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Perturbation Analysis of TK Method for Harmonic Retrieval Problems

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***Abstract*—This paper presents a first-order perturbation analysis of the Tufts–Kumaresan (TK) method used to estimate frequencies of complex sinusoids in small additive noise. Several fundamental properties are presented and proved. Further illustrations are provided through numerical examples.**

I. Introduction

THIS paper is concerned with harmonic retrieval from a finite data sequence contaminated by additive noise. The data sequence y_k is modeled as

$$\hat{y}_k = \sum_{i=1}^{M} a_i e^{j\omega_i k} + n_k, \qquad k = 1, 2, \cdots, N \tag{1}$$

where a_i is the complex amplitude with unknown magnitude $|a_i|$ and phase $\phi_i \cdot \omega_i$ is the unknown angular frequency to be estimated. M is the number of complex sinusoids. n_k is the kth noise component. The hat ^ means that the corresponding variable is affected by noise or estimated under noise. For a noiseless quantity, the hat ^ is dropped. This notation will be used throughout this paper.

There are numerous methods [2] proposed by many authors in the past years to estimate ω_i (and even a_i and M for more general problems). Among them, the noniterative TK method [1] seems to have the second best performance next to the maximum likelihood (ML) method which is usually computed in an iterative way [3], [4] as it is a nonlinear optimization problem.

In this paper, we present the first-order perturbation analysis of the TK method for estimating frequencies ω_i (or $f_i = \omega_i/2\pi$) under relatively small noise. It is assumed that the number M of signals is known and $\omega_i \neq \omega_j$ for $i \neq j$.

In Section II, the TK method is briefly described and discussed. In Section III, the perturbation analysis is performed, and various properties are shown. In Section IV, numerical examples are illustrated.

Some mathematical details are included in Appendixes.

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II. TK Method

Step 1: Form the FBLP (forward-and-backward linear prediction) data matrix $\hat{A}_{FB}$ and data vector $\hat{h}_{FB}$, respectively, as follows:

$$\underset{2(N-L)\times L}{\hat{A}_{FB}} = \begin{bmatrix} \hat{A}_F \\ \hat{A}_B \end{bmatrix} \tag{2}$$

$$\underset{(N-L)\times L}{\hat{A}_F} = \begin{bmatrix} \hat{y}_L & \hat{y}_{L-1} & \cdots & \hat{y}_1 \\ \hat{y}_{L+1} & \hat{y}_L & \cdots & \hat{y}_2 \\ \vdots & \vdots & & \vdots \\ \hat{y}_{N-1} & \hat{y}_{N-2} & \cdots & \hat{y}_{N-L} \end{bmatrix} \tag{3}$$

$$\underset{(N-L)\times L}{\hat{A}_B} = \begin{bmatrix} \hat{y}_2^* & \hat{y}_3^* & \cdots & \hat{y}_{L+1}^* \\ \hat{y}_3^* & \hat{y}_4^* & & \hat{y}_{L+2}^* \\ \vdots & \vdots & & \vdots \\ \hat{y}_{N-L+1}^* & \hat{y}_{N-L+2}^* & \cdots & \hat{y}_N^* \end{bmatrix} \tag{4}$$

$$\hat{h}_{FB} = \begin{bmatrix} \hat{h}_F \\ \cdots \\ \hat{h}_B \end{bmatrix} = [\hat{y}_{L+1}, \cdots, \hat{y}_N; \hat{y}_1^*, \cdots, \hat{y}_{N-L}^*]^T \tag{5}$$

where * means complex conjugate; T means transpose; and L should satisfy $M \leq L \leq N - M/2$ (according to Kumaresan) which will be discussed later.

Step 2: Form the coefficients vector $\hat{g}$ of a polynomial of order L by

$$\hat{g} = -[\hat{A}_{FB}]_T^+ \cdot \hat{h}_{FB} \tag{6}$$

where we call $[\hat{A}_{FB}]_T^+$ the "truncated rank M" pseudoinverse of $\hat{A}_{FB}$, which is defined with the use of SVD [5] as

$$[\hat{A}_{FB}]_T^+ = \sum_{i=1}^{M} \frac{1}{\hat{\sigma}_i} \hat{u}_i \hat{v}_i^H \tag{7}$$

where $\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \cdots \geq \hat{\sigma}_M \geq \cdots$ are singular values of $\hat{A}_{FB}$. $\hat{u}_i$ and $\hat{v}_i$ are the corresponding right and left singular vectors, respectively; and the superscript "H" denotes the conjugate transpose.

It is clear that (for noiseless case) $[A_{FB}]_T^+ = [A_{FB}]^+$, which is the pseudoinverse [7] (the one which satisfies the Moore–Penrose definition) of A_{FB}, since A_{FB} has rank M.

Step 3: Find the zeros of the polynomial equation

$$1 + \sum_{l=1}^{L} \hat{g}_l \hat{z}^{-l} = 0 \tag{8}$$

where $\hat{g}_l$ is the lth element of $\hat{\boldsymbol{g}}$.

The M zeros ($\hat{z}_i$, $i = 1, 2, \cdots, M$), which are the closest to the unit circle, are chosen as the estimates of zeros $z_i = \exp(j\omega_i)$.

Then the frequency estimates are

$$\hat{\omega}_i = [\mathrm{Im}\,[\ln \hat{z}_i]]_{\mathrm{mod}[-\pi, \pi]} \tag{9}$$

where Im means the imaginary part.

It is known that for the noiseless case, all the estimates are exact and the $L - M$ extraneous zeros are inside the unit circle as long as A_{FB} has rank M. One can show that A_{FB} has rank M if $M \leq L \leq N - M$. However, if $N - M + 1 \leq L \leq N - M/2$ (which is part of the interval $M \leq L \leq N - M/2$ proposed in [2]), or equivalently, $M/2 \leq N - L \leq M - 1$, A_{FB} may have rank less than M in which case the TK method fails to work. For instance, as we show in Appendix A, A_{FB} has rank $N - L$ ($\leq M - 1$) if $N - L \leq M - 1$, and frequencies ω_i and phases ϕ_i are such that

$$(\omega_i - \omega_l)(N + 1) + 2(\phi_i - \phi_l) = 2m\pi \quad \text{for all } i \neq l \tag{10}$$

where m is any integer. For details, see Appendix A.

Although A_{FB} does have rank M for $M/2 \leq N - L \leq M - 1$ in "most" cases (note that the chance for ω_i and ϕ_i to satisfy (10) is very small), the performance of the TK method will not be good if (10) is approximately true for all $i \neq l$. This situation was recently explained in [9] for the special case of one real sinusoidal signal (i.e., $M = 2$).

The TK procedure utilizes both forward and backward linear prediction (FBLP). If we replace $\hat{A}_{FB}$ and $\hat{\boldsymbol{h}}_{FB}$ in (6) by $\hat{A}_F$ and $\hat{\boldsymbol{h}}_F$, respectively, then we say that forward linear prediction (FLP) is used. Similarly for backward linear prediction (BLP), $\hat{A}_{FB}$ and $\hat{\boldsymbol{h}}_{FB}$ in (6) are replaced by $\hat{A}_B$ and $\hat{\boldsymbol{h}}_B$. Since FLP and BLP have the same performance, we will only mention FLP in comparison to FBLP. In the discussion, FBLP will be implied if FBLP or FLP is not stated explicitly. It is clear that for FLP we must assume $N - L \geq M$.

III. Perturbation Analysis

In this paper, we derive the first-order perturbation (due to noise n_k) in the estimated zeros and frequencies, and investigate their several fundamental properties. We denote perturbations by preceding the corresponding noisy quantity by Δ. The following theorem is important in our derivation, while the proof is given in Appendix B.

Theorem: Assume

$$\hat{A} = A + \Delta\hat{A} \tag{11}$$

$$\hat{A}_T^+ = A^+ + \Delta\hat{A}_T^+ \tag{12}$$

where A has rank M. $\Delta\hat{A}$ is a small perturbation matrix. $\hat{A}_T^+$ is the "truncated rank M" pseudoinverse of $\hat{A}$ as defined by (7). A^+ is the pseudoinverse of A of rank M. $\Delta\hat{A}_T^+$ is the corresponding perturbation matrix.

Then we have

$$\boldsymbol{u}_o^H \Delta\hat{A}_T^+ \boldsymbol{v}_o = -\boldsymbol{u}_o^H A^+ \Delta\hat{A} A \boldsymbol{v}_o \tag{13}$$

where $\boldsymbol{u}_o^H$ is any row vector in the row space (span of rows) of A. $\boldsymbol{v}_o$ is any column vector in the column space (span of columns) of A.

Using this theorem, we shall show the following. The perturbations in the estimated zeros and frequencies are

$$\Delta\hat{z}_{i,\mathrm{FBLP}} = \frac{-1}{|a_i|} \frac{\boldsymbol{p}_{i,FB}^H \Delta\hat{A}'_{FB} \boldsymbol{g}'}{\sum_{l=1}^{L} l g_l z_i^{-l-1}} \tag{14}$$

$$\Delta\hat{\omega}_{i,\mathrm{FBLP}} = \frac{-1}{|a_i|} \mathrm{Im} \left[\frac{\boldsymbol{p}_{i,FB}^H \Delta\hat{A}'_{FB} \boldsymbol{g}'}{\sum_{l=1}^{L} l g_l z_i^{-l}} \right] \tag{15}$$

$$\Delta\hat{z}_{i,\mathrm{FLP}} = \frac{-1}{|a_i|} \frac{\boldsymbol{p}_{i,F}^H \Delta\hat{A}'_F \boldsymbol{g}'}{\sum_{l=1}^{L} l g_l z_i^{-l-1}} \tag{16}$$

$$\Delta\hat{\omega}_{i,\mathrm{FLP}} = \frac{-1}{|a_i|} \mathrm{Im} \left[\frac{\boldsymbol{p}_{i,F}^H \Delta\hat{A}'_F \boldsymbol{g}'}{\sum_{l=1}^{L} l g_l z_i^{-l}} \right] \tag{17}$$

where $\boldsymbol{p}_{i,FB}^H$ is the ith row of the pseudoinverse $Z_L^+ = (Z_L^H Z_L)^{-1} Z_L^H$ in which Z_L is defined by (A.5) in Appendix A. $\boldsymbol{p}_{i,F}^H$ is the ith row of the pseudoinverse $Z_{LF}^+ = (Z_{LF}^H Z_{LF})^{-1} Z_{LF}^H$ in which Z_{LF} is defined by (A.5). $\boldsymbol{g}' = [{}^1_{\boldsymbol{g}}]$; and $\Delta\hat{A}'_{FB}$ and $\Delta\hat{A}'_F$ are matrices filled with noise components, i.e.,

$$\Delta\hat{A}'_{FB} = [\Delta\hat{\boldsymbol{h}}_{FB}, \Delta\hat{A}_{FB}] = \begin{bmatrix} \Delta\hat{A}'_F \\ \cdots \\ \Delta\hat{A}'_B \end{bmatrix} = \begin{bmatrix} n_{L+1} & n_L & \cdots & n_1 \\ \vdots & & & \vdots \\ n_N & n_{N-1} & \cdots & n_{N-L} \\ n_1^* & n_2^* & \cdots & n_{L+1}^* \\ \vdots & & & \vdots \\ n_{N-L}^* & n_{N-L+1}^* & \cdots & n_N^* \end{bmatrix}. \tag{18}$$

To show (14) and (15), differentiating (8) yields

$$\sum_{l=1}^{L} \Delta\hat{g}_l \hat{z}_i^{-l} - \sum_{l=1}^{L} l g_l z_i^{-l-1} \Delta\hat{z}_i = 0. \tag{19}$$

Then

$$\Delta\hat{z}_i = \frac{\sum_{l=1}^{L} \Delta\hat{g}_l z_i^{-l}}{\sum_{l=1}^{L} l g_l z_i^{-l-1}} = \frac{z_i^H \Delta\hat{g}}{\sum_{l=1}^{L} l g_l z_i^{-l-1}} \tag{20}$$

where

$$z_i^H = [z_i^{-1}, z_i^{-2}, \cdots, z_i^{-L}].$$

Differentiating (6) yields

$$\Delta\hat{g} = -\Delta[\hat{A}_{FB}]_T^+ h_{FB} - A_{FB}^+ \Delta\hat{h}_{FB}. \tag{21}$$

Since z_i^H is the ith row vector of Z_R defined by (A.4) in Appendix A, it is in the row space of A_{FB}; and h_{FB} is a linear combination of columns of Z_L so that it is in the column space of A_{FB}. Therefore, according to the theorem,

$$\begin{aligned} z_i^H \Delta\hat{g} &= z_i^H [A_{FB}^+ \Delta\hat{A}_{FB} A_{FB}^+ h_{FB} - A_{FB}^+ \Delta\hat{h}_{FB}] \\ &= -z_i^H A_{FB}^+ (\Delta\hat{A}_{FB} g + \Delta\hat{h}_{FB}) \\ &= -z_i^H A_{FB}^+ \Delta A'_{FB} g'. \end{aligned} \tag{22}$$

It can be shown that the pseudoinverse of A_{FB} as in (A.2) is

$$\begin{aligned} A_{FB}^+ &= Z_R^+ \Lambda^{-1} Z_L^+ \\ &= Z_R^H (Z_R Z_R^H)^{-1} \Lambda^{-1} (Z_L^H Z_L)^{-1} Z_L^H \end{aligned} \tag{23}$$

where Λ and Z_R are defined by (A.3) and (A.4), respectively. Since z_i^H is the ith row of Z_R so that

$$z_i^H Z_R^+ = [\underbrace{0, \cdots, 0}_{i-1}, 1, 0, \cdots, 0]^T \tag{24}$$

then

$$z_i^H A_{FB}^+ = \frac{1}{|a_i|} p_{i,FB}^H. \tag{25}$$

Combining (25), (22), and (20) yields (14). Then (15) comes from (9) easily. Equations (16) and (17) can be shown similarly.

Based on (14)–(17), several fundamental properties of the TK method are shown next.

Property 1: For both FBLP and FLP, $\Delta\hat{z}_i$ and $\Delta\hat{\omega}_i$ are independent of the noise components n_k for $L + 1 \le k \le N - L$ given $L + 1 \le N - L$.

Proof: It suffices to show that $p_{i,FB}^H \Delta\hat{A}'_{FB} g'$ is independent of n_k for $L + 1 \le k \le N - L$. Let $p_{i,m}^*$ be the mth element of the vector $p_{i,FB}^H$. Then one can verify that, given $L + 1 \le N - L$,

$$\begin{aligned} p_{i,FB}^H \Delta\hat{A}'_{FB} g' &= \sum_{m=1}^{N-L} \sum_{l=0}^{L} p_{i,m}^* n_{L+m-l} g_l \\ &\quad + \sum_{m=1}^{N-L} \sum_{l=0}^{L} p_{i,m+N-L}^* n_{m+l}^* g_l \\ &= \sum_{k=1}^{N} (n_k x_{i,k} + n_k^* y_{i,k}) \end{aligned} \tag{26}$$

where

$$x_{i,k} = \begin{cases} \sum_{l=L+1-k}^{L} p_{i,k+l-L}^* g_l, & 1 \le k \le L \\ \sum_{l=0}^{L} p_{i,k+l-L}^* g_l, & L+1 \le k \le N-L \\ \sum_{l=0}^{N-k} p_{i,k+l-L}^* g_l, & N-L+1 \le k \le N \end{cases} \tag{27}$$

$$y_{i,k} = \begin{cases} \sum_{l=L+1-k}^{L} p_{i,N-L+k+l-L}^* g_{L-l}, & 1 \le k \le L \\ \sum_{l=0}^{L} p_{i,N-L+k+l-L}^* g_{L-l}, & L+1 \le k \le N-L \\ \sum_{l=0}^{N-K} p_{i,N-L+k+l-L}^* g_{L-l}, & N-L+1 \le k \le N \end{cases} \tag{28}$$

where $g_0 = 1$ and g_l for $l > 0$ is the lth element of g. Now we need to show that $x_{i,k} = y_{i,k} = 0$ for $L + 1 \le k \le N - L$. Observing that $\Sigma_{l=0}^{L} e^{-j\omega_i l} g_l = 0$, we form the vectors

$$g_{x,j} = [\underbrace{\overbrace{0, \cdots, 0,}^{j} g_0, \cdots, g_L, 0, \cdots, 0,}_{N-L} \underbrace{0, \cdots, 0}_{N-L}]^T \tag{29}$$

$$g_{y,j} = [\underbrace{0, \cdots, 0,}_{N-L} \underbrace{\overbrace{0, \cdots, 0,}^{j} g_L, \cdots, g_0, 0, \cdots, 0}_{N-L}]^T \tag{30}$$

then it can be shown that $g_{x,j}$ and $g_{y,j}$ ($j = 0, 1, \cdots, N - 2L - 1$) are orthogonal to all columns of Z_L as in (A.5). Since $p_{i,FB}$ is a vector in the column space of Z_L, then for

$L + 1 \leq k \leq N - L$,

$$x_{i,k} = p_{i,FB}^H \cdot g_{x,k-L-1} = 0 \tag{31}$$

$$y_{i,k} = p_{i,FB}^H \cdot g_{y,k-L-1} = 0. \tag{32}$$

Comment: This property implies that the estimated zeros and frequencies are more sensitive to the noise components in the first and last several data samples than in the middle data samples if SNR is moderately high (and $L + 1 \leq N - L$).

Next we investigate the variances of $\Delta\hat{z}_i$ and $\Delta\hat{\omega}_i$. We assume that the zero mean noise n_k are uncorrelated and equally powerful, and the real and imaginary parts of each n_k are uncorrelated and have variance σ for each part. In other words,

$$E\{n_k\} = 0 \tag{33}$$

$$E\{n_k n_l\} = 0 \tag{34}$$

$$E\{n_k n_l^*\} = 2\sigma^2 \delta_{k,l} \tag{35}$$

where $\delta_{k,l}$ is the Kronecker delta function.

Then one can verify from (14) and (16) that the variance of $\Delta\hat{z}_i$ is

$$\text{Var}\,(\Delta\hat{z}_i)_{\text{FBLP}} = \frac{1}{\text{SNR}_i} \frac{p_{i,FB}^H R p_{i,FB}}{\left|\sum_{l=1}^{L} l g_l z_i^{-l-1}\right|^2} \tag{36}$$

$$\text{Var}\,(\Delta\hat{z}_i)_{\text{FLP}} = \frac{1}{\text{SNR}_i} \frac{p_{i,F}^H R_g p_{i,F}}{\left|\sum_{l=1}^{L} l g_l z_i^{-l-1}\right|^2} \tag{37}$$

where

$$\text{SNR}_i = \frac{|a_i|^2}{2\sigma^2} \tag{38}$$

$$\underset{2(N-L)\times 2(N-L)}{R} = \begin{bmatrix} R_g & 0 \\ 0 & R_g^T \end{bmatrix} \tag{39}$$

$$(R_g)_{i,j} = (R_g)_{i-j} = (R_g)_{j,i}^* = \begin{cases} \sum_{l=i-j}^{L} g_l g_{l-(i-j)}^*, & 0 \leq i - j \leq L \\ 0 & i - j > L. \end{cases} \tag{40}$$

In fact, $(R_g)_{i,j}$ is the correlation function of the coefficient sequence g_l. Now we can show the following.

Property 2:

1) $\text{Var}\,(\Delta\hat{z}_i)_{\text{FLP}}$ is invariant to the phases ϕ_j for $j = 1, 2, \cdots, M$, while $\text{Var}\,(\Delta\hat{z}_i)_{\text{FBLP}}$ is not, in general.

2) $$\text{Var}\,(\Delta\hat{z}_i)_{\text{FBLP}} = \tfrac{1}{2}\,\text{Var}\,(\Delta\hat{z}_i)_{\text{FLP}} \tag{41}$$

if any one of the following is true.

a) $$(\omega_i - \omega_j)(N + 1) + 2(\phi_i - \phi_j) = 2m\pi, \quad \text{for all } i \neq j \tag{42}$$

b) $$\omega_i - \omega_j = \frac{2m\pi}{N - L}, \quad \text{for all } i \neq j \tag{43}$$

c) $N - L \gg 1$

d) $M = 1$, i.e., one signal case,

where m is some integer. Note that (42) is the same as (10), and for FLP, $N - L$ must be larger than or equal to M.

Proof: For FLP, one can show that $p_{i,F}^H$ is independent of ϕ_j for $j \neq i$, but is proportional to the complex exponential $e^{-j\phi_i}$. So that $p_{i,F}^H R_g p_{i,F}$ and $\text{Var}\,(\Delta\hat{z}_i)_{\text{FLP}}$ are independent of ϕ_j for all j.

To prove the second part, let us consider $Z_L^+ R Z_L^{+H}$, of which the (i, i)th element is $p_{i,FB}^H R p_{i,FB}$ [the numerator in (36)]. From (A.6) in Appendix A,

$$Z_L = \begin{bmatrix} Z_{LF} \\ P Z_{LF} E_N \end{bmatrix} \tag{44}$$

where P is the permutation matrix as in (A.7), and E_N is the diagonal matrix as in (A.8). Then one can show that

$$Z_L^+ R Z_L^{+H} = [Z_L^H Z_L]^{-1} [Z_{LF}^H R_g Z_{LF} + E_N^H Z_{LF}^H R_g Z_{LF} E_N] [Z_L^H Z_L]^{-1} \tag{45}$$

with

$$Z_L^H Z_L = Z_{LF}^H Z_{LF} + E_N^H Z_{LF}^H Z_{LF}^H E_N \tag{46}$$

where $P R_g^T P = R_g$ is used since R_g is the Hermitian and Toeplitz matrix.

If a) is true, then $E_N = I \cdot \exp[-j\omega_1(N + 1) - j2\phi_1]$, where I is the identity matrix; and then

$$Z_L^+ R Z_L^{+H} = \tfrac{1}{2}[Z_{LF}^H Z_{LF}]^{-1} [Z_{LF}^H R_g Z_{LF}] [Z_{LF}^H Z_{LF}]^{-1} = \tfrac{1}{2} Z_{LF}^+ R_g Z_{LF}^{+H}. \tag{47}$$

Substituting the (i, i)th element of $Z_L^+ R Z_L^{+H}$ as in (47) into (36) for $p_{i,FB}^H R p_{i,FB}$ yields (41).

If b) or c) is true, all columns of Z_L are orthogonal so that $[Z_L^H Z_L]^{-1} = \frac{1}{2}[Z_{LF}^H Z_{LF}]^{-1} = [1/2(N - L)]I$. In a similar way, one can show that (41) is true.

If d) is true, then again $(Z_L^+ Z_L)^{-1} = [1/2(N - L)]I$, so that (41) is true.

Comment: One can show that if the phase pair (ϕ_i, ϕ_j) satisfies (42), the regular inner product of the two corresponding columns of Z_L has the largest magnitude or, in other words, the two columns are the least orthogonal. Furthermore, one can show (see Appendix C) that the condition number of Z_L defined as the ratio of the largest singular value σ_1 of Z_L over the smallest nonzero singular value σ_M of Z_L reaches maximum when (42) is true for all $i \neq j$. Therefore, one may expect that phase variables ϕ_i which satisfy (42) for all $i \neq j$ provide the worst situation (the largest $\text{Var}\,(\Delta\hat{z}_i)_{\text{FBLP}}$) for FBLP, or more concisely,

$$\text{Var}\,(\Delta\hat{z}_i)_{\text{FBLP}} \leq \tfrac{1}{2}\,\text{Var}\,(\Delta\hat{z}_i)_{\text{FLP}} \tag{48}$$

with equality when any of conditions a)–d) in property 2 is met. Although the proof of (48) has not been obtained, the numerical computations have supported this conjecture.

As for Var $(\Delta\hat{\omega}_i)_{\text{FBLP}}$ and Var $(\Delta\hat{\omega}_i)_{\text{FLP}}$, we cannot in general find such simple relationships as (41) or (48). They are much more complicated. However, we do have the following result relating Var $(\Delta\hat{\omega}_i)$ and Var $(\Delta\hat{z}_i)$.

Property 3:

1) For FLP,

$$\text{Var}(\Delta\hat{\omega}_i)_{\text{FLP}} = \tfrac{1}{2}\text{Var}(\Delta\hat{z}_i)_{\text{FLP}}. \tag{49}$$

2) For FBLP, if $L = M$, i.e., the order of polynomial is chosen to be the number of signals, then

$$\text{Var}(\Delta\hat{\omega}_i) = \text{Var}(\Delta\hat{z}_i). \tag{50}$$

Proof: For FLP, one can see that $\Delta\hat{z}_i$ is a linear combination of n_k $(k = 1, 2, \cdots, N)$; so that with the assumption of (33)–(35) one can show that $E\{(\Delta\hat{z}_i/z_i)^2\} = 0$, and hence,

$$\text{Var}(\Delta\hat{\omega}_i) = \text{Var}\left(\text{Im}\left(\frac{\Delta\hat{z}_i}{z_i}\right)\right) = \frac{1}{2}\text{Var}\left(\frac{\Delta\hat{z}_i}{z_i}\right) = \frac{1}{2}\text{Var}(\Delta\hat{z}_i). \tag{51}$$

For FBLP and $L = M$, it is sufficient to show that

$$E\left\{\left(\frac{\Delta z_i}{z_i}\right)^2\right\} = -E\{|\Delta z_i|^2\}. \tag{52}$$

From (14),

$$E\left\{\left(\frac{\Delta\hat{z}_i}{z_i}\right)^2\right\} = \frac{1}{|a_i|^2}\frac{E\{(p_i^H\Delta\hat{A}'_{FB}g')^2\}}{\left(\sum_{l=1}^{L} lg_l z_i^{-l}\right)^2} \tag{53}$$

$$E\{|\Delta\hat{z}_i|^2\} = \frac{1}{|a_i|^2}\frac{E\{|p_i^H\Delta\hat{A}'_{FB}g'|^2\}}{\left|\sum_{l=1}^{L} lg_l z_i^{-l}\right|^2} \tag{54}$$

where $p_i^H = p_{i,FB}^H$ for notational simplicity. It is well known that $g_l = g_L g_{L-l}^*$ and $|g_L| = 1$ (also $g_o = 1$), since for $L = M$, all zeros are on the unit circle. Therefore,

$$\begin{aligned}\left(\sum_{l=1}^{L} lg_l z_i^{-l}\right)^2 &= \left(\sum_{l=1}^{L} lg_l z_i^{-l}\right)\cdot\left(\sum_{l=0}^{L}(L-l)g_{L-l}z_i^{-(L-l)}\right)\\ &= \left(\sum_{l=0}^{L} lg_l z_i^{-l}\right)\cdot\left(\sum_{l=0}^{L}(L-l)g_l^* z_i^l\right)g_L z_i^{-L}\\ &= -\left|\sum_{l=0}^{L} lg_l z_i^{-l}\right|^2 g_L z_i^{-L}\end{aligned} \tag{55}$$

where $\Sigma_{l=0}^{L} g_l^* z_i^l = 0$ is used. One can also write

$$Z_L = \begin{bmatrix} Z_{LF} \\ Z_{LF}^* E_L \end{bmatrix} \tag{56}$$

where

$$E_L = \begin{bmatrix} e^{j\omega_1 L} & & & \\ & e^{j\omega_2 L} & & \\ & & \ddots & \\ & & & e^{j\omega_M L}\end{bmatrix} \tag{57}$$

so that $Z_L^+ = (Z_L^H Z_L)^{-1}(Z_{LF}^H, E_L^H Z_{LF}^T)$, then the elements of the ith row vector p_i^H of Z_L^+ satisfy the relationship

$$p_{i,l}^* = p_{i,N-L+l}\exp(-j\omega_i L), \quad \text{for } 1 \le l \le N - L, \tag{58}$$

comparing the $y_{i,k}$ in (28) to the $x_{i,k}$ in (27) yields

$$y_{i,k} = x_{i,k}^* e^{-j\omega_i L} g_L. \tag{59}$$

From (26),

$$\begin{aligned}E\{(p_i^H\Delta\hat{A}'_{FB}g')^2\} &= 2\sigma^2\sum_{k=1}^{N} 2x_{i,k}y_{i,k}\\ &= 2\sigma^2\left[\sum_{k=1}^{N} 2|x_{i,k}|^2\right]e^{-j\omega_i L}g_L\\ &= E\{|p_i^H\Delta\hat{A}'_{FB}g'|e^{-j\omega_i L}g_L\}.\end{aligned} \tag{60}$$

Substituting (60) and (55) into (53) leads to (52).

Comment: In general $(L > M)$, the relationship between Var $(\Delta\hat{z}_i)$ and Var $(\Delta\hat{\omega}_i)$ is very complicated for FBLP. Numerical computations have shown (see Fig. 7) that the ratio of Var $(\Delta\hat{\omega}_i)$ over Var $(\Delta\hat{z}_i)$ decreases (not completely monotonically) toward 0.5 as L increases.

For the special case $L = M$, the property implies that the perturbation in estimated zero tends to move along the unit circle without the radial variance (for FBLP).

For one signal case and $L = M = 1$, properties 2 and 3 give that

$$\text{Var}(\Delta\hat{\omega}_i)_{\text{FBLP}} = \text{Var}(\Delta\hat{\omega}_i)_{\text{FLP}}; \tag{61}$$

this means that, for frequency estimation and the one signal Prony's case (i.e., $L = M = 1$), FBLP does not introduce improvement over FLP. In fact, it can be shown [10] that FLP is the most efficient for one signal Prony's case and $N = 2$ or 3. Finally the last property is as follows.

Property 4: For either FBLP or FLP, Var $(\Delta\hat{z}_i)$ and Var $(\Delta\hat{\omega}_i)$ are

1) independent of $|a_j|$ for $j \neq i$ but proportional to $1/|a_i|^2$ or $1/\text{SNR}_i$; and

2) invariant to the group shift of phases or/and frequencies, where by group shift of phases we mean that all phases are increased or decreased by a (additive) constant; similarly, group shift of the frequencies implies that all the frequencies are changed by a constant value.

Proof: The first part comes directly from (14)–(17). To show the second part, it is sufficient to consider FBLP, we denote the shifted frequencies and phases by

$$\tilde{\omega}_i = \omega_i + C_\omega \tag{62}$$

$$\tilde{\phi}_i = \phi_i + C_\phi. \tag{63}$$

All variables with ˜ denote the ones after the shift.

Then, it is well known (easy to show) that $\tilde{z}_i = z_i e^{jC_\omega}$ and $\tilde{g}_l = g_l e^{jC_\omega l}$, so that

$$\sum_{l=1}^{L} l\tilde{g}_l\tilde{z}_i^{-l} = \sum_{l=1}^{L} l g_l z_i^{-l} \tag{64}$$

which is the denominator in (14) and (15). (Note that the extra z_i^{-1} in (14) does not contribute to the variance of $\Delta\hat{z}_i$.)

It is easy to show that

$$\tilde{Z}_L = E_C Z_L \tag{65}$$

$$E_C = \begin{bmatrix} \exp\left[jC_\omega(L+1) + jC_\phi\right] & & & & \\ & \ddots & & & \\ & & \exp\left(jC_\omega N + jC_\phi\right) & & \\ & & & \exp\left(-jC_\omega N - jC_\phi\right) & \\ & & & & \ddots \\ & & & & & \exp\left[-jC_\omega(N-L) - jC\phi\right] \end{bmatrix}. \tag{66}$$

Therefore,

$$\tilde{p}_i^H = p_i^H E_C^{-1}. \tag{67}$$

Then, one can verify that

$$\tilde{p}_i^H \Delta\hat{\tilde{A}}'_{FB}\tilde{g}' = p_i^H \Delta\hat{A}'_{FB} g' \tag{68}$$

where $\Delta\hat{\tilde{A}}'_{FB}$ is the same as in (18) with n_k replaced by

$$\tilde{n}_k = n_k \exp\left(-jC_\omega k - jC_\phi\right). \tag{69}$$

Now it is clear that $\Delta\hat{z}_i$ is a linear combination of n_k and n_k^* $(k = 1, 2, \cdots, N)$ and $\Delta\hat{\tilde{z}}_i$ is the same linear combination [see (14), (64), and (68)] of $\tilde{n}_k$ and $\tilde{n}_k^*$ $(k = 1, 2, \cdots, N)$; but both n_k and $\tilde{n}_k$ satisfy (33)–(35) so that Var $(\Delta\hat{z}_i)$ = Var $(\Delta\hat{\tilde{z}}_i)$. Similarly, Var $(\Delta\hat{\omega}_i)$ = Var $(\Delta\hat{\tilde{\omega}}_i)$.

Comment: It is known [8] that the Cramer–Rao lower bound also has the same property. In fact, Var $(\Delta\hat{\omega}_i)$ can be very close to the C–R bound as will be seen next. In the next section, we show several examples of Var $(\Delta\hat{f}_i)$ $= 1/(2\pi)^2$ Var $(\Delta\hat{\omega}_i)$ and the corresponding C–R bound.

IV. Numerical Examples

In this section we only consider examples for FBLP, since the feature of FLP is simpler than FBLP.

Based on (14) and (15) with (33)–(35), one can compute Var $(\Delta\hat{z}_i)$ and Var $(\Delta\hat{\omega}_i)$. Although (28) can be used to calculate Var $(\Delta\hat{z}_i)$, Appendix D gives the detailed formula for computing both Var $(\Delta\hat{z}_i)$ and Var $(\Delta\hat{\omega}_i)$.

Examples 1 and 2 show the consistency between our theoretical results and the simulation results by Tufts and Kumaresan [1].

Example 1: As in [1], we assume that there are two signals present ($M = 2$). The number of data samples is $N = 25$. The frequency difference is $\omega_1 - \omega_2 = 2\pi(f_1 - f_2) = 2\pi(0.02)$ (instead of saying that $f_1 = 0.52$ and $f_2 = 0.5$), and the phase difference is $\phi_1 - \phi_2 = 45°$.

Fig. 1 shows the normalized inversed Cramer–Rao bound and the normalized inversed variance of the estimate $\hat{f}_i$ ($i = 1$ or 2; since the number of signals is two, the normalized variances and the bounds of $\hat{f}$ and $\hat{f}_2$ are the same) versus the order of the polynomial. That is,

$$10 \log_{10}\left[\frac{1}{\text{Bound} \cdot \text{SNR}_i}\right] \text{ versus } L$$

and

$$10 \log_{10}\left[\frac{1}{\text{VAR}\,(\hat{f}_i)\,\text{SNR}_i}\right] \text{ versus } L.$$

In this example, from the plot, the optimal order of the polynomial is $L_{\text{opt}} = 19 = 0.76N$.

Example 2: The same assumptions as in example 1 are given, except that $L = 18$, which clearly is a good choice according to Fig. 1, and $\phi_1 - \phi_1$ is varied. Fig. 2 shows the C–R bound and the variance Var $(\hat{f}_i)$ versus the phase difference $\phi_1 - \phi_2$. That is,

$$10 \log_{10} \frac{1}{\text{Bound} \cdot \text{SNR}_i} \text{ versus } \phi_1 - \phi_2$$

and

$$10 \log_{10} \frac{1}{\text{VAR}\,(\hat{f}_i) \cdot \text{SNR}_i} \text{ versus } \phi_1 - \phi_2.$$

Examples 1 and 2 are consistent with the simulation results presented in [1] where SNR_i = 15 dB (see Figs. 10 and 12 in [1]). However, our results are much smoother. Also note that in Fig. 2 the phase difference $\phi_1 - \phi_2 = 86.4°$ when $1/\text{Var}\,(\hat{f}_i)$ reaches minimum as predicted by (34).

Examples 3 and 4 demonstrate the dependence of the optimal order of polynomial on the phase difference.

Example 3: This example is the same as example 1, except that $\phi_1 - \phi_2 = 86.4°$ which satisfies (42), so that Z_L has the largest condition number (with respect to other phases). In Fig. 3, there are two humps. One is below $L = N/2$ and the other is above $L = N/2$. But they are not

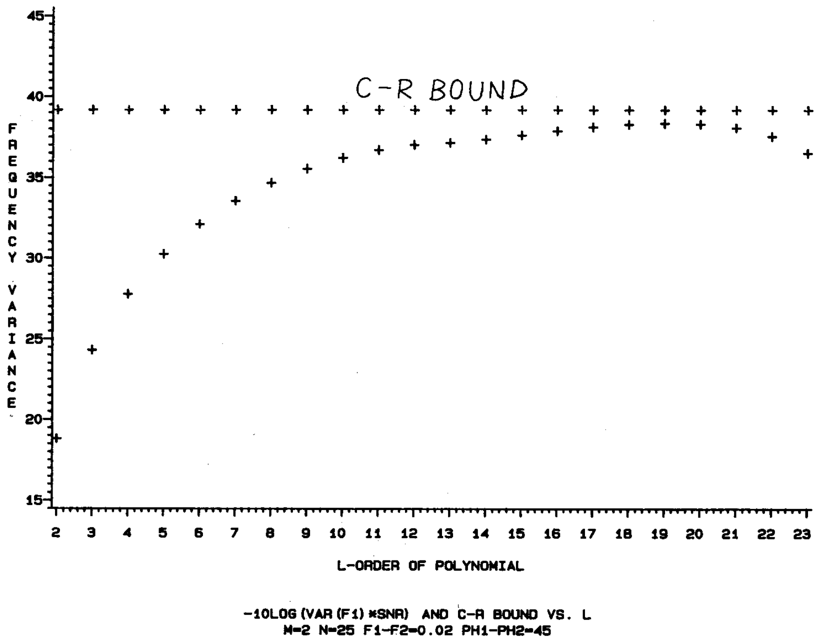

Fig. 1. Performance and C-R bound for $\phi_1 - \phi_2 = 45°$.

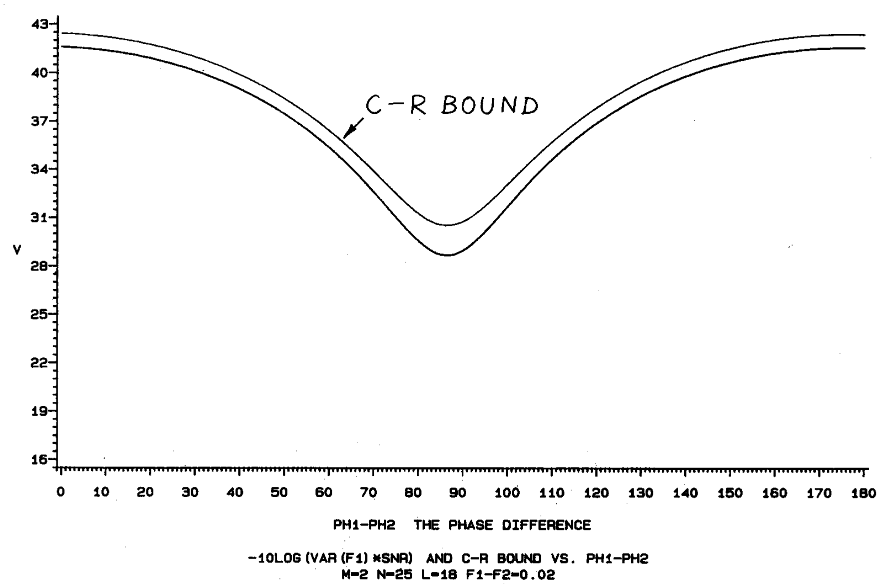

Fig. 2. Performance and C-R bound for $L = 18$.

exactly symmetrical about $L = N/2$. For this example, however, one may choose either $L_{opt} = 15 = 0.6N$ or $L_{opt} = 10 = 0.4N$.

Note that if we let $L = N - M/2 = 24$ in this example, Var $(\hat{f}_i)$ will be infinite (since the condition number of Z_L or Z_{FB} will be infinite).

Example 4: This example is the same as example 3, except that $\phi_1 - \phi_1 = -3.6°$ (or equivalently $176.4°$) predicted by

$$\phi_1 - \phi_2 + \pi(f_1 - f_2)(N + 1) = \frac{\pi}{2} \qquad (70)$$

which, as one can show, causes the two columns of Z_L orthogonal to each other, i.e., Z_L, to be best conditioned. Note that, in general ($M \geq 3$), one cannot find such phases ϕ_i that cause all columns of Z_L to be orthogonal.

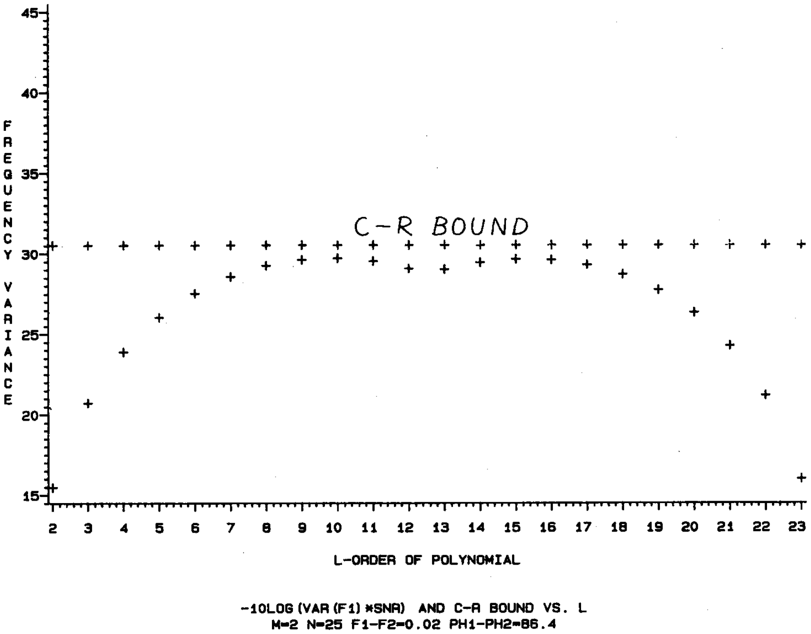

Fig. 3. Performance and C-R bound for $\phi_1 - \phi_2 = 86.4°$.

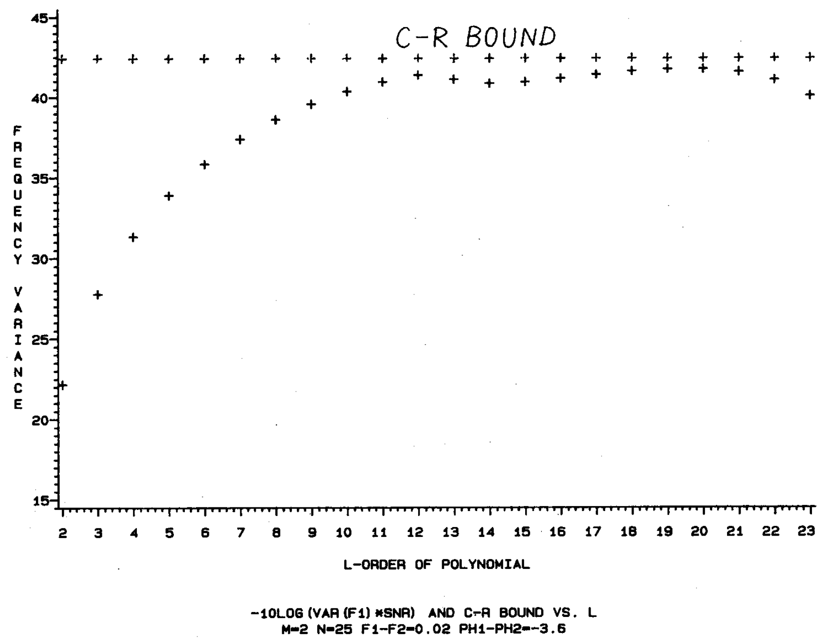

Fig. 4. Performance and C-R bound for $\phi_1 - \phi_2 = 3.6°$.

In Fig. 4, the performance for $L \geq \frac{1}{2}N$ is better than that for $L < \frac{1}{2}N$.

Now we show an example that combines the different features caused by different phase differences.

Example 5: The parameters are assumed to be the same as in example 3 or 4, except that $\phi_2 - \phi_2$ takes values from $-3.6°$ to $86.4°$ in steps of $15°$. The plot is the inversed efficiency in dB versus L, namely,

$$10 \log_{10} \left[\frac{\text{Bound}}{\text{Var}\,(\hat{f}_i)} \right] \text{ versus } L.$$

Note that if $\phi_1 - \phi_2$ takes values from $86.4°$ to $176.4°$,

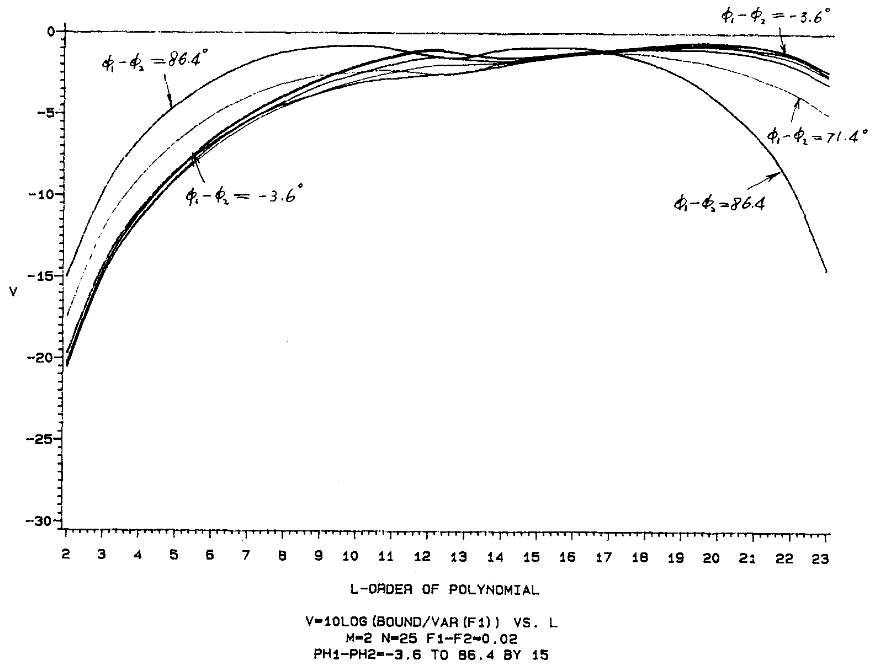

Fig. 5. Efficiency (inversed).

the pattern of Fig. 5 will be repeated since, as one can show, for $M = 2$ Var $(\hat{f}_i)$ is a periodic function of $\phi_1 - \phi_2$ with period 180°.

Based on this example, one can assume that

$$L_{\text{opt}} = 17 = 0.68N. \tag{71}$$

To show that the pattern of Var $(\hat{f}_i)$ follows the C-R bound, we have the following example.

Example 6: This example is the same as example 2, except that $f_1 - f_2 = 0.01$ instead of 0.02. Fig. 6 shows that there are two extremes. The minimum of $[1/\text{Var}(\hat{f}_1)]$ occurs at $\phi_1 - \phi_2 = 133.2°$ which is predicted by (42). The maximum of $1/\text{Var}(\hat{f}_i)$ occurs at $\phi_1 - \phi_2 = 43.2°$ which is predicted by (70).

The last example shows the complicated character of the ratio of Var $(\hat{\omega}_i)$ over Var $(\hat{z}_i)$.

Example 7: All parameters are given as in example 1. The plot shows

$$\frac{\text{Var}(\hat{\omega}_i)}{\text{Var}(\hat{z}_i)} \quad \text{versus } L.$$

We see that the ratio is decreasing (but not completely monotonically) with L toward 0.5. In fact, we have found numerically that, in most cases, the ratio is larger than 0.5 and approaches 0.5 when L is close to $N - M$.

V. Conclusion

The first-order perturbation analysis of the TK method is performed. Several fundamental properties are shown (and proved). Also, numerical examples are presented to illustrate some of the features.

Appendix A
Rank of A_{FB}

In this appendix, we discuss the rank of (noiseless) A_{FB}. As in (6), the noiseless coefficients vector $\boldsymbol{g}$ is

$$\boldsymbol{g} = -A_{FB}^{+}\boldsymbol{h}_{FB} \tag{A.1}$$

where A_{FB}^{+} is defined as in (7) without the hat ^.

Clearly, the existence of A_{FB}^{+} requires that A_{FB} has rank M. In fact, if and only if A_{FB} has rank M, the M correct zeros $z_i = e^{j\omega_i}$ $(i = 1, 2, \cdots, M)$ can be extracted from the polynomial formed by the coefficients vector $\boldsymbol{g}$ given in (A.1).

The following decomposition is useful for our discussion:

$$A_{FB} = Z_L \Lambda Z_R \tag{A.2}$$

$$\Lambda = \begin{bmatrix} |a_1| & & & \\ & |a_2| & & \\ & & \ddots & \\ & & & |a_M| \end{bmatrix} \tag{A.3}$$

$$Z_R = \begin{bmatrix} e^{-j\omega_1} & e^{-j\omega_1 2} & \cdots & e^{-j\omega_1 L} \\ e^{-j\omega_2} & e^{-j\omega_2 2} & \cdots & e^{-j\omega_1 L} \\ \vdots & & & \\ e^{-j\omega_M} & e^{-j\omega_M 2} & \cdots & e^{-j\omega_M L} \end{bmatrix} \tag{A.4}$$

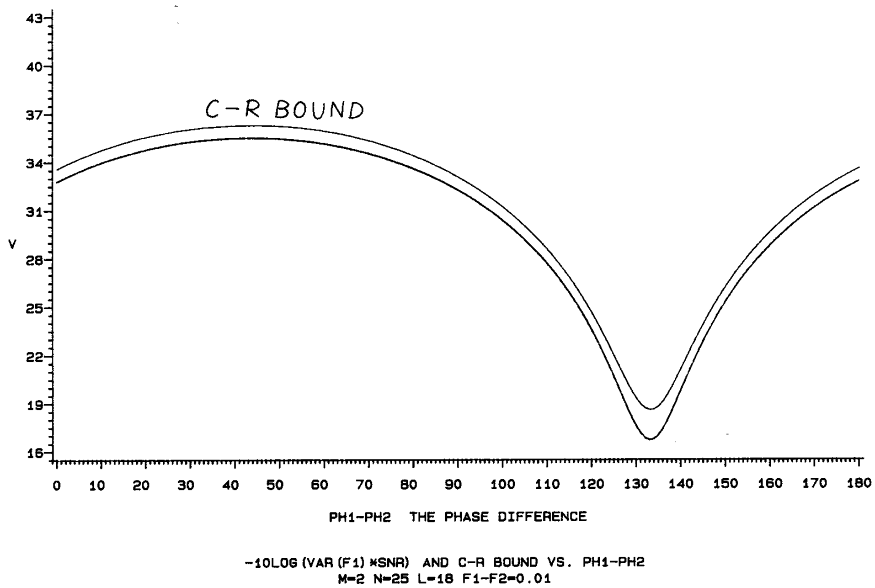

Fig. 6. Performance and C-R bound for $L = 18$ ($f_1 - f_2 = 0.01$).

$$Z_L = \begin{bmatrix} Z_{LF} \\ --- \\ Z_{LB} \end{bmatrix} = \begin{bmatrix} \exp\left[j\omega_1(L+1) + j\phi_1\right] & \cdots & \exp\left[j\omega_M(L+1) + j\phi_M\right] \\ \vdots & & \vdots \\ \exp\left[j\omega_1(N) + j\phi_1\right] & \cdots & \exp\left[j\omega_M(N) + j\phi_M\right] \\ \hline \exp\left(-j\omega_1 - j\phi_1\right) & \cdots & \exp\left(-j\omega_M - j\phi_M\right) \\ \vdots & & \vdots \\ \exp\left[-j\omega_1(N-L) - j\phi_1\right] & \cdots & \exp\left[-j\omega_M(N-L) - j\phi_M\right] \end{bmatrix}. \tag{A.5}$$

It is clear that Λ is nonsingular (rank M), Z_R has rank M if and only if $L \geq M$, and Z_L has rank M if $L \leq N - M$, so that A_{FB} has rank M if $M \leq L \leq N - M$. It is also clear that A_{FB} has rank less than M if $L \leq M$ or $L > N - M/2$.

What about $N - M + 1 \leq L \leq N - M/2$, or equivalently, $M/2 \leq N - L \leq M - 1$, which are "valid" values for L proposed in [2]? It turns out that Z_L (or consequently A_{FB}) does not always have rank M for $M/2 \leq N - L \leq M - 1$.

Notice that Z_L can be written as

$$Z_L = \begin{bmatrix} Z_{LF} \\ PZ_{LF}E_N \end{bmatrix} \tag{A.6}$$

where

$$P = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \tag{A.7}$$

which is a permutation matrix;

$$E_N = \begin{bmatrix} e_1 & & & \\ & e_2 & & \\ & & \ddots & \\ & & & e_M \end{bmatrix} \tag{A.8}$$

$$e_i \triangleq \exp\left(-j\omega_i(N+1) - j2\phi_i\right). \tag{A.9}$$

In fact, if $e_i = e_l$ for all $i \neq l$, or equivalently, (10) is true for all $i \neq l$, then $E_N = I \cdot e_1$, where I is identity matrix so that

$$Z_L = \begin{bmatrix} Z_{LF} \\ PZ_{LF}e_1 \end{bmatrix} \sim \begin{bmatrix} Z_{LF} \\ Z_{LF} \end{bmatrix} \sim \begin{bmatrix} Z_{LF} \\ 0 \end{bmatrix} \tag{A.10}$$

where $\sim$ means that one side is nonsingularly transformed from the other so that both sides have the same rank.

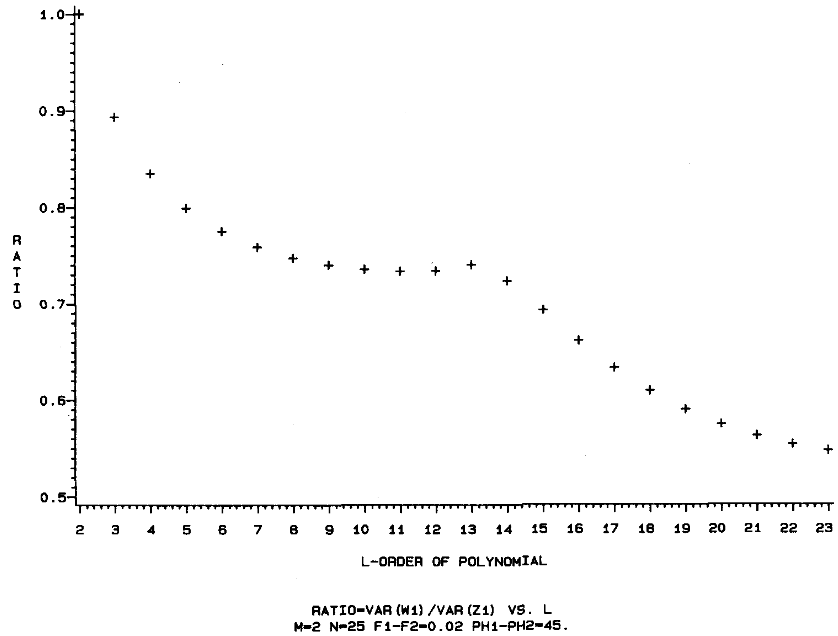

Fig. 7. Ratio of frequency variance over zero variance.

Clearly, Z_L above (or A_{FB}) has the same rank as Z_{LF} which is of rank $N - L\ (\leq M - 1)$ if $M/2 \leq N - L \leq M - L$.

More generally, we have the following theorem.

Theorem A.1: We partition the set $\{e_1, e_2, \cdots, e_M\}$ into groups $G_1, G_2, \cdots, G_r$, such that elements in each group are equal and elements from different groups are not. Let N_{Gj} be the number of elements in group G_j. Clearly, $\Sigma_{j=1}^{r} N_{Gj} = M$. Without loss of generality, we let $N_{G1} \geq N_{G2} \geq \cdots \geq N_{Gr}$.

Assume $M/2 \leq N - L \leq M - 1$; then

1) if $N_{G_1} \geq N - L + 1$, then

$$\text{Rank}\ (A_{FB}) = \text{Rank}\ (Z_L) = M - N_{G_1} + N - L \leq M - 1; \quad \text{(A.11)}$$

2) if $N_{G_1} \leq N - L$ and

a) if $M - N_{G_1} - N - L + 1$, then

$$N_{G_1} + N - L \leq \text{Rank}\ (A_{FB}) = \text{Rank}\ (Z_L) \leq M; \quad \text{(A.12)}$$

b) if $M - N_{G_1} \leq N - L$, then

$$\text{Rank}\ (A_{FB}) = \text{Rank}\ (Z_L) = M. \quad \text{(A.13)}$$

Note that there is an inherent assumption that $N \geq \frac{3}{2}M$, otherwise, no choice of L can be used to estimate the M unknown frequencies.

Proof: Since $L \geq N - M + 1 \geq M + 1$, Z_R has full rank M so that Rank (Z_L) = Rank (A_{FB}). Changing the order of the last $N - L$ rows of Z_L leads to

$$Z_L \sim \begin{bmatrix} Z_{LF} \\ Z_{LF} E_N \end{bmatrix}. \quad \text{(A.14)}$$

Reordering the columns of Z_L according to $G_1, G_2, \cdots, G_r$ yields

$$Z_L \sim \begin{bmatrix} Z_{LF1} & Z_{LF2} & \cdots & Z_{LFr} \\ e_{G_1} Z_{LF1} & e_{G_2} Z_{LF2} & \cdots & e_{G_r} Z_{LFr} \end{bmatrix} \quad \text{(A.15)}$$

where e_{G_j} is an element in group G_j.

Multiplying the first $N - L$ rows by e_{G_1} and subtracting it from the last $N - L$ rows yields

$$Z_L \sim \begin{bmatrix} Z_{LF_1} & Z_{LF_2} & \cdots & Z_{LF_r} \\ \underbrace{0}_{N_{G_1}} & (e_{G_1} - e_{G_2}) Z_{LF_2} & \cdots & (e_{G_r} - e_{G_1}) Z_{LF_r} \end{bmatrix}. \quad \text{(A.16)}$$

1) If $N_{G_1} \geq N - L + 1$, then columns of Z_{LF_1} span the complex vector space $C^{(N-L) \times 1}$ so that

$$Z_L \sim \begin{bmatrix} Z_{LF_1} & 0 & \cdots & 0 \\ 0 & (e_{G_1} - e_{G_2}) Z_{LF_2} & \cdots & (e_{G_r} - e_{G_1}) Z_{LF_r} \end{bmatrix}. \quad \text{(A.17)}$$

Since $M \leq 2(N - L)$, then $M - N_{G_1} \leq N - L - 1$ so that the last $M - N_{G_1}$ columns are independent. Also, Rank $(Z_{LF_1}) = N - L$. Therefore,

$$\text{Rank}\ (Z_L) = M - N_{G_1} + N - L \leq M - 1. \quad \text{(A.18)}$$

2) If $N_{G_1} \leq N - L$ and

a) if $M - N_{G_1} \geq N - L + 1$, then the last $M - N_{G_1}$ columns of Z_L in (A.16) have rank at least $N - L$ (e.g., $e_{G_2} = \cdots = e_{G_r}$), or at most $M - N_{G_1}$ (e.g.,

$e_{G_i} \neq e_{G_j}$ for $2 \leq i \neq j \leq r$) so that

$$N_{G_1} + N - L \leq \text{Rank}(Z_L) \leq M; \quad (A.19)$$

b) if $M - N_{G_1} \leq N - L$, then as can be shown similarly,

$$\text{Rank}(Z_L) = N_{G_1} + (M - N_{G_1}) = M. \quad (A.20)$$

Comment: If ω_i and ϕ_i are independent (random) unknown parameters, it may be reasonable to say that Rank $(Z_L) = M$ in "almost" all cases since e_i are unequal in "almost" all cases for $M/2 \leq N - L \leq M - 1$.

Appendix B
Perturbation in Truncated Pseudoinverse

In this appendix, we prove (13).

Proof: We rewrite (11)

$$\hat{A} = A + \Delta\hat{A} \quad (B.1)$$

where A has rank M. $\hat{A}$ has rank $\hat{M} \geq M$; and, for the moment, $\Delta\hat{A}$ is not necessarily a matrix with very small elements.

One can verify the identity [6]

$$\hat{A}^+ - A^+ = -\hat{A}^+\Delta\hat{A}A^+ - (\hat{A}^H\hat{A})^+\Delta\hat{A}^H P_A^\perp + R_{\hat{A}}^\perp \Delta\hat{A}^H (AA^H)^+ \quad (B.2)$$

where $P_A^\perp = I - AA^+$ is the projector onto the orthogonal complement of the column space of A. $R_{\hat{A}}^\perp = I - \hat{A}^+\hat{A}$ is the projector onto the orthogonal complement of the column space of $\hat{A}^H$.

By SVD,

$$\hat{A}^+ = \sum_{i=1}^{\hat{M}} \frac{1}{\hat{\sigma}_i} \hat{\boldsymbol{u}}_i \hat{\boldsymbol{v}}_i^H \quad (B.3)$$

$$\hat{A}_T^+ = \sum_{i=1}^{M} \frac{1}{\hat{\sigma}_i} \hat{\boldsymbol{u}}_i \hat{\boldsymbol{v}}_i^H \quad (B.4)$$

where $\hat{\sigma}_i$, $\hat{\boldsymbol{u}}_i$, and $\hat{\boldsymbol{v}}_i$ are defined as in (7).

Let $\hat{\boldsymbol{u}}_o$ be a vector from the space spaned by $\hat{\boldsymbol{u}}_i$, $\hat{\boldsymbol{u}}_2$, $\cdots$, $\hat{\boldsymbol{u}}_M$. Clearly, as $\Delta\hat{A}$ approaches zero, this space approaches the column space of A^H so that $\hat{\boldsymbol{u}}_o$ approaches a vector, $\boldsymbol{u}_o$, from the column space of A^H; and $\boldsymbol{v}_o$ is a vector from the column space of A.

Then, we know that

$$P_A^\perp \boldsymbol{v}_o = 0 \quad (B.5)$$

$$R_{\hat{A}}^\perp \hat{\boldsymbol{u}}_o = 0 \quad (B.6)$$

and

$$\hat{\boldsymbol{u}}_o^H \hat{A}^+ = \hat{\boldsymbol{u}}_o^H \hat{A}_T^+. \quad (B.7)$$

Therefore, from (B.2),

$$\hat{\boldsymbol{u}}_o^H(\hat{A}_T^+ - A^+)\boldsymbol{v}_o = \hat{\boldsymbol{u}}_o^H \hat{A}^H \Delta\hat{A} A^+ \boldsymbol{v}_o. \quad (B.8)$$

Now let $\Delta\hat{A}$ approach zero, so we have

$$\boldsymbol{u}_o^H \Delta\hat{A}_T^+ \boldsymbol{v}_o = -\boldsymbol{u}_o^H A^+ \Delta\hat{A} A^+ \boldsymbol{v}_o. \quad (B.9)$$

Appendix C
Condition Number

In this appendix, we show the following theorem.

Theorem C.1: Let Z_F be a matrix of rank equal to the number M of its columns; let E be a unitary matrix; and let

$$Z = \begin{bmatrix} Z_F \\ Z_F E \end{bmatrix} \quad (C.1)$$

$$Z' = \begin{bmatrix} Z_F \\ Z_F \end{bmatrix}. \quad (C.2)$$

Denote by σ_1 and σ_M, respectively, the largest and smallest (nonzero) singular values of Z, and similarly, for σ_1' and σ_M'.

Then

$$\sigma_1 \leq \sigma_1' \quad (C.3)$$

$$\sigma_M \geq \sigma_M'. \quad (C.4)$$

Therefore, the condition numbers k and k', of Z and Z', satisfy

$$k \triangleq \frac{\sigma_1}{\sigma_M} \leq k' \triangleq \frac{\sigma_1'}{\sigma_M'}. \quad (C.5)$$

Proof: It is well known [5] that

$$\sigma_1^2 = \max_{\|x\|_2=1} [\boldsymbol{x}^H Z^H Z \boldsymbol{x}] \quad (C.6)$$

then

$$\begin{aligned} \sigma_1^2 &= \max_{\|x\|_2=1} [\boldsymbol{x}^H Z_F^H Z_F \boldsymbol{x} + \boldsymbol{x}^H E^H Z_F^H Z_F E \boldsymbol{x}] \\ &\leq \max_{\|x\|_2=1} [\boldsymbol{x}^H Z_F^H Z_F \boldsymbol{x} + x^H Z_F^H Z_F \boldsymbol{x}] \\ &= \sigma_1'^2 \end{aligned} \quad (C.7)$$

with equality when $E = I \cdot C$, where C is a complex number.

Similarly, we can show

$$\sigma_M^2 \geq \sigma_M'^2. \quad (C.8)$$

Appendix D
Computation of Var $(\Delta\hat{z}_i)$ and Var $(\Delta\hat{\omega}_i)$

This appendix gives the expressions of Var $(\Delta\hat{z}_i)$ and Var $(\Delta\hat{\omega}_i)$ for numerical computation.

One can verify that [from (14)]

$$\boldsymbol{p}_i^H \Delta\hat{A}_{FB}' \boldsymbol{g}' = \sum_{k=1}^{N} (n_k x_{i,k} + n_k^* y_{i,k}) \quad (D.1)$$

where

$$x_{i,k} = \begin{cases} \sum_{l=L+1-k}^{L} p_{i,k+l-L}^{*} g_l & 1 \le k \le k_1 \\ S_{L,N-L} \sum_{l=L+1-k}^{N-k} p_{i,k+l-L}^{*} g_l & k_1 \le k \le k_2 \\ \sum_{l=0}^{N-k} p_{i,k+l-L}^{*} g_l & k_2 + 1 \le k \le N \end{cases} \quad \text{(D.2)}$$

$$y_{i,k} = \begin{cases} \sum_{l=L+1-k}^{L} p_{i,N-2L+k+l}^{*} g_{L-l} & 1 \le k \le k_1 \\ S_{L,N-L} \sum_{l=L+1-k}^{N-k} p_{i,N-2L+k+l}^{*} g_{L-l} & k_1 + 1 \le k \le k_2 \\ \sum_{l=0}^{N-k} p_{i,N-2L+k+l}^{*} g_{L-l} & k_2 + 1 \le k \le N \end{cases} \quad \text{(D.3)}$$

$$k_1 = \min (L, N - L) \quad \text{(D.4)}$$

$$k_2 = \max (L, N - L) \quad \text{(D.5)}$$

and

$$S_{L,N-L} = \begin{cases} 0 & L > N - L \\ 1 & L < N - L. \end{cases} \quad \text{(D.6)}$$

Note that, if $L = N - L$ or $L = \frac{1}{2}N$, there is no such k that satisfies $k_1 + 1 \le k \le k_2$, so that the second terms in $x_{i,k}$ and $y_{i,k}$ do not exist. Then from (14), (15), and (33)–(35),

$$\mathrm{Var}\,(\Delta\hat{\omega}_i) = \frac{1}{\mathrm{SNR}_i} \frac{1}{2} \sum_{k=1}^{N} \left[\mathrm{Im}^2 \left(\frac{x_{i,k} + y_{i,k}}{D_i} \right) + \mathrm{Re}^2 \left[\frac{x_{i,k} - y_{i,k}}{D_i} \right] \right] \quad \text{(D.7)}$$

$$\mathrm{Var}\,(\Delta\hat{z}_i) = \frac{1}{\mathrm{SNR}_i \, |D_i|^2} \sum_{k=1}^{N} \left[|x_{i,k}|^2 + |y_{i,k}|^2 \right] \quad \text{(D.8)}$$

where

$$\mathrm{SNR}_i = |a_i|^2 / 2\sigma^2 \quad \text{(D.9)}$$

$$D_i = \sum_{l=1}^{L} l g_l z_i^{-l}. \quad \text{(D.10)}$$

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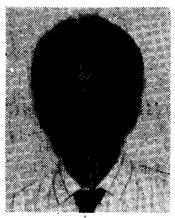

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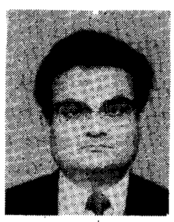

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