A New Criterion of Delay-Dependent Asymptotic Stability for Hopfield Neural Networks With Time Delay

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Abstract—In this brief, the problem of global asymptotic stability for delayed Hopfield neural networks (HNNs) is investigated. A new criterion of asymptotic stability is derived by introducing a new kind of Lyapunov–Krasovskii functional and is formulated in terms of a linear matrix inequality (LMI), which can be readily solved via standard software. This new criterion based on a delay fractioning approach proves to be much less conservative and the conservatism could be notably reduced by thinning the delay fractioning. An example is provided to show the effectiveness and the advantage of the proposed result.

Index Terms—Global asymptotic stability, Hopfield neural network (HNN), linear matrix inequality (LMI), Lyapunov functional.

I. INTRODUCTION

In recent years, Hopfield neural networks (HNNs) have found many applications in a broad range of areas such as associative memory, repetitive learning, classification of patterns, and optimization problems. Thus, considerable attention has been devoted to the research on HNNs [2], [6]. In particular, the stability analysis of HNNs has become a topic of both theoretical and practical importance since stability is one of the most important issues related to their dynamic behaviors. There have been extensive results on global asymptotic stability in the literature (see [3], [12]–[14], [20], and the references therein).

In practice, due to the finite switching speed of amplifiers or finite speed of information processing, time delays are often encountered in hardware implementation [1], [8], [9], which may be a source of oscillation, divergence, and instability in HNNs. Therefore, the stability problems of HNNs with time delay have gained great research interest. Based on different assumptions and different approaches, a great number of stability criteria for delayed HNNs have been proposed (see [16], [17], [19], and [23]–[25]). Among these results, Xu and Lam obtained an improved stability condition in [22] over the existing criteria in [21] and [23].

In this brief, we revisit the problem of stability analysis for delayed HNNs. It is shown that the result in [22] can be further improved by constructing a new Lyapunov–Krasovskii functional with the idea of delay fractioning. The condition is formulated in the form of a linear matrix inequality (LMI) and proves to be much less conservative, shown via a numerical example. The rest of this brief is organized as follows. In Section II, the problem of asymptotic stability analysis for delayed HNNs is formulated. Section III presents our main result. A numerical example is provided in Section IV, and we conclude the brief in Section V.

Notation: The notation used throughout this brief is fairly standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and the notation $P > 0 (\geq 0)$ means that P is real symmetric and positive definite

(semidefinite). I and 0 denote the identity matrix and zero matrix with compatible dimensions, and $\mathrm{diag}\{\ldots\}$ stands for a block-diagonal matrix. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

II. PROBLEM FORMULATION

Consider the following continuous HNN with time delay:

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau)) + c_i \qquad (1)$$

or, equivalently

$$\dot{x}(t) = -Ax(t) + Bf(x(t-\tau)) + c. \qquad (2)$$

Here, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is the neural state vector; $f(x(t-\tau)) = [f_1(x_1(t-\tau)), f_2(x_2(t-\tau)), \ldots, f_n(x_n(t-\tau))]^T$ denotes the activation function; $c = [c_1, c_2, \ldots, c_n]$ is the constant external input; and $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\} > 0 \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix. The scalar $\tau > 0$ is a constant time delay. The interconnection matrix B represents the delayed weight coefficients of the neurons. Furthermore, we assume that $f_j, j = 1, 2, \ldots, n$, satisfies the following assumption [22].

Assumption 1: The activation function $f(x)$ is continuous and bounded and it satisfies the following inequality:

$$0 \leq \frac{f_j(s_1) - f_j(s_2)}{s_1 - s_2} \leq l_j, \qquad j = 1, 2, \ldots, n.$$

for all $s_1, s_2 \in \mathbb{R}, s_1 \neq s_2$.

It should be pointed out that Assumption 1 guarantees there is an equilibrium point for HNN (2). This can be easily verified by employing the well-known Brouwer's fixed point theorem. Let $x^* = [x_1^*, x_2^*, \ldots, x_n^*]$ be the equilibrium point. Then, in order to simplify the equation, we make the following transformation by the change of variables:

$$y(t) = x(t) - x^*.$$

Under this transformation, HNN (2) is rewritten as

$$\dot{y}(t) = -Ay(t) + Bg(y(t-\tau)) \qquad (3)$$

where

$$g_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*). \qquad (4)$$

By (4) and Assumption 1, it is not difficult to verify that

$$g_j(0) = 0, 0 \leq \frac{g_j(y_j)}{y_j} \leq l_j \qquad \forall y_j \neq 0, \qquad j = 1, 2, \ldots, n. \qquad (5)$$

III. MAIN RESULT

In this section, we present our new delay-dependent asymptotic stability criterion for delayed HNNs.

Theorem 1: Given an integer $m \geq 1$, the origin of the delayed HNN in (2) is the unique equilibrium point and it is globally asymptotically stable, if there exist positive–definite matrices $P \in \mathbb{R}^{n \times n}, Q \in \mathbb{R}^{mn \times mn}, R \in \mathbb{R}^{n \times n}$, and $S = \mathrm{diag}\{s_1, s_2, \ldots, s_n\}$ such that

$$\Theta = W_P^T \bar{P} W_P + W_R^T \bar{R} W_R + W_Q^T \bar{Q} W_Q + W_S^T \bar{S} W_S < 0 \qquad (6)$$

where

$$\bar{P} = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \qquad \bar{R} = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}$$

$$\bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} \qquad \bar{S} = \begin{bmatrix} 0 & S \\ S & -S \end{bmatrix}$$

$$W_R = \begin{bmatrix} -\sqrt{\dfrac{\tau}{m}}A & 0_{n,mn} & \sqrt{\dfrac{\tau}{m}}B \\ -\sqrt{\dfrac{m}{\tau}}I_n & \sqrt{\dfrac{m}{\tau}}I_n & 0_{n,mn} \end{bmatrix}$$

$$W_P = \begin{bmatrix} -A & 0_{n,mn} & B \\ I_n & 0_{n,mn} & 0_n \end{bmatrix}$$

$$W_S = \begin{bmatrix} 0_{n,mn} & \dfrac{1}{\sqrt{2}}L & 0_n \\ 0_{n,mn} & 0_n & \sqrt{2}I_n \end{bmatrix}$$

$$W_Q = \begin{bmatrix} I_{mn} & 0_{mn,n} & 0_{mn,n} \\ 0_{mn,n} & I_{mn} & 0_{mn,n} \end{bmatrix}$$

$$L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\}.$$

Proof: The uniqueness of the equilibrium point can be proved by the contradiction method similar in [22]. Now, we are in the position to show that the equilibrium point is globally asymptotically stable. At first, we choose a Lyapunov–Krasovskii functional candidate as

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{7}$$

where

$$V_1(t) = y(t)^T P y(t),$$
$$V_2(t) = \int_{-\frac{\tau}{m}}^{0} \int_{t+v}^{t} \dot{y}^T(\omega) R \dot{y}(\omega) d\omega dv$$
$$V_3(t) = \int_{t-\frac{\tau}{m}}^{t} \Upsilon(\omega)^T Q \Upsilon(\omega) d\omega$$

and

$$\Upsilon(\omega) = \begin{bmatrix} y(\omega) \\ y\left(\omega - \dfrac{1}{m}\tau\right) \\ \vdots \\ y\left(\omega - \dfrac{m-1}{m}\tau\right) \end{bmatrix}.$$

The derivatives of $V_i(t), i = 1, 2, 3$, are given by

$$\dot{V}_1(t) = 2y(t)^T P \dot{y}(t)$$
$$\dot{V}_2(t) = \frac{\tau}{m} \dot{y}^T(t) R \dot{y}(t) - \int_{t-\frac{\tau}{m}}^{t} \dot{y}^T(\omega) R \dot{y}(\omega) d\omega$$
$$\dot{V}_3(t) = \Upsilon(t)^T Q \Upsilon(t) - \Upsilon\left(t - \frac{\tau}{m}\right)^T Q \Upsilon\left(t - \frac{\tau}{m}\right). \tag{8}$$

From Jensen's inequality, we can easily get

$$-\int_{t-\frac{\tau}{m}}^{t} \dot{y}^T(\omega) R \dot{y}(\omega) d\omega \leq -\frac{m}{\tau}$$
$$\times \left[y(t) - y\left(t - \frac{\tau}{m}\right) \right]^T R \left[y(t) - y\left(t - \frac{\tau}{m}\right) \right]. \tag{9}$$

From (5), for any scalar $s_j \geq 0$, it is clear that

$$2 \sum_{j=1}^{n} s_j g_j(y_j(t-\tau))[l_j y_j(t-\tau) - g_j(y_j(t-\tau))] \geq 0$$

or, equivalently

$$2g(y(t-\tau))^T SL y(t-\tau) - 2g(y(t-\tau))^T S g(y(t-\tau)) \geq 0. \tag{10}$$

Using (3), (8), (9), and (10), we have

$$\dot{V}(t) \leq 2y(t)^T P[-Ay(t) + Bg(y(t-\tau))]$$
$$+ \frac{\tau}{m}[-Ay(t) + Bg(y(t-\tau))]^T R[-Ay(t) + Bg(y(t-\tau))]$$
$$- \frac{m}{\tau} \left[y(t) - y\left(t - \frac{\tau}{m}\right) \right]^T R \left[y(t) - y\left(t - \frac{\tau}{m}\right) \right]$$
$$+ \Upsilon(t)^T Q \Upsilon(t) - \Upsilon\left(t - \frac{\tau}{m}\right)^T Q \Upsilon\left(t - \frac{\tau}{m}\right)$$
$$+ 2g(y(t-\tau))^T SL y(t-\tau) - 2g(y(t-\tau))^T S g(y(t-\tau))$$
$$= \zeta(t)^T \Theta \zeta(t)$$

where

$$\zeta(t) = \begin{bmatrix} \Upsilon(t) \\ y(t-\tau) \\ g(y(t-\tau)) \end{bmatrix} \tag{11}$$

and Θ is defined in (6).

On the other hand, condition (6) implies that there exists a positive ε such that

$$\Theta < \mathrm{diag}\{-\varepsilon I, 0, \ldots, 0, 0\}.$$

We pre- and postmultiply this inequality by $\zeta^T(t)$ and $\zeta(t)$, respectively. Then, one can easily achieve

$$\dot{V}(t) \leq \zeta(t)^T \Theta \zeta(t) < -\varepsilon \|y(t)\|^2$$

which shows that the delayed HNN in (2) is asymptotically stable. This completes the proof. □

Remark 1: Theorem 1 presents a new delay-dependent stability criterion for delayed HNNs by using a more general Lyapunov–Krasovskii functional in (7). Note that, even for $m = 1$, the proposed result still demonstrates superiority over the main classical results in the literature. An illustrative example will be provided to show this in Section IV.

Remark 2: The reduced conservatism of Theorem 1 benefits from the construction of the new Lyapunov–Krasovskii functional in (7). The main idea is to fraction the delay, which constitutes the major difference from most existing results in the literature. Moreover, the conservatism reduction increases as the delay fractioning becomes thinner.

Remark 3: The delay fractioning idea can be extended to the time-varying delay case. We just need to modify our Lyapunov–Krasovskii functional by replacing τ with $\tau(t)(0 \leq \tau(t) \leq h, \dot{\tau} \leq \mu)$ to get LMI-based sufficient condition for HNN with time-varying delay.

If we choose the Lyapunov–Krasovskii functional as

$$V(t) = y(t)^T P y(t) + \int_{t-\frac{\tau}{m}}^{t} \Upsilon(\omega)^T Q \Upsilon(\omega) d\omega \tag{12}$$

then it is easy to obtain the following delay-independent stability condition.

Corollary 1: Given an integer $m \geq 1$, the origin of the delayed HNN in (2) is the unique equilibrium point and it is delay-independent globally asymptotically stable, if there exist positive–definite matrices $P \in \mathbb{R}^{n \times n}, Q \in \mathbb{R}^{mn \times mn}$, and $S = \mathrm{diag}\{s_1, s_2, \ldots, s_n\}$ such that

$$\Omega = W_P^T \bar{P} W_P + W_Q^T \bar{Q} W_Q + W_S^T \bar{S} W_S < 0 \tag{13}$$

where

$$\bar{P} = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \quad \bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}$$

$$\bar{S} = \begin{bmatrix} 0 & S \\ S & -S \end{bmatrix} \quad L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\},$$

$$W_P = \begin{bmatrix} -A & 0_{n,mn} & B \\ I_n & 0_{n,mn} & 0_n \end{bmatrix}$$

$$W_S = \begin{bmatrix} 0_{n,mn} & \dfrac{1}{\sqrt{2}}L & 0_n \\ 0_{n,mn} & 0_n & \sqrt{2}I_n \end{bmatrix}$$

$$W_Q = \left[\begin{array}{ccc} I_{mn} & 0_{mn,n} & 0_{mn,n} \\ \hline 0_{mn,n} & I_{mn} & 0_{mn,n} \end{array} \right].$$

Proof: The derivative of (12) is

$$\dot{V}_1(t) = 2y(t)^T P \dot{y}(t) + \Upsilon(t)^T Q \Upsilon(t) - \Upsilon\left(t - \frac{\tau}{m}\right)^T Q \Upsilon\left(t - \frac{\tau}{m}\right)$$

$$\leq 2y(t)^T P[-Ay(t) + Bg(y(t-\tau))]$$

$$+ \Upsilon(t)^T Q \Upsilon(t) - \Upsilon\left(t - \frac{\tau}{m}\right)^T Q \Upsilon\left(t - \frac{\tau}{m}\right)$$

$$+ 2g(y(t-\tau))^T S L y(t-\tau) - 2g(y(t-\tau))^T S g(y(t-\tau))$$

$$\leq \zeta(t)^T \Omega \zeta(t)$$

where $\zeta(t)$ is as defined in (11) and Ω in (13). Then, using similar method in Theorem 1, we complete the proof. $\qquad\square$

Remark 4: Though not in ordinary form of LMI, Theorem 1 and Collary 1 are indeed in the standard LMI form, which can be easily solved by the standard software. Moreover, this form simplified as $W_1^T X W_1 + W_2^T Y W_2 < 0$ is more laconic. It expresses the LMI in several parts, each of which has a symmetric structure with the matrix variable to be determined in center.

IV. ILLUSTRATIVE EXAMPLES

In this section, an example is provided to illustrate the advantage of Theorem 1 by comparing it with recently reported results on delay-dependent asymptotic stability of delayed HNNs.

Example 1: Consider the following third-order delayed HNN:

$$A = \begin{bmatrix} 4.1989 & 0 & 0 \\ 0 & 0.7160 & 0 \\ 0 & 0 & 1.9985 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.1052 & -0.5069 & -0.1121 \\ -0.0257 & -0.2808 & 0.0212 \\ 0.1205 & -0.2153 & 0.1315 \end{bmatrix}$$

with

$$L = \mathrm{diag}\{0.4129, 3.8993, 1.0160\}.$$

Our purpose is to find the maximum allowable delay $\tau_{\max}$ such that the delayed HNN in (2) is globally asymptotically stable. Computational results are shown in Table I, which summarizes the obtained maximum allowable delays by using the previously published methods and our new result for various fractionings.

Table I shows that for no fractioning $(m = 1)$, the new criterion given in Theorem 1 is still less conservative than the previous results, which corresponds to Remark 1. Moreover, for $m > 1$, the conservatism reduction proves to be more obvious. However, it should be noted that although conservatism is reduced as the fractioning becomes thinner, there is no significant improvement after $m = 5$.

TABLE I
MAXIMUM ALLOWABLE DELAYS COMPARISON

Method	$\tau_{\max}$
Ye and Michel, 1994, [23]	0.4121
Xu and Lam, 2005, [21]	1.7484
Xu and Lam, 2006, [22]	1.7644
Theorem 1, $m = 1$	2.14
Theorem 1, $m = 2$	2.48
Theorem 1, $m = 3$	2.54
Theorem 1, $m = 4$	2.57
Theorem 1, $m = 5$	2.581
Theorem 1, $m = 10$	2.596
Theorem 1, $m = 15$	2.597

V. CONCLUSION

By defining a new Lyapunov–Krasovskii functional, an improved delay-dependent asymptotic stability criterion has been obtained for a class of delayed HNNs. The proposed condition is given in terms of LMIs and thus can be readily solved via standard numerical software. The merit of the proposed condition lies in its reduced conservatism, which is based on a time delay fractioning approach. The result proves to become less conservative as the fractioning goes thinner. Finally, a numerical example has been provided to demonstrate the effectiveness of the proposed criterion. The method is expected to be further extended to other neural networks or complex networks [4], [5], [7], [10], [11], [15], [18], which is under our investigation.

REFERENCES

[1] E.-K. Boukas and N. F. Al-Muthairi, "Delay-dependent stabilization of singular linear systems with delays," *Int. J. Innov. Comput. Inf. Control*, vol. 2, no. 2, pp. 283–291, 2006.

[2] J. Cao, "An estimation of the domain of attraction and convergence rate for Hopfield continuous feedback neural networks," *Physica A*, vol. 325, no. 5–6, pp. 370–374, 2004.

[3] J. Cao and J. Wang, "Global asymptotic stability of a general class of recurrent neural networks with time-varying delays," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 1, pp. 34–44, Jan. 2003.

[4] W. Chen and X. Lu, "New delay-dependent exponential stability criteria for neural networks with variable delays," *Physica A*, vol. 351, no. 1–2, pp. 53–58, 2006.

[5] W. Chen and W. Zheng, "Global asymptotic stability of a class of neural networks with distributed delays," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 53, no. 3, pp. 644–652, Mar. 2006.

[6] J. Farrell and A. Michel, "A synthesis procedure for Hopfield's continuous-time associative memory," *IEEE Trans. Circuits Syst.*, vol. 37, no. 7, pp. 877–884, Jul. 1990.

[7] H. Gao, J. Lam, and G. Chen, "New criteria for synchronization stability of general complex dynamical networks with coupling delays," *Phys. Lett. A*, vol. 360, no. 2, pp. 263–273, 2006.

[8] H. Gao and C. Wang, "Comments and further results on 'A descriptor system approach to H_∞ control of linear time-delay systems'," *IEEE Trans. Autom. Control*, vol. 48, no. 3, pp. 520–525, Mar. 2003.

[9] H. Gao and C. Wang, "A delay-dependent approach to robust H_∞ filtering for uncertain discrete-time state-delayed systems," *IEEE Trans. Signal Process.*, vol. 52, no. 6, pp. 1631–1640, Jun. 2004.

[10] Y. He, Q. Wang, and W. Zheng, "Global robust stability for delayed neural networks with polytopic type uncertainties," *Chaos Solitons Fractals*, vol. 26, pp. 1349–1354, 2005.

[11] Y. He, M. Wu, and J. H. She, "An improved global asymptotic stability criterion for delayed cellular neural networks," *IEEE Trans. Neural Netw.*, vol. 17, no. 1, pp. 250–252, Jan. 2006.

[12] H. Huang, D. W. C. Ho, and J. Lam, "Stochastic stability analysis of fuzzy Hopfield neural networks with time-varying delays," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 5, pp. 251–255, May 2005.

[13] X. Liu, K. L. Teo, and B. Xu, "Exponential stability of impulsive high-order Hopfield-type neural networks with time-varying delays," *IEEE Trans. Neural Netw.*, vol. 16, no. 6, pp. 1329–1339, Jun. 2005.

[14] X. Y. Lou and B. Cui, "New LMI conditions for delay-dependent asymptotic stability of delayed Hopfield neural networks," *Neurocomputing*, vol. 69, no. 16–18, pp. 2374–2378, 2006.

[15] V. Singh, "A generalized LMI-based approach to the global asymptotic stability of delayed cellular neural networks," *IEEE Trans. Neural Netw.*, vol. 15, no. 1, pp. 223–225, Jan. 2004.

[16] Z. Wang, Y. Liu, K. Fraser, and X. Liu, "Stochastic stability of uncertain Hopfield neural networks with discrete and distributed delays," *Phys. Lett. A*, vol. 354, no. 4, pp. 288–297, 2006.

[17] Z. Wang, Y. Liu, and X. Liu, "On global asymptotic stability of neural networks with discrete and distributed delays," *Phys. Lett. A*, vol. 345, no. 4–6, pp. 299–308, 2005.

[18] Z. Wang, Y. Liu, and X. Liu, "Stability analysis for stochastic Cohen-Grossberg neural networks with mixed time delays," *IEEE Trans. Neural Netw.*, vol. 17, no. 3, pp. 814–820, May 2006.

[19] Z. Wang, H. Shu, J. Fang, and X. Liu, "Robust stability for stochastic Hopfield neural networks with time delays," *Nonlinear Anal.: Real World Appl.*, vol. 7, no. 5, pp. 1119–1128, 2006.

[20] B. Xu, X. Liu, and X. Liao, "Global exponential stability of high order Hopfield type neural networks," *Appl. Math. Comput.*, vol. 174, no. 1, pp. 98–116, 2006.

[21] S. Xu, J. Lam, D. W. C. Ho, and Y. Zou, "Novel global asymptotic stability criteria for delayed cellular neural networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 6, pp. 349–353, Jun. 2005.

[22] S. Xu, J. Lam, and D. W. C. Ho, "A new LMI condition for delay-dependent asymptotic stability of delayed Hopfield neural networks," *IEEE Trans. Circuits Syst. II. Exp. Briefs*, vol. 53, no. 3, pp. 230–234, Mar. 2006.

[23] H. Ye, A. Michel, and K. Wang, "Global stability and local stability of Hopfield neural networks with delays," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 50, pp. 4026–4213, 1994.

[24] F. Zhang and H. Huo, "Global stability of delayed Hopfield neural networks under dynamical thresholds," *Discrete Dyn. Nature Soc.*, vol. 1, pp. 1–17, 2005, 2005.

[25] H. Zhao, "Global asymptotic stability of Hopfield neural network involving distributed delays," *Neural Netw.*, vol. 17, no. 1, pp. 47–53, 2004.

Energy Function and Energy Evolution on Neuronal Populations

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Abstract—Based on the principle of energy coding, an energy function of a variety of electric potentials of a neural population in cerebral cortex is formulated. The energy function is used to describe the energy evolution of the neuronal population with time and the coupled relationship between neurons at the subthreshold and the suprathreshold states. The Hamiltonian motion equation with the membrane potential is obtained from the neuroelectrophysiological data contaminated by Gaussian white noise. The results of this research show that the mean membrane potential is the exact solution of the motion equation of the membrane potential developed in a previously published paper. It also shows that the Hamiltonian energy function derived in this brief is not only correct but also effective. Particularly, based on the principle of energy coding, an interesting finding is that in some subsets of neurons, firing action potentials at the suprathreshold and some others simultaneously perform activities at the subthreshold level in neural ensembles. Notably, this kind of coupling has not been found in other models of biological neural networks.

Index Terms—Coupled neural population, energy coding, energy evolution, Hamiltonian function.

I. INTRODUCTION

Due to the limitations in current biophysical models of neural coding, research into the mechanisms of neural information processing remains a challenge [1]–[3], thereby the fundamental principles of neural information processing underlying cognitive processes in the brain are still not completely understood today.

Regarding neural information processing, the basic theory of energy coding has received strong supports from many neuroelectrophysiological experiments [4]–[11], [13]–[16], and a significant expansion on energy coding appear in [7]. Two concerned issues pertaining to the aforementioned research results are considered in this brief.

1) Neuronal activities at the subthreshold and the suprathreshold states are separately described and discussed in their mathematical models. The fact is that most activities of neurons at both of these threshold states are mutually coupled, so this separate description does not agree with the real neuronal activities. For this reason, the coupled relationship between neurons at the subthreshold and suprathreshold states is taken into account in this brief, and under such coupling configuration some quantitative expressions of energy coding are obtained.

2) The Hamiltonian energy function describing the collective activities of electric potentials of a whole neural population is derived and analyzed, improving the previously published results devoted only to a single neuron [5], [7]. Therefore, the results reported in this brief are quite universal and significant.