministic PIMAC and the Gaussian PIMAC. Then, we characterized regimes of channel parameters where TIN is sum-capacity optimal for the deterministic PIMAC, and sum-capacity optimal within a constant gap for the Gaussian one. It turns out that one has to combine TIN with TDMA in order to improve the performance of TIN, and make it optimal for a wider range of parameters. This combination, denoted TDMA-TIN, strictly outperforms naive-TIN in the Gaussian PIMAC. This leads to the following conclusion: The naive-TIN scheme where all transmitters transmit simultaneously and all receivers treat interference as noise is always a suboptimal scheme in the PIMAC (except for a special case). This conclusion is in contrast to the 2-user interference channel where naive-TIN is sum-capacity optimal in the so-called noisy interference regime. We have also shown that TDMA-TIN is outperformed by a combination of private and common signaling with interference alignment in some cases. Interestingly, this includes cases where both receivers experience very-weak interference.

Surprisingly, although TIN is optimal (within a constant gap) in some regimes of the Gaussian PIMAC with very-weak interference, there exists regimes also with very-weak interference where TIN is not optimal. In these regimes, interference alignment leads to rate improvement. Furthermore, there exist regimes where not all interference is very-weak, but still TIN is optimal.

3.A An Example for Choosing Power Allocation Parameters

Here, we explain how to use the insight of the linear deterministic case, for choosing the power allocation parameters for the Gaussian case. To do this, we explain the power allocation IA-CP for sub-regime 3B. First, we recall the graphical illustration of the received signal in the LD-PIMAC for this sub-regime. This is shown in Fig. 3.8. Now, we need to replace the bit levels in the deterministic setup with the power levels in the Gaussian setup. To do this, we replace n_k by α_k for all $k \in \{d1, c1, d2, c2, d3, c3\}$. Doing this, we get Fig. 3.11. Notice that, while the length of each block in Fig. 3.8 (for the LD-PIMAC) represents the rate of the corresponding signal, in the Gaussian case it represents the DoF achieved by each signal. As an example, the length of the block which represents $x_{3,c}$ is given by $d_{3,c}$ in Fig. 3.11. Notice that the length of the blocks which represent $x_{1,a}$ and $x_{3,a}$ are the same and $d_{1,a} = d_{3,a} = d_a$.

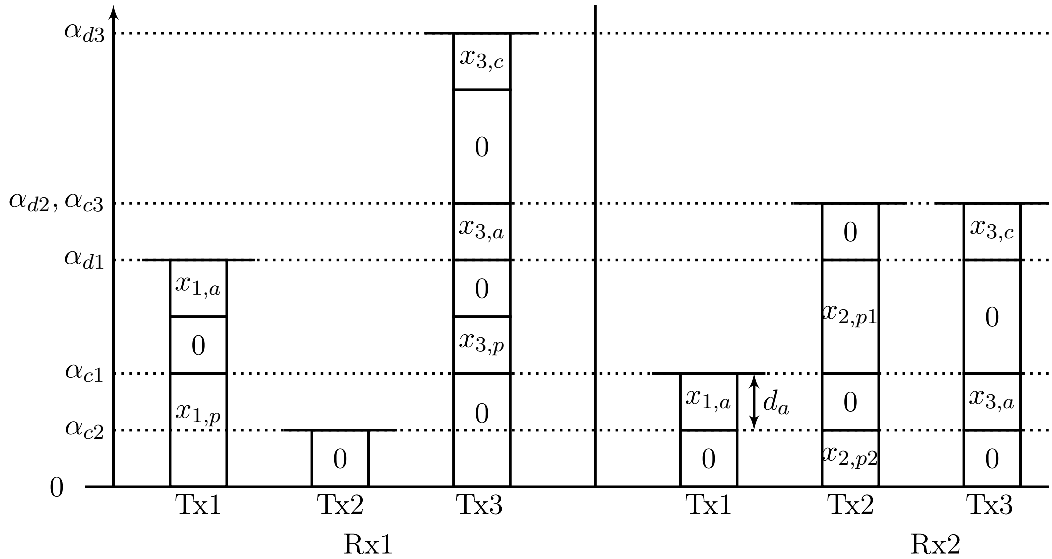


Figure 3.11: A graphical illustration showing the received signals at receivers 1 and 2 of the Gaussian PIMAC for sub-regime 3B.

Now, we are ready to choose the power allocation parameters. In what follows, first we choose the power allocation parameters of the common signal, next alignment signals and finally we deal with private signals. First, consider $x_{3,c}$. As it is shown in Figure 3.11, this signal is received at Rx1 at power level α_{d3}. Roughly speaking, this power level is a logarithmic representation of $P|h_{d3}|^2$. By dividing this received power by $|h_{d3}|^2$ which represents the channel from Tx3 to Rx1, we obtain the transmit power of $x_{3,c}$. Hence, at the moment we set the power of $x_{3,c}$ to P. Similarly, we can set the power of $x_{1,a}$ to P. Since $x_{3,a}$ and $x_{1,a}$ have to be aligned at Rx2, the alignment condition

$$P_{3,a}|h_{c3}|^2 \overset{!}{=} P_{1,a}|h_{c1}|^2$$

has to be satisfied. Hence, we set the power of $x_{3,a}$ to $\frac{P|h_{c1}|^2}{|h_{c3}|^2}$. Now, we need to choose the power of the private signals. First consider $x_{1,p}$, $x_{2,p1}$, and $x_{3,p}$. All these signals are received at the noise level at the undesired Rx. Hence, we set the power of $x_{1,p}$, $x_{2,p1}$, and $x_{3,p}$ to $\frac{1}{|h_{c1}|^2}$, $\frac{1}{|h_{c2}|^2}$, and $\frac{1}{|h_{c3}|^2}$, respectively. Finally, we set the power of $x_{2,p2}$. This signal is received at Rx2 at power level $\alpha_{c1} - d_a$. We can obtain d_a easily from

$$R_a = \min\{(n_{d3} - n_{c3}) - (n_{d1} - n_{c1}), n_{d1} + n_{c3} - n_{d3}\}$$

(given for the linear deterministic case). To obtain d_a, we replace the n-parameters in R_a with the α-parameters. Hence, we write

$$d_a = \min\{(\alpha_{d3} - \alpha_{c3}) - (\alpha_{d1} - \alpha_{c1}), \alpha_{d1} + \alpha_{c3} - \alpha_{d3}\}.$$

Hence, $x_{2,p2}$ is received at Rx2 at power level

$$\alpha_{c1} - d_a = \max\{(\alpha_{d3} - \alpha_{c3}) - (\alpha_{d1} - \alpha_{c1}), \alpha_{d1} - (\alpha_{d3} - \alpha_{c3})\}.$$

Writing this power level in linear scale, we obtain

$$\max\left\{\frac{P|h_{d3}|^2 P|h_{c1}|^2}{P|h_{c3}|^2 P|h_{d1}|^2}, \frac{P|h_{d1}|^2 P|h_{c3}|^2}{P|h_{d3}|^2}\right\}.$$

Note that this is the received power of $x_{2,p2}$ at Rx2. To obtain the transmit power of $x_{2,p2}$, we divide this expression by $|h_{d2}|^2$. Doing this, the allocated power to $x_{2,p2}$ is

$$\max\left\{\frac{P|h_{d3}|^2 P|h_{c1}|^2}{P|h_{c3}|^2 P|h_{d1}|^2 |h_{d2}|^2}, \frac{P|h_{d1}|^2 P|h_{c3}|^2}{P|h_{d3}|^2 |h_{d2}|^2}\right\}.$$

It is obvious that the chosen powers violate the power constraint P. To fix this, we scale the allocated powers by a constant such that the power constraints are satisfied. Hence, we write

$$P_{1,a} = aP, \qquad P_{1,p} = a\frac{1}{|h_{c1}|^2}, \qquad P_{2,p1} = a\frac{1}{|h_{c2}|^2},$$

$$P_{3,c} = aP, \qquad P_{3,a} = a\frac{P|h_{c1}|^2}{|h_{c3}|^2}, \qquad P_{3,p} = a\frac{1}{|h_{c3}|^2}$$

$$P_{2,p2} = a \max\left\{\frac{|h_{d3}|^2 |h_{c1}|^2}{|h_{c3}|^2 |h_{d1}|^2 |h_{d2}|^2}, \frac{P|h_{d1}|^2 |h_{c3}|^2}{|h_{d3}|^2 |h_{d2}|^2}\right\}$$

with

$$\begin{cases} P_{1,a} + P_{1,p} \overset{!}{\leq} P, \\ P_{2,p1} + P_{2,p2} \overset{!}{\leq} P \\ P_{3,c} + P_{3,a} + P_{3,p} \overset{!}{\leq} P \end{cases}.$$

All three power constraints will be satisfied if $a \leq \frac{1}{3}$. For sake of simplicity, we choose a such that its binary logarithm is integer. Hence, here we use $a = \frac{1}{4}$. Notice that since a does not grow with ρ, this scaling does not have any impact on the GDoF. Now, we want to obtain the power allocation parameter r for each signal. For instance, consider signal $x_{3,c}$ with power $P_{3,c} = \frac{P}{4}$. Then, we can write

$$r_{3,c} = \frac{\log_2\left(\frac{P_{3,c}}{P}\right)}{\log_2 \rho} = \frac{-2}{\log_2 \rho}$$

Similarly, for all other signals, we can write

$$\begin{aligned}
r_{1,a} &= \frac{-2}{\log_2 \rho}, & r_{1,p} &= \frac{-2}{\log_2 \rho} - \alpha_{c1}, & r_{2,p1} &= -\frac{2}{\log_2 \rho} - \alpha_{c2}, \\
r_{3,c} &= \frac{-2}{\log_2 \rho}, & r_{3,a} &= -\frac{2}{\log_2 \rho} + \alpha_{c1} - \alpha_{c3}, & r_{3,p} &= \frac{-2}{\log_2 \rho} - \alpha_{c3},
\end{aligned}$$

$$r_{2,p2} = \max\{(\alpha_{d3} - \alpha_{c3}) - (\alpha_{d1} - \alpha_{c1}), \alpha_{d1} - (\alpha_{d3} - \alpha_{c3})\} - \alpha_{d2} - \frac{2}{\log_2 \rho}.$$

3.B Suboptimality of TIN when $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$

Here, we show the suboptimality of TDMA-TIN when $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$ holds. To do this, we propose a scheme which outperforms TDMA-TIN in term of GDoF. This scheme is similar to IA-CP (proposed in subsection 3.3.3.2) from this aspect that both schemes are based on common and private signaling with interference alignment. The difference of the schemes is that while in IA-CP the interference alignment is done in the signal level space, in this scheme, the phase alignment is required [CJW10]. This scheme is called PA-CP (phase alignment with common and private signaling).

Before we present the scheme in details, we simplify our model as follows. In Gaussian PIMAC, the received signals of two receivers are given by

$$
\begin{aligned}
y_1 &= |h_{d1}|\mathrm{e}^{\mathrm{j}\varphi_{d1}}x_1 + |h_{c2}|\mathrm{e}^{\mathrm{j}\varphi_{c2}}x_2 + |h_{d3}|\mathrm{e}^{\mathrm{j}\varphi_{d3}}x_3 + z_1 \\
y_2 &= |h_{c1}|\mathrm{e}^{\mathrm{j}\varphi_{c1}}x_1 + |h_{d2}|\mathrm{e}^{\mathrm{j}\varphi_{d2}}x_2 + |h_{c3}|\mathrm{e}^{\mathrm{j}\varphi_{c3}}x_3 + z_2.
\end{aligned}
$$

Now, by defining $\tilde{x}_1 = \mathrm{e}^{\mathrm{j}\varphi_{d1}}x_1$, $\tilde{x}_3 = \mathrm{e}^{\mathrm{j}\varphi_{d3}}x_3$, $\tilde{y}_2 = \mathrm{e}^{-\mathrm{j}(\varphi_{c1}-\varphi_{d1})}y_2$, and $\tilde{z}_2 = \mathrm{e}^{-\mathrm{j}(\varphi_{c1}-\varphi_{d1})}z_2$, we write

$$
\begin{aligned}
y_1 &= |h_{d1}|\tilde{x}_1 + |h_{c2}|\mathrm{e}^{\mathrm{j}\varphi_{c2}}x_2 + |h_{d3}|\tilde{x}_3 + z_1 \\
\tilde{y}_2 &= |h_{c1}|\tilde{x}_1 + |h_{d2}|\mathrm{e}^{\mathrm{j}(\varphi_{d2}-\varphi_{c1}+\varphi_{d1})}x_2 + |h_{c3}|\mathrm{e}^{\mathrm{j}(\varphi_{c3}-\varphi_{d3}-\varphi_{c1}+\varphi_{d1})}\tilde{x}_3 + \tilde{z}_2.
\end{aligned}
$$

We proceed by defining $\tilde{x}_2 = \mathrm{e}^{\mathrm{j}(\varphi_{d2}-\varphi_{c1}+\varphi_{d1})}x_2$, $\theta = \varphi_{c2} - \varphi_{d2} + \varphi_{c1} - \varphi_{d1}$, and $\varphi = \varphi_{c3} - \varphi_{d3} - \varphi_{c1} + \varphi_{d1}$. Doing this, we obtain

$$
\begin{aligned}
y_1 &= |h_{d1}|\tilde{x}_1 + |h_{c2}|\mathrm{e}^{\mathrm{j}\theta}\tilde{x}_2 + |h_{d3}|\tilde{x}_3 + z_1 \\
\tilde{y}_2 &= |h_{c1}|\tilde{x}_1 + |h_{d2}|\tilde{x}_2 + |h_{c3}|\mathrm{e}^{\mathrm{j}\varphi}\tilde{x}_3 + \tilde{z}_2.
\end{aligned}
$$

As it is shown above the input-output relationship of any PIMAC can be rewritten such that all channels except two of them are real. Hence, without loss of generality, we present the transmission scheme for a simplified PIMAC with the input output relationship given as follows

$$
y_1 = |h_{d1}|x_1 + |h_{c2}|\mathrm{e}^{\mathrm{j}\theta}x_2 + |h_{d3}|x_3 + z_1 \tag{3.179}
$$

$$
y_2 = |h_{c1}|x_1 + |h_{d2}|x_2 + |h_{c3}|\mathrm{e}^{\mathrm{j}\varphi}x_3 + z_2. \tag{3.180}
$$

Note that all input and output signals in (3.179) and (3.180) are complex. Now, by writing the complex numbers in an alternative vector form with real entries (as in [CJW10]), we obtain

$$
\begin{bmatrix} y_1^R \\ y_1^I \end{bmatrix} = |h_{d1}| \begin{bmatrix} x_1^R \\ x_1^I \end{bmatrix} + |h_{c2}| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_2^R \\ x_2^I \end{bmatrix} + |h_{d3}| \begin{bmatrix} x_3^R \\ x_3^I \end{bmatrix} + \begin{bmatrix} z_1^R \\ z_1^I \end{bmatrix} \tag{3.181}
$$

$$
\begin{bmatrix} y_2^R \\ y_2^I \end{bmatrix} = |h_{c1}| \begin{bmatrix} x_1^R \\ x_1^I \end{bmatrix} + |h_{d2}| \begin{bmatrix} x_2^R \\ x_2^I \end{bmatrix} + |h_{c3}| \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} x_3^R \\ x_3^I \end{bmatrix} + \begin{bmatrix} z_2^R \\ z_2^I \end{bmatrix}, \tag{3.182}
$$

where x^R and x^I represent the real and imaginary part of signal x, respectively. Now, we are ready to present the transmission scheme. The transmitters split their messages as follows:

- Tx1 splits its message W_1 into $W_{1,p}^R$, $W_{1,p}^I$, and $W_{1,a}^I$ with rates $R_{1,p}^R$, $R_{1,p}^I$, and $R_{1,a}^I$, respectively.
- Tx2 splits its message W_2 into $W_{2,p}^R$ and $W_{2,p}^I$ with rates $R_{2,p}^R$, $R_{2,p}^I$, respectively.
- Tx3 splits its message W_3 into $W_{3,c}^R$, $W_{3,c}^I$, and $W_{3,a}^R$ with rates $R_{3,c}^R$, $R_{3,c}^I$, and $R_{3,a}^{R,n}$, respectively.

Note that in what follows we set $R_{1,a}^I = R_{3,a}^R = R_a$.

Encoding: Similar to the scheme presented in subsection 3.3.3.2, while the alignment messages are encoded using nested-lattice codes (Λ_f, Λ_c) with power 1 and rate R_a, other messages are encoded using Gaussian random codebooks. Encoding the alignment signals is done in the same way as discussed in subsection 3.3.3.2 for IA-CP. For example, the message $W_{1,a}^I$ is encoded into a length-n codeword $\lambda_{1,a}^I$ using the nested-lattice codebook (Λ_f, Λ_c). Then, the signal

$$x_{1,a}^{I,n} = \sqrt{P_{1,a}^I}\left[(\lambda_{1,a}^I - d_{1,a}^I) \mod \Lambda_c\right]$$

is constructed, where $P_{1,a}^I$ is the power allocated to this signal and $d_{1,a}^I$ is an n-dimensional random dither vector which is also known at the receivers. Similarly, Tx3 generates $x_{3,a}^R$. The generated signals and their powers are summarized in Table 3.7.

Then, each transmitter generates its signal as follows[6]

$$\begin{bmatrix} x_1^R \\ x_1^I \end{bmatrix} = \begin{bmatrix} x_{1,p}^R \\ x_{1,p}^I \end{bmatrix} + \begin{bmatrix} 0 \\ x_{1,a}^I \end{bmatrix}, \quad \begin{bmatrix} x_2^R \\ x_2^I \end{bmatrix} = \begin{bmatrix} x_{2,p}^R \\ x_{2,p}^I \end{bmatrix}, \quad \begin{bmatrix} x_3^R \\ x_3^I \end{bmatrix} = \begin{bmatrix} x_{3,c}^R \\ x_{3,c}^I \end{bmatrix} + x_{3,a}^R \begin{bmatrix} \sin(\varphi) \\ \cos(\varphi) \end{bmatrix}. \tag{3.183}$$

Note that the assigned powers given in Table 3.7 satisfy the power constraint.

Decoding: First, we present the decoding at Rx1. By using (3.183), we rewrite the received signal (3.181) as follows

$$\begin{aligned}\begin{bmatrix} y_1^R \\ y_1^I \end{bmatrix} =& |h_{d1}| \left(\begin{bmatrix} x_{1,p}^R \\ x_{1,p}^I \end{bmatrix} + \begin{bmatrix} 0 \\ x_{1,a}^I \end{bmatrix} \right) + |h_{c2}| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_{2,p}^R \\ x_{2,p}^I \end{bmatrix} \\ &+ |h_{d3}| \left(\begin{bmatrix} x_{3,c}^R \\ x_{3,c}^I \end{bmatrix} + x_{3,a}^R \begin{bmatrix} \sin(\varphi) \\ \cos(\varphi) \end{bmatrix} \right) + \begin{bmatrix} z_1^R \\ z_1^I \end{bmatrix}.\end{aligned}$$

Note that y_1^R and y_1^I are received over two orthogonal dimensions, i.e., real and imaginary part of the received signal y_1. Hence, Rx1 can decode each dimension

[6] We drop the superscript n since all sequences have the length n.

Encoded Message	Generated Signal	Power	Encoding
$W_{1,p}^{R}$	$x_{1,p}^{R,n}$	$P_{1,p}^{R} = \frac{1}{4}\frac{1}{\lvert h_{c1}\rvert^2}$	Gaussian random codebook
$W_{1,p}^{I}$	$x_{1,p}^{I,n}$	$P_{1,p}^{I} = \frac{1}{4}\frac{1}{\lvert h_{c1}\rvert^2}$	Gaussian random codebook
$W_{1,a}^{I}$	$x_{1,a}^{I,n}$	$P_{1,a}^{I} = \frac{P}{4}\min\left\{1, \frac{\lvert h_{c3}\rvert^2}{\lvert h_{c1}\rvert^2}\right\}$	nested-lattice codebook
$W_{2,p}^{R}$	$x_{2,p}^{R,n}$	$P_{2,p}^{R} = \frac{1}{4}\frac{1}{\lvert h_{c2}\rvert^2}$	Gaussian random codebook
$W_{2,p}^{I}$	$x_{2,p}^{I,n}$	$P_{2,p}^{I} = \frac{1}{4}\frac{1}{\lvert h_{c2}\rvert^2}$	Gaussian random codebook
$W_{3,c}^{R}$	$x_{3,c}^{R,n}$	$P_{3,c}^{R} = \begin{cases} \frac{P}{4} & \text{if } \frac{\lvert h_{d2}\rvert^2}{\lvert h_{c2}\rvert^2} \leq P\lvert h_{c3}\rvert^2 \\ 0 & \text{otherwise} \end{cases}$	Gaussian random codebook
$W_{3,c}^{I}$	$x_{3,c}^{I,n}$	$P_{3,c}^{I} = \begin{cases} \frac{P}{4} & \text{if } \frac{\lvert h_{d2}\rvert^2}{\lvert h_{c2}\rvert^2} \leq P\lvert h_{c3}\rvert^2 \\ 0 & \text{otherwise} \end{cases}$	Gaussian random codebook
$W_{3,a}^{R}$	$x_{3,a}^{R,n}$	$P_{3,a}^{R} = \frac{P}{4}\min\left\{1, \frac{\lvert h_{c1}\rvert^2}{\lvert h_{c3}\rvert^2}\right\}$	nested-lattice codebook

Table 3.7: The message encoding and the allocated power to each signal are given in this table.

without suffering from any interference caused by the other dimension. Here, Rx1 decodes first y_1^R and then y_1^I. Rx1 decodes y_1^R in the following order: $W_{3,c}^R \to W_{3,a}^R \to W_{1,p}^R$. The receiver decodes each of these signals while the remaining signals in y_1^R are treated as noise, then it removes the contribution of the decoded signal, and proceeds with the decoding. Similar to subection 3.3.3.2, we can write the conditions for the reliable decoding of $x_{3,c}^R$, $x_{3,a}^R$, and $x_{1,p}^R$ as follows

$$R_{3,c}^R \leq \frac{1}{2}\log_2\left(1 + \frac{|h_{d3}|^2 P_{3,c}^R}{\frac{1}{2} + I_{3,c}^{R,(1)}}\right) \tag{3.184}$$

$$R_a \leq \frac{1}{2}\log_2\left(1 + \frac{|h_{d3}|^2 P_{3,a}^R \sin^2(\varphi)}{\frac{1}{2} + |h_{d1}|^2 P_{1,p}^R + |h_{c2}|^2(\cos^2(\theta)P_{2,p}^R + \sin^2(\theta)P_{2,p}^I)}\right) \tag{3.185}$$

$$R_{1,p}^R \leq \frac{1}{2}\log_2\left(1 + \frac{|h_{d1}|^2 P_{1,p}^R}{\frac{1}{2} + |h_{c2}|^2(\cos^2(\theta)P_{2,p}^R + \sin^2(\theta)P_{2,p}^I)}\right), \tag{3.186}$$

where

$$I_{3,c}^{R,(1)} = |h_{d3}|^2 P_{3,a}^R \sin^2(\varphi) + |h_{d1}|^2 P_{1,p}^R + |h_{c2}|^2(\cos^2(\theta)P_{2,p}^R + \sin^2(\theta)P_{2,p}^I).$$

As long as the rates of the messages satisfy the conditions (3.184)-(3.186), Rx1 is able to decode $x_{3,c}^R$, $x_{3,a}^R$, and $x_{1,p}^R$ successfully. Hence, Rx1 is able to remove the interference caused by $x_{3,a}^R$ before decoding y_1^I. Doing this, Rx1 obtains

$$y_1^I - \cos(\varphi)x_{3,a}^R = |h_{d1}|(x_{1,p}^I + x_{1,a}^I) + |h_{c2}|(\sin(\theta)x_{2,p}^R + \cos(\theta)x_{2,p}^I) + |h_{d3}|x_{3,c}^I + z_1^I.$$

Next, Rx1 decodes in the following order: $W^I_{3,c} \rightarrow W^I_{1,a} \rightarrow W^I_{1,p}$. This successive decoding is done similar to above. The successful decoding can be accomplished as long as

$$R^I_{3,c} \leq \frac{1}{2}\log_2\left(1+\frac{|h_{d3}|^2 P^I_{3,c}}{\frac{1}{2}+I^{I,(1)}_{3,c}}\right) \tag{3.187}$$

$$R_a \leq \frac{1}{2}\log_2\left(1+\frac{|h_{d1}|^2 P^I_{1,a}}{\frac{1}{2}+|h_{d1}|^2 P^I_{1,p}+|h_{c2}|^2(\sin^2(\theta)P^R_{2,p}+\cos^2(\theta)P^I_{2,p})}\right) \tag{3.188}$$

$$R^I_{1,p} \leq \frac{1}{2}\log_2\left(1+\frac{|h_{d1}|^2 P^I_{1,p}}{\frac{1}{2}+|h_{c2}|^2(\sin^2(\theta)P^R_{2,p}+\cos^2(\theta)P^I_{2,p})}\right), \tag{3.189}$$

where

$$I^{I,(1)}_{3,c} = |h_{d1}|^2(P^I_{1,p}+P^I_{1,a}) + |h_{c2}|^2(\sin^2(\theta)P^R_{2,p}+\cos^2(\theta)P^I_{2,p}).$$

Now, we explain the decoding at Rx2. The received signal at Rx2 in (3.182) can be rewritten as

$$\begin{aligned}\begin{bmatrix} y^R_2 \\ y^I_2 \end{bmatrix} =& |h_{c1}|\left(\begin{bmatrix} x^R_{1,p} \\ x^I_{1,p} \end{bmatrix} + \begin{bmatrix} 0 \\ x^I_{1,a} \end{bmatrix}\right) + |h_{d2}|\begin{bmatrix} x^R_{2,p} \\ x^I_{2,p} \end{bmatrix} \\ &+ |h_{c3}|\left(\underbrace{\begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}}_{\boldsymbol{U}}\begin{bmatrix} x^R_{3,c} \\ x^I_{3,c} \end{bmatrix} + \begin{bmatrix} 0 \\ x^R_{3,a} \end{bmatrix}\right) + \begin{bmatrix} z^R_2 \\ z^I_2 \end{bmatrix}.\end{aligned}$$

Note that due to the rotation matrix $\boldsymbol{U}$, signals $x^R_{3,c}$ and $x^I_{3,c}$ are received in both components y^R_2 and y^I_2. In order to separate these two signals in two orthogonal dimensions, we rotate the vector $\begin{bmatrix} y^R_2 & y^I_2 \end{bmatrix}^T$ by multiplying $\boldsymbol{U}^T$ from right hand side to it. Note that $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}_2$. Hence, we have

$$\begin{aligned}\boldsymbol{U}^T\begin{bmatrix} y^R_2 \\ y^I_2 \end{bmatrix} =& \boldsymbol{U}^T\left(|h_{c1}|\left(\begin{bmatrix} x^R_{1,p} \\ x^I_{1,p} \end{bmatrix} + \begin{bmatrix} 0 \\ x^I_{1,a} \end{bmatrix}\right) + |h_{d2}|\begin{bmatrix} x^R_{2,p} \\ x^I_{2,p} \end{bmatrix} + |h_{c3}|\begin{bmatrix} 0 \\ x^R_{3,a} \end{bmatrix} + \begin{bmatrix} z^R_1 \\ z^I_1 \end{bmatrix}\right) \\ &+ |h_{c3}|\begin{bmatrix} x^R_{3,c} \\ x^I_{3,c} \end{bmatrix}.\end{aligned} \tag{3.190}$$

Now, Rx2 decodes $W^R_{3,c}$ and $W^I_{3,c}$ separately. This can be done successfully as long as

$$R^R_{3,c} \leq \frac{1}{2}\log_2\left(1+\frac{|h_{c3}|^2 P^R_{3,c}}{\frac{1}{2}+I^{R,(2)}_{3,c}}\right) \tag{3.191}$$

$$R^I_{3,c} \leq \frac{1}{2}\log_2\left(1+\frac{|h_{c3}|^2 P^I_{3,c}}{\frac{1}{2}+I^{I,(2)}_{3,c}}\right), \tag{3.192}$$

where

$$\begin{aligned} I_{3,c}^{R,(2)} =& \cos^2(\varphi)[|h_{c1}|^2 P_{1,p}^R + |h_{d2}|^2 P_{2,p}^R] \\ &+ \sin^2(\varphi)[|h_{c1}|^2 (P_{1,p}^I + P_{1,a}^I) + |h_{d2}|^2 P_{2,p}^I + |h_{c3}|^2 P_{3,a}^R], \\ I_{3,c}^{I,(2)} =& \sin^2(\varphi)[|h_{c1}|^2 P_{1,p}^R + |h_{d2}|^2 P_{2,p}^R] \\ &+ \cos^2(\varphi)[|h_{c1}|^2 (P_{1,p}^I + P_{1,a}^I) + |h_{d2}|^2 P_{2,p}^I + |h_{c3}|^2 P_{3,a}^R]. \end{aligned}$$

After successful decoding of $x_{3,c}^R$ and $x_{3,c}^I$, Rx2 rotates the vector in (3.190) back to $\begin{bmatrix} y_2^R & y_2^I \end{bmatrix}^T$ by multiplying $\boldsymbol{U}$ from right hand side to (3.190). Next, it removes the interference caused by $x_{3,c}^R$ and $x_{3,c}^I$ and obtains $\begin{bmatrix} \tilde{y}_2^R & \tilde{y}_2^I \end{bmatrix}^T$ given by

$$\begin{bmatrix} \tilde{y}_2^R \\ \tilde{y}_2^I \end{bmatrix} = |h_{c1}| \left(\begin{bmatrix} x_{1,p}^R \\ x_{1,p}^I \end{bmatrix} + \begin{bmatrix} 0 \\ x_{1,a}^I \end{bmatrix} \right) + |h_{d2}| \begin{bmatrix} x_{2,p}^R \\ x_{2,p}^I \end{bmatrix} + |h_{c3}| \begin{bmatrix} 0 \\ x_{3,a}^R \end{bmatrix} + \begin{bmatrix} z_1^R \\ z_1^I \end{bmatrix}.$$

Now, Rx2 proceeds by decoding $x_{2,p}^R$ while $x_{1,p}^R$ is treated as noise. Notice that all signals which are contained in $\tilde{y}_2^I$ do not cause any interference during decoding $x_{2,p}^R$. Reliable decoding of $x_{2,p}^R$ is possible as long as

$$R_{2,p}^R \leq \frac{1}{2} \log_2 \left(1 + \frac{|h_{d2}|^2 P_{2,p}^R}{\frac{1}{2} + |h_{c1}|^2 P_{1,p}^R} \right). \tag{3.193}$$

Next, Rx2 decodes $W_{2,p}^I \to f(W_{3,a}^R, W_{1,a}^I)$, where $f(W_{3,a}^R, W_{1,a}^I)$ is the sum $\left(\lambda_{3,a}^R + \lambda_{1,a}^I\right)$ mod Λ_c. Rx2 can decode $W_{2,p}^I$ successfully if

$$R_{2,p}^I \leq \frac{1}{2} \log_2 \left(1 + \frac{|h_{d2}|^2 P_{2,p}^I}{\frac{1}{2} + |h_{c1}|^2 P_{1,p}^I + |h_{c1}|^2 P_{1,a}^I + |h_{c3}|^2 P_{3,a}^R} \right). \tag{3.194}$$

Next, Rx2 removes the interference caused by $x_{2,p}^I$ and decodes $f(W_{3,a}^R, W_{1,a}^I)$. Note that $x_{3,a}^R$ and $x_{1,a}^I$ are aligned at Rx2 since the transmit power of $x_{3,a}^R$ and $x_{1,a}^I$ satisfy

$$|h_{c1}|^2 P_{1,a}^I = |h_{c3}|^2 P_{3,a}^R.$$

The decoding of $f(W_{3,a}^R, W_{1,a}^I)$ is done successfully as long as

$$R_a \leq \frac{1}{2} \left[\log_2 \left(\frac{1}{2} + \frac{|h_{c1}|^2 P_{1,a}^I}{\frac{1}{2} + |h_{c1}|^2 P_{1,p}^I} \right) \right]^+. \tag{3.195}$$

This schemes achieves

$$R_{\Sigma,\text{PA-CP}} = R_{1,p}^R + R_{1,p}^I + R_a + R_{2,p}^R + R_{2,p}^I + R_{3,c}^R + R_{3,c}^I + R_a, \tag{3.196}$$

where all rates above satisfy (3.184)-(3.189) and (3.191)-(3.195). By dividing the sum-rate in (3.196) by $\log_2 \rho$ and letting $\rho \to \infty$, we write the achievable GDoF

$$d_{\Sigma,\text{PA-CP}}(\boldsymbol{\alpha}) = d_{1,p}^R + d_{1,p}^I + d_a + d_{2,p}^R + d_{2,p}^I + d_{3,c}^R + d_{3,c}^I + d_a, \tag{3.197}$$

where

$$d^C_{1,p} = \frac{R^C_{1,p}}{\log_2 \rho}, \quad d_a = \frac{R_a}{\log_2 \rho}, \quad d^C_{2,p} = \frac{R^C_{2,p}}{\log_2 \rho}, \quad d^C_{3,c} = \frac{R^C_{3,c}}{\log_2 \rho}, \quad C \in \{R, I\}$$

as $\rho \to \infty$. The terms above can be obtained by substituting the powers of each signal given in Table 3.7 into the rate constraints in (3.184)-(3.189) and (3.191)-(3.195). Hence, we write

$$d^R_{1,p} = d^I_{1,p} = \frac{1}{2}(\alpha_{d1} - \alpha_{c1}) \tag{3.198}$$

$$d_a = \frac{1}{2}\min\{\alpha_{c1}, \alpha_{c3}\} \quad \text{if } \varphi \mod \pi \neq 0 \tag{3.199}$$

$$d^R_{2,p} = \frac{1}{2}(\alpha_{d2} - \alpha_{c2}) \tag{3.200}$$

$$d^I_{2,p} = \frac{1}{2}[(\alpha_{d2} - \alpha_{c2}) - \min\{\alpha_{c1}, \alpha_{c3}\}] \tag{3.201}$$

$$d^R_{3,c} = d^I_{3,c} = \frac{1}{2}(\alpha_{c3} - (\alpha_{d2} - \alpha_{c2}))^+. \tag{3.202}$$

Now, by substituting (3.198)-(3.202) into (3.197), we see that this schemes achieves a GDoF of

$$\begin{aligned} d_{\Sigma,\text{PA-CP}}(\boldsymbol{\alpha}) = & \underbrace{\alpha_{d1} - \alpha_{c1} + \alpha_{d2} - \alpha_{c2} + (\alpha_{c3} - [\alpha_{d2} - \alpha_{c2})]^+}_{d_{\Sigma,\text{TDMA-TIN}}(\boldsymbol{\alpha})} \\ & + \frac{1}{2}\min\{\alpha_{c1}, \alpha_{c3}\} \quad \text{if } \varphi \mod \pi \neq 0. \end{aligned} \tag{3.203}$$

Since PA-CP achieves a higher GDoF than TDMA-TIN as long as $\varphi \mod \pi \neq 0$, we conclude that TDMA-TIN cannot achieve the GDoF of PIMAC when $\alpha_{d3} - \alpha_{c3} = \alpha_{d1} - \alpha_{c1}$ except over a subset of channel coefficient values of measure 0.

4 TIN in the X-channel

In the previous chapter, we have studied the optimality of treating interference as noise (TIN) in the PIMAC. As it is shown, switching some transmitters off can expand the GDoF optimal regime of TIN in PIMAC. Apart from this, the new established upper bounds on the capacity of the PIMAC has shown the optimality of TIN with respect to the GDoF in some regimes which are outside the traditional definition of very-weak interference regime based on [GNAJ15, GSJ15]. The question which arises is whether this observation on optimality of TIN is only caused from the structure of the PIMAC or it might be partially extendable to a more general networks. To answer this question, we generalize the system model of the PIMAC by increasing the number of transmitters to M and considering independent messages between each transmitter to each receiver. The generated setup is known as an $M \times 2$ X-channel. Obviously, all achievable sum-rates in PIMAC are also achievable in the $M \times 2$ X-channel. However, evaluating the optimality of TIN in this channel based on the results in the PIMAC is not so trivial. The goal of this chapter is to study the performance of TIN in the $M \times 2$ X-channel from the GDoF perspective. To do this, we borrow the key ideas in establishing the bounds on the capacity from the analysis in PIMAC in order to extend them to the X-channel. Hence, here we do not study the linear deterministic model of the X-channel as an approximation but we will directly consider the Gaussian setup. In what follows, we present first the system model in details.

4.1 System Model of the $M \times 2$ Gaussian X-channel

The system we consider is an $M \times 2$ Gaussian X-channel which consists of M transmitters and two receivers (Fig. 4.1). Each transmitter wants to communicate with both receivers. Namely, transmitter i (Txi) wants to send the messages W_{ji} to receiver j (Rxj), where $i \in \{1, \ldots, M\}$ and $j \in \{1, 2\}$.

The message W_{ji} is a random variable, uniformly distributed over the message set

$$\mathcal{W}_{ji} = \left\{1, \ldots, \lfloor 2^{nR_{ji}} \rfloor\right\}, \tag{4.1}$$

where R_{ji} denotes the rate of the message. Txi uses an encoding function f_i to encode its message (W_{1i}, W_{2i}) into a length n complex-valued symbols

$$X_i^n = f_i(W_{1i}, W_{2i}). \tag{4.2}$$

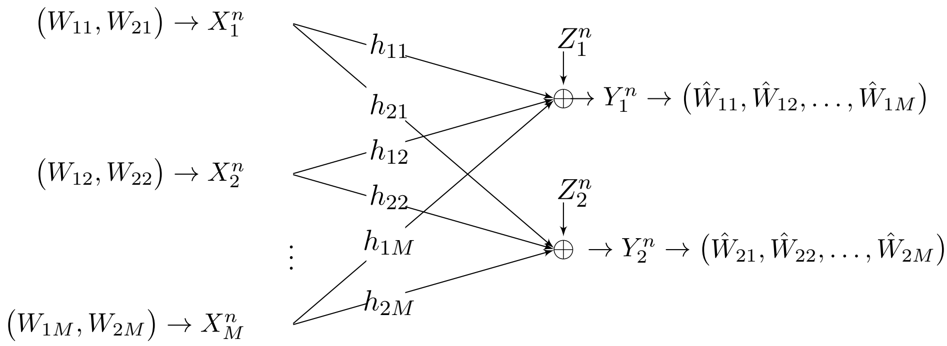


Figure 4.1: System model of the $M \times 2$ Gaussian X-channel.

The codeword X_i^n satisfies the power constraint of Txi, which is given by

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}[|X_i[t]|^2] = P_i \leq P. \tag{4.3}$$

Now, consider the received signal. At time instant $t \in \{1, \cdots, n\}$, Rxj receives[1]

$$y_j[t] = \sum_{i=1}^{M} h_{ji} x_i[t] + z_j[t], \tag{4.4}$$

where $z_j[t]$, $j \in \{1, 2\}$ is a realization of a random variable $Z_j \sim \mathbb{C}\mathcal{N}(0, 1)$ which is independent from all other random variables and independent and identically distributed (i.i.d.) over time, and the constant h_{ji} represents the complex (static) channel coefficient between Txi and Rxj which is assumed to be known at all nodes. Here, we are interested in the interference limited scenario and thus, we have the following assumption

$$P \min\{|h_{ji}|^2\} > 1, \quad \forall\, i \in \{1, \ldots, M\}, j \in \{1, 2\}. \tag{4.5}$$

After n transmissions, Rxj obtains Y_j^n and uses the decoding function g_j to decode W_{ji}, $i \in \{1, \ldots, M\}$. Hence, we have

$$(\hat{W}_{j1}, \ldots, \hat{W}_{jM}) = g_j(Y_j^n). \tag{4.6}$$

The messages sets, encoding functions, and decoding functions constitute a code for the channel which is denoted by an $\left(\left(2^{nR_{11}}, 2^{nR_{12}}, \ldots, 2^{nR_{2M}}\right), n\right)$ code.

[1]The time index t will be suppressed henceforth for clarity unless necessary.

An error E_{ji} occurs if for some $i \in \{1, \ldots, M\}$ and $j \in \{1, 2\}$, $\hat{W}_{ji} \neq W_{ji}$. We assume that the message tuple $(W_{11}, W_{12}, \ldots, W_{2M})$ is uniformly distributed over $[1 : 2^{nR_{11}}] \times [1 : 2^{nR_{12}}] \times \ldots \times [1 : 2^{nR_{2M}}]$. The average error probability $\mathbb{P}^{(n)}$ defined as

$$\mathbb{P}^{(n)} = \text{Prob}\left((\hat{W}_{11}, \hat{W}_{12}, \ldots, \hat{W}_{2M}) \neq (W_{11}, W_{12}, \ldots, W_{2M})\right). \quad (4.7)$$

Reliable communication takes place if the error probability can be made arbitrarily small by increasing n. A rate tuple $(R_{11}, R_{12}, \ldots, R_{2M})$ is said to be achievable if there exists a sequence of $\left(\left(2^{nR_{11}}, 2^{nR_{12}}, \ldots, 2^{nR_{2M}}\right), n\right)$ codes such that $\mathbb{P}^{(n)} \to 0$ as $n \to \infty$. The set of all achievable rate tuples is the capacity region of the $M \times 2$ X-channel denoted by $\mathcal{C}$. Further, the sum-capacity is defined as the maximum achievable sum-rate, i.e.,

$$C_\Sigma = \max_{(R_{11}, R_{21}, \ldots, R_{2M}) \in \mathcal{C}} R_\Sigma, \quad (4.8)$$

where $R_\Sigma = \sum_{i=1}^{M} \sum_{j=1}^{2} R_{ji}$. The sum-capacity C_Σ is the maximum achievable sum-rate for all rate tuples in the capacity region $\mathcal{C}$. Furthermore, we define

$$\alpha_{ji} = \frac{\log_2(P|h_{ji}|^2)}{\log_2(\rho)}, \quad (4.9)$$

where ρ denotes the received SNR for a reference point-to-point (P2P) channel. Using the α-parameter, we define the generalized degrees-of-freedom (GDoF) of the channel as in the previous chapter

$$d_\Sigma(\boldsymbol{\alpha}) = \lim_{\rho \to \infty} \frac{C_\Sigma(\rho, \boldsymbol{\alpha})}{\log_2(\rho)}, \quad (4.10)$$

where $\boldsymbol{\alpha}$ is a vector which contains all α_{ji}.

The focus of this chapter is studying GDoF optimality of TIN for the $M \times 2$ Gaussian X-channel. Next, we introduce the transmission strategy.

4.2 TIN in the $M \times 2$ X-Channel

In this section, we want to study the performance of TIN in the $M \times 2$ X-channel. After a brief discussion on the achievable sum-rate of TIN, we restrict ourself to its achievable GDoF in the $M \times 2$ X-channel. Based on the achievable GDoF, we will classify TIN into two groups which will be introduced at the end of this section.

4.2.1 Achievable sum-rate of TIN

Here, we want to find the achievable sum-rate using TIN at the receiver side and Gaussian encoding at the transmitter side. Suppose that Txi ($1 \leq i \leq M$) encodes its message W_{ji}, $j = \{1, 2\}$, into X_{ji}^n using a Gaussian codebook with power P_{ji} and rate R_{ji} and sends the sum of the codewords. Hence, we can write

$$X_i^n = \sum_{j=1}^{2} X_{ji}^n, \; P_i = \sum_{j=1}^{2} P_{ji}. \tag{4.11}$$

Rxj decodes its desired messages $W_{j1}, \dots, W_{jM}$ as in a multiple access channel (MAC) by treating the undesired signals, i.e., $X_{j'1}^n, \dots, X_{j'M}^n$ ($j' \in \{1, 2\}$, $j' \neq j$), as noise. Doing this, we can write the achievable sum-rate as follows

$$R_{\Sigma,\text{TIN}}(\boldsymbol{P}) = \log_2\left(1 + \frac{\sum_{i=1}^{M} P_{1i}|h_{1i}|^2}{1 + \sum_{i=1}^{M} P_{2i}|h_{1i}|^2}\right) + \log_2\left(1 + \frac{\sum_{i=1}^{M} P_{2i}|h_{2i}|^2}{1 + \sum_{i=1}^{M} P_{1i}|h_{2i}|^2}\right), \tag{4.12}$$

where $\boldsymbol{P}$ is a vector which contains all P_{ji}. All achievable sum-rates of TIN is given in the following proposition.

Proposition 9. *Any sum-rate*

$$\begin{aligned} R_\Sigma \leq \max_{\boldsymbol{P}} \quad & R_{\Sigma,TIN}(\boldsymbol{P}) \\ \text{subject to} \quad & \sum_{j=1}^{2} P_{ji} \leq P, \quad \forall i \in \{1, \dots, M\} \end{aligned} \tag{4.13}$$

is achievable by using TIN at the receivers and Gaussian codebook at the transmitters.

Since our goal is to study the GDoF optimality of TIN, we present next its maximum achievable GDoF.

4.2.2 Achievable GDoF of TIN

Here, we establish the achievable GDoF in an $M \times 2$ X-channel if the decoding strategy at the receiver side is restricted to TIN and the transmitters use only Gaussian encoding. Further, power allocation at the transmitter side is utilized to maximize the achievable GDoF. In the following lemma, we present the achievable GDoF using TIN.

Lemma 17. *The maximum achievable GDoF in an $M \times 2$ X-channel using TIN at the receiver side and Gaussian encoding at the transmitter side is given by*

$$d_{\Sigma,TIN}(\boldsymbol{\alpha}) = \max_{i_1,i_2} \max\{\alpha_{1i_1}, \alpha_{2i_2}, (\alpha_{1i_1} - \alpha_{1i_2})^+ + (\alpha_{2i_2} - \alpha_{2i_1})^+\}. \tag{4.14}$$

Proof. To find the maximum achievable GDoF using TIN at the receiver side and Gaussian encoding at the transmitter side, we need to first translate the achievable sum-rate in (4.12) into the achievable GDoF. To do this, we use the following definitions

$$\beta_{ji} = \frac{\log_2(P_{ji}|h_{ji}|^2)}{\log_2 \rho} \qquad \gamma_{ji} = \frac{\log_2(P_{j'i}|h_{ji}|^2)}{\log_2 \rho}. \tag{4.15}$$

In words, while β_{ji} indicates the strength of the desired signal received at Rxj from Txi, γ_{ji} indicates the strength of the interference signal received at Rxj from Txi. Using these definitions and that of the GDoF, we obtain

$$d_{\Sigma,\text{TIN}} = \left[\max_i\{\beta_{1i}\} - \left(\max_i\{\gamma_{1i}\}\right)^+\right]^+ + \left[\max_i\{\beta_{2i}\} - \left(\max_i\{\gamma_{2i}\}\right)^+\right]^+ \quad (4.16)$$

from the achievable sum-rate in (4.12). Let $\max_i\{\beta_{1i}\} = \beta_{1i_1}$, $\max_i\{\beta_{2i}\} = \beta_{2i_2}$, $\max_i\{\gamma_{1i}\} = \gamma_{1i_3}$, and $\max_i\{\gamma_{2i}\} = \gamma_{2i_4}$. Now, consider the case that Txi_1 sends only to receiver 1 and Txi_2 sends only to receiver 2 and the remaining transmitters are silent. In this case, we achieve

$$d_{\Sigma,\text{TIN}} = \left[\beta_{1i_1} - (\gamma_{1i_2})^+\right]^+ + \left[\beta_{2i_2} - (\gamma_{2i_1})^+\right]^+, \quad (4.17)$$

which is larger than the achievable GDoF in (4.16) since $\gamma_{1i_2} \leq \gamma_{1i_3}$ and $\gamma_{2i_1} \leq \gamma_{2i_4}$ by assumption. Hence, we conclude that as long as the decoding is restricted to TIN and Gaussian codebooks are used at the Tx's, we do not gain from the GDoF point of view, if more than one Tx communicates with one receiver or one transmitter communicates with more than one receiver in an $M \times 2$ X-channel. Hence, the optimal scheme is to have maximum two active transmitter while the receivers use TIN. Now, from the definitions in (4.15), we have

$$\beta_{1i_1} = \frac{\log_2\left(P_{1i_1}|h_{1i_1}|^2\right)}{\log_2\rho} = \frac{\log_2\left(\frac{P_{1i_1}P|h_{1i_1}|^2}{P}\right)}{\log_2\rho} = \alpha_{1i_1} + \frac{\log_2\left(\frac{P_{1i_1}}{P}\right)}{\log_2\rho}. \quad (4.18)$$

Similarly, we can write

$$\gamma_{2i_1} = \alpha_{2i_1} + \frac{\log_2\left(\frac{P_{1i_1}}{P}\right)}{\log_2\rho}. \quad (4.19)$$

Hence, we have $\gamma_{2i_1} = \beta_{1i_1} + (\alpha_{2i_1} - \alpha_{1i_1})$. Similarly, we obtain $\gamma_{1i_2} = \beta_{2i_2} + (\alpha_{1i_2} - \alpha_{2i_2})$. Therefore, we can rewrite (4.17) as follows

$$d_{\Sigma,\text{TIN}} = \left[\beta_{1i_1} - (\beta_{2i_2} + (\alpha_{1i_2} - \alpha_{2i_2}))^+\right]^+ + \left[\beta_{2i_2} - (\beta_{1i_1} + (\alpha_{2i_1} - \alpha_{1i_1}))^+\right]^+. \quad (4.20)$$

Now, we want to choose the β-parameters such that we obtain the maximum achievable GDoF. Notice that since $0 < P_{1i_1}, P_{2i_2} \leq P$ (cf. (4.3)), the β-parameters satisfy $-\infty < \beta_{1i_1} \leq \alpha_{1i_1}$ and $-\infty < \beta_{2i_2} \leq \alpha_{2i_2}$. Hence, we have the following optimization problem

$$\begin{aligned} \max_{\beta_{1i_1},\beta_{2i_2}} \quad & \left[\beta_{1i_1} - (\beta_{2i_2} + k_1)^+\right]^+ + \left[\beta_{2i_2} - (\beta_{1i_1} + k_2)^+\right]^+ \\ \text{subject to} \quad & \beta_{1i_1} \leq \alpha_{1i_1}, \quad \beta_{2i_2} \leq \alpha_{2i_2}, \end{aligned} \quad (4.21)$$

where $k_1 = \alpha_{1i_2} - \alpha_{2i_2}$ and $k_2 = \alpha_{2i_1} - \alpha_{1i_1}$ are both constants and independent of the optimization parameters β_{1i_1}, β_{2i_2}. In what follows, we want to show that choosing $(\beta_{1i_1}, \beta_{2i_2}) = (\beta_{1i_1}^{\text{opt}}, \beta_{2i_2}^{\text{opt}})$, with $\beta_{1i_1}^{\text{opt}} \in \{-\infty, \alpha_{1i_1}\}$ and $\beta_{2i_2}^{\text{opt}} \in \{-\infty, \alpha_{2i_2}\}$, maximizes the objective function of optimization problem (4.21). In other words, the optimal power control at the transmitter side is simply binary, in which transmitters either send with full power or they are inactive. To prove this statement, we need to show that the GDoF obtained by setting $(\beta_{1i_1}, \beta_{2i_2}) = (\beta_1, \beta_2)$, where $\beta_1 \in (-\infty, \alpha_{1i_1})$ and $\beta_2 \in (-\infty, \alpha_{2i_2})$, i.e.,

$$d_{\Sigma,\text{TIN}}(\beta_1, \beta_2) = \Bigg[\underbrace{\beta_1 - \Big(\underbrace{\beta_2 + k_1}_{\Lambda_1}\Big)^+}_{\Gamma_1}\Bigg]^+ + \Bigg[\underbrace{\beta_2 - \Big(\underbrace{\beta_1 + k_2}_{\Lambda_2}\Big)^+}_{\Gamma_2}\Bigg]^+ \tag{4.22}$$

is less than or equal to $d_{\text{TIN}}^{\text{opt}}$, where

$$d_{\Sigma,\text{TIN}}^{\text{opt}} = \max\{d_{\text{TIN}}(-\infty, \alpha_{2i2}), d_{\text{TIN}}(\alpha_{1i_1}, \infty), d_{\text{TIN}}(\alpha_{1i_1}, \alpha_{2i_2})\}. \tag{4.23}$$

To this end, we fix the parameters β_{1i_1} and β_{2i_2} to arbitrary values. Next, we show that by either increasing or decreasing only one of these parameter (β_{1i_1} or β_{2i_2}), the GDoF cannot deteriorate. Suppose that $(\beta_{1i_1}, \beta_{2i_2}) = (\beta_1, \beta_2)$ is one possible optimal solution for the optimization problem (4.21), where $\beta_1 \in (-\infty, \alpha_{1i_1})$ and $\beta_2 \in (-\infty, \alpha_{2i_2})$. Depending on the values of β_1 and β_2, one can distinguish between two cases: $\Gamma_1 \geq 0$ or $\Gamma_1 < 0$ which will be discussed in what follows.

- $\Gamma_1 \geq 0$: If Γ_1 and Γ_2 are nonnegative, then increasing β_1 does not change the objective value. If at some value of β_1, Γ_2 becomes negative, then increasing β_1 further increases the objective function. Therefore, for all (β_1, β_2) where $\Gamma_1 \geq 0$, setting $\beta_1 = \alpha_{1i_1}$ maximizes the objective function.
- $\Gamma_1 < 0$: For (β_1, β_2) for which $\Gamma_1 < 0$, decreasing β_1 does not decrease the objective function. Hence, setting $\beta_1 = -\infty$ leads to either the same value or a larger one for the objective function.

From the discussion above, we conclude that for any $\beta_{2i_2} = \beta_2$, the optimal value of β_{1i_1} (for problem (4.21)) is at α_{1i_1} or $-\infty$. Similarly, we can show that for any β_1 the optimal value of β_{2i_2} is from the set $\{\alpha_{2i_2}, -\infty\}$. Hence, one solution for the optimization problem (4.21) is given by

$$\max\left\{\alpha_{1i_1}, \alpha_{2i_2}, (\alpha_{1i_1} - \alpha_{1i_2})^+ + (\alpha_{2i_2} - \alpha_{2i_1})^+\right\}. \tag{4.24}$$

Now, in order to write the maximum achievable GDoF using TIN in an $M \times 2$ X-channel, we need to choose i_1, i_2 such that the expression in (4.24) is maximized. Hence, we write

$$d_{\text{TIN}}(\boldsymbol{\alpha}) = \max_{i_1, i_2} \max\{\alpha_{1i_1}, \alpha_{2i_2}, (\alpha_{1i_1} - \alpha_{1i_2})^+ + (\alpha_{2i_2} - \alpha_{2i_1})^+\}. \tag{4.25}$$

This completes the proof. □

Variants of TIN

Now, we are ready to introduce the different variants of TIN in the $M \times 2$ X-channel based on the achievable GDoF presented in Lemma 17. In this lemma, it is shown that the GDoF optimal transmission strategy combined with TIN at the receivers is to let at most two transmitters to be active. Moreover, each transmitter should serve only one receiver. It means that, it is GDoF optimal to reduce an $M \times 2$ X-channel to either a 2-user IC or a P2P channel when the receivers use TIN for decoding. Henceforth, based on Lemma 17, we consider two different variants of TIN. While in the first variant, the $M \times 2$ X-channel is reduced to a P2P channel, in the second variant, the X-channel is decomposed to a 2-user IC. In what follows, the achievable sum-rates of these two variants are presented.

- P2P-TIN[2] ($M \times 2$ X-channel $\rightarrow$ P2P channel): In this variant, $M-1$ Tx's are inactive while only one Tx communicates with one Rx. Here, one has to consider $2M$ different P2P channels and pick the one which provides the highest GDoF. Hence, we obtain

$$d_{\Sigma,\mathrm{P2P-TIN}}(\boldsymbol{\alpha}) = \max_{i_1,j_1}\{\alpha_{j_1 i_1}\}. \tag{4.26}$$

- 2-IC-TIN ($M \times 2$ X-channel $\rightarrow$ 2-user IC with TIN): In this variant, we switch $M-2$ users off and we dedicate only one Tx to each receiver. Doing this, the $M \times 2$ X-channel is decomposed into a 2-user IC. Here, one has to consider $M(M-1)$ different 2-user IC's in an $M \times 2$ X-channel. Among all these IC's, the one which achieves the highest GDoF determines the maximum achievable GDoF. Hence, the following GDoF is achievable by using TIN in an $M \times 2$ X-channel

$$d_{\Sigma,\text{2-IC-TIN}} = \max_{i_1,i_2}\left\{(\alpha_{1i_1} - \alpha_{1i_2})^+ + (\alpha_{2i_2} - \alpha_{2i_1})^+\right\}. \tag{4.27}$$

While the optimality of the IC-TIN, has been studied in [GSJ15], the optimality of the other type, i.e., P2P-TIN, has not been considered. However, there are cases in which P2P-TIN outperforms 2-IC-TIN. As an example, suppose $M = 3$ and

$$(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}) = (5, 2, 3, 2, 3, 3).$$

In this case, the index of the active transmitter and receiver for the best P2P channel is $i_1 = 1$ and $j_1 = 1$. Moreover, the index of the transmitters and receivers of the best 2-user IC is $i_1, j_1 = 1$ and $i_2, j_2 = 2$. Hence, $d_{\Sigma,\text{2-IC-TIN}} = 4$ which is less than the achievable GDoF of the P2P channel between Tx1 to Rx1, i.e., $d_{\Sigma,\text{P2P-TIN}} = 5$.

[2] Similar to PIMAC in X-channel, we cannot exclude P2P-TIN from the variants of TIN, since here the decoding strategy at the receiver is as in a P2P channel without taking the impact of interference into account.

4.3 Sum-capacity Upper Bounds

To show the GDoF optimality of TIN, we need to establish some upper bounds which coincide with the achievable GDoF. To do this, we need to first bound the capacity of the $M \times 2$ X-channel. In the following, we present some upper bounds on the sum-capacity of the $M \times 2$ X-channel. All these upper bounds are genie-aided bounds. It means that to establish them, some side information is provided to some users. Notice that providing extra information to the users can not decrease the sum-capacity of this network. Hence, the capacity of the enhanced channel (i.e., the channel after providing the side information) serves as an upper bound for the original channel. The following lemma presents the first upper bound.

Lemma 18. *Suppose $j', j \in \{1, 2\}$, $j \neq j'$, and $\tilde{W}_j = \{W_{j1}, W_{j2}, \ldots, W_{jM}\}$. The sum-capacity of the Gaussian $M \times 2$ X-channel is upper bounded by*

$$C_\Sigma \leq \log_2 \left(1 + \sum_{i=1}^{M} |h_{ji}|^2 P\right) + \lim_{n\to\infty} \frac{1}{n}\Theta_j \tag{4.28}$$

where

$$\Theta_j = h\left(\sum_{i=1}^{M} h_{j'i} X_i^n + Z_{j'}^n | \tilde{W}_j\right) - h\left(\sum_{i=1}^{M} h_{ji} X_i^n + Z_j^n | \tilde{W}_j\right). \tag{4.29}$$

Proof. In order to obtain the upper bound in (4.28), we provide $\tilde{W}_j$ as side information to Rxj', where $j, j' \in \{1, 2\}$, $j \neq j'$, and $\tilde{W}_j = \{W_{j1}, W_{j2}, \ldots, W_{jM}\}$. In words, $\tilde{W}_j$ is the set of messages desired at Rxj. Now, we use Fano's inequality to write

$$n(R_\Sigma - \epsilon_n) \leq I(\tilde{W}_j; Y_j^n) + I(\tilde{W}_{j'}; Y_{j'}^n, \tilde{W}_j), \tag{4.30}$$

where $\epsilon_n \to 0$ as $n \to \infty$. For the sake of simplicity suppose that $j = 1$, the similar upper bound can be established for the case that $j = 2$. Next, by using the chain rule and the fact that the messages are independent from each other, we obtain

$$\begin{aligned}
n(R_\Sigma - \epsilon_n) \leq& I(\tilde{W}_1; Y_1^n) + I(\tilde{W}_2; Y_2^n | \tilde{W}_1) \\
=& h(Y_1^n) - h\left(\sum_{i=1}^{M} h_{1i} X_i^n + Z_1^n | \tilde{W}_1\right) \\
&+ h\left(\sum_{i=1}^{M} h_{2i} X_i^n + Z_2^n | \tilde{W}_1\right) - h\left(\sum_{i=1}^{M} h_{2i} X_i^n + Z_2^n | \tilde{W}_1, \tilde{W}_2\right) \\
\stackrel{(a)}{=}& h(Y_1^n) - h\left(\sum_{i=1}^{M} h_{1i} X_i^n + Z_1^n | \tilde{W}_1\right) \\
&+ h\left(\sum_{i=1}^{M} h_{2i} X_i^n + Z_2^n | \tilde{W}_1\right) - h\left(Z_2^n | \tilde{W}_1, \tilde{W}_2\right),
\end{aligned} \tag{4.31}$$

where step (a) follows since knowing $\tilde{W}_1$ and $\tilde{W}_2$, X_i, $i = 1, \ldots, M$ is known and can be removed. To get an upper bound on (4.31), we bound first $h(Y_1^n) - h(Z_2^n|\tilde{W}_1, \tilde{W}_2)$ as follows

$$\begin{aligned} h(Y_1^n) - h(Z_2^n|\tilde{W}_1, W_2) &\overset{(b)}{\leq} n[h(Y_{1G}) - h(Z_2)] \\ &= n \log_2 \left(1 + \sum_{i=1}^{M} |h_{1i}|^2 P_i\right) \\ &\overset{(c)}{\leq} n \log_2 \left(1 + \sum_{i=1}^{M} |h_{1i}|^2 P\right), \end{aligned} \tag{4.32}$$

where the subscript G indicates that the inputs are i.i.d. and Gaussian distributed i.e., for all $i \in \{1, \ldots, M\}$, $X_{iG} \sim \mathbb{C}\mathcal{N}(0, P)$. Here, step (b) holds since Z_2 is i.i.d. and independent of all other random variables, and we used [CT06, corllary to Theorem 8.6.2] and the fact that the Gaussian distribution maximizes the differential entropy for a fixed variance [CT06, Therorem 8.6.5], i.e.,

$$\mathbb{E}\left[\left(\sum_{i=1}^{M} h_{1i} X_{iG} + Z_1\right)^2\right] = \sum_{i=1}^{M} |h_{1i}|^2 P_i + 1. \tag{4.33}$$

Moreover, step (c) follows since $P_i \leq P$ (cf. (4.3)). Now, by substituting (4.32) into (4.31), we obtain the following upper bound for the sum-rate

$$n(R_\Sigma - \epsilon_n) \leq n \left[\log_2 \left(1 + \sum_{i=1}^{M} |h_{1i}|^2 P\right) + \Theta_1\right], \tag{4.34}$$

where Θ_j with $j \in \{1, 2\}$ is defined as in (4.29). By applying the similar steps as above to (4.30) for $j = 2$, we obtain

$$n(R_\Sigma - \epsilon_n) \leq n \left[\log_2 \left(1 + \sum_{i=1}^{M} |h_{2i}|^2 P\right) + \Theta_2\right]. \tag{4.35}$$

Dividing (4.34), (4.35) by n and letting $n \to \infty$, we obtain the bound given in (4.28). □

Notice that we can show the optimality of reducing an $M \times 2$ X-channel to a P2P channel, if we can bound the expression Θ_j in (4.28) by a constant. Before doing this, we will present the upper bounds which are required for showing the optimality of 2-IC-TIN.

Lemma 19. *Let $\mathcal{M} = \{1, 2, ..., M\}$. Furthermore, let $\{i_1, i_2\}$, $\mathcal{I}^1$, $\mathcal{I}^2$, and $\mathcal{I}^w$ be four arbitrarily chosen but disjoint subsets of $\mathcal{M}$, such that $\{i_1, i_2\} \cup \mathcal{I}^1 \cup \mathcal{I}^2 \cup \mathcal{I}^w = \mathcal{M}$*

and $2+|\mathcal{I}^1|+|\mathcal{I}^2|+|\mathcal{I}^w|=M$*. Then it holds that the sum-capacity of the Gaussian* $M\times 2$ *X-channel is upper bounded by*

$$C_\Sigma \leq \sum_{j=1}^{2}\log_2\left(1+P|h_{ji_{j'}}|^2+\sum_{k\in\{\mathcal{I}^{j'},\mathcal{I}^w\}}P|h_{jk}|^2+\frac{|h_{ji_j}|^2}{|h_{j'i_j}|^2}\right)+\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{2}\Delta_j, \tag{4.36}$$

where $j'\in\{1,2\}$*,* $j\neq j'$*, and*

$$\begin{aligned}\Delta_j =& h\left(h_{j'i_j}X_{i_j}^n+\sum_{k\in\mathcal{I}^j}\frac{h_{j'i_j}h_{jk}}{h_{ji_j}}X_k^n+N_j^n|W_{j'i_j},W_{j'\mathcal{I}^j}\right)\\ &-h\left(h_{j'i_j}X_{i_j}^n+\sum_{k\in\mathcal{I}^j}h_{j'k}X_k^n+Z_{j'}^n|W_{j'i_j},W_{j'\mathcal{I}^j}\right).\end{aligned}$$

Proof. In this upper bound, we split the transmitters $\{1,\ldots,M\}$ arbitrarily into four non-intersecting sets, i.e., $\{i_1,i_2\}$, $\mathcal{I}^1$, $\mathcal{I}^2$, $\mathcal{I}^w$. We provide (V_1,S_1^n) and (V_2,S_2^n) as side information to Rx1 and Rx2, respectively, where $V_1=\{W_{2i_1},W_{2\mathcal{I}^1}\}$, $V_2=\{W_{1i_2},W_{1\mathcal{I}^2}\}$, and for $j=1,2$

$$S_j^n=c_j\left(h_{ji_j}X_{i_j}^n+\sum_{k\in\mathcal{I}^j}h_{jk}X_k^n\right)+N_j^n, \tag{4.37}$$

$$c_j=\begin{cases}\frac{h_{2i_1}}{h_{1i_1}} & \text{if } j=1\\ \frac{h_{1i_2}}{h_{2i_2}} & \text{if } j=2\end{cases}, \tag{4.38}$$

where $N_1,N_2\sim\mathbb{C}\mathcal{N}(0,1)$ are independent of all other random variables and each other and are i.i.d. over time. Suppose that $\tilde{W}_j$ is the set of messages desired at Rxj, i.e., $\{W_{j1},\ldots,W_{jM}\}$. Notice that for $j=1,2$, $V_j\cap\tilde{W}_j=\varnothing$, and hence V_j and $\tilde{W}_j$ are independent from each other. The generated channel after providing the side information to Rx's is shown in Fig. 4.2. Now, by using Fano's inequality we obtain

$$n(R_\Sigma-\epsilon_n)\leq\sum_{j=1}^{2}I(\tilde{W}_j;Y_j^n,S_j^n,V_j), \tag{4.39}$$

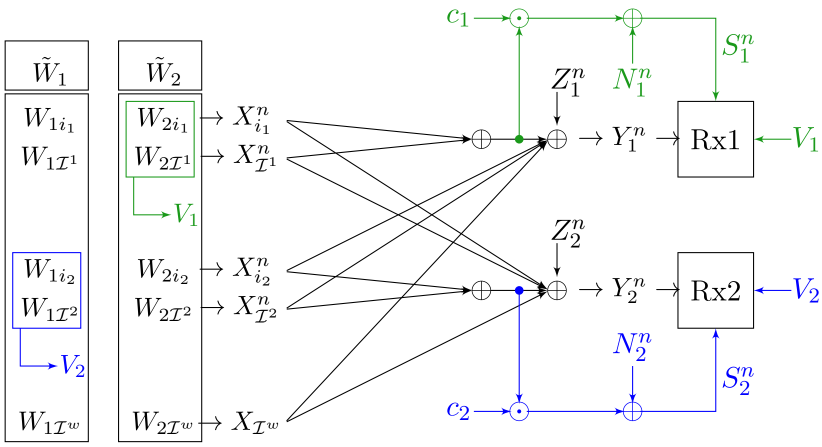


Figure 4.2: This figure shows the obtained channel after providing (V_j, S_j^n) to the jth receiver of the $M \times 2$ X-channel. The capacity of this channel serves as an upper bound on the capacity of the original channel, i.e., an $M \times 2$ X-channel. Note that $\mathcal{I}^1$, $\mathcal{I}^2$, and $\mathcal{I}^w$ are non-intersecting and $\mathcal{I}^1 \cup \mathcal{I}^2 \cup \mathcal{I}^w \cup \{i_1, i_2\} = \{1, \dots, M\}$ (cf. Lemma 19).

where $\epsilon_n \to 0$ as $n \to \infty$. Now, by using the chain rule, we write

$$
\begin{aligned}
&n(R_\Sigma - \epsilon_n)\\
&\quad = \sum_{j=1}^{2} I(\tilde{W}_j; V_j) + I(\tilde{W}_j; S_j^n | V_j) + I(\tilde{W}_j; Y_j^n | S_j^n, V_j)\\
&\quad \overset{(a)}{=} \sum_{j=1}^{2} I(\tilde{W}_j; S_j^n | V_j) + I(\tilde{W}_j; Y_j^n | S_j^n, V_j)\\
&\quad = \sum_{j=1}^{2} h(S_j^n | V_j) - h(S_j^n | V_j, \tilde{W}_j) + h(Y_j^n | S_j^n, V_j) - h(Y_j^n | S_j^n, V_j, \tilde{W}_j)\\
&\quad \overset{(b)}{=} \sum_{j=1}^{2} h(S_j^n | V_j) - h(S_j^n | V_j, \tilde{W}_j, X_{i_j}^n, X_{\mathcal{I}^j}^n) + h(Y_j^n | S_j^n, V_j) - h(Y_j^n | S_j^n, V_j, \tilde{W}_j)\\
&\quad \overset{(c)}{=} \sum_{j=1}^{2} h(S_j^n | V_j) - h(N_j^n) + h(Y_j^n | S_j^n, V_j) - h(Y_j^n | S_j^n, V_j, \tilde{W}_j), \qquad (4.40)
\end{aligned}
$$

where (a) follows from the fact that $\tilde{W}_j$ and V_j are independent from each other, (b) follows since knowing V_j and $\tilde{W}_j$, the signals $X_{i_j}^n$ and $X_{\mathcal{I}^j}^n$ can be reconstructed, and (c) follows since knowing $X_{i_j}^n$ and $X_{\mathcal{I}^j}^n$, the only randomness remaining in S_j^n is that originating from N_j^n. In order to bound (4.40), we will bound first

$$T_j = h(Y_j^n | S_j^n, V_j) - h(N_j^n)$$

as follows.

$$\begin{aligned}
T_j &= h(Y_j^n|S_j^n, V_j) - h(N_j^n) \\
&\overset{(d)}{=} h\left(Y_j^n - \frac{1}{c_j} S_j^n | S_j^n, V_j\right) - h(N_j^n) \\
&\overset{(e)}{\leq} h\left(Y_j^n - \frac{1}{c_j} S_j^n\right) - h(N_j^n) \\
&\overset{(f)}{=} h\left(h_{ji_{j'}} X_{i_{j'}}^n + \sum_{k\in\{\mathcal{I}^{j'},\mathcal{I}^w\}} h_{jk} X_k^n + Z_j^n - \frac{1}{c_j} N_j^n\right) - h(N_j^n) \\
&\overset{(g)}{\leq} h\left(h_{ji_{j'}} X_{i_{j'}G} + \sum_{k\in\{\mathcal{I}^{j'},\mathcal{I}^w\}} h_{jk} X_{kG} + Z_j - \frac{1}{c_j} N_j\right) - h(N_j) \\
&\leq nC\left(P_{i_{j'}}|h_{ji_{j'}}|^2 + \sum_{k\in\{\mathcal{I}^{j'},\mathcal{I}^w\}} P_k |h_{jk}|^2 + \frac{1}{|c_j|^2}\right) \\
&\overset{(h)}{\leq} nC\left(P|h_{ji_{j'}}|^2 + \sum_{k\in\{\mathcal{I}^{j'},\mathcal{I}^w\}} P|h_{jk}|^2 + \frac{|h_{ji_j}|^2}{|h_{j'i_j}|^2}\right),
\end{aligned} \tag{4.41}$$

where $j' \in \{1, 2\}$, $j' \neq j$ and the subscript G indicates that the inputs are i.i.d. and Gaussian distributed. Moreover, step (d) follows since $h(A|B) = h(A - cB|B)$, where c is a constant, in step (e) we used the fact that conditioning does not increase the entropy, step (f) hold since

$$Y_j^n = h_{ji_1} X_{i_1}^n + h_{ji_2} X_{i_2}^n + \sum_{k\in\{\mathcal{I}^1,\mathcal{I}^2,\mathcal{I}^w\}} h_{jk} X_k^n + Z_1^n,$$

and in step (g), we used [CT06, Theorem 8.6.2], the fact that N_1 is i.i.d. over the time, and Gaussian distribution maximizes the differential entropy for a fixed variance,

$$\begin{aligned}
&\mathbb{E}\left[\left(h_{ji_{j'}} X_{i_{j'}} + \sum_{k\in\{\mathcal{I}^{j'},\mathcal{I}^w\}} h_{jk} X_k + Z_j - \frac{1}{c_j} N_j\right)^2\right] \\
&\quad = P_{i_{j'}}|h_{ji_{j'}}|^2 + \sum_{k\in\{\mathcal{I}^{j'},\mathcal{I}^w\}} P_k |h_{jk}|^2 + 1 + \frac{1}{|c_j|^2},
\end{aligned} \tag{4.42}$$

finally step (h) is due to the power constraint in (4.3). Now, consider the remaining terms in (4.40) i.e.,

$$\Delta_1 + \Delta_2 = h(S_1^n|V_1) - h(Y_1^n|S_1^n, V_1, \tilde{W}_1) + h(S_2^n|V_2) - h(Y_2^n|S_2^n, V_2, \tilde{W}_2), \tag{4.43}$$

where $\Delta_j = h(S_j^n|V_j) - h(Y_{j'}^n|S_{j'}^n, V_{j'}, \tilde{W}_{j'})$. To bound this expression, we consider the following upper bound on Δ_j

$$\begin{aligned}\Delta_j =& h(S_j^n|V_j) - h(Y_{j'}^n|S_{j'}^n, V_{j'}, \tilde{W}_{j'}) \\ \overset{(i)}{\leq}& h(S_j^n|V_j) - h(Y_{j'}^n|S_{j'}^n, V_{j'}, \tilde{W}_{j'}, W_{j\mathcal{I}^w})\end{aligned}$$

where step (i) follows since conditioning does not increase the entropy. Now, by keeping in mind that

$$\{V_{j'}, \tilde{W}_{j'}, W_{j\mathcal{I}^w}\} = \{W_{ji_{j'}}, W_{j\mathcal{I}^{j'}}, W_{j'i_1}, W_{j'i_2}, W_{j'\mathcal{I}^1}, W_{j'\mathcal{I}^2}, W_{j'\mathcal{I}^w}, W_{j\mathcal{I}^w}\},$$

we proceed

$$\begin{aligned}\Delta_j \overset{(j)}{\leq}& h(S_j^n|V_j) - h\left(\sum_{k\in\{i_j,\mathcal{I}^j\}} h_{j'k}X_k^n + Z_{j'}^n|N_{j'}^n, V_{j'}, \tilde{W}_{j'}, W_{j\mathcal{I}^w}, W_{j\mathcal{I}^w}\right) \\ \overset{(k)}{=}& h(S_j^n|V_j) - h\left(\sum_{k\in\{i_j,\mathcal{I}^j\}} h_{j'k}X_k^n + Z_{j'}^n|W_{j'i_j}, W_{j'\mathcal{I}^j}\right)\end{aligned}$$

where step (j) follows since knowing $W_{ji_{j'}}, W_{j'i_1}, W_{j'i_2}$, we know $X_{i_{j'}}^n$ and from $W_{j\mathcal{I}^{j'}}$, $W_{j'\mathcal{I}^1}$, $W_{j'\mathcal{I}^2}$, we can reconstruct $X_{\mathcal{I}^{j'}}^n$. Knowing $W_{j\mathcal{I}^w}$, $W_{j'\mathcal{I}^w}$, we know $X_{\mathcal{I}^w}^n$. Additionally, knowing $X_{i_{j'}}^n$, $X_{\mathcal{I}^{j'}}^n$, and $X_{\mathcal{I}^w}^n$, the remaining randomness of $S_{j'}^n$ and $Y_{j'}^n$ is originating from $N_{j'}^n$ and $\sum_{k\in\{i_j,\mathcal{I}^j\}} h_{j'k}X_k^n + Z_{j'}^n$, respectively. Moreover, step (k) holds since X_{i_j} and $X_{\mathcal{I}^j}$ are independent from $N_{j'}$ and the messages of Txk, $k \in \{i_{j'}, \mathcal{I}^{j'}, \mathcal{I}^w\}$. Now, due to the definition of S_j given in (4.37), we obtain

$$\begin{aligned}\Delta_j \leq& h\left(h_{j'i_j}X_{i_j}^n + \sum_{k\in\mathcal{I}^j} \frac{h_{j'i_j}h_{jk}}{h_{ji_j}} X_k^n + N_j^n|W_{j'i_j}, W_{j'\mathcal{I}^j}\right) \\ &- h\left(h_{j'i_j}X_{i_j}^n + \sum_{k\in\mathcal{I}^j} h_{j'k}X_k^n + Z_{j'}^n|W_{j'i_j}, W_{j'\mathcal{I}^j}\right). \end{aligned} \tag{4.44}$$

By substituting (4.41) and (4.44) into (4.40) and dividing the expression by n and letting $n \to \infty$, we obtain the upper bound given in (4.36). □

We still need an upper bound for showing the optimality of 2-IC-TIN. This is presented in the following lemma.

Lemma 20. *The sum-capacity of the Gaussian $M \times 2$ X-channel is upper bounded by*

$$C_\Sigma \leq \log_2\left(1 + \sum_{\substack{i=1\\ i\neq i_1}}^{M} P|h_{1i}|^2 + \frac{|h_{1i_1}|^2}{|h_{2i_1}|^2}\right) + \log_2\left(1 + \sum_{\substack{i=1\\ i\neq i_2}}^{M} P|h_{2i}|^2 + \frac{|h_{2i_2}|^2}{|h_{1i_2}|^2}\right). \tag{4.45}$$

Proof. This upper bound is established in a similar way as in [ETW08, HCJ12]. To establish this bound, we provide (S_1^n, W_{2i_1}) and (S_2^n, W_{1i_2}) to Rx1 and Rx2, respectively, where $i_1, i_2 \in \{1, \dots M\}$, $i_1 \neq i_2$. Moreover,

$$\begin{aligned} S_1^n &= h_{2i_1} X_{i_1}^n + N_1^n \\ S_2^n &= h_{1i_2} X_{i_2}^n + N_2, \end{aligned}$$

where $N_1, N_2 \sim \mathbb{C}\mathcal{N}(0,1)$ are i.i.d over the time and independent of other random variables and each other. Hence, we bound the sum-rate as follows

$$n(R_\Sigma - \epsilon_n) \leq I(\tilde{W}_1; Y_1^n, S_1^n, W_{2i_1}) + I(W_2; Y_2^n, S_2^n, W_{1i_2}), \tag{4.46}$$

where $\tilde{W}_j = \{W_{j1}, W_{j2}, \dots, W_{jM}\}$. By using the chain rule and the fact that the messages are independent, we obtain

$$\begin{aligned} n(R_\Sigma - \epsilon_n) \leq & I(\tilde{W}_1; S_1^n | W_{2i_1}) + I(\tilde{W}_1; Y_1^n | S_1^n, W_{2i_1}) \\ & + I(\tilde{W}_2; S_2^n | W_{1i_2}) + I(\tilde{W}_2; Y_2^n | S_2^n, W_{1i_2}). \end{aligned} \tag{4.47}$$

In what follows, we bound each term in (4.47) separately. Now, consider the first term

$$\begin{aligned} I(\tilde{W}_1; S_1^n | W_{2i_1}) &= h(S_1^n | W_{2i_1}) - h(S_1^n | W_{2i_1}, \tilde{W}_1) \\ &\overset{(a)}{=} h(S_1^n | W_{2i_1}) - h(N_1^n). \end{aligned} \tag{4.48}$$

In step (a), we used the fact that knowing W_{2i_1} and W_{1i_1}, the signal $X_{i_1}^n$ can be reconstructed and its contribution can be removed from S_1^n. Similarly, we bound the third term in (4.47) as follows

$$I(\tilde{W}_2; S_2^n | W_{1i_2}) = h(S_2^n | W_{1i_2}) - h(N_2^n). \tag{4.49}$$

Now, we bound the second term in (4.47)

$$\begin{aligned} I(\tilde{W}_1; Y_1^n | S_1^n, W_{2i_1}) &= h(Y_1^n | S_1^n, W_{2i_1}) - h(Y_1^n | S_1^n, W_{2i_1}, \tilde{W}_1) \\ &\overset{(b)}{\leq} h(Y_1^n | S_1^n) - h(Y_1^n | S_1^n, W_{2i_1}, \tilde{W}_1, X_{\mathcal{M} \setminus \{i_2\}}^n) \\ &\overset{(c)}{=} h(Y_1^n | S_1^n) - h(h_{1i_2} X_{i_2}^n + Z_1^n | W_{1i_2}), \end{aligned} \tag{4.50}$$

where $\mathcal{M} = \{1, \dots, M\}$. Step (b) follows since conditioning does not increase the entropy [CT06]. Notice that in step (c), we used the condition that $i_2 \neq i_1$. Similarly, by bounding the fourth term in (4.47) we obtain

$$I(\tilde{W}_2; Y_2^n | S_2^n, W_{1i_2}) \leq h(Y_2^n | S_2^n) - h(h_{2i_1} X_{i_1}^n + Z_2^n | W_{2i_1}). \tag{4.51}$$

Now, by substituting (4.48)-(4.51) into (4.47) and considering the fact that $h(S_1^n | W_{2i_1}) - h(h_{2i_1} X_{i_1}^n + Z_2^n | W_{2i_1}) = 0$ and $h(S_2^n | W_{1i_2}) - h(h_{1i_2} X_{i_2}^n + Z_1^n | W_{1i_2}) = 0$, we write

$$n(R_\Sigma - \epsilon_n) \leq h(Y_1^n | S_1^n) - h(N_1^n) + h(Y_2^n | S_2^n) - h(N_2^n). \tag{4.52}$$

Next, we bound the difference of the first two terms in (4.52) as follows

$$\begin{aligned}
h(Y_1^n|S_1^n) - h(N_1^n) &\overset{(d)}{\leq} \sum_{t=1}^{n} h(Y_1[t]|S_1[t]) - h(N_1[t]) \\
&\overset{(e)}{\leq} n\left[h(Y_{1G}|S_{1G}) - h(N_1)\right] \\
&= n\left[h\left(Y_{1G} - \frac{h_{1i_1}}{h_{2i_1}} S_{1G}|S_{1G}\right) - h(N_1)\right] \\
&\leq n\left[h\left(\sum_{\substack{i=1\\ i\neq i_1}}^{M} h_{1i}X_i + Z_1 - \frac{h_{1i_1}}{h_{2i_1}} N_1\right) - h(N_1)\right] \\
&\overset{(f)}{=} n\log_2\left(1 + \sum_{\substack{i=1\\ i\neq i_1}}^{M} P_i|h_{1i}|^2 + \frac{|h_{1i_1}|^2}{|h_{2i_1}|^2}\right) \\
&\overset{(g)}{\leq} n\log_2\left(1 + \sum_{\substack{i=1\\ i\neq i_1}}^{M} P|h_{1i}|^2 + \frac{|h_{1i_1}|^2}{|h_{2i_1}|^2}\right), \qquad (4.53)
\end{aligned}$$

where the subscript G indicates that the inputs are i.i.d. and Gaussian distributed, i.e., for all $i \in \{1, \ldots, i_1\}$, $X_{iG} \sim \mathbb{C}\mathcal{N}(0, P)$. In step (d), we used the chain rule, the fact that conditioning does not increase entropy and N_1 is i.i.d.. In step (e), we used [AV09, Lemma 1] which shows that a circularly symmetric complex Gaussian distribution maximizes the conditional differential entropy under a covariance constraint. Step (f) follows since the signals sent by different transmitters are independent of each other. Moreover, step (g) is due to the fact that $\log_2(1 + x)$ is a monotonically increasing function with respect to x.

Similarly, we can upper bound the difference of the last two terms in (4.52) as follows

$$h(Y_2^n|S_2^n) - h(N_2^n) \leq n\log_2\left(1 + \sum_{\substack{i=1\\ i\neq i_2}}^{M} P|h_{2i}|^2 + \frac{|h_{2i_2}|^2}{|h_{1i_2}|^2}\right). \qquad (4.54)$$

By substituting (4.53) and (4.54) into (4.52), and dividing the obtained expression by n and letting $n \to \infty$, we complete the proof of Lemma 20. □

At this point, it is worth mentioning that we need to modify the presented upper bounds to show the GDoF optimality of TIN. These modifications are listed bellow.

- **Bounding some terms:** In order to obtain tight upper bounds on the GDoF which coincide with the achievable GDoF of TIN, we need to bound the second terms in (4.28) and (4.36) (Θ_1, Θ_2 Δ_1, Δ_2) by a constant independent of P.

- **Fixing the parameter:** While all the upper bounds in Lemmata 18, 19, and 20 are upper bounds on the sum-capacity of the Gaussian $M \times 2$ X-channel, only an appropriate choice of the parameters, i.e., i_1, i_2, $\mathcal{I}^1$, $\mathcal{I}^2$, $\mathcal{I}^w$ can lead to a tight and useful bound.

To bound the expressions Θ_1, Θ_2, Δ_1, and Δ_2 in (4.28) and (4.36), we introduce here a novel lemma which bounds the difference of two entropy terms. Fixing the parameters i_1, i_2, $\mathcal{I}^1$, $\mathcal{I}^2$, $\mathcal{I}^w$ for obtaining useful upper bounds for showing the optimality of TIN will be discussed in the next Section. Using this lemma, we are able to tighten the upper bounds in (4.28) and (4.36) under certain conditions.

Lemma 21. *Consider two signals*

$$S_1^n = \sum_{\ell=1}^{L} g_\ell A_\ell^n + N_1^n \text{ and } S_2^n = \sum_{\ell=1}^{L} f_\ell A_\ell^n + N_2^n,$$

where A_ℓ^n with $\ell = 1, \ldots, L$ is the sequence of complex random variables and $A_{\ell_1}^n$ is independent of $A_{\ell_2}^n$ as long as $\ell_1 \neq \ell_2$. The variance of A_ℓ^n satisfies

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[|A_\ell[t]|^2\right] \leq P.$$

Moreover, N_1^n, N_2^n are the noise sequences with distribution $\mathbb{CN}(0,1)$, independent of all other random variables, and i.i.d. over time. Moreover, suppose that $W_1, W_2, \ldots, W_L$ are independent messages, where A_i is independent of W_j as long as $i \neq j$. Furthermore, we define $W_c = \{W_1, W_2, \ldots, W_L\}$ and the parameters

$$\beta_\ell = \frac{\log_2(P|g_\ell|^2)}{\log_2 \rho}, \quad \gamma_\ell = \frac{\log_2(P|f_\ell|^2)}{\log_2 \rho}, \tag{4.55}$$

where for all $\ell \in \{1 \ldots, L\}$, γ_ℓ and β_ℓ are non-negative. As long as, the following constraint is satisfied

$$\gamma_\ell - \gamma_{\ell-1} \geq \beta_\ell, \text{ for all } \ell \in \{1, \ldots, L\}, \tag{4.56}$$

where $\gamma_0 = 0$, the difference between the differential entropies of random variables S_1^n and S_2^n conditioned on W_c is upper bounded by

$$h(S_1^n|W_c) - h(S_2^n|W_c) \leq n \log_2 L!. \tag{4.57}$$

Proof. Supposing that $\Delta = h(S_1^n|W_c) - h(S_2^n|W_c)$, we write

$$\begin{aligned}\Delta &\overset{(a)}{=} h(S_1^n|W_c) - h(S_2^n|W_c) - h(N_1^n|W_c, A_1^n, \ldots, A_L^n) + h(N_2^n|W_c, A_1^n, \ldots, A_L^n) \\ &= I(A_1^n, \ldots, A_L^n; S_1^n|W_c) - I(A_1^n, \ldots, A_L^n; S_2^n|W_c),\end{aligned}$$

where in step (a), we used the fact that N_1^n and N_2^n are independent of all other random variables, i.i.d over time and they have the same distribution. Now, by using the chain rule and the fact that the mutual information is non-negative [CT06], we bound Δ as follow

$$\Delta \leq \sum_{l=1}^{L} I(A_l^n; S_1^n | W_c, A_{l+1}^n, \ldots, A_L^n) - \sum_{l=1}^{L} I(A_l^n; S_2^n | W_c, A_{l+1}^n, \ldots, A_L^n)$$
$$+ \sum_{l=1}^{L} I(A_l^n; A_1^n, \ldots, A_{l-1}^n | W_c, S_1^n, A_{l+1}^n, \ldots, A_L^n).$$

Now, by using the chain rule, we obtain

$$\Delta \leq \sum_{l=1}^{L} I(A_l^n; S_1^n, A_1^n, \ldots, A_{l-1}^n | W_c, A_{l+1}^n, \ldots, A_L^n) - \sum_{l=1}^{L} I(A_l^n; S_2^n | W_c, A_{l+1}^n, \ldots, A_L^n).$$

Now, by using the chain rule and the fact that A_l^n and $(A_1^n, ..., A_{l-1}^n)$ are conditionally independent given $(A_{l+1}^n, ..., A_L^n)$, we write

$$\begin{aligned}
\Delta &\leq \sum_{l=1}^{L} I(A_l^n; S_1^n | W_c, A_1^n, \ldots, A_{l-1}^n, A_{l+1}^n, \ldots, A_L^n) - \sum_{l=1}^{L} I(A_l^n; S_2^n | W_c, A_{l+1}^n, \ldots, A_L^n) \\
&= \sum_{l=1}^{L} h(S_1^n | W_c, A_1^n, \ldots, A_{l-1}^n, A_{l+1}^n, \ldots, A_L^n) - \sum_{l=1}^{L} h(S_1^n | W_c, A_1^n, \ldots, A_L^n) \\
&\quad - \sum_{l=1}^{L} I(A_l^n; S_2^n | W_c, A_{l+1}^n, \ldots, A_L^n) \\
&\overset{(b)}{=} \sum_{l=1}^{L} h(g_l A_l^n + N_1^n | W_l) - \sum_{l=1}^{L} h(N_1^n | W_l) - \sum_{l=1}^{L} I(A_l^n; S_2^n | W_c, A_{l+1}^n, \ldots, A_L^n) \\
&\leq \sum_{l=1}^{L} I(A_l^n; g_l A_l^n + N_1^n | W_l) - I(A_l^n; S_2^n | W_c, A_{l+1}^n, \ldots, A_L^n),
\end{aligned} \tag{4.58}$$

where in (b), we used the fact that knowing $(A_1^n, \ldots, A_{l-1}^n, A_{l+1}^n, \ldots, A_L^n)$, the randomness of S_1^n is caused from A_l^n and N_1^n and that $g_l A_l + N_1$ is independent of $A_1, \ldots, A_{l-1}, A_{l+1}, \ldots, A_L$ given W_l. To upper bound the expression in (4.58), we lower bound

$$T_l = I(A_l^n; S_2^n | W_c, A_{l+1}^n, \ldots, A_L^n) \tag{4.59}$$

for all $l = 1, \ldots, L$ as follows

$$
\begin{aligned}
T_l &= h\left(\sum_{i=1}^{L} f_i A_i^n + N_2^n | W_c, A_{l+1}^n, \ldots, A_L^n\right) - h\left(\sum_{i=1}^{L} f_i A_i^n + N_2^n | W_c, A_l^n, \ldots, A_L^n\right) \\
&\overset{(c)}{=} h\left(\sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c\right) - h\left(\sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c, A_l^n\right) \\
&= I\left(A_l^n; \sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c\right). \qquad (4.60)
\end{aligned}
$$

In (c), we removed the randomness of the known signals and used the fact that the signals $A_1^n, \ldots, A_L^n, N_2^n$ are independent of each other. Now, using the fact that scaling does not affect mutual information, we can write

$$
T_l = I\left(A_l^n; \left(\sum_{i=1}^{l} f_i A_i^n + N_2^n\right) \frac{1}{\sqrt{P}|f_{l-1}|} \middle| W_c\right), \qquad (4.61)
$$

where we define $f_0 = \frac{1}{\sqrt{P}}$. Due to the assumption in Lemma 21, the γ-parameters are non-negative and hence, $\sqrt{P}|f_{l-1}| \geq 1$. Therefore, we obtain a lower bound for (4.61) by increasing the variance of the noise N_2^n from 1 to $P|f_{l-1}|^2$. Thus,

$$
\begin{aligned}
T_l \geq& I\left(A_l^n; \frac{1}{\sqrt{P}|f_{l-1}|} \sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c\right) \\
=& h\left(\frac{1}{\sqrt{P}|f_{l-1}|} \sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c\right) - h\left(\frac{1}{\sqrt{P}|f_{l-1}|} \sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c, A_l^n\right) \\
\overset{(d)}{\geq}& h\left(\frac{1}{\sqrt{P}|f_{l-1}|} \sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c, A_1^n, \ldots, A_{l-1}^n\right) \\
&- h\left(\frac{1}{\sqrt{P}|f_{l-1}|} \sum_{i=1}^{l} f_i A_i^n + N_2^n | W_c, A_l^n\right) \\
\overset{(e)}{=}& h\left(\frac{f_l}{\sqrt{P}|f_{l-1}|} A_l^n + N_2^n | W_l\right) - h\left(\frac{1}{\sqrt{P}|f_{l-1}|} \sum_{i=1}^{l-1} f_i A_i^n + N_2^n | W_c\right) \\
&- h(N_2^n | W_l, A_l^n) + h(N_2^n | W_c, A_1^n, \ldots, A_{l-1}^n) \\
=& I\left(A_l^n; \frac{f_l}{\sqrt{P}|f_{l-1}|} A_l^n + N_2^n | W_l\right) - c_l, \qquad (4.62)
\end{aligned}
$$

where

$$
c_l = I\left(A_1^n, \ldots, A_{l-1}^n; \frac{1}{\sqrt{P}|f_{l-1}|} \sum_{i=1}^{l-1} f_i A_i^n + N_2^n | W_c\right). \qquad (4.63)
$$

In step (d), we used the fact that conditioning does not increase the entropy. Step (e) follows since variables $A_1, \ldots, A_l, N_2$ are all independent of each other. Moreover, we dropped the conditions on W_j, $(j \neq l)$ in the first entropy term of step (e), since these messages are all independent of A_l and N_2. Now, by substituting (4.62) into the second term of (4.58), the expression $h(S_1^n|W_c) - h(S_2^n|W_c)$ is upper bounded as follows

$$\Delta \leq \sum_{l=1}^{L} I(A_l^n; g_l A_l^n + N_1^n|W_l) - I\left(A_l^n; \frac{f_l}{\sqrt{P}|f_{l-1}|} A_l^n + N_2^n|W_l\right) + c_l. \tag{4.64}$$

Due to the condition in (4.56), we know that for all $l = 1, \ldots, L$, $\frac{|f_l|^2}{|g_l|^2 P |f_{l-1}|^2} \geq 1$. Hence, we obtain an upper bound for the expression in (4.64), by increasing the variance of the noise in the second mutual information term in (4.64) from 1 to $\frac{|f_l|^2}{|g_l|^2 P |f_{l-1}|^2}$ and write

$$\begin{aligned}
\Delta &\leq \sum_{l=1}^{L} I(A_l^n; g_l A_l^n + N_1^n|W_l) - I\left(A_l^n; \frac{f_l}{\sqrt{P}|f_{l-1}|} A_l^n + \frac{|f_l|}{|g_l|\sqrt{P}|f_{l-1}|} N_2^n|W_l\right) + c_l \\
&\overset{(f)}{=} \sum_{l=1}^{L} I(A_l^n; g_l A_l^n + N_1^n|W_l) \\
&\qquad - I\left(A_l^n; \left(\frac{f_l}{\sqrt{P}|f_{l-1}|} A_l^n + \frac{|f_l|}{|g_l|\sqrt{P}|f_{l-1}|} N_2^n\right) \frac{g_l\sqrt{P}|f_{l-1}|}{f_l} |W_l\right) + c_l \\
&= \sum_{l=1}^{L} I(A_l^n; g_l A_l^n + N_1^n|W_l) - I\left(A_l^n; g_l A_l^n + \tilde{N}_2^n|W_l\right) + c_l \\
&\overset{(g)}{=} \sum_{l=1}^{L} c_l,
\end{aligned} \tag{4.65}$$

where $\tilde{N}_2 \sim \mathbb{C}\mathcal{N}(0,1)$ is i.i.d. over time and independent of all other random variables except N_2. In second term of step (f), we used the fact that scaling does not affect mutual information and step (g) follows since the expressions $I(A_l^n; g_l A_l^n + N_1^n)$ and $I\left(A_l^n; g_l A_l^n + \tilde{N}_2^n\right)$ are equal since they are the mutual information between input and output of statistically identical P2P channels.

Now, we need to upper bound c_l given in (4.63). To do this, we write

$$
\begin{aligned}
c_l =& I\left(A_1^n,\dots,A_{l-1}^n;\frac{1}{\sqrt{P}|f_{l-1}|}\sum_{i=1}^{l-1} f_iA_i^n + N_2^n|W_c\right)\\
=& h\left(\frac{1}{\sqrt{P}|f_{l-1}|}\sum_{i=1}^{l-1} f_iA_i^n + N_2^n|W_c\right) - h\left(N_2^n|W_c,A_1^n,\dots,A_{l-1}^n\right)\\
\overset{(h)}{\leq}& h\left(\frac{1}{\sqrt{P}|f_{l-1}|}\sum_{i=1}^{l-1} f_iA_i^n + N_2^n\right) - h\left(N_2^n\right)\\
\overset{(i)}{\leq}& n\left[h\left(\frac{1}{\sqrt{P}|f_{l-1}|}\sum_{i=1}^{l-1} f_iA_{iG} + N_2\right) - h\left(N_2\right)\right],
\end{aligned}
\tag{4.66}
$$

where the subscript G indicates that the random variable is i.i.d. and Gaussian distributed, i.e., $A_{iG} \sim \mathbb{C}\mathcal{N}(0,P)$. Step (h) follows since conditioning does not increase the entropy and N_2 is independent from all other random variables. In step (i), we used [CT06, corollary to Theorem 8.6.2], the fact that N_2^n is i.i.d. over time, and choosing Gaussian distribution with maximum variance for the random variables $A_1,\dots,A_{l-1}$, maximizes the differential entropy [CT06]. Hence, c_l is bounded as follows

$$
c_l \leq n\log_2\left(1+\sum_{i=1}^{l-1}\frac{|f_i|^2}{|f_{l-1}|^2}\right). \tag{4.67}
$$

Due to the condition in (4.56), we can write $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{l-1}$, which is equivalent to $|f_1|^2 \leq |f_2|^2 \leq \dots \leq |f_{l-1}|^2$. Hence, for all $i = 1,\dots,l-1$, we have $\frac{|f_i|^2}{|f_{l-1}|^2} \leq 1$. Therefore, we can upper bound the expression (4.67) as follows

$$
c_l \leq n\log_2 l. \tag{4.68}
$$

By substituting (4.68) into (4.65), we obtain the following upper bound for $h(S_1^n|W_c) - h(S_2^n|W_c)$

$$
\begin{aligned}
\Delta &\leq n\sum_{l=1}^{L}\log_2 l\\
&= n\log_2 L!
\end{aligned}
\tag{4.69}
$$

which completes the proof. □

4.4 GDoF Optimality of TIN in an $M \times 2$ X-channel

In this section, we want to introduce the regimes in which TIN is GDoF optimal. As it is discussed before, the maximum achievable GDoF using TIN in an $M \times 2$

X-channel is achieved when the setup is reduced to a P2P channel or a 2-user IC. In what follows, we show the optimality of these two schemes separately. To do this, we use the established upper bounds on the capacity of the $M \times 2$ X-channel presented in Section 4.3. We translate these upper bounds into the GDoF upper bounds. Depending on the scheme whose optimality needs to be shown, we fix the parameters i_1, i_2, $\mathcal{I}^1$, $\mathcal{I}^2$, $\mathcal{I}^w$. Furthermore, we tighten the upper bounds on the GDoF by using Lemma 21. Finally, by comparing the achievable GDoF of TIN with the obtained upper bound on the GDoF, we characterize the regime in which TIN is GDoF optimal.

4.4.1 Optimality of P2P-TIN

Here, we want to establish the conditions for the GDoF optimality of reducing an $M \times 2$ X-channel to a P2P channel. In other words, we want to find the conditions for the optimality of the GDoF expression given in (4.26). Without loss of generality, suppose that the optimal j_1 for (4.26) is $j_1 = 1$. Hence, we achieve the following GDoF by considering a P2P channel between Txi_1 to Rx1

$$d_{\Sigma,\mathrm{P2P-TIN}}(\boldsymbol{\alpha}) = \max_{i_1}\{\alpha_{1i_1}\}. \tag{4.70}$$

Before presenting the optimality conditions mathematically, some intuitive comments on the optimality of P2P-TIN might be useful. In fact, one can distinguish between the following cases in which P2P-TIN (with GDoF in (4.70)) is optimal:

- **Dominant MAC:** The $M \times 2$ X-channel can be reduced to two different multiple access channels (MAC1, MAC2), where in MACj with $j = 1, 2$, all transmitters want to communicate with Rxj. Without loss of generality, consider the case that the GDoF of MAC1 is significantly larger than MAC2 such that we do not benefit from an active MAC2 in terms of GDoF. Hence, in this case the GDoF optimal strategy is that all transmitters serve only Rx1, i.e., a multiple access channel.

- **Dominant BC:** Here, we consider M different broadcast channels (BC1, ... BCM), where in BCi with $i = 1, \ldots, M$, Txi serves both receivers. Suppose that the GDoF of BC1 is sufficiently larger than the GDoF of all remaining BC's such that we cannot obtain any GDoF gain by letting Tx2, ..., TxM be active. Therefore, in this case, it is GDoF optimal to let only Tx1 serve all receivers, which decomposes the X-channel to a broadcast channel.

Note that, the GDoF resulting in a MAC and BC are achievable by reducing these setups to a P2P channel. Hence, both aforementioned cases lead to optimality of P2P-TIN. The following theorems summarize the conditions for the optimality of P2P-TIN. While the first theorem considers optimality of P2P-TIN for a dominant MAC, the second theorem presents the optimality of P2P-TIN based on a dominant BC.

Theorem 4. *Let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_M)$ be a permutation of $(1, \dots, M)$. Then, it is GDoF optimal to reduce the $M \times 2$ X-channel to a P2P channel between Txπ_1 to Rxj with $j \in \{1, 2\}$ if*

$$\alpha_{j\pi_i} - \alpha_{j\pi_{i+1}} \geq \alpha_{j'\pi_i} \tag{4.71}$$

for all $i = 1, \dots, M$, where $\alpha_{j\pi_{M+1}} = 0$ and $j' \neq j$.

Proof. To prove this theorem, we use Lemma 21 for bounding Θ_j in (4.28). Doing this, we obtain a tight upper bound if (4.71) is satisfied. The details of the proof are as follows. Suppose that we have a permutation of transmitters $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_M)$ which satisfies the condition (4.71). Hence, we have

$$\alpha_{j\pi_i} - \alpha_{j\pi_{i+1}} \geq \alpha_{j'\pi_i}, \tag{4.72}$$

where $\alpha_{j\pi_{M+1}} = 0$. In what follows, we focus on $j = 1$ and a specific permutation of $\boldsymbol{\pi}$ which is $\boldsymbol{\pi} = (1, 2, ..., M)$, since all other cases follow similarly. Hence, the condition in (4.72) can be rewritten as follows

$$\begin{aligned} \alpha_{11} - \alpha_{12} &\geq \alpha_{21} \\ \alpha_{12} - \alpha_{13} &\geq \alpha_{22} \\ &\vdots \\ \alpha_{1M} &\geq \alpha_{2M}. \end{aligned}$$

Therefore, the best achievable GDoF using P2P-TIN is α_{11}. In what follows, we show that as long as (4.71) is satisfied, α_{11} is the best achievable GDoF. To this end, we use the upper bound on the capacity of $M \times 2$ X-channel given in (4.28) for $j = 1$. However, first we need to bound the expression Θ_1. To do this, we use Lemma 21. For the sake of simplicity, we substitute the parameters of Θ_1 into the parameters given in Lemma 21 as follows

$$\begin{aligned} &L \triangleq M, \quad W_c \triangleq \tilde{W}_1 \quad A_\ell \triangleq X_m, \quad f_\ell \triangleq h_{1m}, \\ &\gamma_\ell \triangleq \alpha_{1m}, \quad g_\ell \triangleq h_{2m}, \quad \beta_\ell \triangleq \alpha_{2m}, \quad N_1 \triangleq Z_2, \quad N_2 \triangleq Z_1, \end{aligned}$$

where the relationship between ℓ and m is given in in Table 4.1. Moreover, $\gamma_0 \triangleq \alpha_{1(M+1)} = 0$. Hence, we can write the expression $h(S_1^n|W_c) - h(S_2^n|W_c)$ in Lemma

ℓ	L	$L-1$	$\dots$	2	1
m	1	2	$\dots$	$M-1$	M

Table 4.1: The relationship between the parameters ℓ (used in Lemma 21) and m.

21 as follows

$$h(S_1^n|W_c) - h(S_2^n|W_c) \triangleq h\left(\sum_{i=1}^{M} h_{2i}X_i^n + Z_2^n|\tilde{W}_1\right) - h\left(\sum_{i=1}^{M} h_{1i}X_i^n + Z_1^n|\tilde{W}_1\right), \tag{4.73}$$

which is Θ_1 in (4.28). Moreover the condition (4.56) can be rewritten as

$$\alpha_{1k} - \alpha_{1(k+1)} \geq \alpha_{2k}, \quad \forall k \in \{1, \ldots, M\}. \tag{4.74}$$

Notice that this condition is satisfied in Theorem 4. Since we can use Lemma 21 to bound the expression Θ_1 by $n \log_2 M!$, we rewrite the upper bound (4.28) with $j = 1$ as follows

$$C_\Sigma \leq \log_2 \left(1 + \sum_{i=1}^{M} |h_{1i}|^2 P\right) + \log_2 M!. \tag{4.75}$$

Now, by using (4.9), we obtain

$$C_\Sigma \leq \log_2 \left(1 + \sum_{i=1}^{M} \rho^{\alpha_{1i}}\right) + \log_2 M!. \tag{4.76}$$

Dividing this expression by $\log_2 \rho$, we translate this upper bound to a GDoF upper bound. Hence, we write

$$d(\boldsymbol{\alpha}) \leq \max\left\{0, \alpha_{1i}|_{i=1}^{M}\right\} \stackrel{(a)}{=} \alpha_{11}, \tag{4.77}$$

where in (a), we used the condition in (4.71). The upper bound in (4.77) coincides with the achievable GDoF in (4.70) for $i_1 = 1$.

□

Theorem 5. *Let $i_1 \in \{1, \cdots, M\}$, $j, j' \in \{1, 2\}$, $j \neq j'$, and define*

$$i_2 = \underset{i \in \{1,\ldots,M\} \setminus \{i_1\}}{\operatorname{argmax}} \alpha_{j'i}.$$

Then, it is GDoF optimal to reduce the $M \times 2$ X-channel to a P2P channel between Txi_1 and Rxj if

$$\alpha_{ji_1} - \alpha_{j'i_1} \geq \max_{i \in \{1,\cdots,M\} \setminus \{i_1\}} \alpha_{ji} \tag{4.78}$$

$$\alpha_{j'i_1} \geq \max\{\alpha_{j'i_2} - \alpha_{ji_2}, \max_{i \in \{1,\cdots,M\} \setminus \{i_1,i_2\}} \alpha_{j'i}\}. \tag{4.79}$$

Proof. To prove Theorem 5, we focus on the case that $j = 1$. The other case can be shown similarly. By using the upper bound given in (4.45) and the definition in (4.9), we can rewrite this upper bound as follows

$$C_\Sigma \leq \log_2 \left(1 + \sum_{\substack{i=1 \\ i \neq i_1}}^{M} \rho^{\alpha_{1i}} + \rho^{\alpha_{1i_1} - \alpha_{2i_1}}\right) + \log_2 \left(1 + \sum_{\substack{i=1 \\ i \neq i_2}}^{M} \rho^{\alpha_{2i}} + \rho^{\alpha_{2i_2} - \alpha_{1i_2}}\right), \tag{4.80}$$

where $i_1, i_2 \in \{1, \ldots, M\}$ and $i_1 \neq i_2$. Now, by dividing (4.80) by $\log_2 \rho$ and letting $\rho \to \infty$, we can bound the GDoF of the $M \times 2$ X-channel as follows

$$d(\boldsymbol{\alpha}) \leq \max\left\{(\alpha_{1i_1} - \alpha_{2i_1})^+, \max_{i\in\{1,\ldots,M\}\setminus\{i_1\}} \alpha_{1i}\right\} + \max\left\{(\alpha_{2i_2} - \alpha_{1i_2})^+, \max_{i\in\{1,\ldots,M\}\setminus\{i_2\}} \alpha_{2i}\right\}. \tag{4.81}$$

Now, if there exist distinct i_1 and $i_2 = \arg\max_{i\in\{1,\cdots,M\}\setminus\{i_1\}} \alpha_{2i}$ which satisfy (4.78) and (4.79) for $j = 1$, we have

$$\alpha_{1i_1} - \alpha_{2i_1} \geq \max_{i\in\{1,\cdots,M\}\setminus\{i_1\}} \alpha_{1i} \tag{4.82}$$

$$\alpha_{2i_1} \geq \max\{\alpha_{2i_2} - \alpha_{1i_2}, \max_{i\in\{1,\cdots,M\}\setminus\{i_1,i_2\}} \alpha_{2i}\}. \tag{4.83}$$

Hence, we can write the upper bound in (4.81) as follows

$$d(\boldsymbol{\alpha}) \leq \alpha_{1i_1}. \tag{4.84}$$

This coincides with the achievable GDoF for the case that Txi_1 communicates with Rx1. □

Example 1. *Based on Theorem 4, 5, it is GDoF optimal to reduce the* 3×2 *X-channels shown in Fig. 4.3a and 4.3b to a P2P channel between Tx*1 *to Rx*1*. In more details, while the GDoF optimality of P2P-TIN for the setup in Fig. 4.3a can be shown using Theorem 4, we need Theorem 5 to show the optimality of P2P-TIN for setup in Fig. 4.3b. As shown in Fig. 4.3a, the channels to Rx*1 *are stronger than the channels to Rx*2 *and thus we cannot obtain any GDoF gain by letting Rx*2 *be active. Moreover, in Fig. 4.3b, the channels from Tx*1 *are so strong that we can switch the other transmitters off without any loss in the GDoF.*

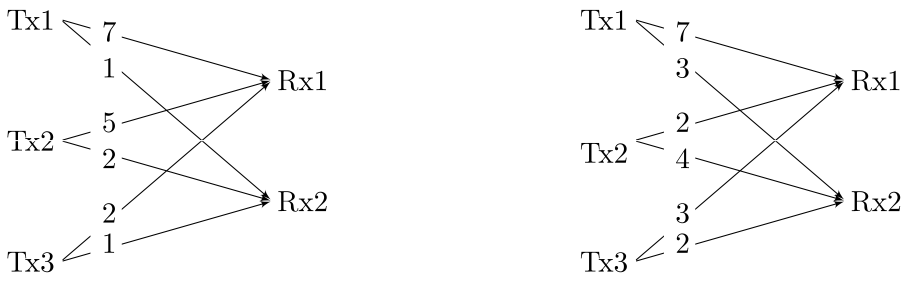
4.4.2 Optimality of 2-IC-TIN

Here, we study the GDoF optimality of reducing an $M \times 2$ X-channel to a 2-user IC where the receivers use TIN. Before we present the conditions on the optimality of this variant of TIN, we need to discuss the scheme in more details and introduce some definitions. Consider using TIN in a 2-user IC in which Txi_1 and Txi_2 communicate with Rx1 and Rx2, respectively.

Remark 10. *Notice that if* $\alpha_{1i_1} < \alpha_{1i_2}$ *or* $\alpha_{2i_2} < \alpha_{2i_1}$*, P2P-TIN achieves*

$$\max\{\alpha_{1i_1}, \alpha_{2i_2}\}$$

which is larger than the achievable GDoF using 2-IC-TIN (cf. 4.14). Hence, in this case, the 2-IC-TIN (in which Txi_1 *and Tx*i_2 *communicate with Rx*1 *and Rx*2*, respectively) is suboptimal. Therefore, in what follows, we exclude this case, and assume that* $\alpha_{1i_1} \geq \alpha_{1i_2}$ *and* $\alpha_{2i_2} \geq \alpha_{2i_1}$*.*



(a) There is no GDoF gain if the transmitters communicate with Rx2.

(b) We cannot obtain any GDoF gain by letting Tx2 and Tx3 be active.

Figure 4.3: P2P-TIN (from Tx1 to Rx1) is optimal in both setups.

In the following, the interference observed at Rx1 and Rx2 from each Tx excluding Txi_1 and Txi_2 is compared with a threshold called GNAJ-level inspired by [GNAJ15]. This threshold is called after the initials of the authors of [GNAJ15]. The applied thresholds are

$$\alpha^{(1)}_{\text{GNAJ-level}} = (\alpha_{1i_1} - \alpha_{2i_1})^+ \quad \text{and} \quad \alpha^{(2)}_{\text{GNAJ-level}} = (\alpha_{2i_2} - \alpha_{1i_2})^+$$

at Rx1 and Rx2, respectively. Based on this comparison, we classify the remaining Tx's which are not part of the 2-user IC (i.e., Txi $i \in \{1, 2, \ldots, M\} \setminus \{i_1, i_2\}$) into four groups as follow.

- $\mathcal{I}^1(i_1, i_2) = \{i \,|i \in \mathcal{M} \setminus \{i_1, i_2\}, \alpha_{2i} \leq (\alpha_{2i_2} - \alpha_{1i_2})^+, \alpha_{1i} > (\alpha_{1i_1} - \alpha_{2i_1})^+\}$
- $\mathcal{I}^2(i_1, i_2) = \{i \,|i \in \mathcal{M} \setminus \{i_1, i_2\}, \alpha_{2i} > (\alpha_{2i_2} - \alpha_{1i_2})^+, \alpha_{1i} \leq (\alpha_{1i_1} - \alpha_{2i_1})^+\}$
- $\mathcal{I}^w(i_1, i_2) = \{i \,|i \in \mathcal{M} \setminus \{i_1, i_2\}, \alpha_{2i} \leq (\alpha_{2i_2} - \alpha_{1i_2})^+, \alpha_{1i} \leq (\alpha_{1i_1} - \alpha_{2i_1})^+\}$
- $\mathcal{I}^s(i_1, i_2) = \{i \,|i \in \mathcal{M} \setminus \{i_1, i_2\}, \alpha_{2i} > (\alpha_{2i_2} - \alpha_{1i_2})^+, \alpha_{1i} > (\alpha_{1i_1} - \alpha_{2i_1})^+\}$

where $\mathcal{M} = \{1, 2, \ldots, M\}$. Roughly speaking, $\mathcal{I}^j$ with $j = 1, 2$ is the set of Tx's with a strong link to Rxj and a weak link to the other Rx. Moreover, $\mathcal{I}^w$ and $\mathcal{I}^s$ are the sets of the Tx's with weak and strong interference to both Rx's, respectively. It is worth mentioning that these sets are defined such that $\mathcal{I}^1(i_1, i_2)$, $\mathcal{I}^2(i_1, i_2)$, $\mathcal{I}^w(i_1, i_2)$, $\mathcal{I}^s(i_1, i_2)$ together with $\{i_1, i_2\}$ are complementary, which means that

$$\mathcal{I}^1(i_1, i_2) \cup \mathcal{I}^2(i_1, i_2) \cup \mathcal{I}^w(i_1, i_2) \cup \mathcal{I}^s(i_1, i_2) \cup \{i_1, i_2\} = \{1, 2, \ldots, M\}$$
$$|\mathcal{I}^1(i_1, i_2)| + |\mathcal{I}^2(i_1, i_2)| + |\mathcal{I}^w(i_1, i_2)| + |\mathcal{I}^s(i_1, i_2)| + 2 = M.$$

Without loss of generality, we fix $i_1 = 1$ and $i_2 = 2$ for further discussions and we focus on the optimality of TIN in the 2-user IC consists of Tx1, Tx2 and Rx1, Rx2.

Therefore, since the input of the functions $\mathcal{I}^j$, $\mathcal{I}^w$, and $\mathcal{I}^s$ will be always $(1,2)$, we drop the input unless it is required. In the following, we study the conditions on optimality of decomposing $M \times 2$ X-channel to a 2-user IC and using TIN which achieves

$$d_{\Sigma,\text{2-IC-TIN}}(\boldsymbol{\alpha}) = \alpha_{11} - \alpha_{12} + \alpha_{22} - \alpha_{21}. \tag{4.85}$$

Before going into further details, it is worthy to recall the known results on optimality of this type of TIN. By applying the conditions on optimality of TIN in [GSJ15] to our setup, we conclude that the achievable GDoF in (4.85) is optimal as long as $\mathcal{I}^1 = \mathcal{I}^2 = \mathcal{I}^s = \varnothing$. In what follows, we study the GDoF optimality of TIN when this condition is not necessarily satisfied. To this end, we use first the following lemma to exclude an important case in which 2-IC-TIN is suboptimal.

Lemma 22. *Reducing an $M \times 2$ X-channel to a 2-user IC in which Tx1 and Tx2 communicate with Rx1 and Rx2, respectively while the Rx's use TIN is GDoF suboptimal, if there exists an $i \in \{3, \ldots, M\}$ such that*

$$\alpha_{1i} \geq (\alpha_{11} - \alpha_{21})^+, \tag{4.86}$$
$$\alpha_{2i} \geq (\alpha_{22} - \alpha_{12})^+. \tag{4.87}$$

Proof. To prove suboptimality of a scheme, we need to show that this scheme can be outperformed by another scheme. In Appendix 4.A, we present a scheme which achieves a GDoF larger than that achieved by 2-IC-TIN (given in (4.85)) if there exists an $i \in \{1, \ldots, M\}$ that satisfies (4.86) and (4.87). □

From Lemma 22, we conclude that if the set $\mathcal{I}^s$ is not empty, then TIN is suboptimal. Hence, in order to characterize the GDoF optimal regime of TIN, we focus on the remaining case, i.e., $\mathcal{I}^s = \varnothing$. The following theorem introduces the complete GDoF optimal regime of 2-IC-TIN.

Theorem 6. *Reducing an $M \times 2$ X-channel to a 2-user IC in which Tx1 and Tx2 communicate with Rx1 and Rx2, respectively while the Rx's use TIN achieves (4.85) and performs GDoF optimally if and only if there is a permutation of senders in which $\mathcal{I}^1 = \{3, \ldots, 2 + |\mathcal{I}^1|\}$, $\mathcal{I}^2 = \{3 + |\mathcal{I}^1|, \ldots, 2 + |\mathcal{I}^1| + |\mathcal{I}^2|\}$, and $\mathcal{I}^w = \{3 + |\mathcal{I}^1| + |\mathcal{I}^2|, \ldots, M\}$, where*

$$\forall i \in \mathcal{I}^1,\ \alpha_{11} - \alpha_{21} < \alpha_{1i},\ \alpha_{22} - \alpha_{12} \geq \alpha_{2i},\ \alpha_{2(i+1)} \geq \alpha_{2i},\ \alpha_{23} \geq \alpha_{21}, \tag{4.88}$$
$$\forall i \in \mathcal{I}^2,\ \alpha_{11} - \alpha_{21} \geq \alpha_{1i},\ \alpha_{22} - \alpha_{12} < \alpha_{2i},\ \alpha_{1,(i+1)} \geq \alpha_{1i},\ \alpha_{1(3+|\mathcal{I}^1|)} \geq \alpha_{12}, \tag{4.89}$$
$$\forall i \in \mathcal{I}^w,\ \alpha_{11} - \alpha_{21} \geq \alpha_{1i},\ \alpha_{22} - \alpha_{12} \geq \alpha_{2i}, \tag{4.90}$$

and the following conditions are satisfied

$$\alpha_{11} - \alpha_{21} \geq \max \left\{ \begin{array}{c} \alpha_{12} \\ \alpha_{13} - (\alpha_{23} - \alpha_{21}) \\ \max\limits_{i \in \{4,\dots,2+|\mathcal{I}^1|\}} \alpha_{1i} - (\alpha_{2i} - \alpha_{2(i-1)}) \end{array} \right\} \tag{4.91}$$

$$\alpha_{22} - \alpha_{12} \geq \max \left\{ \begin{array}{c} \alpha_{21} \\ \alpha_{2(3+|\mathcal{I}^1|)} - (\alpha_{1(3+|\mathcal{I}^1|)} - \alpha_{12}) \\ \max\limits_{i \in \{3+|\mathcal{I}^1|,\dots,2+|\mathcal{I}^1|+|\mathcal{I}^2|\}} \alpha_{2i} - (\alpha_{1i} - \alpha_{1(i-1)}) \end{array} \right\}. \tag{4.92}$$

Proof. The proof of this theorem is done in two steps. First, we need to show that the presented conditions are sufficient for the optimality of (4.85). To do this, we show that the upper bound in (4.36) coincides with (4.85) as long as the given conditions are satisfied. The details of this is given in Appendix 4.B. Next, we show that these conditions are not only sufficient but also necessary. To do this, we show that there exists a scheme which achieves a GDoF larger than (4.85), if one of the given optimality conditions in this theorem is violated. The details of this step, is presented in Appendix 4.C. □

Example 2. *In Fig. 4.4, the received signals at receivers are illustrated for a permutation of transmitters of a* 7×2 *X-channel. In this figure, the length of the blocks represent the strength of the corresponding link (α-parameters). As shown in this figure, the received signals at Rx*1 *from Tx*3 *and Tx*4 *are stronger than* $\alpha_{11} - \alpha_{21}$*. On the other hand, the signals received at Rx*2 *from Tx*5 *and Tx*6 *are stronger than* $\alpha_{22} - \alpha_{12}$*. Hence, we have* $\mathcal{I}^1 = \{3, 4\}$ *and* $\mathcal{I}^2 = \{5, 6\}$*. Moreover,* $\mathcal{I}^w = \{7\}$*. Notice that this channel satisfies the condition given in Theorem 6 and hence, it is optimal to reduce this X-channel to a* 2*-user IC where the receivers treat interference as noise.*

4.4.3 Discussion

Suppose that in an $M \times 2$ X-channel the sets $\mathcal{I}^1$ and $\mathcal{I}^2$ are both empty. Then the conditions on the GDoF optimality of TIN in Theorem 6 are reduced to $\alpha_{11} - \alpha_{21} \geq \alpha_{12}$ and $\alpha_{22} - \alpha_{12} \geq \alpha_{21}$. These are equivalent to the conditions of the GDoF optimality of TIN in [GSJ15]. Notice that based on the results in [GSJ15], if the set $\mathcal{I}^1$ or $\mathcal{I}^2$ is not empty, we cannot judge about the optimality of TIN. However, from Theorem 6, we know that there are cases in which although $\mathcal{I}^1$ and $\mathcal{I}^2$ are not empty, TIN still performs optimally.

In the following, we explain the required conditions for the optimality of 2-IC-TIN based on the blockwise representation of the channels as in example 2. To this end, consider Txi, $i \in \mathcal{I}^1$. For this transmitter, the condition given in Theorem 6 can be explained as follows. The top-most part of the signal from Txi which goes beyond the threshold $\alpha^{(1)}_{\text{GNAJ-level}}$ needs to appear at Rx2 without any overlap with the signals from all Tx's in $\{1, \mathcal{I}^1\} \setminus \{i\}$ which cause interference at Rx2 lower than

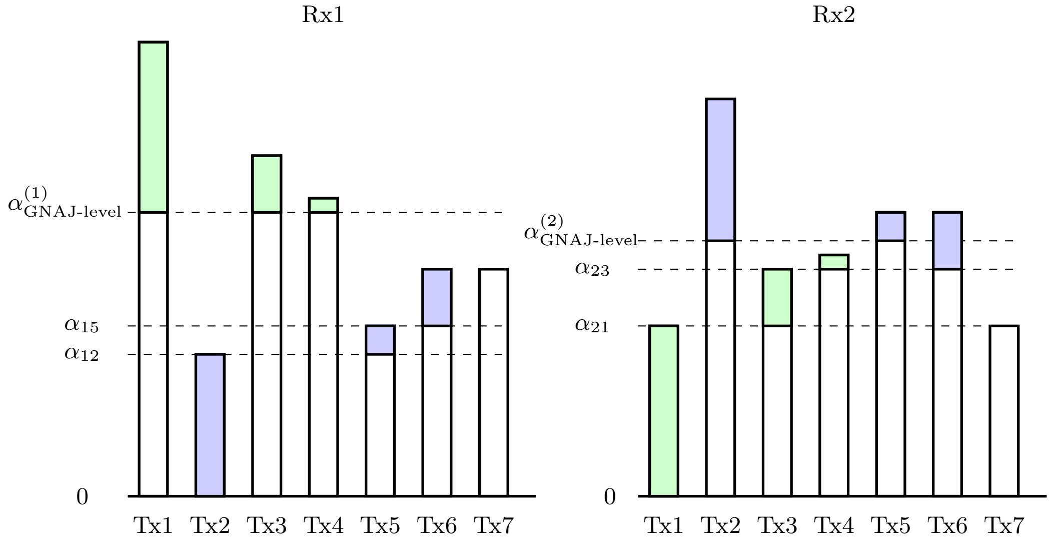
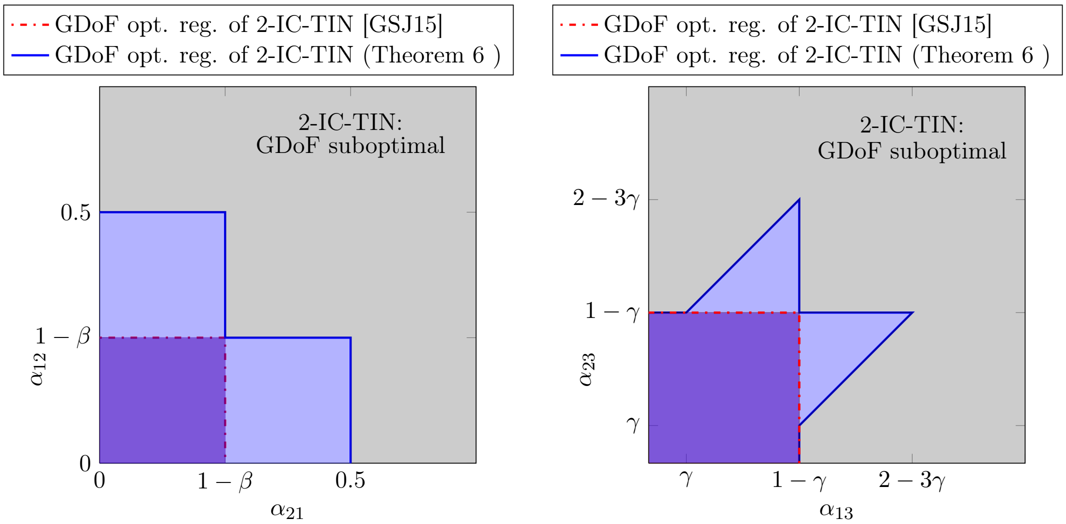


Figure 4.4: The received signal at both receivers are illustrated for a network with 7 transmitters and two receivers. Notice that in this example $\mathcal{I}^1 = \{3, 4\}$, $\mathcal{I}^2 = \{5, 6\}$, and $\mathcal{I}^w = \{7\}$. In this example, the conditions given in Theorem 6 are satisfied and hence the GDoF expression in (4.27) is optimal.

α_{2i}. A similar condition needs to be also satisfied for all Tx's in $\mathcal{I}^2$ for the optimality of 2-IC-TIN in an $M \times 2$ X-channel.

In Fig. 4.5a, the GDoF optimal regime of 2-IC-TIN is visualized for a 3×2 X-channel with $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{13} = \alpha_{23} = \beta$, where β is larger than 0.5 and smaller than 1. In the considered 2-IC-TIN, Tx1 and Tx2 communicate with Rx1 and Rx2, respectively, while the receivers use TIN. The obtained GDoF optimal regime of 2-IC-TIN from Theorem 6 is given by the union of the rectangle defined by $(\alpha_{21}, \alpha_{12}) \in [0, 0.5] \times [0, 1 - \beta]$ and the rectangle defined by $(\alpha_{21}, \alpha_{12}) \in [0, 1 - \beta] \times [0, 0.5]$. Moreover, we know that the proposed 2-IC-TIN is suboptimal if we are outside this regime. On the other hand, by applying the conditions on optimality of TIN given in [GSJ15], we obtain the intersection of these two rectangles. Obviously, the new GDoF optimal regime of TIN does not only subsume the previously known regime from [GSJ15] but also extends it. Note that if the channel gains from Tx3 to the receivers decrease, then the intersection of the two rectangles increases. At the point $\beta = 1/2$, both regimes will coincide and the regime becomes a rectangle with width and height of 1/2. Now, consider Fig. 4.5b. The channel strengths of the 3×2 X-channel considered in this figure are given as follows: $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = \gamma$, where $\gamma \leq 0.5$. By applying the result of [GSJ15] to this setup, we know that the 2-IC-TIN is optimal if $\max\{\alpha_{13}, \alpha_{23}\} - (1 - \gamma) \leq 0$. By using Theorem 6, this condition is relaxed to $\max\{\alpha_{13}, \alpha_{23}\} - (1 - \gamma) \leq (\min\{\alpha_{13}, \alpha_{23}\} - \gamma)^+ \leq 1 - 2\gamma$. As it is also shown in this figure, the GDoF optimal regime of 2-IC-TIN obtained from Theorem 6 subsumes and extends that of [GSJ15]. It is worth mentioning that the extended regime gets larger by decreasing the parameter γ.



(a) $\alpha_{11} = \alpha_{22} = 1$, and $\alpha_{13} = \alpha_{23} = \beta$, where $0.5 < \beta < 1$

(b) $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = \gamma$, where $\gamma \leq 0.5$

Figure 4.5: The GDoF optimal regime of 2-IC-TIN for 3×2 X-channel. Here, we focus on the optimality of using TIN in a 2-user IC in which Tx1 and Tx2 communicate with Rx1 and Rx2, respectively. It means that when the X-channel operates outside the GDoF optimal regime of 2-IC-TIN (shown in these figures), it might be the case that TIN performs GDoF optimally in another 2-user IC.

Remark 11. *As it can be seen in Fig. 4.5, the GDoF optimal regime of 2-IC-TIN in which Tx*1 *and Tx*2 *communicate with Rx*1 *and Rx*2*, respectively, and the receivers use TIN is in general a non-convex regime.*

4.5 Summary

The optimality of treating interference as noise (TIN) in $M \times 2$ X-channel is studied from the generalized degrees of freedom (GDoF) point of view. It turns out that the best transmission scheme alongside TIN is to let at most two transmitters send with full power while other transmitters are silent. Furthermore, it is shown that the best achievable GDoF of TIN is achieved by assigning at most one desired transmitter to each receiver. This leads us to two variants of TIN: P2P-TIN and 2-IC-TIN. While in P2P-TIN the setup is reduced to a point-to-point (P2P) channel, in 2-IC-TIN, the X-channel is decomposed into 2-user IC in which the receivers use TIN. The GDoF performance of these schemes are compared with new upper bounds provided in this chapter. It turned out that P2P-TIN performs GDoF optimally as long as either the channels to one receiver or from one transmitter are significantly large. Moreover, the sufficient conditions for the GDoF optimality of 2-IC-TIN were introduced. The presented conditions relax the optimality conditions derived from [GSJ15] and thus our obtained GDoF optimal regime of TIN subsumes and expands the known optimality regime of TIN. Moreover, it is shown that each particular 2-IC-TIN is outperformed by a better scheme from the GDoF point of view as long as we are outside its optimality regime. This proves the necessity of the optimality conditions.

4.A Proof of Lemma 22

Suppose that Tx1 and Tx2 communicate with Rx1 and Rx2, respectively. By using TIN at the receiver side, the achievable GDoF is given by (4.85). Now, suppose that there exists a user (e.g. Tx3) which satisfies the following conditions

$$(\alpha_{11} - \alpha_{21})^+ < \alpha_{13} \tag{4.93}$$

$$(\alpha_{22} - \alpha_{12})^+ < \alpha_{23}. \tag{4.94}$$

To show that the achievable GDoF in (4.85) is suboptimal, we propose a transmission scheme which achieves a higher GDoF. In this scheme, Tx1 and Tx3 send W_{11} and W_{13} to Rx1 and Tx2 sends W_{22} to Rx2. While W_{13} is a common message which is decoded by both Rx's, W_{11} and W_{22} are the private messages which are decoded only at the desired receivers. The messages W_{11}, W_{22}, W_{13} are encoded into x_1^n, x_2^n, x_3^n using random Gaussian codebooks with powers $P_1 = \frac{1}{|h_{21}|^2}$, $P_2 = \frac{1}{|h_{12}|^2}$, $P_3 = P$, and rates R_{11}, R_{12}, R_{13}, respectively. Note that P_1, P_2, and P_3 satisfy the power constraint due to the condition in (4.5). In time slot t, Txi, $i \in \{1, 2, 3\}$ sends $x_i[t]$ and Rxj, $j \in \{1, 2\}$ receives

$$y_j[t] = h_{j1}x_1[t] + h_{j2}x_2[t] + h_{j3}x_3[t] + z_j[t]. \tag{4.95}$$

At the end of the nth channel use, both Rx's decode first W_{13} by treating the remaining signals as noise. This can be done reliably as long as

$$\begin{aligned} R_{13} \leq & \min_j \quad \log_2\left(1 + \frac{P_3|h_{j3}|^2}{P_2|h_{j2}|^2 + P_1|h_{j1}|^2 + 1}\right) \\ = & \min\left\{\log_2\left(1 + \frac{P|h_{13}|^2}{2 + \frac{|h_{11}|^2}{|h_{21}|^2}}\right), \log_2\left(1 + \frac{P|h_{23}|^2}{2 + \frac{|h_{22}|^2}{|h_{12}|^2}}\right)\right\}. \end{aligned} \tag{4.96}$$

Then, Rxj reconstructs x_3^n and removes its contribution from (4.95). Next, it decodes W_{jj} reliably as long as

$$R_{jj} \leq \log_2\left(1 + \frac{P_j|h_{jj}|^2}{1 + P_{j'}|h_{jj'}|^2}\right),$$

where $j = 1, 2$, $j' = 1, 2$, and $j \neq j'$. Note that $P_{j'}|h_{jj'}|^2 = 1$. Therefore, we obtain

$$R_{11} \leq \log_2\left(1 + \frac{|h_{11}|^2}{2|h_{21}|^2}\right), \quad R_{22} \leq \log_2\left(1 + \frac{|h_{22}|^2}{2|h_{12}|^2}\right). \tag{4.97}$$

Transforming the achievable rate in (4.96), and (4.97) into achievable GDoF we obtain

$$\begin{aligned} d = & \min\{(\alpha_{13} - (\alpha_{11} - \alpha_{21})^+)^+, (\alpha_{23} - (\alpha_{22} - \alpha_{12})^+)^+\} \\ & + (\alpha_{11} - \alpha_{21})^+ + (\alpha_{22} - \alpha_{12})^+ \\ > & \, d_{\text{TIN,2-IC}}. \end{aligned} \tag{4.98}$$

As it is shown, our proposed transmission scheme outperforms (4.85) which completes the proof of Lemma 22.

4.B Poof of Theorem 6: Sufficient Optimality Conditions of 2-IC-TIN

Here, we want to show the optimality of reducing an $M \times 2$ X-channel to a 2-user IC where the receivers use TIN. Now, suppose that there exists a permutation as mentioned in Theorem 6. The goal is to show that the achievable GDoF in (4.85) is optimal. To do this, we use the upper bound in (4.36) by substituting $i_1 = 1$, $i_2 = 2$, and the sets $\mathcal{I}^1$, $\mathcal{I}^2$, $\mathcal{I}^w$ as in (4.88)-(4.90). First, we need to bound the expressions for Δ_1 and Δ_2. In what follows, we bound Δ_1 which is given by

$$\begin{aligned}\Delta_1 =& h\left(h_{21}X_1^n + \sum_{k\in\mathcal{I}^1}\frac{h_{21}h_{1k}}{h_{11}}X_k^n + N_1^n|W_{21},W_{2\mathcal{I}^1}\right)\\ &- h\left(h_{21}X_1^n + \sum_{k\in\mathcal{I}^1}h_{2k}X_k^n + Z_2^n|W_{21},W_{2\mathcal{I}^1}\right).\end{aligned} \tag{4.99}$$

Notice that Δ_1 is the difference of two entropy terms, which can be bounded using the result of Lemma 21, as long as the required conditions are satisfied. In order to apply Lemma 21, we substitute first the parameters of (4.99) into the parameters given in Lemma 21 as follows

$$\begin{aligned}
&A_1 \triangleq X_1, \quad A_\ell \triangleq X_m \\
&f_1 \triangleq h_{21}, \quad f_\ell \triangleq h_{2m} \qquad \gamma_1 = \alpha_{21}, \quad \gamma_\ell \triangleq \alpha_{2m}, \\
&g_1 \triangleq h_{21}, \quad g_\ell \triangleq \frac{h_{21}h_{1m}}{h_{11}}, \quad \beta_1 \triangleq \alpha_{21}, \quad \beta_\ell \triangleq \alpha_{21} + \alpha_{1m} - \alpha_{11}, \\
&N_1 \triangleq N_1, \quad N_2 \triangleq Z_2, \qquad W_c \triangleq \{W_{21}, W_{2\mathcal{I}^1}\}
\end{aligned}$$

where the relationship between ℓ and m for $\ell \geq 2$ is given by $m = \ell + 1$ and $\gamma_0 \triangleq 0$. Now, we need to show that the conditions given in Lemma 21, i.e. for all $\ell \in \{1, \ldots, L\}$, $\gamma_\ell - \gamma_{\ell-1} \geq \beta_\ell$ are satisfied. To do this, consider the following cases

$$\ell = 1: \quad \gamma_1 - \gamma_0 \geq \beta_1 \Rightarrow \quad \alpha_{21} \geq \alpha_{21} \tag{4.100}$$
$$\ell = 2: \quad \gamma_2 - \gamma_1 \geq \beta_2 \Rightarrow \quad \alpha_{23} - \alpha_{21} \geq \alpha_{21} + \alpha_{13} - \alpha_{11} \tag{4.101}$$
$$\ell \geq 3: \quad \gamma_\ell - \gamma_{\ell-1} \geq \beta_\ell \Rightarrow \quad \alpha_{2m} - \alpha_{2(m-1)} \geq \alpha_{21} + \alpha_{1m} - \alpha_{11} \tag{4.102}$$

Notice that given the condition (4.91) in Theorem 6, all the conditions (4.100)-(4.102) are satisfied. Hence, we can write

$$\Delta_1 \leq n \log_2((1 + |\mathcal{I}^1|)!). \tag{4.103}$$

Similarly, the expression Δ_2 is bounded by

$$\Delta_2 \leq n \log_2((1 + |\mathcal{I}^2|)!). \tag{4.104}$$

Now, we can write the upper bound in (4.36) as follows

$$C_\Sigma \leq \log_2((1+|\mathcal{I}^1|)!) + \log_2\left(1 + P|h_{12}|^2 + \sum_{k\in\{\mathcal{I}^2,\mathcal{I}^w\}} P|h_{1k}|^2 + \frac{|h_{11}|^2}{|h_{21}|^2}\right)$$
$$+ \log_2((1+|\mathcal{I}^2|)!) + \log_2\left(1 + P|h_{21}|^2 + \sum_{k\in\{\mathcal{I}^1,\mathcal{I}^w\}} P|h_{2k}|^2 + \frac{|h_{22}|^2}{|h_{12}|^2}\right).$$

Now, by using the definition of the parameter α in (4.9), we can write

$$C_\Sigma \leq \log_2((1+|\mathcal{I}^1|)!) + \log_2\left(1 + \rho^{\alpha_{12}} + \sum_{k\in\{\mathcal{I}^2,\mathcal{I}^w\}} \rho^{\alpha_{1k}} + \rho^{\alpha_{11}-\alpha_{21}}\right)$$
$$+ \log_2((1+|\mathcal{I}^2|)!) + \log_2\left(1 + \rho^{\alpha_{21}} + \sum_{k\in\{\mathcal{I}^1,\mathcal{I}^w\}} \rho^{\alpha_{2k}} + \rho^{\alpha_{22}-\alpha_{12}}\right).$$

Further, by dividing the upper bound expression by $\log_2 \rho$ and letting $\rho \to \infty$, we obtain the following upper bound for the GDoF

$$d(\boldsymbol{\alpha}) \leq \max\left\{\alpha_{12}, (\alpha_{11}-\alpha_{21})^+, \max_{i\in\{\mathcal{I}^2,\mathcal{I}^w\}} \alpha_{1i}\right\}$$
$$+ \max\left\{\alpha_{21}, (\alpha_{22}-\alpha_{12})^+, \max_{i\in\{\mathcal{I}^1,\mathcal{I}^w\}} \alpha_{2i}\right\}. \quad (4.105)$$

Now, due to the conditions (4.88)-(4.90), and (4.91), (4.92) and the fact that the all parameters α are non-negative, we can write the following upper bound on the GDoF

$$d(\boldsymbol{\alpha}) \leq \alpha_{11} - \alpha_{21} + \alpha_{22} - \alpha_{12}. \quad (4.106)$$

This coincides with (4.85) which completes the proof.

4.C Proof of Theorem 6: Necessary Optimality Conditions of 2-IC-TIN

Here, we want to show that the conditions on the GDoF optimality of 2-IC TIN in Theorem 6 are necessary. Notice that here the focus is on the 2-IC-TIN which is defined in Theorem 6. In this scheme, the $M \times 2$ X-channel is reduced to the 2-user IC in which Tx1 and Tx2 serve Rx1 and Rx2, respectively and the receivers use TIN. More precisely, we show that if only one of the conditions in (4.88)-(4.92) is not satisfied, then the achievable GDoF in (4.85) is suboptimal. Notice that as it is justified in Remark 10, here we assume that $\alpha_{11} \geq \alpha_{21}$ and $\alpha_{22} \geq \alpha_{12}$. Notice that if it is not the case, P2P-TIN outperforms the proposed 2-IC-TIN (cf. (4.14)).

We know from Lemma 22 that if there exists a transmitter with channels to Rx1 and Rx2 larger that $\alpha_{11} - \alpha_{21}$ and $\alpha_{22} - \alpha_{12}$, respectively, then the achievable GDoF in (4.85) is suboptimal. Hence, in all $M \times 2$ X-channels in which the proposed 2-IC-TIN performs GDoF optimally, Txi with $i \in \{3, \dots, M\}$ has to be either a member of $\mathcal{I}^1$ or $\mathcal{I}^2$ or $\mathcal{I}^w$. This justifies the necessity of splitting of Txi with $i \in \{3, \dots, M\}$ into these three groups for optimality of 2-IC-TIN (as in (4.88)-(4.90)). Notice that the third condition in (4.88) and (4.89) represent the order of the transmitters which can be satisfied by the permutation.

Now, we need to justify the necessity of the fourth condition in (4.88) and (4.89) and the conditions in (4.91), (4.92). Consider the first condition in (4.91), i.e., $\alpha_{11} - \alpha_{21} \geq \alpha_{12}$. Notice that if this condition is violated, the P2P channel between Tx2 and Rx2 achieves α_{22} which is larger than the achievable GDoF of 2-IC-TIN in (4.85) (since $\alpha_{11} - \alpha_{21} < \alpha_{12}$). The necessity of the first condition in (4.92) can be shown similarly.

Now, we show the necessity of the fourth condition in (4.88), (4.89) and the second condition in (4.91) and (4.92) as follows. Consider a 3×2 X-channel consisting of Tx1, Tx2, Tx3 and Rx1, Rx2 which is illustrated in Fig. 4.6a. Obviously, all achievable rates in this 3×2 X-channel are also achievable in the original $M \times 2$ X-channel. Suppose that Tx3 $\in \mathcal{I}^1$ and the channel parameters of the 3×2 X-channel satisfy the (so far known) necessary conditions for optimality of 2-IC-TIN. Hence, we have

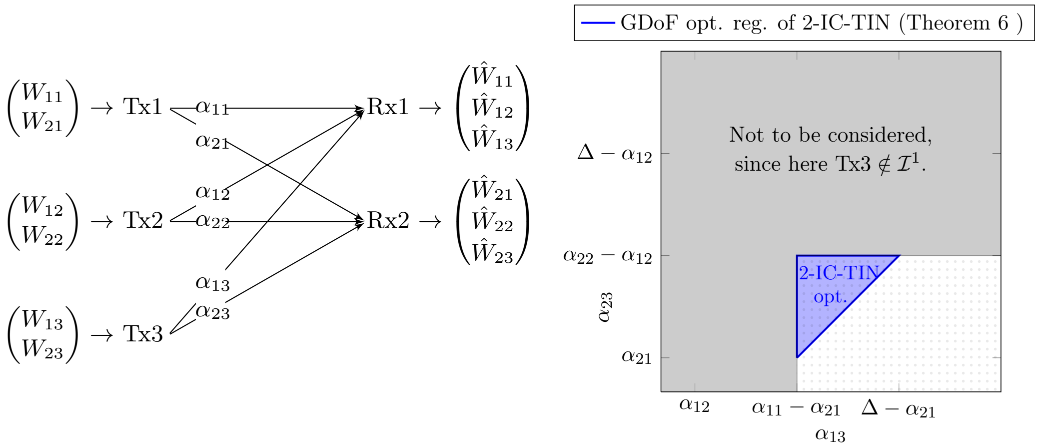
$$\alpha_{11} - \alpha_{21} \geq \alpha_{12}, \quad \alpha_{22} - \alpha_{12} \geq \alpha_{21}, \quad \alpha_{13} > \alpha_{11} - \alpha_{21}, \quad \alpha_{23} \leq \alpha_{22} - \alpha_{12}.$$

In Fig. 4.6b, the dotted area illustrates the allowed parameters for α_{13} and α_{23}. From Theorem 6, we know that 2-IC-TIN performs GDoF optimally in a part of the dotted area which is given as follows

$$\alpha_{13} \leq \alpha_{11} - 2\alpha_{21} + \alpha_{23}. \tag{4.107}$$

This area is separated by a solid line in Fig. 4.6b. In what follows, we need to show the suboptimality of 2-IC-TIN for the remaining area which is given by

$$\alpha_{13} > \max\{\alpha_{11} - 2\alpha_{21} + \alpha_{23}, \alpha_{11} - \alpha_{21}\}, \text{ and } \alpha_{23} \leq \alpha_{22} - \alpha_{12}. \tag{4.108}$$



(a) 3×2 X-channel.

(b) The GDoF (sub)-optimal regimes of 2-IC-TIN.

Figure 4.6: The GDoF (sub)-optimal regimes of 2-IC-TIN for 3×2 X-channel, where $\Delta = \alpha_{11} - \alpha_{21} + \alpha_{22} - \alpha_{12}$.

As it is also shown in this figure, this area includes all cases where $\alpha_{23} < \alpha_{21}$. Hence, by showing the suboptimality of 2-IC-TIN in the area defined by (4.108), we do not only show the necessity of the second condition in (4.91) but also the fourth condition in (4.88). By setting the rate of messages W_{12}, W_{21}, W_{23} to zero, the 3×2 X-channel in Fig. 4.6a is reduced to the PIMAC which is a P2P channel interfered with a MAC. The system model of the PIMAC is shown in Fig.4.7. Notice that the capacity of

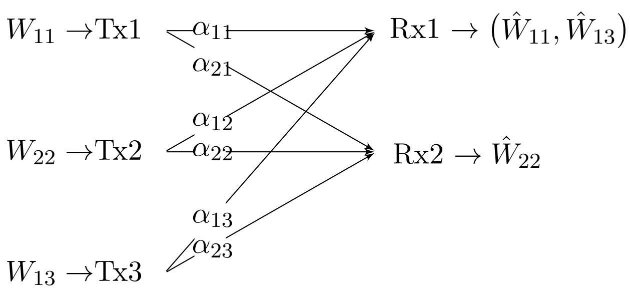


Figure 4.7: The system model of the PIMAC.

the 3×2 X-channel is an upper for the capacity of the PIMAC. Hence, all achievable rates in the PIMAC are also achievable in the 3×2 X-channel. Therefore, if the proposed 2-IC-TIN performs GDoF suboptimally in PIMAC, it is also suboptimal in the original X-channel. The optimality of TIN in PIMAC whose channels satisfy $\alpha_{11} - \alpha_{21} \geq \alpha_{12}$ and $\alpha_{22} - \alpha_{12} \geq \alpha_{21}$ was studied in Chapter 3. It turned out that reducing the PIMAC to the 2-user IC in which Tx1 and Tx2 communicate with Rx1 and Rx2, respectively, and the receivers use TIN is GDoF suboptimal if (4.108) (this corresponds to regimes 3C, 2A, 2B, and 2C in Fig 3.7) is satisfied. Hence, we conclude that the proposed 2-IC-TIN is also GDoF suboptimal in the 3×2 X-channel if (4.108) holds. Similarly, we can show the GDoF suboptimality of the 2-IC-TIN in the X-channel if the second condition in (4.92) or the fourth condition in (4.89) is not satisfied.

In order to complete the proof, we need to also show the necessity of the third condition in (4.91) and (4.92). Without loss of generality, suppose that with $i = 4$, the third condition in (4.91) is violated. Now, consider the 4×2 X-channel consisting of Tx1, Tx2, Tx3, Tx4 and Rx1, Rx2, where

$$\min\{\alpha_{13}, \alpha_{14}\} > \alpha_{11} - \alpha_{21} \geq \max\{\alpha_{12}, \alpha_{13} - (\alpha_{23} - \alpha_{21})\}, \tag{4.109}$$

$$\alpha_{22} - \alpha_{12} \geq \alpha_{24} \geq \alpha_{23} \geq \alpha_{21}. \tag{4.110}$$

Notice that the necessity of these conditions for the optimality of 2-IC-TIN has been already shown. Now, we want to show that if in this 4×2 X-channel,

$$\alpha_{11} - \alpha_{21} < \alpha_{14} - (\alpha_{24} - \alpha_{23}) \tag{4.111}$$

holds, 2-IC-TIN is GDoF suboptimal. The following lemma, summarizes the achievable GDoF in this X-channel.

Lemma 23. *As long as* (4.109)- (4.111) *are satisfied in a* 4×2 *X-channel consisting Tx*1*, Tx*2*, Tx*3*, Tx*4 *and Rx*1*, Rx*2*, the following GDoF*

$$d(\boldsymbol{\alpha}) = \alpha_{11} - \alpha_{21} + \alpha_{22} - \alpha_{12} + \ell, \tag{4.112}$$

with $\ell = \min\{\mu - \nu, \nu - (\alpha_{11} - \alpha_{21}), \alpha_{23}\}$*,* $\mu = \max\{\alpha_{14} - (\alpha_{24} - \alpha_{23}), \alpha_{13}\}$*, and* $\nu = \min\{\alpha_{14} - (\alpha_{24} - \alpha_{23}), \alpha_{13}\}$ *is achievable.*

Proof. To achieve this GDoF, we propose a scheme which is a combination of private signaling with alignment. The details of the proof can be found in Appendix 4.D. □

Due to the condition in (4.109) and (4.111), ℓ is positive excluding the special case that $\alpha_{14} - \alpha_{13} = \alpha_{24} - \alpha_{23}$. Hence, the achievable GDoF in (4.112) is almost surely strictly larger than that of the proposed 2-IC-TIN given in (4.85). Similarly, we can show the suboptimality of 2-IC-TIN if the third condition in (4.91) is violated by any Txi with $5 \leq i \leq 2 + |\mathcal{I}^1|$. In that case, we reduce the $M \times 2$ X-channel to a 4×2 X-channel consists of Tx1 , Tx2, Tx$(i-1)$, Txi, Rx1, and Rx2. Next, we use the transmission scheme introduced in Appendix 4.D with a slight difference that Tx3 and Tx4 in this scheme needs to be replaced by Tx$(i-1)$ and Txi, respectively. This shows the necessity of the third condition in (4.91). Similarly, we can also show the necessity of the third condition in (4.92). This completes the necessity of conditions in (4.88)-(4.92) for the optimality of the proposed 2-IC-TIN in Theorem 6.

4.D Proof of Lemma 23

Here, we propose a scheme which achieves (4.112) as long as (4.109)- (4.111) hold. In this scheme, Tx1, Tx3, and Tx4 communicate with Rx1 while only Tx2 communicates with Rx2. The messages of Tx1 and Tx2 are private messages, which are denoted by $W_{1,p}$ and $W_{2,p}$. The messages of Tx3 and Tx4 are alignment message and represented by $W_{3,a}$ and $W_{4,a}$. The rates of messages $W_{i_p,p}$ and $W_{i_a,a}$ with $i_p \in \{1,2\}$ and $i_a \in \{3,4\}$, are denoted by $R_{i_p,p}$ and $R_{i_a,a}$, respectively. In what follows, we suppose that $R_{3,a} = R_{4,a} = R_a$. In this scheme, Tx2 splits its message $W_{2,p}$ into two sub-messages $W_{2,p1}$ and $W_{2,p2}$ with rates $R_{2,p1}$ and $R_{2,p2}$, where $R_{2,p1} + R_{2,p2} = R_{2,p}$.

Encoding: Tx1 and Tx2 use Gaussian random codebooks to encode their messages $W_{1,p}$ and $W_{2,p1}$, $W_{2,p2}$ into complex-valued symbols $x^n_{1,p}$ and $x^n_{2,p1}$, $x^n_{2,p2}$ with powers

$$P_{1,p} = \frac{1}{|h_{21}|^2}, \quad P_{2,p1} = \frac{1}{2|h_{12}|^2}, \quad P_{2p2} = \frac{\rho^{(\alpha_{23}-\ell)}}{2|h_{22}|^2}, \tag{4.113}$$

respectively, where

$$\ell = \min\{\mu - \nu, \nu - (\alpha_{11} - \alpha_{21}), \alpha_{23}\}, \tag{4.114}$$

$$\mu = \max\{\alpha_{14} - (\alpha_{24} - \alpha_{23}), \alpha_{13}\}, \tag{4.115}$$

$$\nu = \min\{\alpha_{14} - (\alpha_{24} - \alpha_{23}), \alpha_{13}\}. \tag{4.116}$$

Tx3 and Tx4 use the same nested-lattice codebook (Λ_f, Λ_c) with rate R_a and power 1 to encode their messages.

Remark 12. *Notice that since $\alpha_{23} \le \alpha_{22} - \alpha_{12}$ (cf. (4.109)), and $\ell \ge 0$ (due to (4.109) and (4.111)), $P_{2,p2} \le P_{2,p1}$.*

Λ_c and Λ_f denote the coarse and fine lattices, respectively. In more details, Txi_a with $i_a \in \{3,4\}$ uses (Λ_f, Λ_c) to encode $W_{i_a,a}$ into length-n codeword λ_{i_a} with rate R_a. Then, Txi_a constructs the following complex-valued signal

$$x^n_{i_a,a} = \sqrt{P_{i_a,a}}\left[(\lambda_{i_a} - d_{i_a}) \bmod \Lambda_c\right],$$

where d_{i_a} is an n-dimensional random dither vector [NG11] known also at the receivers. Moreover,

$$P_{3,a} = P, \quad P_{4,a} = \frac{P|h_{23}|^2}{|h_{24}|^2}. \tag{4.117}$$

Since the length of all sequences in this section is n, we drop the superscript n in the rest of the section. Then the transmitters send the signals

$$x_1 = x_{1,p}, \quad x_2 = x_{2,p1} + x_{2,p2}, \quad x_3 = \mathrm{e}^{-\mathrm{j}\,\varphi_{23}} x_{3,a} \quad x_4 = \mathrm{e}^{-\mathrm{j}\,\varphi_{24}} x_{4,a}$$

where $\mathrm{j} = \sqrt{-1}$ and φ_{ji} represents the phase of the channel h_{ji}. Notice that since we consider interference limited scenario and $(\alpha_{24} \ge \alpha_{23})$ (cf. (4.110)), the assigned powers in (4.113) and (4.117) fulfill the power constraints, i.e.,

$$P_1 = P_{1,p} \le P, \quad P_2 = P_{2,p1} + P_{2,p2} \overset{(a)}{\le} 2P_{2,p1} \le P, \quad P_3 = P_{3,a} \le P, \quad P_4 = P_{4,a} \le P,$$

where (a) follows from Remark 12.

Decoding: Now, we write the received signals at the receivers as follows

$$y_j = h_{j1}x_{1,p} + h_{j2}(x_{2,p1} + x_{2,p2}) + h_{j3}\mathrm{e}^{-\mathrm{j}\varphi_{23}}x_{3,a} + h_{j4}\mathrm{e}^{-\mathrm{j}\varphi_{24}}x_{4,a} + z_j.$$

in what follows, we explain the decoding process at each receiver separately.

First, consider Rx1. The decoding order at Rx1 depends on the channel strength. If the parameter $\mu = \alpha_{14} - (\alpha_{24} - \alpha_{23})$, Rx1 decodes first $W_{4,a}$ otherwise it decodes first W_{3a}.

Consider first the case that $\mu = \alpha_{14} - (\alpha_{24} - \alpha_{23})$. Then, Rx1 decodes $W_{4,a}$ while it treats x_1, x_2, and x_3 as noise. It is shown in [EZ04] that nested-lattice codes achieve the capacity of the point-to-point AWGN channel. Therefore, the rate constraint for reliable decoding of $W_{4,a}$ is given by

$$R_{4,a} = R_a \leq \log_2\left(1 + \frac{P_4|h_{14}|^2}{P_1|h_{11}|^2 + P_2|h_{12}|^2 + P_3|h_{13}|^2 + 1}\right) \text{ if } \mu = \alpha_{14} - (\alpha_{24} - \alpha_{23}). \tag{4.118}$$

After decoding $W_{4,a}$, Rx1 removes the interference caused by x_4 from the received signal. Then it decodes x_{3a} reliably as long as

$$R_{3,a} = R_a \leq \log_2\left(1 + \frac{P_3|h_{13}|^2}{P_1|h_{11}|^2 + P_2|h_{12}|^2 + 1}\right) \text{ if } \mu = \alpha_{14} - (\alpha_{24} - \alpha_{23}). \tag{4.119}$$

Decoding $W_{3,a}$, $W_{4,a}$ can be done similarly if $\mu = \alpha_{13}$. For this case, the rate constraints can be written as follows

$$R_{3,a} = R_a \leq \log_2\left(1 + \frac{P_3|h_{13}|^2}{P_1|h_{11}|^2 + P_2|h_{12}|^2 + P_4|h_{14}|^2 + 1}\right) \quad \text{if } \mu = \alpha_{13} \tag{4.120}$$

$$R_{4,a} = R_a \leq \log_2\left(1 + \frac{P_4|h_{14}|^2}{P_1|h_{11}|^2 + P_2|h_{12}|^2 + 1}\right) \quad \text{if } \mu = \alpha_{13}. \tag{4.121}$$

Next, Rx1 decodes $W_{1,p}$ by treating $W_{2,p}$ as noise. This can be done reliably, as long as

$$R_{1,p} \leq \log_2\left(1 + \frac{P_1|h_{11}|^2}{P_2|h_{12}|^2 + 1}\right). \tag{4.122}$$

Now, consider the decoding at the second receiver. Rx2 decodes the messages in following orders $W_{2,p1} \to f(W_{3,a}, W_{4,a}) \to W_{2,p2}$, where $f(W_{3,a}, W_{4,a})$ is a function of $W_{3,a}$, $W_{4,a}$. The reliable decoding of $W_{2,p1}$ is possible as long as

$$R_{2,p1} \leq \log_2\left(1 + \frac{P_{2,p1}|h_{22}|^2}{P_3|h_{23}|^2 + P_4|h_{24}|^2 + P_{2,p2}|h_{22}|^2 + P_1|h_{21}|^2 + 1}\right). \tag{4.123}$$

Rx2 proceeds with decoding $(|h_{23}|\lambda_{3,a} + |h_{24}|\lambda_{4,a}) \mod \Lambda_c$. Notice that $P_{3,a}$ and $P_{4,a}$ are chosen in a way such that $\lambda_{3,a}$ and $\lambda_{4,a}$ are received aligned at Rx2. Reliable decoding of $(|h_{23}|\lambda_{3,a} + |h_{24}|\lambda_{4,a}) \mod \Lambda_c$ is possible as long as

$$R_a \leq \left[\log_2\left(1 + \frac{P_3|h_{23}|^2}{P_{2,p2}|h_{22}|^2 + P_1|h_{21}|^2 + 1} - \frac{1}{2}\right)\right]^+. \tag{4.124}$$

Next, Rx2 reconstructs $|h_{23}|x_3 + |h_{24}|x_4$ from the decoded sum $(|h_{23}|\lambda_{3,a} + |h_{24}|\lambda_{4,a})$ mod Λ_c and removes the interference caused by it. Doing this, Rx2 observes $x_{2,p2}$ interference-free. The reliable decoding of $W_{2,p2}$ is possible as long as

$$R_{2,p2} \leq \log_2\left(1 + \frac{P_{2,p2}|h_{22}|^2}{P_1|h_{21}|^2 + 1}\right). \tag{4.125}$$

This scheme achieves

$$R_\Sigma = R_{3,a} + R_{4,a} + R_{1,p} + R_{2,p1} + R_{2,p2}, \tag{4.126}$$

as long as the constraints in (4.118)-(4.125) are satisfied. In order to complete the prove, we need to translate all rate constraints and the achievable rate to GDoF constraints and achievable GDoF, respectively. To do this, consider first (4.118). By substituting the powers (4.113) and (4.117) into (4.118) and keeping in mind that $P_{2,p2} \leq P_{2,p1}$ (cf. Remark 12), we can write that all rates

$$R_a \leq \log_2\left(1 + \frac{\rho^{\alpha_{14}+\alpha_{23}-\alpha_{24}}}{\rho^{\alpha_{11}-\alpha_{21}} + \frac{1}{|h_{12}|^2}|h_{12}|^2 + \rho^{\alpha_{13}} + 1}\right) \text{ if } \mu = \alpha_{14} - (\alpha_{24} - \alpha_{23}). \tag{4.127}$$

are achievable. By dividing (4.127) with $\log_2 \rho$ and letting $\rho \to \infty$, we conclude that the achievable GDoF using the alignment message needs to satisfy the following condition

$$\begin{aligned} d_a &\leq (\alpha_{14} + \alpha_{23} - \alpha_{24} - \max\{\alpha_{11} - \alpha_{21}, \alpha_{13}, 0\})^+ \\ &\overset{(b)}{=} \alpha_{14} + \alpha_{23} - \alpha_{24} - \alpha_{13}, \quad \text{if } \mu = \alpha_{14} - (\alpha_{24} - \alpha_{23}), \end{aligned} \tag{4.128}$$

where step (b) follows since Tx3 $\in \mathcal{I}^1$ (cf. (4.109)) and in this case $\mu = \alpha_{14} - (\alpha_{24} - \alpha_{23})$. Similarly, we can write (4.119) as a GDoF constraint as follows

$$d_a \leq \alpha_{13} - (\alpha_{11} - \alpha_{21}), \quad \text{if } \mu = \alpha_{14} - (\alpha_{24} - \alpha_{23}). \tag{4.129}$$

One can combine the conditions (4.128) and (4.129) as follows

$$d_a \leq \min\left\{\alpha_{14} + \alpha_{23} - \alpha_{24} - \alpha_{13}, \alpha_{13} - (\alpha_{11} - \alpha_{21})\right\}, \quad \text{if } \mu = \alpha_{14} - (\alpha_{24} - \alpha_{23}) \tag{4.130}$$

Similarly, we can write the constraints (4.120) and (4.121) as the following constraints on the GDoF

$$\begin{aligned} d_a &\leq \alpha_{13} - \max\{\alpha_{11} - \alpha_{21}, \alpha_{14} + \alpha_{23} - \alpha_{24}\} \\ &\overset{(c)}{=} \alpha_{13} - \alpha_{14} - \alpha_{23} + \alpha_{24}, \quad \text{if } \mu = \alpha_{13}, \end{aligned} \tag{4.131}$$

$$d_a \leq \alpha_{14} + \alpha_{23} - \alpha_{24} - (\alpha_{11} - \alpha_{21}), \quad \text{if } \mu = \alpha_{13}, \tag{4.132}$$

where step (c) follows from the condition in (4.111). By combining the conditions (4.131) and (4.132), we obtain

$$d_a \leq \min\{\alpha_{13} - \alpha_{14} - \alpha_{23} + \alpha_{24}, \alpha_{14} + \alpha_{23} - \alpha_{24} - (\alpha_{11} - \alpha_{21})\}, \quad \text{if } \mu = \alpha_{13}. \tag{4.133}$$

The conditions in (4.129) and (4.133) can be summarized as follows

$$d_a \leq \min\{\mu - \nu, \nu - (\alpha_{11} - \alpha_{21})\}. \tag{4.134}$$

Now, consider (4.122). This constraint can be converted to the following GDoF condition

$$d_{1,p} \leq \alpha_{11} - \alpha_{21}. \tag{4.135}$$

Now, we use the fact that $P_{2,p2}|h_{22}|^2 \leq P_3|h_{23}|^2 = P_4|h_{24}|^2$, we write the rate constraint in (4.123) as the following GDoF constraint

$$d_{2,p1} \leq \alpha_{22} - \alpha_{12} - \alpha_{23}. \tag{4.136}$$

Consider the constraint in (4.124). This constraint can be converted to following GDoF constraint

$$d_a \leq \alpha_{23} - (\alpha_{23} - \ell)^+ \stackrel{(d)}{=} \ell, \tag{4.137}$$

where step (d) follows since $\ell \leq \alpha_{23}$. Notice that given (4.137), the condition in (4.134) is always satisfied. Finally, we convert (4.125) into the following GDoF constraint

$$d_{2,p2} \leq \alpha_{23} - \ell. \tag{4.138}$$

Now, we can write the achievable GDoF as follows

$$d(\boldsymbol{\alpha}) = 2d_a + d_{1,p} + d_{2,p1} + d_{2,p2}, \tag{4.139}$$

where the achievable GDoF of private and alignment messages need to satisfy the conditions in (4.135)-(4.138). In order to achieve the maximum GDoF, these constraints need to be fulfilled with equality. Hence, using the proposed scheme, the following GDoF is achievable

$$d(\boldsymbol{\alpha}) = \alpha_{11} - \alpha_{21} + \alpha_{22} - \alpha_{12} + \ell. \tag{4.140}$$

This completes the proof of Lemma 23.

5 Conclusion

In this chapter, we summarize the contribution and provide some new problems as extension of this thesis.

5.1 Summary of Contributions

In this thesis, the optimality of employing treating interference as noise (TIN) at receivers together with Gaussian encoding at transmitters was studied for elemental networks; namely a point-to-point channel interfering with a multiple access channel (PIMAC) and $M \times 2$ X-channel. The main focus was on characterizing the optimal regime of TIN from the generalized degrees of freedom (GDoF) point of view. Our approach towards finding the GDoF optimal regime of TIN for the PIMAC has been initiated with the capacity analysis of the linear deterministic model of PIMAC (LD-PIMAC). The capacity optimal regime of TIN for the LD-PIMAC has been completely characterized. Furthermore, the capacity results obtained for the LD-PIMAC have been translated to the the GDoF results for the Gaussian counterpart. The capacity optimal regime of TIN for the LD-PIMAC corresponded completely to the GDoF optimal regime of TIN for the Gaussian PIMAC.

Interestingly, it turned out that in some regimes TIN is suboptimal although all the undesired links are very weak. The main reason of this fact is that in PIMAC, in contrast to the interference channel (IC), more than one transmitter can serve a single receiver. However by using TIN, no GDoF gain is attained by dedicating multiple transmitters to a receiver. Therefore, as it is also shown in Chapter 3 that the desired links which correspond to the inactive transmitters need to be considered as interference links in the optimality condition of TIN. On the other hand, it is shown that the intuitive condition which restricts the optimality of TIN to the regimes with very weak interference links compared to the desired links is generally not necessary. These insights obtained on the optimality of TIN in the Gaussian PIMAC have been further extended to the $M \times 2$ X-channel. In order to obtain a complete characterization of the GDoF optimal regime of TIN, the following issues were considered individually for PIMAC and $M \times 2$ X-channel.

Different variants of TIN: The achievable performance of using TIN at the receiver side has been studied while the transmitters were allowed to use Gaussian encoding with arbitrary power control. Depending on the transmit power, different sum-rates were achievable. Interestingly, it has been shown that in PIMAC and $M \times 2$ X-channel as long as the receivers use TIN, the GDoF optimal power control is a binary power control in which the transmitters either send with full power or are completely inactive. Moreover, it turned out that no additional GDoF gain can be attained by allowing more than one desired transmitter for a single receiver or multiple desired receivers for a single transmitter, as long as the receivers employ TIN. These facts led us to consider different variants of TIN for each setup. While in all those variants, the decoding strategy was restricted to simple TIN, their difference was mainly originated from different message flows in those variants in a sense that the original channel was reduced to different interference channels (IC) or point-to-point (P2P) channels. Obviously, by increasing the variants of TIN the achievable sum-rate increases and subsequently the regime where TIN is GDoF-optimal.

Tighter upper bounds: Different upper bounds on the GDoF performance of each channel have been established. All of the bounds were genie-aided in which additional information were provided to the receivers. Some of them were inspired from the upper bound presented in [ETW08] for the 2-user IC in which the noisy version of the interference signal caused by each transmitter is provided to the desired receiver. In this thesis, a more general type of upper bound has been also established in which a linear combinations of signals was given as side information to a receiver. Additionally, in order to tighten the obtained upper bounds, we proposed a novel lemma on the maximum difference between differential entropies of noisy linear combinations of signals under power constraints. This difference is bounded by a constant independent of power and hence does not appear in the GDoF expression. Due to our new upper bound, the obtained GDoF optimal regime of TIN does not only subsume the known regime in the literature but also extends them significantly.

Suboptimality of TIN: It was shown for all variants of TIN in PIMAC and $M \times 2$ X-channel (excluding P2P-TIN in which the $M \times 2$ X-channel is reduced to a P2P channel) that the characterized GDoF optimal regime of TIN is complete. In other words, the suboptimality of TIN has been shown for the cases that the channel operates outside the GDoF optimal regime of TIN. To do this, transmission schemes based on interference alignment combined with common and private signaling which required interference decoding have been introduced. It turned out that TIN can be outperformed by such schemes from the GDoF perspective as long as the channel performs outside of the proposed GDoF optimal regime of TIN.

5.2 Future Work

There are still some further interesting open problems on the optimality of TIN and different directions to extend the result of this thesis. For instance, the complete GDoF optimal regime of P2P-TIN in the $M \times 2$ X-channel remained still open. It is interesting to know whether there exists a tighter GDoF upper bound than the bounds introduced in this work for showing the optimality of P2P-TIN. If this is the case, then one can relax the proposed conditions on optimality of P2P-TIN and extend the GDoF optimal regime of TIN for the $M \times 2$ X-channel. On the other hand, to show that the proposed conditions on GDoF optimality of TIN are not only sufficient but also necessary, the suboptimality of TIN needs to be shown from the GDoF point of view.

Another interesting direction to extend the results of the thesis is to know whether TIN performs GDoF optimally in the X-channel if the role of transmitters and receivers will be swapped. Notice that based on the proposed GDoF optimal regime of TIN in [GSJ15], if TIN is GDoF optimal in an $M \times 2$ X-channel, it performs also optimally in the $2 \times M$ X-channel. It is interesting to know whether our extended GDoF optimal regime of TIN for the $M \times 2$ X-channel is also valid for the backward channel. To get some insight knowledge on this problem, one need to study the GDoF optimality of TIN in a $2 \times M$ X-channel. We do believe that this study provides interesting insights on the duality of the GDoF optimality conditions of TIN in the X-channel.

The GDoF result is an intermediate step towards characterizing the capacity of a channel. For instance, the optimality of TIN for the 2-user IC operating in very weak interference regime has been initially shown in [ETW08] with respect to the GDoF. Knowing this approximated result, a capacity optimal regime of TIN has been found in [MK09, AV09, SKC09]. Similarly, we know a capacity optimal regime of TIN for the 2×2 X-channel from [HCJ12] which is also found through studying the GDoF optimality of TIN. It is interesting to extend the results on the GDoF optimal regime of TIN for the $M \times 2$ X-channel to the capacity optimal regime of TIN. This extension becomes more interesting by focusing on the new characterized GDoF optimal regime of TIN. The question is whether any sub-regime of our extended GDoF optimal regime might appear in the capacity optimal regime of TIN. To answer this question, our proposed lemma (Lemma 21) needs to be refined. This lemma bounds the difference between differential entropies of noisy linear combinations of random variables under some constraints by a constant. Obviously, since this constant does not scale with transmit power, it does not have any impact on the GDoF analysis. However, for the capacity analysis, we need to be able to bound the difference of the entropy terms by zero to obtain a tight upper bound.

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