The equidistribution of some Mahonian statistics over permutations avoiding a pattern of length three

Phan Thuan Do *   Thi Thu Huong Tran †   Vincent Vajnovszki ‡

August 12, 2021

Abstract

We prove the equidistribution of several multistatistics over some classes of permutations avoiding a 3-length pattern. We deduce the equidistribution, on the one hand of inv and foze″ statistics, and on the other hand that of maj and makl statistics, over these classes of pattern avoiding permutations. Here inv and maj are the celebrated Mahonian statistics, foze″ is one of the statistics defined in terms of generalized patterns in the 2000 pioneering paper of Babson and Steingrímsson, and makl is one of the statistics defined by Clarke, Steingrímsson and Zeng in 1997. These results solve several conjectures posed by Amini in 2018.

1 Introduction

Babson and Steingrímsson [2] introduced generalized permutation patterns and, based on linear combinations of functions counting occurrences of such patterns, they defined various statistics over permutations and identified many of them with well-known Mahonian statistics. They made several conjectures regarding the Mahonian nature of new generalized pattern-based statistics. These have since been proved, some of them refined and generalized in several different ways, see for instance [7, 8, 11].

In [6] the notion of Wilf equivalence is generalized in the following way: for a statistic st, the sets of classical patterns Π and Σ are st-Wilf equivalent if st has the same distribution over the set of permutations avoiding each pattern in Π and over the set of those avoiding each pattern in Σ. Among the results proved in [6] are: the inversion number, inv, has the same distribution over 132-avoiding (resp. 231-avoiding) permutations and over 213-avoiding (resp. 312-avoiding) permutations; and the major index, maj, has the same distribution over 132-avoiding (resp. 213-avoiding) permutations and over 231-avoiding (resp. 312-avoiding) permutations.

In [1], Amini exhaustively investigated quadruples $(\mathrm{st}_1, \mathrm{st}_2; \sigma, \tau)$ where st_1 and st_2 are Babson-Steingrímsson's Mahonian statistics, and σ and τ are 3-length classical patterns. These quadruples must satisfy an equidistribution property, namely st_1 over the set of σ-avoiding permutations

*Department of Computer Science, Hanoi University of Science and Technology, 01 Dai Co Viet, Hanoi, Vietnam, email: thuandp@soict.hust.edu.vn

†Vietnamese-German University, Le Lai street, Hoa Phu ward, Thu Dau Mot city, Binh Duong, Vietnam, email: huong.ttt@vgu.edu.vn

‡LIB, Université de Bourgogne Franche-Comté, B.P. 47 870, 21078 Dijon-Cedex, France, email: vvajnov@u-bourgogne.fr

is equidistributed with st_2 over the set of τ-avoiding permutations, or alternatively,

$$\sum_{\pi \text{ avoids } \sigma} q^{\mathrm{st}_1 \pi} t^{|\pi|} = \sum_{\pi \text{ avoids } \tau} q^{\mathrm{st}_2 \pi} t^{|\pi|},$$

where $|\pi|$ is the length of the permutation π.

For instance, with this terminology, an aforementioned result in [6] is that $(\mathrm{inv}, \mathrm{inv}; 132, 213)$ satisfies the equidistribution property. In [1] the equidistribution property is proved for many such quadruples, and for others it is still conjectured. Chen [4] based on a preprint version of [1] settled the conjectures corresponding to $\mathrm{st}_1 = \mathrm{maj}$ and $\mathrm{st}_2 = \mathrm{bast}$. Note that Chen used the notation STAT adopted from [2] (and used in other papers [8, 11]) instead of bast as in [1], and through this paper we adhere to Amini's [1] notations for Babson-Steingrímsson's Mahonian statistics.

Other statistic in [2] is foze'', whose Mahonian nature (up to reverse operation) is proved in [7, Theorem 2] and makl introduced in [5] in the context of Mahonian statistics on words. In the present paper we solve the conjectures in [1] corresponding to $\mathrm{st}_1 = \mathrm{foze}''$ and $\mathrm{st}_2 = \mathrm{inv}$, and to $\mathrm{st}_1 = \mathrm{makl}$ and $\mathrm{st}_2 = \mathrm{maj}$. In both cases we are able to find more refined equidistributions, and other conjectures in [1] are solved by combining these results with other known equidistributions.

2 Notations and definitions

We denote by $\mathfrak{S}_n$ the set of n-length permutations, and $\mathfrak{S} = \cup_{n \geq 0} \mathfrak{S}_n$. Let $\sigma \in \mathfrak{S}_k$ and $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$, $1 \leq k \leq n$, be two permutations. One says that π contains σ if π contains a subsequence $\pi_{i_1} \pi_{i_2} \ldots \pi_{i_k}$, $i_1 < i_2 < \cdots < i_k$, order isomorphic to σ; otherwise one says that π avoids σ, or π is σ-avoiding. In this context σ is called a (classical) *pattern*. For a pattern σ, $\mathtt{Av}_n(\sigma)$ denotes the set of σ-avoiding permutations in $\mathfrak{S}_n$, and $\mathtt{Av}(\sigma) = \cup_{n \geq 0} \mathtt{Av}_n(\sigma)$. *Generalized patterns* have been introduced in [2] and they were extensively studied since then. See for instance Chapter 7 in [10] for a comprehensive description of results on these patterns. A particular case of generalized patterns is that where two adjacent letters may be underlined, which means that the corresponding letters in the permutation must be adjacent. For example, the pattern $\underline{31}2$ occurs in the permutation 361524 four times, namely, as the subsequences 615, 612, 614 and 524. Note that the subsequences 312 and 624 are not occurrences of $\underline{31}2$ because their first two letters are not adjacent in the permutation.

A combinatorial *statistic* over $\mathfrak{S}_n$ is simply a function defined on $\mathfrak{S}_n$. Here we consider statistics whose codomains are integers, sets of integers or sets of pairs of integers. The distribution of an integer valued statistic st over the set $S \subset \mathfrak{S}_n$ is given by the coefficients of the generating function $\sum_{\sigma \in S} q^{\mathrm{st}\, \sigma}$. The overall number of occurrences of the patterns $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ in the permutation π becomes an integer valued statistic, denoted $(\sigma_1 + \sigma_2 + \cdots + \sigma_\ell)\, \pi$. The *descent number* statistic is defined as $\mathrm{des}\, \pi = (\underline{21})\, \pi$. Another classical example of integer valued statistic is inv, the *inversion number*: an inversion in a permutation π is a pair (i, j) with $i < j$ and $\pi_i > \pi_j$, and $\mathrm{inv}\, \pi$ is the number of inversions of π. Alternatively, in terms of functions counting occurrences of generalized patterns, it is easily seen that

$$\mathrm{inv}\, \pi = (\underline{23}1 + \underline{31}2 + \underline{32}1 + \underline{21})\, \pi,$$

and any integer valued statistic over $\mathfrak{S}_n$ which has the same distribution as inv is called Mahonian. The first statistic proved to be equidistributed with inv (and thus Mahonian) is maj

[12], defined as $\mathrm{maj}\,\pi = \sum_{\pi_i > \pi_{i+1}} i$, or alternatively, by counting for each descent the number of entries in π to its left,

$$\mathrm{maj}\,\pi = (1\underline{32} + 2\underline{31} + 3\underline{21} + \underline{21})\,\pi.$$

Babson-Steingrímsson's [2] statistics considered in [1] are (with the notations in [1]):

$$\begin{array}{ll}
\mathrm{mad}\,\pi = (2\underline{31} + 2\underline{31} + \underline{31}2 + \underline{21})\,\pi & \mathrm{foze}\,\pi = (\underline{21}3 + 3\underline{21} + \underline{13}2 + \underline{21})\,\pi \\
\mathrm{mak}\,\pi = (1\underline{32} + \underline{31}2 + \underline{32}1 + \underline{21})\,\pi & \mathrm{foze}'\,\pi = (1\underline{32} + 2\underline{31} + 2\underline{31} + \underline{21})\,\pi \\
\mathrm{makl}\,\pi = (1\underline{32} + 2\underline{31} + \underline{32}1 + \underline{21})\,\pi & \mathrm{foze}''\,\pi = (\underline{23}1 + \underline{31}2 + \underline{31}2 + \underline{21})\,\pi \\
\mathrm{bast}\,\pi = (\underline{13}2 + \underline{21}3 + \underline{32}1 + \underline{21})\,\pi & \mathrm{sist}\,\pi = (\underline{13}2 + \underline{13}2 + 2\underline{13} + \underline{21})\,\pi \\
\mathrm{bast}'\,\pi = (\underline{13}2 + \underline{31}2 + \underline{32}1 + \underline{21})\,\pi & \mathrm{sist}'\,\pi = (\underline{13}2 + \underline{13}2 + 2\underline{31} + \underline{21})\,\pi \\
\mathrm{bast}''\,\pi = (1\underline{32} + 3\underline{12} + 3\underline{21} + \underline{21})\,\pi & \mathrm{sist}''\,\pi = (\underline{13}2 + 2\underline{31} + 2\underline{31} + \underline{21})\,\pi,
\end{array}$$

see [5] for alternative definitions of some of these statistics.

For a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, its *reverse* $\mathrm{r}(\pi)$ is the permutation $\pi_n\pi_{n-1}\ldots\pi_1$, and its *complement* $\mathrm{c}(\pi)$ is the permutation $(n-\pi_1+1)(n-\pi_2+1)\ldots(n-\pi_n+1)$.

A *left-to-right maximum* of π is a pair (i, π_i) with $\pi_i > \pi_j$ for all $j < i$; i is the *position* and π_i is the *letter* of the left-to-right maximum. We denote by $\mathrm{Lrmax}\,\pi$ the set of left-to-right maxima of π and by $\mathrm{Lrmaxl}\,\pi$ the set of letters of left-to-right maxima of π. We define similarly a *left-to-right minimum*, *right-to-left maximum* and *right-to-left minimum* of π, and Lrminl, Rlmaxl, Rlmin and Rlminl have obvious meaning.

In any occurrence $\pi_i\pi_{i+1}$ of the pattern $\underline{21}$ in the permutation π (thus, $\pi_i > \pi_{i+1}$), the position i is called *descent*, the value π_i is called *descent top* and π_{i+1} *descent bottom*; $\mathrm{Des}\,\pi$, $\mathrm{Dtop}\,\pi$ and $\mathrm{Dbot}\,\pi$ denotes, respectively, the set of descents, descent tops and descent bottoms of π. Similarly, if $\pi_i < \pi_{i+1}$, then the position i is called *ascent* and the value π_{i+1} is called *ascent top*; $\mathrm{Asc}\,\pi$ and $\mathrm{Atop}\,\pi$ denotes, respectively, the set of ascents and of ascent tops of π.

For a permutation π, $|\pi|$ is its length (and so, $\pi \in \mathfrak{S}_{|\pi|}$), and for two permutations α and β, their *direct sum*, denoted $\alpha \oplus \beta$, is the permutation π of length $|\alpha| + |\beta|$ with

$$\pi_i = \left\{ \begin{array}{ll} \alpha_i & \text{if} \quad 1 \le i \le |\alpha| \\ \beta_{i-|\alpha|} + |\alpha| & \text{if} \quad |\alpha| + 1 \le i \le |\alpha| + |\beta|, \end{array} \right.$$

and their *skew sum*, denoted $\alpha \ominus \beta$, is the permutation π of length $|\alpha| + |\beta|$ with

$$\pi_i = \left\{ \begin{array}{ll} \alpha_i + |\beta| & \text{if} \quad 1 \le i \le |\alpha| \\ \beta_{i-|\alpha|} & \text{if} \quad |\alpha| + 1 \le i \le |\alpha| + |\beta|. \end{array} \right.$$

The following lemma regarding the structure of $\mathtt{Av}(231)$ is part of the folklore of pattern avoidance (see for instance [10]).

Lemma 1. *Any non-empty permutation π is 231-avoiding if and only if it can be written in a unique way as $\pi = (1 \ominus \alpha) \oplus \beta$, for some 231-avoiding permutations α and β.*

See the Figure 1 for the diagram representation of such a permutation π.

3 Equidistributions involving foze'' and inv

The main results of this section are Theorems 1 and 2. Some of their consequences are summarized in Table 1.

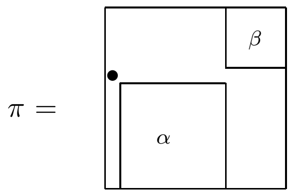


Figure 1: The decomposition $\pi = (1 \ominus \alpha) \oplus \beta$ of a non-empty 231-avoiding permutation π.

3.1 The equidistribution of foze'' and inv over $\mathtt{Av}_n(231)$

We define recursively the map ϕ on $\mathtt{Av}(231)$ as:

(i) if $\pi = \lambda$, then $\phi(\pi) = \lambda$, where λ is the empty permutation,

(ii) if $\pi = 1 \oplus \beta$, then $\phi(\pi) = 1 \oplus \phi(\beta)$, and

(iii) if $\pi = (1 \ominus \alpha) \oplus \beta$ with $\alpha \neq \lambda$, then $\phi(\pi) = (1 \ominus ((1 \ominus \delta) \oplus \gamma)) \oplus \phi(\beta)$, where γ and δ are such that $\phi(\alpha) = (1 \ominus \gamma) \oplus \delta$.

See Figure 2 for the diagram representation of $\phi(\pi)$ in the case (iii).

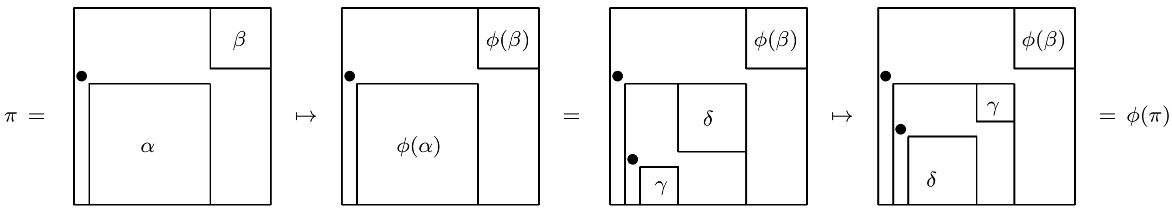


Figure 2: The construction of the image of $\pi = (1 \ominus \alpha) \oplus \beta$ through ϕ when α is not empty.

Remark 1. By construction, ϕ is length preserving and $\phi(\pi) \in \mathtt{Av}(231)$ whenever $\pi \in \mathtt{Av}(231)$. Moreover, ϕ keeps the first entry and the statistic Lrmax of a permutation in its preimage and image.

Our map ϕ is defined on $\mathtt{Av}(231)$, and by a slight abuse of notation we denote also by ϕ its restriction to $\mathtt{Av}_n(231)$. By induction on n it can be seen that ϕ on $\mathtt{Av}_n(231)$ is invertible, so ϕ is a bijection (but in general ϕ is not an involution).

In the proof of the next theorem we need the following remark where the equality involving foze'' statistic is based on the observation that if p is one of the patterns $\underline{23}1$, $\underline{31}2$ or $\underline{21}$, then $(p)\,\alpha \oplus \beta = (p)\,\alpha + (p)\,\beta$.

Remark 2. For any two permutations α and β we have that $\mathrm{inv}\,\alpha \oplus \beta = \mathrm{inv}\,\alpha + \mathrm{inv}\,\beta$ and $\mathrm{foze}''\,\alpha \oplus \beta = \mathrm{foze}''\,\alpha + \mathrm{foze}''\,\beta$.

Theorem 1. *The bistatistics* $(\mathrm{foze}'', \mathrm{Lrmax})$ *and* $(\mathrm{inv}, \mathrm{Lrmax})$ *have the same distribution over* $\mathtt{Av}_n(231)$.

Proof. We show by induction on n that $(\mathrm{inv}, \mathrm{Lrmax})\, \phi(\pi) = (\mathrm{foze}'', \mathrm{Lrmax})\, \pi$ for any $\pi \in \mathtt{Av}_n(231)$. By Remark 1 it is sufficient to show that $\mathrm{inv}\, \phi(\pi) = \mathrm{foze}''\, \pi$, and by Remark 2 it is sufficient to prove that if $\tau = 1 \ominus \alpha$, with α non-empty, then $\mathrm{inv}\, \phi(\tau) = \mathrm{foze}''\, \tau$. Note that τ is the first term of π in point (iii) defining ϕ, and thus $\phi(\tau) = 1 \ominus ((1 \ominus \delta) \oplus \gamma)$, where γ and δ are such that $\phi(\alpha) = (1 \ominus \gamma) \oplus \delta$. We refer the reader to Figure 2 reading it from right to left, disregarding β and $\phi(\beta)$. We have

$$\begin{aligned}
\mathrm{inv}\, \phi(\tau) &= \mathrm{inv}\, 1 \ominus ((1 \ominus \delta) \oplus \gamma) \\
&= \mathrm{inv}\, \delta + \mathrm{inv}\, \gamma + 2|\delta| + |\gamma| + 1 \\
&= \mathrm{inv}\, (1 \ominus \gamma) \oplus \delta + 2|\delta| + 1, \\
&= \mathrm{inv}\, \phi(\alpha) + 2|\delta| + 1.
\end{aligned}$$

Since ϕ preserves the first element of a permutation, it follows that if $\phi(\alpha) = (1 \ominus \gamma) \oplus \delta$, then $\alpha = (1 \ominus \gamma') \oplus \delta'$ with $|\delta| = |\delta'|$ (and $|\gamma| = |\gamma'|$). With these notations, by the induction hypothesis we have

$$\mathrm{inv}\, \phi(\alpha) + 2|\delta| + 1 = \mathrm{foze}''\, \alpha + 2|\delta'| + 1.$$

For any permutation α we have $(\underline{23}1)\, 1 \ominus \alpha = (\underline{23}1)\, \alpha$, and $(\underline{21})\, 1 \ominus \alpha = (\underline{21})\, \alpha + 1$. In addition, since $\alpha = (1 \ominus \gamma') \oplus \delta'$ it follows that $(\underline{31}2)\, 1 \ominus \alpha = (\underline{31}2)\, \alpha + |\delta'|$. Finally, since foze'' is a 'linear combination' of $(\underline{23}1)$, $(\underline{31}2)$ and $(\underline{21})$ we have

$$\begin{aligned}
\mathrm{foze}''\, \alpha + 2|\delta'| + 1 &= \mathrm{foze}''\, 1 \ominus \alpha \\
&= \mathrm{foze}''\, \tau,
\end{aligned}$$

and the statement holds. □

Example 1. If $\pi = 321$, then $\phi(\pi) = 312$, and if $\pi = 321654$, then $\phi(\pi) = 312645$. In the last case $\mathrm{foze}''\, \pi = \mathrm{inv}\, \phi(\pi) = 4$ and $\mathrm{Lrmax}\, \pi = \mathrm{Lrmax}\, \phi(\pi) = \{(1,3),(4,6)\}$.

3.2 The equidistribution of foze'' over $\mathtt{Av}_n(312)$ and inv over $\mathtt{Av}_n(321)$

We recall the definition of the well-known bijection $\psi : \mathtt{Av}_n(312) \to \mathtt{Av}_n(321)$ due to Simion and Schmidt [14]. The map $\pi \overset{\psi}{\mapsto} \sigma$ is defined as: keep the left-to-right maxima of π fixed, and write all the other entries in increasing order. Clearly, σ is 321-avoiding as it is the union of two increasing subsequences, one of which is the sequence of left-to-right maxima and the other is the increasing sequence of the remaining entries. In [14] it is shown that ψ is a bijection, see also [3, Lemma 4.3] (up to complement operation) and Figure 3 for an example.

Theorem 2. *The bistatistic* $(\mathrm{foze}'', \mathrm{Lrmax})$ *over* $\mathtt{Av}_n(312)$ *has the same distribution as* $(\mathrm{inv}, \mathrm{Lrmax})$ *over* $\mathtt{Av}_n(321)$.

Proof. Bijection ψ preserves Lrmax statistic, thus it is sufficient to prove that $\mathrm{foze}''\, \pi = \mathrm{inv}\, \psi(\pi)$ for $\pi \in \mathtt{Av}_n(312)$; and since π is 312-avoiding it is equivalent to prove that $(\underline{23}1 + \underline{21})\, \pi = \mathrm{inv}\ \psi(\pi)$. We proceed by induction on n. Let $\pi = \pi_1 \pi_2 \ldots \pi_n$ be a permutation in $\mathtt{Av}_n(312)$, $n \geq 2$. We distinguish two cases: (i) $\pi_2 > \pi_1$, and (ii) otherwise.

(i) Let π' be the permutation in $\mathtt{Av}_{n-1}(312)$ obtained from π after deleting its first entry and re-scaling the remaining entries to a permutation, and let $\sigma' = \psi(\pi')$. Clearly, $\sigma = \psi(\pi)$ is obtained from σ' by inserting π_1 in front of it, and adding 1 to all entries in σ' larger than or equal to π_1. We have

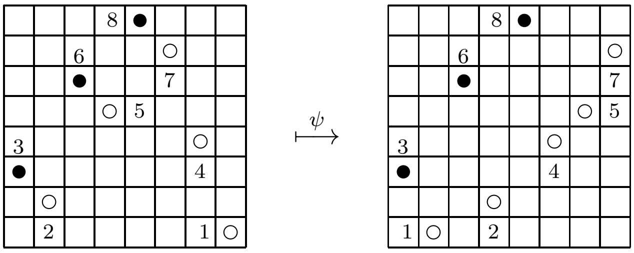


Figure 3: The bijection ψ transforms the permutation $\pi = 32658741 \in \texttt{Av}(312)$ into $\sigma = 31628457 \in \texttt{Av}(321)$. Left-to-right maxima are black.

- $(\underline{23}1 + \underline{21})\,\pi = (\underline{23}1 + \underline{21})\,\pi' + u$, where u is the number of occurrences of $\underline{23}1$ in π where the role of 2 is played by π_1, or equivalently, the number of entries in π less than π_1, and
- inv $\sigma =$ inv $\sigma' + v$, where v is the number of inversions in σ involving $\sigma_1 = \pi_1$.

Clearly, $u = v$, and by the induction hypothesis $(\underline{23}1 + \underline{21})\,\pi' = \text{inv } \sigma'$, and the statement holds.

(ii) Since π avoids 312, it follows that $\pi_1 = \pi_2 + 1$. Let π' be the permutation obtained from π by deleting π_2 and re-scaling the remaining entries to a permutation, and let $\sigma' = \psi(\pi')$. The permutation $\sigma = \psi(\pi)$ is obtained from σ' by adding 1 to each of its entries, then by inserting 1 between the entries σ'_1 and σ'_2. We have

- $(\underline{23}1 + \underline{21})\,\pi = (\underline{23}1 + \underline{21})\,\pi' + 1$ because π and π' have the same numbers of occurrences of $\underline{23}1$ but π has one more descent than π', namely $\pi_1\pi_2$, and
- inv $\sigma =$ inv $\sigma' + 1$.

Again, by the induction hypothesis, the statement holds. ☐

Example 2. If π and σ are the permutations in Figure 3, then $\text{foze}''\,\pi = \text{inv}\,\sigma = 8$ and $\text{Lrmax}\,\pi = \text{Lrmax}\,\sigma = \{(1,3),(3,6),(5,8)\}$.

3.3 Some consequences

We conclude this section with a couple of consequences of Theorems 1 and 2.

Since inv has the same distribution over $\texttt{Av}_n(231)$ and $\texttt{Av}_n(312)$, see [6], we have the next corollary.

Corollary 1. *The statistic* foze″ *over* $\texttt{Av}_n(231)$ *has the same distribution as* inv *over* $\texttt{Av}_n(312)$*.*

Corollary 2. *The statistic* foze″ *over* $\texttt{Av}_n(312)$ *(resp. over* $\texttt{Av}_n(231)$*) has the same distribution as* mad *over* $\texttt{Av}_n(231)$ *(resp. over* $\texttt{Av}_n(312)$*).*

Proof. By Theorem 2, foze″ over $\texttt{Av}(312)$ has the same distribution as inv over $\texttt{Av}_n(321)$, which in turn by [1, Corollary 24], has the same distribution as mad over $\texttt{Av}_n(231)$.
By Theorem 1, foze″ over $\texttt{Av}_n(231)$ has the same distribution as inv over $\texttt{Av}_n(231)$, or equivalently applying $\text{r} \circ \text{c}$ operation, as inv over $\texttt{Av}_n(312)$, which in turn has the same distribution as mad over $\texttt{Av}_n(312)$, see [9]. (Recall that $\text{r} \circ \text{c}$ is the reverse-complement operation, and it preserves inv statistic.) ☐

$\mathrm{inv}, \mathrm{foze}''$	231, 231 321, 312 312,231	Th. 1 Th. 2 Cor. 1
$\mathrm{mad}, \mathrm{foze}''$	231, 312 312, 231	Cor. 2
$\mathrm{foze}', \mathrm{foze}''$	132, 231	Cor. 3.1
$\mathrm{sist}, \mathrm{foze}''$	213, 231 132, 312	Cor. 3.2
$\mathrm{sist}', \mathrm{foze}''$	132, 312 231, 231	Cor. 3.3
$\mathrm{sist}'', \mathrm{foze}''$	132, 231 231, 312	Cor. 3.4

Table 1: Equidistribution of foze$''$ and other Babson-Steingrímsson's Mahonian statistic conjectured in [1] and proved in Section 3. It must be read, for instance, as: inv and foze$''$ have the same distribution over $\mathtt{Av}_n(231)$; and inv over $\mathtt{Av}_n(321)$ has the same distribution as foze$''$ over $\mathtt{Av}_n(312)$. Computer experiments have shown that these are, up to trivial transformations, the only such equidistributions involving foze$''$.

It is worth mentioning that the bistatistic $(\mathrm{foze}'', \mathrm{mad})$ over $\mathtt{Av}_n(312)$ does not have the same distribution as $(\mathrm{foze}'', \mathrm{mad})$ nor as $(\mathrm{mad}, \mathrm{foze}'')$ over $\mathtt{Av}_n(231)$.

Combining Theorems 1 and 2 with equidistributions in [1] and reasoning as above we have the next corollary.

Corollary 3. *The statistic* foze$''$

1. *over* $\mathtt{Av}_n(231)$ *has the same distribution as* foze$'$ *over* $\mathtt{Av}_n(132)$*,*
2. *over* $\mathtt{Av}_n(231)$ *(resp. over* $\mathtt{Av}_n(312)$*) has the same distribution as* sist *over* $\mathtt{Av}_n(213)$ *(resp. over* $\mathtt{Av}_n(132)$*),*
3. *over* $\mathtt{Av}_n(312)$ *(resp. over* $\mathtt{Av}_n(231)$*) has the same distribution as* sist$'$ *over* $\mathtt{Av}_n(132)$ *(resp. over* $\mathtt{Av}_n(231)$*),*
4. *over* $\mathtt{Av}_n(231)$ *(resp. over* $\mathtt{Av}_n(312)$*) has the same distribution as* sist$''$ *over* $\mathtt{Av}_n(132)$ *(resp. over* $\mathtt{Av}_n(231)$*).*

4 Equidistributions involving maj and makl

In this section, we turn our attention to the statistics maj and makl to prove similar equidistribution results. The main results of this section are Theorem 3, and Propositions 4 and 5. Some of their consequences are summarized in Table 3.

4.1 The equidistribution of maj and makl over $\mathtt{Av}_n(231)$

For a permutation $\pi \in \mathfrak{S}_n$, a maximal interval $\{i, i+1, i+2, \ldots, k\} \subset [n]$ is called *descent run* of π if $\pi_i \pi_{i+1} \ldots \pi_k$ is a decreasing factor of π, and if an entry π_ℓ does not belong to such a decreasing factor, then $\{\ell\}$ is a singleton descent run. A subset $\{i_1, i_2, \ldots, i_k\}$ of $[n]$ is called *inverse descent run* (or *i.d.r.* for short) if $\{\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}\}$ is a descent run in the (group theoretical) inverse π^{-1} of π. Alternatively, an i.d.r. can be defined as a maximal (possibly singleton) subset $\{i_1, i_2, \ldots, i_k\}$ of $[n]$ with $\pi_{i_\ell} = \pi_{i_{\ell+1}} + 1$, for $1 \leq \ell < k$. For a permutation π in $\mathfrak{S}_n$ with k i.d.r's let $I_1, I_2, \ldots, I_k$ be its i.d.r's. Then $\{I_1, I_2, \ldots, I_k\}$ is a partition of $[n]$ and $\{\pi_i\}_{i \in I_\ell}$ is an interval for every ℓ, $1 \leq \ell \leq k$. In the following we order $I_1, I_2, \ldots, I_k$ decreasingly by their largest element, that is, such that $\max(I_1) > \max(I_2 >) \cdots > \max(I_k)$. For instance, the i.d.r.'s of $\pi = 7615324$ are $\{1, 2, 4, 7\}$, $\{5, 6\}$ and $\{3\}$, and those of $\tau = 7132654$ are $\{1, 5, 6, 7\}$, $\{3, 4\}$ and $\{2\}$. See Figure 4 where permutations are represented by their diagrams.

With the notations above, $\pi_i > \pi_j$ for every $i \in I_\ell$ and $j \in I_{\ell+1}$ and the following remark gives some easy to understand properties of i.d.r.'s of permutations in $\texttt{Av}(231)$.

Remark 3. For $\pi \in \texttt{Av}(231)$ we have:

- I_1 is the set of positions of right-to-left maxima of π. Moreover, if we erase the entries π_i in π for each $i \in I_1$, then I_2 is the set of positions of right-to-left maxima of the resulting permutation, and so on;
- $\{\max(I_1), \max(I_2), \ldots, \max(I_k)\}$ is the set of positions of right-to-left minima of π.

For two finite sets A and B of integers we say that B is *nested* in A if there are no three integers $i < \ell < j$ with $i, j \in B$ and $\ell \in A$.

The next proposition gives the characterization of permutations in $\texttt{Av}(231)$ in terms of i.d.r.'s.

Proposition 1. *The permutation π belongs to $\texttt{Av}(231)$ if and only if for every two i.d.r.'s I_a and I_b of π with $\max(I_a) > \max(I_b)$ we have that I_b is nested in I_a.*

Proof. Clearly, if I_b is nested in I_a whenever $\max(I_a) > \max(I_b)$, then π avoids 231. Conversely, if I_b is not nested in I_a with $\max(I_a) > \max(I_b)$, then there are $i, j \in I_b$ and $\ell \in I_a$ with $i < \ell < j$, and so $\pi_i \pi_\ell \pi_j$ is an occurrence of 231 in π. □

The following technical definition will be used in the next two propositions.

Definition 1. For $1 \leq k \leq n$ the pair of sequences $c_1, c_2, \ldots, c_k$ and $m_1, m_2, \ldots, m_k$ of positive integers is *consistent* (with respect to k and n) if:

$c_1 \geq 2$,

$c_1 + c_2 + \cdots + c_k = n$,

$n = m_1 > m_2 > \cdots > m_k$, and

$m_\ell \geq c_\ell + c_{\ell+1} + \cdots + c_k + 1$ for every ℓ, $1 < \ell \leq k$.

For example $c_1, c_2, c_3 = 4, 2, 1$ and $m_1, m_2, m_3 = 7, 6, 5$ is a pair of consistent sequences with $k = 3$ and $n = 7$.

We denote by $\texttt{Av}'_n(231)$ the set of permutations in $\texttt{Av}_n(231)$ beginning by n, and $\texttt{Av}'(231) = \cup_{n \geq 0} \texttt{Av}'_n(231)$.

Proposition 2. *For $1 \leq k \leq n$ let $c_1, c_2, \ldots, c_k$ and $m_1, m_2, \ldots, m_k$ be a pair of consistent sequences. Then there is a unique permutation π in $\texttt{Av}'_n(231)$, where $n = m_1$, having k i.d.r.'s, say $I_1, I_2, \ldots, I_k$ ordered by $\max(I_1) > \max(I_2) > \cdots > \max(I_k)$, such that*

1. $|I_\ell| = c_\ell$, $1 \le \ell \le k$, *and*

2. $\operatorname{Asc}\pi = \{m_k, m_{k-1}, \ldots, m_2\}$.

Proof. We proceed by induction on k, and when $k = 1$ the desired permutation π is the decreasing one. For $k \ge 2$, the pair of sequences $c_1, c_2, \ldots, c_{k-1}$ and $m_1 - c_k, m_2 - c_k, \ldots, m_{k-1} - c_k$ is consistent with respect to $k-1$ and $n - c_k$, and let σ be the permutation in $\mathtt{Av}'(231)$ of length $c_1 + c_2 + \ldots c_{k-1} = n - c_k$ with $k-1$ i.d.r.'s, $I'_1, I'_2, \ldots, I'_{k-1}$, and

- $|I'_\ell| = c_\ell$, $1 \le \ell \le k-1$, and
- $\operatorname{Asc}\sigma = \{m_{k-1} - c_k, m_{k-2} - c_k, \ldots, m_2 - c_k\}$.

The desired permutation π is obtained from σ by adding c_k to each element of σ, then inserting a contiguous i.d.r. $c_k(c_k-1)\cdots 1$ into the slot after its $(m_k - c_k)$th entry. Formally, the permutation π is defined as

$$\pi_i = \begin{cases} \sigma_i + c_k & \text{if} \quad 1 \le i \le m_k - c_k \text{ or } m_k < i \le n \\ m_k - i + 1 & \text{if} \quad m_k - c_k + 1 \le i \le m_k. \end{cases}$$

The permutation π has k i.d.r.'s $I_1, I_2, \ldots, I_k$, and after deleting each entry π_i, $i \in I_k$, in π and re-scaling the remaining entries to a permutation we recover σ. It is routine to check that π is the unique permutation satisfying conditions 1 and 2 in the present proposition, and the statement holds. □

Example 3. For $k = 3$ and $n = 7$, using the example of a pair of consistent sequences $c_1, c_2, c_3 = 4, 2, 1$ and $m_1, m_2, m_3 = 7, 6, 5$ already given, the unique permutation prescribed by Proposition 2 is $\pi = 7653124 \in \mathtt{Av}'_7(231)$ with i.d.r.'s $I_1 = \{1, 2, 3, 7\}$, $I_2 = \{4, 6\}$, $I_3 = \{5\}$ and $\operatorname{Asc}\pi = \{5, 6\}$. See also Table 2, and Figure 4 for the diagram representation of π.

Proposition 3. *For* $1 \le k \le n$ *let* $c_1, c_2, \ldots, c_k$ *and* $m_1, m_2, \ldots, m_k$ *be a pair of consistent sequences. Then there is a unique permutation* τ *in* $\mathtt{Av}'_n(231)$*, where* $n = m_1$*, having* k *i.d.r.'s, say* $I_1, I_2, \ldots, I_k$ *ordered by* $\max(I_1) > \max(I_2) > \cdots > \max(I_k)$*, such that*

1. $|I_\ell| = c_\ell$, $1 \le \ell \le k$, *and*

2. $\operatorname{Atop}\tau = \{m_k, m_{k-1}, \ldots, m_2\}$.

Proof. Reasoning by induction as in the proof of Proposition 2, when $k = 1$ the desired permutation τ is the decreasing one. For $k \ge 2$, let $j \le k$ be the smallest integer such that the sets $1 + c_{j+1} + c_{j+2} + \cdots + c_k, 2 + c_{j+1} + c_{j+2} + \cdots + c_k, \ldots, c_j + c_{j+1} + c_{j+2} + \cdots + c_k$ and $\{m_2, \ldots, m_k\}$ are disjoint. Such a j necessarily exists since $\{1, 2, \ldots, c_k\}$ and $\{m_2, \ldots, m_k\}$ are disjoint. Now, if m_p is the smallest element of $m_1, m_2, \ldots, m_k$ larger than $n - (c_1 + c_2 + \cdots + c_{j-1})$, we denote by $m'_1, m'_2, \ldots, m'_{k-1}$ the sequence $m_1 - c_j, m_2 - c_j, \ldots, m_{p-1} - c_j, m_{p+1}, \ldots, m_k$ and by $c'_1, c'_2, \ldots, c'_{k-1}$ the sequence $c_1, c_2, \ldots, c_{j-1}, c_{j+1}, \ldots, c_k$. The obtained pair of sequences $c'_1, c'_2, \ldots, c'_{k-1}$ and $m'_1 m'_2, \ldots, m'_{k-1}$ is consistent with respect to $k-1$ and $n - c_j$ and let σ be the permutation with $k-1$ i.d.r.'s, $I'_1, I'_2, \ldots, I'_{k-1}$, and

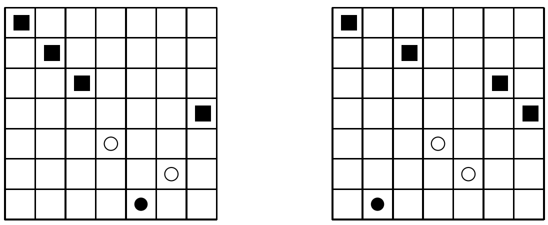
- $|I'_\ell| = c'_\ell$, $1 \le \ell \le k-1$, and
- $\operatorname{Atop}\sigma = \{m'_{k-1}, m'_{k-2}, \ldots, m'_2\}$.

The desired permutation τ is obtained from σ by adding c_j to each entry of σ larger than $c_{j+1}+c_{j+2}+\cdots+c_k$, then inserting a contiguous i.d.r. $(c_j+c_{j+1}+c_{j+2}+\cdots+c_k)(c_j+c_{j+1}+c_{j+2}+\cdots+c_k-1)\cdots(c_{j+1}+c_{j+2}+\cdots+c_k+1)$ of length c_j in the slot before the entry of value m_p. The obtained permutation τ has k i.d.r.'s, namely $I_1, I_2, \ldots, I_k$, and after deleting each entry τ_i, $i \in I_j$, in τ and re-scaling the remaining entries to a permutation we obtain σ. It is routine to check that τ is the unique permutation satisfying conditions 1 and 2 in the present proposition, and the statement holds. □

Example 4. For $k=3$ and $n=7$, using the example of a pair of consistent sequences $c_1, c_2, c_3 = 4, 2, 1$ and $m_1, m_2, m_3 = 7, 6, 5$, the unique permutation prescribed by Proposition 3 is $\tau = 7163254 \in \mathtt{Av}'_7(231)$ with i.d.r.'s $I_1=\{1,3,6,7\}$, $I_2=\{4,5\}$, $I_3=\{2\}$ and $\mathrm{Atop}\,\tau=\{5,6\}$. See also Table 2, and Figure 4 for the diagram representation of τ.

sequences c	sequences m	π corresponding by Pr. 2	sequences c	sequences m	τ corresponding by Pr. 3
$4,2,1$	$7,6,5$	7653**1**24	$4,2,1$	$7,6,5$	716**32**54
$4,2$	$6,5$	654**21**3	$4,1$	$5,4$	5**1**432
4	4	4321	4	4	4321

Table 2: Correspondences in Propositions 2 and 3 between consistent pairs of sequences and permutations. Permutations are constructed by inserting i.d.r.'s, in bold. The image of π through the bijection θ' defined in Theorem 3 is τ; and $\mathrm{Asc}\,\pi = \mathrm{Atop}\,\tau$.



π τ

Figure 4: The 231-avoiding permutations $\pi = 7653124$ and $\tau = \theta'(\pi) = 7163254$. The three i.d.r.'s of π are $\{1,2,3,7\}$, $\{4,6\}$ and $\{5\}$, and those of τ are $\{1,3,6,7\}$, $\{4,5\}$ and $\{2\}$; and $\mathrm{Asc}\,\pi = \mathrm{Atop}\,\tau = \{5,6\}$.

Theorem 3. *For every $n \geq 1$, there is a bijection θ' from $\mathtt{Av}'_n(231)$ into itself which*

1. *preserves the number of i.d.r's and their respective cardinality,*
2. *transforms* Asc *into* Atop*, and*
3. *preserves* Rlmaxl *and* Rlminl *statistics.*

Proof. Let π be a permutation in $\mathtt{Av}'_n(231)$ with k i.d.r's, say $I_1, I_2, \ldots, I_k$ ordered by $\max(I_1) > \max(I_2) > \cdots > \max(I_k)$. Then π determines the consistent (with respect to k and n) pair

$c_1, c_2, \dots, c_k$ and $m_1, m_2, \dots, m_k$ satisfying points 1 and 2 in Proposition 2. In turn, this pair uniquely determines a permutation $\tau \in \mathtt{Av}'_n(231)$ satisfying points 1 and 2 in Proposition 3. Thus, τ has k i.d.r's, say $I'_1, I'_2, \dots, I'_k$, ordered by $\max(I'_1) > \max(I'_2) > \cdots > \max(I'_k)$ and $|I_\ell| = |I'_\ell|$, $1 \le \ell \le k$. In addition $m_\ell \in \operatorname{Asc}\pi$ if and only if $m_\ell \in \operatorname{Atop}\tau$, $2 \le \ell \le k$.

It follows that the transformation $\pi \mapsto \tau$ is a bijection satisfying points 1 and 2 in the present theorem. Moreover, since $c_1 = |I_1| = |I'_1|$ we have

$$\begin{aligned}\operatorname{Rlmaxl}\pi &= \{n - c_1 + 1, n - c_1 + 2, \dots, n\}\\ &= \operatorname{Rlmaxl}\tau,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Rlminl}\pi &= \{\pi_{\max(I_k)}, \pi_{\max(I_{k-1})}, \dots, \pi_{\max(I_1)}\}\\ &= \{n - c_1 + 1, n - c_1 - c_2 + 1, \dots, n - c_1 - c_2 - \cdots - c_k + 1\}\\ &= \{\tau_{\max(I'_k)}, \tau_{\max(I'_{k-1})}, \dots, \tau_{\max(I'_1)}\}\\ &= \operatorname{Rlminl}\tau,\end{aligned}$$

and the statement holds. □

See Table 2 for the construction of $\theta'(\pi)$ in the proof of the previous theorem and Figure 4 for an example.

Since every n-length permutation π satisfies $\operatorname{Des}\pi = [n-1] \setminus \operatorname{Asc}\pi$, and in addition if $\pi_1 = n$, then $\operatorname{Dbot}\pi = [n-1] \setminus \operatorname{Atop}\pi$, it follows that bijection θ' in the proof of Theorem 3 transforms Des statistic into Dbot one, and we have the next corollary.

Corollary 4. *The multistatistics* (Des, Rlmaxl, Rlminl) *and* (Dbot, Rlmaxl, Rlminl) *are equidistributed over the set* $\mathtt{Av}'_n(231)$ *of n-length* 231*-avoiding permutations beginning by n.*

Any permutation $\pi \in \mathtt{Av}_n(231)$ can uniquely be written as a direct sum

$$\pi = \pi^{(1)} \oplus \pi^{(2)} \oplus \cdots \oplus \pi^{(k)} \tag{1}$$

of permutations in $\mathtt{Av}'_n(231)$, for some $k \ge 1$, and we extend θ' defined in Theorem 3 to θ on $\mathtt{Av}_n(231)$ as

$$\theta(\pi^{(1)} \oplus \pi^{(2)} \oplus \cdots \oplus \pi^{(k)}) = \theta'(\pi^{(1)}) \oplus \theta'(\pi^{(2)}) \oplus \cdots \oplus \theta'(\pi^{(k)}), \tag{2}$$

and clearly its inverse is

$$\theta^{-1}(\pi^{(1)} \oplus \pi^{(2)} \oplus \cdots \oplus \pi^{(k)}) = \theta'^{-1}(\pi^{(1)}) \oplus \theta'^{-1}(\pi^{(2)}) \oplus \cdots \oplus \theta'^{-1}(\pi^{(k)}),$$

and since θ' is bijective, so is θ.

For $\pi \in \mathtt{Av}_n(231)$ as in relation (1), the set $\operatorname{Lrmax}\pi$ is determined by the sequence $|\pi^{(1)}|, |\pi^{(2)}|, \dots, |\pi^{(k)}|$ which is the same as $|\theta'(\pi^{(1)})|, |\theta'(\pi^{(2)})|, \dots, |\theta'(\pi^{(k)}|$ (see relation (2)), and so θ preserves Lrmax statistic. In addition, θ (as θ') transforms Des statistic into Dbot one, and π and $\theta(\pi)$ have the same number of i.d.r's with the same respective cardinality. Thus θ preserves in addition Rlmaxl and Rlminl and we have the next consequence of Corollary 4.

Proposition 4. *The bijection θ on $\mathtt{Av}_n(231)$ transforms the multistatistic* (Des, Lrmax, Rlmaxl, Rlminl) *into* (Dbot, Lrmax, Rlmaxl, Rlminl) *one.*

For $\pi \in \texttt{Av}(231)$ it is easy to see that $\operatorname{makl} \pi = \sum_{i \in \operatorname{Dbot} \pi} i$, and since $\operatorname{maj} \pi = \sum_{i \in \operatorname{Des} \pi} i$ we obtain the next result.

Corollary 5. *The statistics* maj *and* makl *are equidistributed over* $\texttt{Av}_n(231)$.

The statistic maj has the same distribution over $\texttt{Av}_n(132)$ and over $\texttt{Av}_n(231)$, see [6], and we have the next consequence of Corollary 5.

Corollary 6. *The statistics* maj *over* $\texttt{Av}_n(132)$ *has the same distribution as* makl *over* $\texttt{Av}_n(231)$.

4.2 The equidistribution of maj and makl over $\texttt{Av}_n(312)$

The complement-reverse transformation $\mathrm{c} \circ \mathrm{r} : \mathfrak{S} \mapsto \mathfrak{S}$ maps a permutation in $\texttt{Av}(231)$ into one in $\texttt{Av}(312)$ and conversely, and in the following we will turn the equidistributions over $\texttt{Av}_n(231)$ in Section 4.1 into ones over $\texttt{Av}_n(312)$. The transformation $\mathrm{c} \circ \mathrm{r}$ is its own inverse, and it satisfies:

Property 1. For every permutation $\pi \in \mathfrak{S}_n$ we have

1. $i \in \operatorname{Des} \pi$ iff $n - i \in \operatorname{Des} \mathrm{c} \circ \mathrm{r}(\pi)$,
2. $i \in \operatorname{Dbot} \pi$ iff $n - i + 1 \in \operatorname{Dtop} \mathrm{c} \circ \mathrm{r}(\pi)$,
3. $(i, j) \in \operatorname{Lrmax} \pi$ iff $(n - i + 1, n - j + 1) \in \operatorname{Rlmin} \mathrm{c} \circ \mathrm{r}(\pi)$,
4. $i \in \operatorname{Rlmaxl} \pi$ iff $n - i + 1 \in \operatorname{Lrminl} \mathrm{c} \circ \mathrm{r}(\pi)$,
5. $i \in \operatorname{Rlminl} \pi$ iff $n - i + 1 \in \operatorname{Lrmaxl} \mathrm{c} \circ \mathrm{r}(\pi)$.

In the next proposition, which is the 312-avoiding counterpart of Proposition 4, we will make use of the following notations

- $n - \operatorname{Des} \pi = \{n - i \,:\, i \in \operatorname{Des} \pi\}$,
- $n - \operatorname{Dbot} \pi = \{n - i \,:\, i \in \operatorname{Dbot} \pi\}$,
- $\operatorname{Dtop} \pi - 1 = \{i - 1 \,:\, i \in \operatorname{Dtop} \pi\}$.

Proposition 5. *The statistics* $(\mathrm{Des}, \mathrm{Rlmin}, \mathrm{Lrminl}, \mathrm{Lrmaxl})$ *and* $(\mathrm{Dtop}{-}1, \mathrm{Rlmin}, \mathrm{Lrminl}, \mathrm{Lrmaxl})$ *are equidistributed over the set* $\texttt{Av}_n(312)$.

Proof. The map $\pi \mapsto \sigma = \mathrm{c} \circ \mathrm{r}\,(\theta(\mathrm{c} \circ \mathrm{r}(\pi)))$ is a bijection from $\texttt{Av}_n(312)$ into itself, where θ is the bijection from $\texttt{Av}_n(231)$ into itself defined in Section 4.1. For $\pi \in \texttt{Av}_n(312)$ we have

$$\begin{array}{rcll} \operatorname{Des} \pi & = & n - \operatorname{Des} \mathrm{c} \circ \mathrm{r}(\pi) & \text{(by point 1 of Property 1)} \\ & = & n - \operatorname{Dbot} \theta(\mathrm{c} \circ \mathrm{r}(\pi)) & \text{(by Proposition 4)} \\ & = & \operatorname{Dtop} \mathrm{c} \circ \mathrm{r}\,(\theta(\mathrm{c} \circ \mathrm{r}(\pi))) - 1, & \text{(by point 2 of Property 1)} \end{array}$$

and thus $\operatorname{Des} \pi = \operatorname{Dtop} \sigma - 1$. By points 3–5 of Property 1 and Proposition 4 it is routine to check that $(\mathrm{Rlmin}, \mathrm{Lrminl}, \mathrm{Lrmaxl})\, \pi = (\mathrm{Rlmin}, \mathrm{Lrminl}, \mathrm{Lrmaxl})\, \sigma$, and the statement holds. ☐

Example 5. If $\pi = 4675321 \in \texttt{Av}(312)$, then

- $\mathrm{c} \circ \mathrm{r}(\pi) = 7653124$,

- $\theta(\mathrm{c} \circ \mathrm{r}(\pi)) = \theta'(\mathrm{c} \circ \mathrm{r}(\pi)) = 7163254$ (see Table 2 and Figure 4), and
- $\mathrm{c} \circ \mathrm{r}(\theta(\mathrm{c} \circ \mathrm{r}(\pi))) = 4365271 \in \mathtt{Av}(312)$.

Moreover, $\mathrm{Des}\, \pi = \mathrm{Dtop}\, \mathrm{c} \circ \mathrm{r}\, (\theta(\mathrm{c} \circ \mathrm{r}(\pi))) - 1 = \{3, 4, 5, 6\}$.

Since $\mathrm{maj}\, \pi = \sum_{i \in \mathrm{Des}\, \pi} i$, and for $\pi \in \mathtt{Av}(312)$ we have $\mathrm{makl}\, \pi = \sum_{i \in \mathrm{Dtop}\, \pi - 1} i$ we obtain the next result.

Corollary 7. *The statistic* maj *and* makl *are equidistributed over* $\mathtt{Av}_n(312)$.

The statistic maj has the same distribution over $\mathtt{Av}_n(312)$ and over $\mathtt{Av}_n(213)$, see [6], and we have the next consequence of Corollary 7.

Corollary 8. *The statistic* maj *over* $\mathtt{Av}_n(213)$ *has the same distribution as* makl *over* $\mathtt{Av}_n(312)$.

4.3 Some consequences

Combining the results in the present section with other equidistributions proved in [1] we obtain the following corollary.

Corollary 9. *The statistic* makl

over $\mathtt{Av}_n(231)$ *has the same distribution as*

1. mak *over* $\mathtt{Av}_n(132)$ *or over* $\mathtt{Av}_n(312)$,
2. bast′ *or* foze *over* $\mathtt{Av}_n(132)$, *and*

over $\mathtt{Av}_n(312)$ *has the same distribution as*

3. mak *over* $\mathtt{Av}_n(213)$ *or over* $\mathtt{Av}_n(231)$,
4. foze *or* bast″ *over* $\mathtt{Av}_n(231)$.

From Corollaries 5–9 together with results in [4] we have the next consequence.

Corollary 10. *The statistic* bast *over*

1. $\mathtt{Av}_n(213)$ *has the same distribution as* makl *over* $\mathtt{Av}_n(231)$,
2. $\mathtt{Av}_n(231)$ *has the same distribution as* makl *over* $\mathtt{Av}_n(312)$,
3. $\mathtt{Av}_n(213)$ *has the same distribution as* mak *over* $\mathtt{Av}_n(132)$ *or over* $\mathtt{Av}_n(312)$,
4. $\mathtt{Av}_n(231)$ *has the same distribution as* mak *over* $\mathtt{Av}_n(213)$ *or over* $\mathtt{Av}_n(231)$,
5. $\mathtt{Av}_n(231)$ *has the same distribution as* bast″ *or* foze *over* $\mathtt{Av}_n(231)$.

Further research directions. It can be of interest to explore the techniques presented in this paper for other cases left open in [1] or for permutations of a multiset.

maj, makl	231, 231 132, 231 312, 312 213, 312	Cor. 5 Cor. 6 Cor. 7 Cor. 8
mak, makl	132, 231 312, 231	Cor. 9.1
	213, 312 231, 312	Cor. 9.3
bast′, makl	132,231	Cor. 9.2
bast″, makl	231,312	Cor. 9.4
foze, makl	132, 231 231, 312	Cor. 9.2 Cor. 9.4
bast, makl	213, 231 231, 312	Cor. 10.1-2

mak, bast	132, 213 312, 213 213, 231 231, 231	Cor. 10.3-4
bast″, bast	231,231	Cor. 10.5
foze, bast	231,231	Cor. 10.5

Table 3: The equidistributions conjectured in [1] that are proved in Section 4.

Acknowledgement

This research was initiated when the authors visited the Vietnam Institute for Advanced Study in Mathematics (VIASM), Hanoi, in the academic year 2019-2020. The authors express their warmest thank to the VIASM for the hospitality and for the wonderful working environment. The research is partly funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.01-2020.23.
The authors are grateful to the anonymous referees for their careful reading of the paper and for providing many useful suggestions. Special thanks to Sergey Kirgizov for fruitful discussion on the topic.

References

[1] N. Amini, Equidistributions of Mahonian statistics over pattern avoiding permutations, *The Electronic Journal of Combinatorics*, **25**(1) 2018, #P1.7.

[2] E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of Mahonian statistics, *Séminaire Lotharingien de Combinatoire* (electronic), **44** (2000), art. B44b.

[3] M. Bóna, Combinatorics of permutations, Chapman & Hall/CRC, 2004.

[4] J.N. Chen, Equidistributions of MAJ and STAT over pattern avoiding permutations, arXiv, 2017.

[5] R.J. Clarke, E. Steingrímsson, J. Zeng, New Euler-Mahonian statistics on permutations and words, *Advances in Applied Mathematics*, **18** (1997), 237–270.

[6] T. Dokos, T. Dwyer, B. Johnson, B. Sagan, K. Selsor, Permutation patterns and statistics, *Discrete Mathematics*, **312**(18) (2012), 2760–2775.

[7] D. Foata, D. Zeilberger, Babson-Steingrímsson statistics are indeed Mahonian (and sometimes even Euler–Mahonian), *Advances in Applied Mathematics*, **27**(2–3) (2001), 390–404.

[8] S. Fu, T. Hua, V. Vajnovszki, Mahonian STAT on rearrangement class of words, *Discrete Applied Mathematics*, **270** (2019), 134–141.

[9] J. Kim, K. Mézáros, G. Panova, D. Wilson, Dyck tilings, linear extensions, descents, and inversions, *DMTCS Proceedings*, AR, 2012, 769–780.

[10] S. Kitaev, Patterns in permutations and words, Springer-Verlag, 2011.

[11] S. Kitaev, V. Vajnovszki, Mahonian STAT on words, *Information Processing Letters*, **116**(2) (2016), 157–162.

[12] P.A. MacMahon, *Combinatory Analysis*, Vols. 1 and 2, Cambridge Univ. Press, Cambridge, 1915 (reprinted by Chelsea, New York, 1955).

[13] B. Sagan, C. Savage, Mahonian pairs, *Journal of Combinatorial Theory, Series A*, **119**(3) (2012), 526–545.

[14] R. Simion, F. Schmidt, Restricted permutations, *Europ. J. Combinatorics*, **6** (1985), 383–406.

[15] V. Vajnovszki, Lehmer code transforms and Mahonian statistics on permutations, *Discrete Mathematics*, **313** (2013), 581–589.