MAXIMUM DIAMETER OF 3- AND 4-COLORABLE GRAPHS

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Abstract. P. Erdős, J. Pach, R. Pollack, and Z. Tuza [*J. Combin. Theory* **B 47** (1989), 279–285] made conjectures for the maximum diameter of connected graphs without a complete subgraph K_{k+1}, which have order n and minimum degree δ. Settling a weaker version of a problem, by strengthening the K_{k+1}-free condition to k-colorable, we solve the problem for $k = 3$ and $k = 4$ using a unified linear programming duality approach. The case $k = 4$ is a substantial simplification of the result of É. Czabarka, P. Dankelmann, and L. A. Székely [*Europ. J. Comb.* **30** (2009), 1082–1089].

1. Introduction

We study the maximum diameter of connected graphs in terms of other graph parameters such as order, minimum degree, etc. Several papers [1, 6, 7, 8] have shown that:

Theorem 1. *For a fixed minimum degree $\delta \geq 2$, every connected graph G of order n satisfies* $\operatorname{diam}(G) \leq \frac{3n}{\delta+1} + O(1)$*, as $n \to \infty$.*

This upper bound is sharp (even for δ-regular graphs [2]), but the constructions have complete subgraphs, whose order increases with n. Erdős, Pach, Pollack, and Tuza [6] conjectured that the upper bound in Theorem 1 can be improved, if large cliques are excluded:

Conjecture 1 ([6]). Let $r, \delta \geq 2$ be fixed integers and let G be a connected graph of order n and minimum degree δ.

(i) If G is K_{2r}-free and δ is a multiple of $(r-1)(3r+2)$ then, as $n \to \infty$,

$$\begin{aligned}\operatorname{diam}(G) &\leq \frac{2(r-1)(3r+2)}{(2r^2-1)} \cdot \frac{n}{\delta} + O(1) \\ &= \left(3 - \frac{2}{2r-1} - \frac{1}{(2r-1)(2r^2-1)}\right)\frac{n}{\delta} + O(1).\end{aligned}$$

(ii) If G is K_{2r+1}-free and δ is a multiple of $3r-1$, then, as $n \to \infty$,

$$\operatorname{diam}(G) \leq \frac{3r-1}{r} \cdot \frac{n}{\delta} + O(1) = \left(3 - \frac{2}{2r}\right)\frac{n}{\delta} + O(1).$$

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Furthermore, they created examples showing that the above conjecture, if true, is sharp, and showed part (ii) of the conjecture for $r = 1$.

Czabarka, Dankelmann and Székely [3] arrived at the conclusion of Conjecture 1 (ii) for $r = 2$ under a stronger hypothesis:

Theorem 2. *For every connected* 4*-colorable graph* G *of order* n *and minimum degree* $\delta \geq 1$, $\mathrm{diam}(G) \leq \frac{5n}{2\delta} - 1$.

Czabarka, Singgih and Székely [4] gave an infinite family of $(2r-1)$-colorable (hence K_{2r}-free) graphs with diameter $\frac{(6r-5)(n-2)}{(2r-1)\delta+2r-3} - 1$, providing a counterexample for Conjecture 1 (i) for every $r \geq 2$ and $\delta > 2(r-1)(3r+2)(2r-3)$. The question whether Conjecture 1 (i) holds in the range $(r-1)(3r+2) \leq \delta \leq 2(r-1)(3r+2)(2r-3)$ remains open. The counterexample led Czabarka *et al.* [4] to the modified conjecture below, which no longer requires cases for the parity of the order of the excluded complete subgraphs:

Conjecture 2 ([4]). For every $k \geq 3$ and $\delta \geq \lceil \frac{3k}{2} \rceil - 1$, if G is a K_{k+1}-free (under a stronger hypothesis, k-colorable) connected graph of order n and minimum degree at least δ, $\mathrm{diam}(G) \leq \left(3 - \frac{2}{k}\right) \frac{n}{\delta} + O(1)$.

Czabarka, Singgih and Székely [5] showed that the extremal graphs for the diameter maximization problem of Conjecture 2 include graphs blown up from some very specific structures, called *canonical clump graphs*. Furthermore, [5] showed using the weak duality theorem of linear programming that providing a sufficiently good solution for a dual problem on canonical clump graphs gives an upper bound for the diameter of graphs blown up from canonical clump graphs (see Theorem 7), and hence an upper bound for the diameter maximization problem of Conjecture 2. Using this method, they proved:

Theorem 3 ([5])**.** *Assume* $k \geq 3$. *If* G *is a connected* k*-colorable graph of minimum degree at least* δ, *then*

$$\mathrm{diam}(G) \leq \frac{3k-4}{k-1} \cdot \frac{n}{\delta} - 1 = \left(3 - \frac{1}{k-1}\right) \frac{n}{\delta} - 1$$

.

Czabarka, Singgih and Székely [5] also made a slight improvement on Theorem 3 for 3-colorable graphs, but with a different argument.

In this paper we give a common short proof of the Conjecture 2 (under the stronger hypothesis) for $k = 3$ and 4 (the latter being Theorem 2 from Dankelmann *et al.* [3]) using the approach above:

Theorem 4. *Assume* $k = 3$ *or* 4. *If* G *is a connected* k*-colorable graph of order* n, *and of minimum degree at least* $\delta \geq 1$, *then* $\mathrm{diam}(G) \leq \left(3 - \frac{2}{k}\right) \frac{n}{\delta} - 1$.

The main tool of the proof is still the use of canonical clump graphs, however, we focus on an even smaller class, *strongly canonical clump graphs*, of which blown up copies are still present among the extremal graphs for the diameter maximization problem of Conjecture 2, as shown in Section 2. We partition the strongly canonical graph into segments of three types. Weighting the vertices such that the total weight of the neighbors of any vertex is at most 1 and the average weight of a layer in each segment is $\frac{k}{3k-2}$ finishes the proof. When $k \in \{3, 4\}$, Type 1 and Type 2 segments (defined in Section 4) have a very limited structure, as shown in Lemma 9. For $k \geq 5$ we have examples of segments that cannot be weighted according to this scheme, so new ideas are needed.

2. Clump Graphs

Given a k-colorable connected graph G of order n and minimum degree at least δ, choose a vertex x whose eccentricity is $\text{diam}(G)$. Take a *fixed* good k-coloring of G. Let *layer* L_i denote the set of vertices at distance i from x, and a *clump* in L_i be the set of vertices in L_i that have the same color. The number of layers is $\text{diam}(G) + 1$. We call a graph *layered*, if such a vertex x and the distance layers $L_0 = \{x\}, L_1, \ldots, L_D$ are given. Let $c(i) \in \{1, 2, \ldots, k\}$ denote the number of colors used in layer L_i by our fixed coloration. We can assume without loss of generality that any two vertices in layer L_i in G, which are differently colored, are joined by an edge in G, and also that two vertices in consecutive layers, which are differently colored, are also joined by an edge in G. We call this assumption *saturation* with respect to the fixed good k-coloring. Assuming saturation does not make loss of generality, as adding these edges does not decrease degrees, keeps the fixed good k-coloration, and does not reduce the diameter, while making the graph more structured for our convenience.

From the layered and saturated graph G above, we create an *unweighted clump graph* $H = H(G)$. Vertices of H correspond to the clumps of G. Two vertices of H are connected by an edge if there were edges between the corresponding clumps in G. H is naturally k-colored and layered, based on the coloration and layering of G. With a slight abuse of notation, we denote the layers of H by L_i as well. To create a *weighted clump graph*, we assign positive integer *weights* to each vertex of the unweighted clump graph. Blowing up vertices of H into as many copies as their weight is, we obtain a bigger k-colorable graph of the same diameter (we do not put edges between successors of the same vertex). In case the weights are the cardinalities of the clumps in G, after the blow-up of $H = H(G)$ we get back G. The degree of a vertex v in a blow-up of H, where v is a successor of a vertex w of H by blow-up, is the sum of the weights of neighbors of the vertex w in H. The number of vertices in a blow-up of H is the sum of the weights of all vertices in H.

The following theorem was proven in [5]:

Theorem 5 ([5]). *Assume $k \geq 3$. Let G' be a k-colorable connected graph of order n, diameter D and minimum degree at least δ. Then there is a saturated k-colored and layered connected graph G of the same parameters n and δ, with layers $L_0, \ldots, L_D$, for which the following hold for every i ($0 \leq i \leq D-1$):*

- (i) *If $c(i) = 1$, then $c(i+1) \leq k-1$.*
- (ii) *The number of colors used to color the set $L_i \cup L_{i+1}$ is $\min(k, c(i) + c(i+1))$. In particular, when $c(i) + c(i+1) \leq k$, then L_i and L_{i+1} do not share any color.*
- (iii) *If $c(i) = k$, then $i \geq 2$ and $c(i+1) \geq 2$.*
- (iv) *If $|L_i| > c(i)$, i.e., L_i contains two vertices of the same color, then $i > 0$ and $c(i) + \max\big(c(i-1), c(i+1)\big) \geq k$.*

Canonical clump graphs were defined in [5] as $H = H(G)$ clump graphs, where G satisfies the conclusions of Theorem 5. Now we define *strongly canonical clump graphs* for $D \geq 2$ as $H = H(G)$ canonical clump graphs (i.e., G satisfies the conclusions of Theorem 5), and in addition, $c(0) = c(D) = 1$.

It is not difficult to see the following: if the graph G' in the assumption of Theorem 5 is layered with $|L'_0| = 1$ and $c'(D) = 1$, then the proof of Theorem 5 in [5] provides a layered G with $|L_0| = 1$ (and hence $c(0) = 1$), and $c(D) = 1$. Based on this observation, the following lemma implies that to resolve Conjecture 2 (or proving Theorem 4), we may assume that G has a strongly canonical clump graph.

Lemma 6. *Assume $k \geq 3$ and $D \geq 2$. Let G' be a k-colored layered connected graph of order n, diameter D, and minimum degree at least δ, with layers $L'_0, \dots, L'_D$. Then there is a k-colored layered connected graph G of the same parameters, with layers $L_0, \dots, L_D$, for which $c(0) = c(D) = 1$, and for each i $(0 \leq i \leq D-2)$, we have $c'(i) = c(i)$ and $L'_i = L_i$.*

Proof. As $|L'_0| = 1$ is necessary in a layered graph, we must have $c'(0) = 1$, and if $c'(D) = 1$, the choice $L'_i = L_i$ suffices. If $c'(D) > 1$, pick a color A in L'_D. If possible, pick such a color that also appears in L'_{D-2}. This ensures that for all colors B in L'_D such that $B \neq A$ there is a color C in L'_{D-2} such that $B \neq C$ (where $C = A$, if A appeared in L_{D-2}, otherwise any color in L_{D-2} works). Create a layered graph graph G from G' by moving all vertices in L'_D that are not colored A to the next-to-last layer, which will be L_{D-1}, and connect them to all vertices in $L_{D-2} = L'_{D-2}$ that have different color. Note that for all vertices of L_{D-1}, there is at least one such vertex. As we only changed the number of vertices in layers $D-1$ and D, and did not change the coloration of the vertices, the claim follows. ☐

3. Duality

Let $\mathcal{H}_{k,D,\delta}$ denote the family of unweighted canonical clump graphs of diameter D that arises from connected k-colorable graphs G with diameter D and minimum degree at least δ, when the order of G is unspecified. We will rely on the following result from [5]:

Theorem 7. ([5]) *Fix $k \geq 3$. Assume that there exists constants $\tilde{u} > 0$ and $C \geq 0$ such that for all D and δ, and for all $H \in \mathcal{H}_{k,D,\delta}$, the optimum of the linear program*

$$\textit{Maximize } \delta \cdot \sum_{y \in V(H)} u(y),$$

subject to the condition

$$\forall x \in V(H) \qquad \sum_{y \in V(H):\, xy \in E(H)} u(y) \leq 1. \tag{1}$$

is at least

$$\tilde{u}\delta D + C.$$

Then for any k-colorable graphs G with minimum degree δ on n vertices, we have

$$\operatorname{diam}(G) \leq \frac{1}{\tilde{u}} \frac{n}{\delta} - \frac{C}{\tilde{u}}.$$

In Theorem 7 and in its proof we may change the family of canonical clump graphs $\mathcal{H}$ to the family of strongly canonical clump graphs $\mathcal{H}'$ keeping all arguments valid.

4. Some definitions and observations

Recall that we use the sloppy notation L_i for the layers of the clump graph $H(G)$ as well, not just for the layers of G. Hence $c(i) = |L_i|$, if L_i denotes a layer of the clump graph. Based on the arguments of Section 2, we have:

Claim 1. *An unweighted k-colorable strongly canonical clump graph with layers $L_0, \ldots, L_D$ satisfies the following properties:*

(i) $|L_0| = |L_D| = 1$,
(ii) *If* $|L_i| = k$, *then* $2 \leq i \leq D-1$ *and* $\min(|L_{i-1}|, |L_{i+1}|) \geq 2$, *and*
(iii) *For* $i \in [D]$, *the edges that do not appear between* L_{i-1} *and* L_i *form a matching of size* $\max(k, |L_{i-1}| + |L_i|) - k$.

For the following definition, and also for the rest of this section, assume that we are given a k-colorable canonical clump graph H with layers $L_0, \ldots, L_D$. We define for convenience two additional layers, as $L_{-1} = L_{D+1} = \emptyset$. For a vertex $x \in V(H)$, let $N(x)$ denote the set of neighbors of x.

Definition 1. For each $i : 0 \leq i \leq D$, define the set $S_i = \{x \in L_i : L_{i-1} \cup L_i \subseteq N(x)\}$. We call a layer L_i *big* if $|S_i| > \frac{k}{2}$. A layer is *small* if it is not big.

Note that if L_i is big, then $i \notin \{0, D\}$. We set $S_{-1} = S_{D+1} = \emptyset$, in accordance with $L_{-1} = L_{D+1} = \emptyset$.

Lemma 8. *Assume $D \geq 2$. Let H be an unweighted k-colorable strongly caonical clump graph with layers $L_0, \ldots, L_D$. The following is true for each $i : 0 \leq i \leq D$:*

(i) $|L_i| \leq k - \max(|S_{i-1}|, |S_{i+1}|)$,
(ii) $|S_i| \leq k - 1$,
(iii) *if* L_i *is big, then* $1 \leq i \leq D-1$ *and* L_{i-1}, L_{i+1} *are small,*
(iv) *if* $|L_i| = 1$, *then* $L_i = S_i$,
(v) $\max(|L_i \setminus S_i|, |L_{i+1} \setminus S_{i+1}|) \leq k - |S_i| - |S_{i+1}|$,
(vi) *if* $|S_i| = k-1$, *then* $L_i = S_i$ *and for* $j = i \pm 1$, $|L_j| = |S_j| = 1$,
(vii) *if* $k \in \{3, 4\}$ *and* L_i *is big, then* $|S_i| = k - 1$.

Proof. (i) follows from the facts that $S_{i-1} \cup L_i$, and also $S_{i+1} \cup L_i$, forms a complete subgraph in k-colorable graph H.

(ii) follows from (i) and the fact that $L_{i-1} \cup L_{i+1}$ contains at least one vertex.

(iii) follows from (i).

(iv) : As $|L_i| = 1$, then from Claim 1 (ii) we get that $\max(|L_{i-1}|, |L_{i+1}|) \leq k-1$. By Claim 1 (iii), the vertex in L_i is adjacent to every vertex in $L_{i-1} \cup L_{i+1}$.

For (v), $S_i \cup S_{i+1} \cup (L_i \setminus S_i)$ forms a complete graph in the k-colorable graph H, and hence $|L_i \setminus S_i| \leq k - |S_i| - |S_{i+1}|$, and similarly, $S_i \cup S_{i+1} \cup (L_{i+1} \setminus S_{i+1})$ forms a complete graph, and hence $|L_{i+1} \setminus S_{i+1}| \leq k - |S_i| - |S_{i+1}|$.

For (vi), if $|S_i| = k-1$, then $1 \le i \le D-1$. By (i), $|L_{i-1}| = |L_{i+1}| = 1$, and by Claim 1 (ii), $|L_i| \le k-1$, and by $k-1 = |S_i| \le |L_i| \le k-1$, $S_i = L_i$.

For (vii), if L_i is big, then by definition $\frac{k}{2} < |S_i|$. By (ii) $|S_i| \le k-1$. For $k \in \{3,4\}$, these give $|S_i| = k-1$. □

Definition 2. Let H be an unweighted k-colorable strongly canonical clump graph with layers $L_0, \ldots, L_D$. If for some $s \ge 1$ the contiguous segment of layers $L_i, L_{i+1}, \ldots, L_{i+2s}$ satisfies all the following conditions:

(i) for each $j : 1 \le j \le s$ the layer L_{i+2j-1} is big (thus, L_{i+2j-2}, L_{i+2j} are small),
(ii) $i = 0$ or L_{i-1} is small,
(iii) $i + 2s = D$ or L_{i+2s+1} is small,

then we say that the contiguous segment is Type 1, if $s = 1$, and Type 2, if $s > 1$.

Definition 3. Let H be an unweighted k-colorable strongly canonical clump graph with layers $L_0, \ldots, L_D$. Assume that $t \ge 0$. We say that the contiguous segment of layers $L_i, L_{i+1}, \ldots, L_{i+t}$ is Type 3, if the following hold:

(i) for each $j : i \le j \le i+t$ the layer L_j is small,
(ii) if $i \ne 0$ then $i > 2$ and L_{i-2} is big (thus, L_{i-1}, L_{i-3} are small),
(iii) if $i + t \ne D$ then $i + t < D - 2$ and L_{i+t+2} is big (thus, L_{i+t+1}, L_{i+t+3} are small).

Observe that in a Type 3 segment every layer is small.

The following Lemma easily follows from the definition of strongly canonical clump graphs and Lemma 8.

Lemma 9. *Let H be an unweighted k-colorable strongly canonical clump graph. Then the layers $L_0, \ldots, L_D$ can be partitioned into segments of Type 1, Type 2 and Type 3. Moreover, if $k \in \{3,4\}$ and L_j is a layer in a Type 1 or Type 2 segment, then $L_j = S_j$ and $|L_j| \in \{1, k-1\}$.*

5. Proof of Theorem 4

Assume $k \in \{3,4\}$, and let H be an unweighted k-colorable strongly canonical clump graph. By Theorem 7, it is enough to find a dual weighting u of the vertices of H, which satisfies the conditions of that theorem and has total weight $(D+1)\frac{k}{3k-2}$. Fix a partition of the layers of H into segments of Type 1, Type 2 and Type 3 according to Lemma 9. For shortness, we will say that layer L_i is of Type j, if L_i falls into a segment of Type j.

Consider a vertex v in a layer L_i. Set $u(v)$ as follows:

- If L_i is of Type 1: $u(v) = \begin{cases} \frac{2}{3k-2}, & \text{if} |L_i| = k-1, \\ \frac{k+2}{2(3k-2)} & \text{otherwise.} \end{cases}$
- If L_i is of Type 2:
$$u(v) = \begin{cases} \frac{1}{2(k-1)}, & \text{if } |L_i| = k-1, \\ \frac{k+2}{2(3k-2)}, & \text{if } |L_i| = 1 \text{ and } L_i \text{ is not the first or last layer of the segment,} \\ \frac{3k+2}{4(3k-2)}, & \text{otherwise.} \end{cases}$$
Note that as $k \ge 3$, $\frac{k+2}{2(3k-2)} < \frac{3k+2}{4(3k-2)} < \frac{k}{3k-2}$.

- If L_i is of Type 3:
$$u(v) = \begin{cases} \frac{k}{(3k-2)|L_i|}, & \text{if } |L_i| \le \frac{k}{2} \\ \frac{2}{3k-2}, & \text{if } |L_i| > \frac{k}{2} \text{ and } v \in S_i \\ \frac{k-2|S_i|}{(3k-2)(|L_i|-|S_i|)}, & \text{otherwise.} \end{cases}$$
Note that as $|S_i| \le \frac{k}{2}$, we get $u(v) \ge 0$. Also, $u(v) \ge \frac{2}{3k-2}$ if $|L_i| \le \frac{k}{2}$ or $v \in S_i$.

We define the weight $u(X)$ of a vertex set X as $\sum_{v \in X} u(v)$. It is easy to check that for any Type 3 layer L_i, $u(L_i) = \frac{k}{3k-2}$. Also, first and last layers in a segment of any type have weight at most $\frac{k}{3k-2}$.

If L_i, L_{i+1}, L_{i+2} is a Type 1 segment, then $u(L_i) + u(L_{i+1}) + u(L_{i+2}) = (k-1) \cdot \frac{2}{3k-2} + 2 \cdot \frac{k+2}{2(3k-2)} = 3 \cdot \frac{k}{3k-2}$.

Assume that $L_i, L_{i+1}, \ldots, L_{i+2s}$ is a Type 2 segment. The total weight of the layers of this segment is $s \cdot \frac{1}{2} + (s-1) \cdot \frac{k+2}{2(3k-2)} + 2 \cdot \frac{3k+2}{4(3k-2)} = (2s+1) \cdot \frac{k}{3k-2}$.

This gives that the total weight of H is $(D+1)\frac{k}{3k-2}$, as required. We need to check condition (1) at every vertex $v \in V(H)$.

Assume first that $v \in L_i$, where L_i is of Type 1.

If L_i is big, then $u(N(v)) = (k-2) \cdot \frac{2}{3k-2} + 2 \cdot \frac{k+2}{2(3k-2)} = 1$. If L_i is small, then it is the first or last layer in its segment. As first and last layers in a segment of any type have weight at most $\frac{k}{3k-2}$, we have $u(N(v)) \le \frac{k}{3k-2} + (k-1) \cdot \frac{2}{3k-2} = 1$.

Assume next that $v \in L_i$, where L_i is of Type 2. If L_i is small, and is not the first or last layer in the segment, then $u(N(v)) = 2 \cdot (k-1) \cdot \frac{1}{2k-1} = 1$. If L_i is the first or last layer in the segment, then $u(N(v)) \le (k-1) \cdot \frac{1}{2(k-1)} + \frac{k}{3k-2} < 1$. If L_i is big, then $u(N(v))$ is the greatest if L_i is the second or next-to-last layer in its segment. Therefore

$$\begin{aligned} u(N(v)) &\le \left(\frac{1}{2} - \frac{1}{2(k-1)}\right) + \frac{k+2}{2(3k-2)} + \frac{3k+2}{4(3k-2)} = \frac{(11k+2)(k-1) - 6k + 4}{4(k-1)(3k-2)} \\ &= \frac{(11k-4)(k-1) - 2}{4(3k-2)(k-1)} \le 1. \end{aligned}$$

Assume that $v \in L_i$, where L_i is of Type 3. Then $\max(u(L_i), u(L_{i-1}), u(L_{i+1})) \le \frac{k}{3k-2}$. If $u(v) \ge \frac{2}{3k-2}$, then $u(N(v)) \le u(L_{i-1}) + u(L_i) + u(L_{i+1}) - u(v) \le \frac{3k}{3k-2} - \frac{2}{3k-2} = 1$. Otherwise, we have that $|L_i| > \frac{k}{2}$ and $v \notin S_i$. Since $v \notin S_i$, there is a $j \in \{i-1, i+1\}$ and a $w \in L_j$ that is not a neighbor of v. As $w \in L_j \setminus S_j$, by Lemma 8 (iv) we have that $|L_j| > 1$, therefore L_j is also of Type 3.

It $u(v)+u(w) \ge \frac{2}{3k-2}$, then we get, as before, that $u(N(v)) \le 1$, as needed. In particular, if $u(w) \ge \frac{2}{3k-2}$ then we are done. So we may further assume that $|L_j| > \frac{k}{2}$ and $w \notin S_j$, Moreover, from Lemma 8(v) we have $\max(|L_i|-|S_i|,|L_j|-|S_j|) \le k-|S_i|-|S_j|$. Therefore

$$\begin{aligned} u(v)+u(w) &= \frac{k-2|S_i|}{(3k-2)(|L_i|-|S_i|)}+\frac{k-2|S_j|}{(3k-2)(|L_j|-|S_j|)} \\ &\ge \frac{k-2|S_i|}{(3k-2)(k-|S_i|-|S_j|)}+\frac{k-2|S_j|}{(3k-2)(k-|S_i|-|S_j|)}=\frac{2}{3k-2}. \end{aligned}$$

This finishes the proof.

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